Basic Signal and Image Processing Knowledge (4)

基礎信號與影像處理知識(四)

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Image and Signal Processing 基礎知識 (4)

(1) 研究進程
(2) Orthogonality
(3) Norm
(4) Moment

(1) 研究進程

多思考

多虛心學習

做好時間規畫

(2) Orthogonality

Suppose that there is a vector set $\{\phi_1[n], \phi_2[n], \ldots., \phi_N[n]\}$.

If

$\langle \phi_m[n], \phi_n[n] \rangle = 0$ when $m \neq n$ $\Rightarrow$ orthogonal

If

$\langle \phi_m[n], \phi_n[n] \rangle = 0$ when $m \neq n$ $\Rightarrow$ orthonormal

$\langle \phi_n[n], \phi_n[n] \rangle = 1$

$$\langle x[n], y[n] \rangle = \sum_{n=n_1}^{n_2} x[n]y[n]$$

Gram-Schmidt Method

Suppose that $\{f_1[n], f_2[n], \ldots., f_N[n]\}$ is a non-orthogonal vector set.

We try to convert them into an orthonormal vector set $\{\phi_1[n], \phi_2[n], \ldots., \phi_N[n]\}$.

$$\phi_1[n] = \frac{f_1[n]}{\sqrt{\langle f_1[n], f_1[n] \rangle}}$$

for $a = 2$ to N

$$g_a[n] = f_a[n] - \sum_{m=1}^{a-1} \langle f_a[n], \phi_m[n] \rangle \phi_m[n]$$

$$\phi_a[n] = \frac{g_a[n]}{\sqrt{\langle g_a[n], g_a[n] \rangle}}$$

end

Q: Express a signal in terms of 1, n, n^2, n^3

$$x[n] \cong c_0 + c_1 n + c_2 n^2 + c_3 n^3$$

to minimize the approximation error

$$error = \left\| x[n] - (c_0 + c_1 n + c_2 n^2 + c_3 n^3) \right\|_2^2$$

$$= \sum_{n=n_1}^{n_2} \left(x[n] - (c_0 + c_1 n + c_2 n^2 + c_3 n^3) \right)^2$$

Method: Converting 1, n, n^2, n^3 into orthonormal vectors

$\{\phi_1[n], \phi_2[n], \phi_3[n], \phi_4[n]\}$

by the Gram-Schmidt method

$$x_1[n] = \sum_{a=1}^{4} d_a \phi_a[n]$$

$$d_a = \langle x[n], \phi_a[n] \rangle$$

Then $x_1[n]$ is a linear combination of 1, n, n^2, n^3 and it can approximate $x[n]$ with the minimal error.

$$\left\| x[n] - x_1[n] \right\|_2^2 \le \left\| x[n] - (c_0 + c_1 n + c_2 n^2 + c_3 n^3) \right\|_2^2$$

(3) Norm

Norm (L_α norm): $$\|x\|_\alpha = \sqrt[\alpha]{\sum_{n=0}^{N-1} |x[n]|^\alpha}$$

$\lim_{\alpha \to 0} (L_\alpha \text{ norm})^\alpha = K$ where K is the number of points such that $x[n] \neq 0$

(Physical meaning: The number of nonzero points)

L_1 norm: $$\|x[n]\|_1 = \sum_{n=0}^{N-1} |x[n]|$$

(Physical meaning: Sum of Amplitudes)

L_2 norm: $$\|x[n]\|_2 = \sqrt{\sum_{n=0}^{N-1} |x[n]|^2}$$

(Physical meaning: Distance)

Norm (L_α norm): $\|x[n]\|_\alpha = \sqrt[\alpha]{\sum_{n=0}^{N-1} |x[n]|^\alpha}$ when $\alpha \to \infty$

L_∞ norm: $\|x[n]\|_\infty = \text{Max}\{|x[n]|\}$ $\sum_{n=0}^{N-1} |x[n]|^\alpha \cong Max\{|x[n]|\}$

(Physical meaning: The maximal amplitude)

The L_2 norm is easier for optimization, but it often happens that using the L_0 or the L_1 norm can obtain even better image processing results.

(4) Moment

Raw Moment: $$m_a = \sum_{n=0}^{N-1} n^a x[n]$$

$$m_{a,b} = \sum_{n_x=0}^{N-1} \sum_{n_y=0}^{N-1} n_x^a n_y^b x[n_x, n_y]$$

Central Moment:

$$\hat{m}_a = \sum_{n=0}^{N-1} (n - \bar{n})^a x[n]$$ where $$\bar{n} = \sum_{n=0}^{N-1} n x[n] / \sum_{n=0}^{N-1} x[n]$$

$$\hat{m}_{a,b} = \sum_{n_x=0}^{N-1} \sum_{n_y=0}^{N-1} (n_x - \bar{n}_x)^a \left(n_y - \bar{n}_y\right)^b x[n_x, n_y]$$

where $$\bar{n}_x = \frac{\sum_{n_x=0}^{N-1} \sum_{n_y=0}^{N-1} n_x x[n_x, n_y]}{\sum_{n_x=0}^{N-1} \sum_{n_y=0}^{N-1} x[n_x, n_y]}$$ $$\bar{n}_y = \frac{\sum_{n_x=0}^{N-1} \sum_{n_y=0}^{N-1} n_y x[n_x, n_y]}{\sum_{n_x=0}^{N-1} \sum_{n_y=0}^{N-1} x[n_x, n_y]}$$

Physical Meanings of Moments:

m_0 :	sum
m_1 :	mean
$\frac{1}{N}\hat{m}_2$:	variance
$\frac{\hat{m}_3}{\hat{m}_2^{3/2}}$:	skewness
$\frac{\hat{m}_4}{\hat{m}_2^{2}}$:	kurtosis

練習程式

(1) Use the Gram-Schmidt method to convert $1, n, n^2, n^3, n^4$ into an orthonormal vector set where $n \in [0, 10]$.

(2) (a) Construct an elliptic image $\dfrac{(n_x - x_0)^2}{r_x^2} + \dfrac{(n_y - y_0)^2}{r_y^2} \le 1$

(b) Determine the $\lim_{\alpha \to 0} L_\alpha^\alpha$, L_1, L_2, and L_∞ norms of the image

(c) Determine the central moments $\hat{m}_{2,0}$, $\hat{m}_{1,1}$, $\hat{m}_{0,2}$ of the image

(3) Write a Matlab program of Harris's corner detection.