Draft

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Theorem 1. *There is an algorithm $A(\epsilon)$ which given an n-vertex weighted graph G computes a partition $\mathcal{S}_\epsilon$ such that*

$$w(\mathcal{S}_\epsilon) \leq w(\mathcal{S}^*) + \epsilon\mu(V). \tag{1}$$

Here $\mathcal{S}^*$ is the minimum normalized cut partition, and G can be embedded in a (D, β)-quasi-random graph[1].

Algorithm $A(\epsilon)$ runs in time polynomial...

Proof. We go along with the way in [2], let $V_1, V_2, \ldots V_k$ be the ϵ-regular partition for G, denote $d_{ij} = \frac{\rho(V_i,V_j)}{\mu(V_i)\mu(V_j)}$, for all partition $\mathcal{S} = \{S_1, S_2, \ldots, S_l\}$, let $S_{i,r} = V_i \cap S_r, T_{i,r} = V_i \backslash S_{i,r}$, then

$$\mathrm{NCut}(\mathcal{S}) = \sum_{r=1}^{l} \frac{\rho(S_r, \bar{S}_r)}{\mu(S_r)}. \tag{2}$$

Since we have

$$\left| \frac{\rho(S_{ir}, T_{jr})}{\mu(S_{ir})\mu(T_{jr})} - \frac{\rho(V_i, V_j)}{\mu(V_i)\mu(V_j)} \right| \leq \epsilon, \tag{3}$$

for all i, j. Hence we have,

$$\begin{aligned}
\sum_{r=1}^{l} \frac{\rho(S_r, \bar{S}_r)}{\mu(S_r)} &\leq \sum_{r=1}^{l} \frac{\sum_{ij=1}^{k}(\epsilon + d_{ij})\mu(S_{ir})\mu(T_{jr})}{\mu(S_r)} \\
&= \sum_{r=1}^{l} \frac{\epsilon}{\mu(S_r)} \sum_{ij=1}^{k} \mu(T_{jr})\mu(S_{ir}) + \sum_{r=1}^{l}\sum_{ij=1}^{k} \frac{d_{ij}\mu(S_{ir})\mu(T_{jr})}{\mu(S_r)} \\
&= \sum_{r=1}^{l} \frac{\epsilon}{\mu(S_r)} (l-1)\left(\mu(S_r)\right)^2 + \sum_{r=1}^{l}\sum_{ij=1}^{k} \frac{d_{ij}\mu(S_{ir})\mu(T_{jr})}{\mu(S_r)} \\
&= \epsilon(l-1)\mu(V) + \sum_{r=1}^{l}\sum_{ij=1}^{k} \frac{d_{ij}\mu(S_{ir})\mu(T_{jr})}{\mu(S_r)}.
\end{aligned} \tag{4}$$

Suppose $V_r := \{v_{1r}, v_{2r}, \ldots\}$ and $\bar{\mu}_{tr} = \sum_{k=1}^{t} v_{kr}$, we define

$$\eta_{sr} := \inf\{\bar{\mu}_{tr} : \bar{\mu}_{tr} - \bar{\mu}_{s-1,r} \geq \epsilon\mu(V_r)\}, \tag{5}$$

and notice that for all r, there are at most $1/\epsilon$ such η_{sr}. For every S_{ir}, we choose the nearest η_{ir} such that $|\mu(S_{ir}) - \eta_{ir}| < \epsilon\mu(V_r)$, denote $\tilde{\eta}_{jr} := \mu(V_r) - \eta_{jr}$, then we have

$$
\begin{aligned}
&\left|\sum_{r=1}^{l}\sum_{ij=1}^{k}\frac{d_{ij}\mu(S_{ir})\mu(T_{jr})}{\mu(S_r)} - \sum_{r=1}^{l}\sum_{ij=1}^{k}\frac{d_{ij}\eta_{ir}\tilde{\eta}_{jr}}{\mu(S_r)}\right| \\
\leq &\sum_{r=1}^{l}\sum_{ij=1}^{k}\frac{d_{ij}|\mu(S_{ir})\mu(T_{jr}) - \eta_{ir}\tilde{\eta}_{jr}|}{\mu(S_r)} \\
= &\sum_{r=1}^{l}\sum_{ij=1}^{k}\frac{d_{ij}}{\mu(S_r)}|\mu(S_{ir})\mu(T_{jr}) - \mu(S_{ir})\tilde{\eta}_{jr} + \mu(S_{ir})\tilde{\eta}_{jr} - \eta_{ir}\tilde{\eta}_{jr}| \\
\leq &\sum_{r=1}^{l}\sum_{ij=1}^{k}\frac{d_{ij}}{\mu(S_r)}(\mu(S_{ir})|\mu(T_{jr}) - \tilde{\eta}_{jr}| + \tilde{\eta}_{jr}|\mu(S_{ir}) - \eta_{ir}|) \\
\leq &\sum_{r=1}^{l}\sum_{ij=1}^{k}\frac{d_{ij}}{\mu(S_r)}(\mu(S_{ir})\epsilon\mu(V_r) + \tilde{\eta}_{jr}\epsilon\mu(V_r)) \\
= &\sum_{r=1}^{l}\frac{\epsilon\mu(V_r)}{\mu(S_r)}\sum_{ij=1}^{k}(\mu(S_{ir}) + \tilde{\eta}_{jr}) \\
= &\sum_{r=1}^{l}\frac{\epsilon\mu(V_r)}{\mu(S_r)}l\mu(S_r) \\
= &\epsilon l\mu(V).
\end{aligned}
$$

Thus to find a cut which maximizes $\mathrm{NCut}(\mathcal{S})$ to within $3\epsilon\mu(V)$ we need only find one which maximizes

$$\sum_{r=1}^{l}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{d_{ij}\eta_{ir}\tilde{\eta}_{jr}}{\mu(S_r)}. \tag{6}$$

This is a simple matter as η_{ir} takes one of at most $1/\epsilon$ values. □

Lemma 2. *There is an algorithm $P(\epsilon)$ which given an ϵ-regular partition of graph G in time polynomial where G can be embedded in a (D, β)-quasi-random-graph.*

References

[1] B Ela and Andr As. Weighted regularity lemma with applications. pages 1–17.

[2] Alan Frieze and Ravi Kannan. The Regularity Lemma and Approximation schemes for dense. pages 12–20, 1996.