EXTRA EXERCISE 4: DETERMINANT CRITERIAS FOR COLINEARITY...

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Problem 1 Given three points $A = (1, 1, -1), B = (-1, 1, 1), C = (1, 1, 1)$ in $\mathbb{R}^3$. Find the plane through A, B, C and find the center of the circumscribed circle of ΔABC (i.e. the circle passing through A, B, C).

Problem 2(Colinear condition)

- Let $P_i = (x_i, y_i),\ i = 1, 2, 3$ be three points on $\mathbb{R}^2$, prove they lie on a line iff

$$\det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = 0$$

- Let $P_i = (x_i, y_i, z_i), i = 1, \cdots, 4$ be four points in $\mathbb{R}^3$. Prove that $P_1, \cdots, P_4$ lie on one plane if and only if

$$\det \begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix} = 0$$

Problem 3(Concyclic condition) Let $P_i = (x_i, y_i),\ i = 1, 2, 3, 4$ be four points on $\mathbb{R}^2$ such that any three of them are not colinear. Find conditions (in terms of determinant) on P_1, P_2, P_3, P_4 such that they lie on the same circle (i.e. concyclic).

$$\det \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 + y_1^2 \\ 1 & x_2 & y_2 & x_2^2 + y_2^2 \\ 1 & x_3 & y_3 & x_3^2 + y_3^2 \\ 1 & x_4 & y_4 & x_4^2 + y_4^2 \end{bmatrix} = 0$$

Explanation: The four points are concyclic if and if there exists $T = (s, t) \in \mathbb{R}^2$ such that

- $\overrightarrow{Q_1T} \perp \overrightarrow{P_1P_2}$
- $\overrightarrow{Q_2T} \perp \overrightarrow{P_2P_3}$
- $\overrightarrow{Q_3T} \perp \overrightarrow{P3P_4}$

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where Q_1, Q_2, Q_3 are the middle points of P_1P_2, P_2P_3, P_3P_4 respectively. Now, we could rewrite the perpendicular conditions in terms of dot product:

$$\overrightarrow{P_1P_2} \cdot \overrightarrow{OT} = \overrightarrow{P_1P_2} \cdot \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2})$$
$$\overrightarrow{P_2P_3} \cdot \overrightarrow{OT} = \overrightarrow{P_2P_3} \cdot \frac{1}{2}(\overrightarrow{OP_2} + \overrightarrow{OP_3})$$
$$\overrightarrow{P_3P_4} \cdot \overrightarrow{OT} = \overrightarrow{P_3P_4} \cdot \frac{1}{2}(\overrightarrow{OP_3} + \overrightarrow{OP_4})$$

i.e.

$$\begin{bmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_2 & y_3 - y_2 \\ x_4 - x_3 & y_4 - y_3 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x_2^2 + y_2^2 - x_1^2 - y_1^2 \\ x_3^2 + y_3^2 - x_2^2 - y_2^2 \\ x_4^2 + y_4^2 - x_3^2 - y_3^2 \end{bmatrix}$$

Since the rank of the left matrix is 2 (this is because any three points are not colinear), the system has solution if and only if

$$rank \begin{bmatrix} x_2 - x_1 & y_2 - y_1 & \frac{1}{2}(x_2^2 + y_2^2 - x_1^2 - y_1^2) \\ x_3 - x_2 & y_3 - y_2 & \frac{1}{2}(x_3^2 + y_3^2 - x_2^2 - y_2^2) \\ x_4 - x_3 & y_4 - y_3 & \frac{1}{2}(x_4^2 + y_4^2 - x_3^2 - y_3^2) \end{bmatrix} = 2$$

i.e. the determinant of this matrix is 0. On the other hand, it is not difficult to observe that the determinant is equal to

$$\frac{1}{2} \det \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 + y_1^2 \\ 1 & x_2 & y_2 & x_2^2 + y_2^2 \\ 1 & x_3 & y_3 & x_3^2 + y_3^2 \\ 1 & x_4 & y_4 & x_4^2 + y_4^2 \end{bmatrix}$$

Further problem (Cospherical condition) Find a condition on $P_i = (x_i, y_i, z_i) \in \mathbb{R}^3, \ \ i = 1, 2, 3, 4, 5$ so that they lie on one common sphere in $\mathbb{R}^3$.