Matrix Multiplication, Inverse, Elimination

1. Matrix multiplication in "dot product"
   The ith component of Ax is $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n$
   In more compact notation: $a_i = \sum_{j=1}^{n} a_{ij}x_j$

2. Matrix multiplication in "column row"
   AB = Columns $1, ..., n$ of A multiply rows $1, ..., n$ of B, and add up.
   Example 1: $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$
   Example 2: $\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$

3. Matrix Operation Laws:
   (1) $AB \neq BA$ in general, with some exceptions. One such is that cI commute with all other matrices. Can you think of another exception?
   Exercise: (Strang 2.4-6)
   Show that $(A+B)^2$ is different from $A^2 + 2AB + B^2$, when $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$.
   Write down the correct rule for $(A+B)(A+B) = A^2 + \_\_\_\_\_ + B^2$.

   Exercise: (Strang 2.3-31)
   Find E_{21}, E_{32}, E_{43} to change K into U: $E_{43}E_{32}E_{21} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

   (2) $(AB)C = A(BC)$
   Exercise (Strang 2.4-37):

   To prove that $(\boldsymbol{AB})\,\boldsymbol{C} = \boldsymbol{A}\,(\boldsymbol{BC})$, use the column vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ of B. First suppose that C has only one column $\boldsymbol{c}$ with entries $c_1, \ldots, c_n$:

   AB has columns $A\boldsymbol{b}_1, \ldots, A\boldsymbol{b}_n$ and then $(AB)\,\boldsymbol{c}$ equals $c_1 A\boldsymbol{b}_1 + \cdots + c_n A\boldsymbol{b}_n$.
   $B\boldsymbol{c}$ has one column $c_1\boldsymbol{b}_1 + \cdots + c_n\boldsymbol{b}_n$ and then $A\,(B\boldsymbol{c})$ equals $A\,(c_1\boldsymbol{b}_1 + \cdots + c_n\boldsymbol{b}_n)$.

   Linearity gives equality of those two sums. This proves $(AB)\boldsymbol{c} = A(B\boldsymbol{c})$. The same is true for all other _____ of C. Therefore $(AB)C = A(BC)$. Apply to inverses:

   If $BA = I$ and $AC = I$, prove that the left-inverse B equals the right-inverse C.

   (3) $A^3 = AAA$. Note that it is A times A then times A. Not taking the power of each individual element of A.
   (4) Block multiplication
   To multiply A and B, make sure the cuts between columns of A match cuts in the rows of B.
   Example: block elimination

$E = \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}$

$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$

Here is a concrete numerical example: $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & x \\ 3 & x & x \\ 4 & x & x \end{bmatrix} = \begin{bmatrix} 1 & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix}$

4. Inverse:
   (1) some tricks on inverses:

   (a) 2 by 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

   (b) diagonal matrices If $A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}$

   (c) use meaning
   $E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
   Caveat: extra if you use meaning:
   Exercise
   Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

   (2)$(AB)^{-1} = B^{-1}A^{-1}$: Think about it in terms of matrices as actors.
   Exercise (Example 2-3 for "Inverse of AB" in Section 2.5 of Strang)
   $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$
   Calculate F^{-1}, FE, $E^{-1}F^{-1}$, and $F^{-1}E^{-1}$

5. Elimination:
   Exercise (Strang 2.2-2.2B): Suppose A is already a triangular matrix (upper triangular or lower triangular). Where do you see its pivots?

More Exercises:

1.(Strang 2.3-12) Multiply these matrices:

(1) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$

2.(Strang 2.2-30) If the last corner entry is $A(5,5) = 11$ and the last pivot of A is $U(5,5) = 4$, what different entry $A(5,5)$ would have made A singular?