Chapter 4:
Four Subspaces, Projections, Least Squres, QR Decomposition

1. Basis
   Definition: Linearly independent vectors that span the space.
   Notes: The bases are not unique.
   Example:
   (1) The pivot columns of A are a basis for its column space.
   (2) What about the basis for row space of A?
   (3) Fourier series

2. Orthogonality of subspaces
   Definition: Two subspaces V and W of a vector space are orthogonal if every vector v in V is perpendicular to every vector w in W.
   Notes: If V and W are to be orthogonal, $\dim(V)+\dim(W)\leq$ dimension of whole space; if V and W are orthogonal complements, the equality holds. (V's orthogonal complement is denoted by $V^{\perp}$)

   Exercise(Strang 4.1-28c) True or False: Two subspaces that meet only in the zero vector are orthogonal.

   Exercise(Strang 4.1-14, 15)

   (a) Find a vector in the column space of both matrices:
   A= $\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$, B= $\begin{bmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{bmatrix}$.
   Hint: This will be a vector $A\boldsymbol{x}$ and also $B\hat{\boldsymbol{x}}$. Think 3 by 4 with the matrix $\begin{bmatrix} A & B \end{bmatrix}$.

   (b) Extend the column space that A spans to a p-dimensional subspace V and a q-dimensional subspace W of $\mathbf{R}^n$. What inequality on p+q guarantees that V intersects W in a nonzero vector? (These subspaces cannot be orthogonal.)

3. The Big Picture of 4 subspaces

   Exercise(Strang 4.1-14, 28b, modified)
   The subspace spanned by (1,1,0,0,0) and (0,0,0,1,1) is the orthogonal complement of the subspace spanned by (1,-1,0,0,0) and (2,-2,3,4,-4).

   (a) Is the statement true or false?
   (b) If false, find the correct basis for the orthogonal complement.

Exercise(Strang 4.1-5) If $Ax = b$ has a solution and $A^T = 0$, is $y^T x = 0$ or $y^T b = 0$?

4. Projection onto a subspace: $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$
   Problem: project $\boldsymbol{b}$ onto the column space of A (subspace spanned by the independent columns of A).
   The projection matrix that does this action of "projection" is $P = A(A^TA)^{-1}A^T$;
   The projection is $\boldsymbol{p} = P\boldsymbol{b} = A\hat{\boldsymbol{x}}$
   Notes:
   When A is an orthogonal matrix Q, $\boldsymbol{p} = Q\hat{\boldsymbol{x}}, \hat{\boldsymbol{x}} = Q^T\boldsymbol{b}, \boldsymbol{p} = QQ^T\boldsymbol{b} = \boldsymbol{q_1}(\boldsymbol{q_1^T b}) + ... + \boldsymbol{q_n}(\boldsymbol{q_n^T b})$
   (When Q is square, then $QQ^T = I$, one can further simplify $\boldsymbol{p} = \boldsymbol{b}$)

5. The Gram-Schmidt Process and QR decomposition

   (a) Key point for simplifying calculation using orthonormal matrix: $Q^TQ = I$.
   When Q is square, its transpose is its inverse (orthogonal matrix): $Q^T = Q^{-1}$
   Notes: One more good property that might come in handy: $\|Qx\| = \|x\|$

   Exercises (Strang 4.4-3)
   (a) If A has three orthogonal columns each of length 4, what is A^TA?
   (b) If A has three orthogonal columns each of length 1, 2, 3, what is A^TA?

   (b) The Gram-Schmidt Process
   Notes: During calculation, check if the new vector is perpendicular to the ones found before.

   (c) QR decomposition: $A = QR$
   $$\begin{bmatrix}\boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c}\end{bmatrix} = \begin{bmatrix}\boldsymbol{q_1} & \boldsymbol{q_2} & \boldsymbol{q_3}\end{bmatrix}\begin{bmatrix}\boldsymbol{q_1^T a} & \boldsymbol{q_1^T b} & \boldsymbol{q_1^T c} \\ & \boldsymbol{q_2^T b} & \boldsymbol{q_2^T c} \\ & & \boldsymbol{q_3^T c}\end{bmatrix}$$
   Least square equation can be simplified to $R\hat{\boldsymbol{x}} = Q^T\boldsymbol{b}$ or $\hat{\boldsymbol{x}} = R^{-1}Q^T\boldsymbol{b}$

   Exercises (Strang 4.4-11):
   Which point is closest to (1,0,0,0,0) on the plane spanned by two vectors $\boldsymbol{a} = (1, 3, 4, 5, 7)$ and $\boldsymbol{b} = (-6, 6, 8, 0, 8)$

   Exercises (Strang 4.4-36):
   If A is m by n with rank n, MATLAB's code `qr(A)` produces (m by m)(m by n), $A = \begin{bmatrix}Q_1 & Q_2\end{bmatrix}\begin{bmatrix}R \\ 0\end{bmatrix}$.
   The n columns of Q_1 are an orthonormal basis for which fundamental subspace? The m-n columns of Q_2 are an orthonormal basis for which fundamental subspace?