Exercise 1: (a) Yes. $Q\Lambda Q^T = A$, $A^T = Q\Lambda Q^T = A$. ✓

(b) $A = Q\Lambda Q^T$

$A^{-1} = (Q\Lambda Q^T)^{-1} = Q\Lambda^{-1}Q^T$ ✓

Exercise 2: This is a diagonal matrix. λ's are on its diagonal.

Find λ's and x's

$\lambda_1 = 1 \Rightarrow (A-\lambda I)x_1 = 0$ $\qquad \begin{bmatrix} 0 & 10^{-15} \\ 0 & 10^{-15} \end{bmatrix} x_1 = 0$ $\qquad$ • choose $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 1+10^{-15} \Rightarrow (A-\lambda_2 I)x_2 = 0$ $\qquad \begin{bmatrix} -10^{-15} & 10^{-15} \\ 0 & 0 \end{bmatrix} x_2 = 0$ $\qquad$ • choose $x_2 = \begin{bmatrix} 1 \\ +1 \end{bmatrix}$

x_1 & x_2 are 45° apart. Not 90° apart.

Exercise 3: Change orthogonal to orthonormal $\Rightarrow Q^TQ = I$

Show that $||\lambda|| = 1$ for eigenvalues of every orthonormal matrix

$||Qy|| = (Qy)^T(Qy) = y^T(Q^TQ)y = y^Ty = ||y||$ for all y.

Then for eigenvectors we must have $||Qx|| = ||x||$?

Also, $Qx = \lambda x \Rightarrow ||\lambda x|| = |x|$ $\quad |\lambda| = 1$

Take away: Symmetric and orthonormal $\Rightarrow |\lambda| = 1$.

Exercise 4: ① Observe that B is orthonormal.

$\Rightarrow$ All λ's have $|\lambda| = 1$

② Also trace $= 0$

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$.

The only way is 2λ's $= 1$ and the other 2λ's $= -1$.

$\Rightarrow B - I = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$ $\qquad \Rightarrow B+I \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

rank $= 2$

$\dim(N) = 2$.

$x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\qquad x_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ $x_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$.

Ask M: Positive eigen = positive pivot in general?

(a) ⟺ (b)

$D_1 > 0$.

$P_2 = \frac{D_2}{D_1} > 0$.

$\Rightarrow D_2 > 0$.

⋮

(d) $A = LDL^T$.

$\underbrace{[x_1\ x_2 \dots x_n]}_{Y_n} L D L^T \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}}$ $\qquad Y = L^TX$

$[Y_1\ Y_2 \dots Y_n] D \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = d_1 Y_1^2 + d_2 Y_2^2 + \dots d_n Y_n^2$

(d) ⟺ (a)

⟺ (c)