Symmetric Matrices, Similar Matrices

1. **Two good facts about symmetric matrices**
   (1) Eigenvalues are real. (2) Eigenvectors are orthonormal. Put formally, this is the **Spectral Theorem**: every symmetric matrix can be written as $A = Q\Lambda Q^{-1} = Q\Lambda Q^T$. For proof, you may skip the ones on Strang and refer to Matthew's handout.

   For a 2 by 2 matrix A, the above equation can be written out as: (now you see clearly the sum of rank 1 **projection** matrices)

   $$A = \begin{bmatrix} \boldsymbol{x_1} & \boldsymbol{x_2} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x_1^T} \\ \boldsymbol{x_2^T} \end{bmatrix} = \lambda_1 \boldsymbol{x_1} \boldsymbol{x_1^T} + \lambda_2 \boldsymbol{x_2} \boldsymbol{x_2^T} \tag{1}$$

   Exercise 1(Strang 6.4-23) True or False: (a) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric. (b) The inverse of a symmetric matrix is symmetric.

   Exercise 2(Strang 6.4-28) This A is nearly symmetric. But its eigenvectors are far from orthogonal: find A's eigenvectors, $A = \begin{bmatrix} 1 & 10^{-15} \\ 0 & 1+10^{-15} \end{bmatrix}$.

2. **One more good fact about symmetric and orthogonal matrices**
   Exercise 3 Show that $\|\lambda\| = 1$ **for the eigenvalues of every orthogonal matrix**.

   Exercise 4(orthogonal and symmetric matrices) If $A = I$ (2 by 2) and $B = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$, find all four eigenvalues and eigenvectors of B.

3. **Positive definite matrices**
   Five equivalent ways to define "Positive definite" for a **symmetric matrix**:

   (a) All n pivots are positive
   (b) All n upper left determinants are positive
   (c) All n eigenvalues are positive
   (d) $\boldsymbol{x}^T A \boldsymbol{x}$ is positive except at $\boldsymbol{x} = 0$.
   To understand the term $\boldsymbol{x}^T A \boldsymbol{x}$, let's write $A = Q\Lambda Q^T$ then,

   $$\boldsymbol{x}^T A \boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & ... & x_n \end{bmatrix} Q\Lambda Q^T \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = \begin{bmatrix} X_1 & X_2 & ... & X_n \end{bmatrix} \Lambda \begin{bmatrix} X_1 \\ X_2 \\ ... \\ X_n \end{bmatrix} = \lambda_1 X_1^2 + \lambda_2 X_2^2 + ... \lambda_n X_n^2 \tag{2}$$

   Now you see the connection to eigenvalues. Use $A = LDL^T$ to see the connection to pivots (See Strang-6.5A)
   Exercise 5 (Strang 6.5-9) Find the 3 by 3 symmetric matrix A and its pivots, rank, eigenvalues and determinant.

   $$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2 \text{ (Hint: use } A = LDL^T\text{)}$$

(e) A equals $R^T R$ for a matrix R with independent columns.

Exercise 6 Can you explain why (e) holds?

Exercise 7 Based on $A = LDL^T = Q\Lambda Q^T$, and you want to find R such that $R^T R = A$, you can let R= _____ or _____. If you want R to be symmetric you can also let R=_____.

Exercise 8 (Strang 6.5-12)For what numbers c and d are A positive definite? $A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}$.

4. **Similar matrices**
Please note that the notation in my handout is different from the one in 6.1 in Strang 5E, but equivalent.
Similar matrices represent the same transformation of n-dimensional space under bases M other than I. In the following A and B are similar.

$$B = M^{-1}AM, \ M \text{ is any invertible matrix.} \tag{3}$$

Two points:

(a) Similar matrices have the same eigenvalues.

(b) If $\boldsymbol{x}$ is an eigenvector of A, then $M^{-1}\boldsymbol{x}$ is an eigenvector of B.

Exercise 9 (Strang 4E 6.6-18) If B is invertible, prove that AB **is similar to** BA. Therefore, AB and BA have the same eigenvalues.

Exercise 10 (Strang 4E 6.6-20e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2, the eigenvalues stay the same. Why? Hint: In this case, M=_____.

More Exercise 1(Strang 6.5-17)
(Before you attempt, try Strang 6.5-16, where it is shown that a positive definite matrix cannot have a zero on its diagonal.)Show that a diagonal entry $a_i i$ of a symmetric matrix cannot be smaller than all the $\lambda's$. Hint: If it were, then $A - a_{jj}I$ would have _____ eigenvalues and would therefore be positive definite. On the other hand, $A - a_{jj}I$ has a _____ on its main diagonal.
The conclusion you draw from Strang 6.5-17 can be used to prove the Hadamard inequality, which states that for any positive definite matrix K, its determinant is less than the product of its diagonal elements, that is,

$$|K| \leq \prod_i K_{ii} \tag{4}$$

More Exercise 2(Strang 6.6-8, 4E)
Suppose $Ax = \lambda x$ and $Bx = \lambda x$ with the same $\lambda's$ and $x's$. With n independent eigenvectors we have $A = B$: why? Find A$\neq$ B when both have eigenvalues 0,0 but only one line of eigenvectors $(x_1, 0)$.
Practice by yourself (such as when you review for quiz) Strang 6.4-12, 14, 32; 6.5-14, 20, 33 (Hint: switch the order of elements in the dot product term.);