Dot Product, Matrix Multiplication, and the Inverse Matrix

1. (Some side notes) Dot product and correlation

   (a) The physics interpretation of the dot product.
   if there is one force $\boldsymbol{a} = (a_1, a_2)$ acting in the displacement vector $\boldsymbol{b} = (b_1, b_2)$ then $\boldsymbol{ab} = a_1 * b_1 + a_2 * b_2$ can be interpreted as work the force does: the sum of that on the x direction plus that on the y direction. This equals $\boldsymbol{ab} = ||a|| * ||b||cos(\theta)$, which can be interpreted by projecting $\boldsymbol{b}$ onto 2 directions: one perpendicular to $\boldsymbol{a}$, and the other in the same direction as $\boldsymbol{a}$. The work done perpendicular to $\boldsymbol{a}$ is 0, while the other is the actual work done.

   (b) With two random variables X and Y with zero mean, the correlation between the two is defined as: $corr(x, y) = \frac{E(XY)}{||X||||Y||} = cos(\theta)$

2. Matrix Multiplication:
   Viewpoint 1: Look in columns
   Viewpoint 2: Matrix acts on a vector. In the following example: A acts on $\boldsymbol{a}$
   Example:

$$A\boldsymbol{a} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} * x_1 + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} * x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} * x_3 = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix} = \boldsymbol{b} \quad (1)$$

3. (Toy example) Transforming in 2D:
   $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$;
   The third example is transforming a 2D point into a 1-D subspace: x-axis.

4. Inverse and invertible matrix:
   The word: inverse, invertible. Another case where the word 'inverse' is used: the inverse function, commonly shown as: $y^{-1}(x)$.

   (a) What is the "inverse of a matrix"
   Suppose I have $\boldsymbol{b}$, which is the right hand side of the equation (1), and the matrix A but not $\boldsymbol{a}$. Now I want to find $\boldsymbol{a}$. This question is the *inverse* of that we explored in section 2 above, where we go from the left of the equation to the right: now we want to go from the transformed $\boldsymbol{a}$, which is $\boldsymbol{b}$, back to $\boldsymbol{a}$. How do we do it? Wouldn't it be great to have something like below that can easily invert the transformation from $\boldsymbol{a}$ to $\boldsymbol{b}$ so that we can get $\boldsymbol{b}$ from $\boldsymbol{a}$ easily?

$$A^{-1}A\boldsymbol{a} = A^{-1}\boldsymbol{b} = \boldsymbol{a} \quad (2)$$

   This something, specifically, A^{-1}, is called the inverse of A.

   (b) the existence of the inverse of a matrix
   This topic will be explored later in the course formally, but here we talk about some intuition.
   Put simply: a transformation brought up by a matrix A is invertible if the transformation is from $\mathbb{R}^n \to \mathbb{R}^n$. Then A will have its inverse matrix A^{-1}, and A is invertible.

Otherwise, dimension reduction occurs, and information is lost forever by the transformation, so the inverse is not attainable.

This is the idea elaborated by p25-27 of the textbook (4th edition). Matrix C in equation 11 on page 25 is not invertible, ie, *singular*, as it essentially brings $\mathbb{R}^3 \to \mathbb{R}^2$. Wait...t, why does it bring $\mathbb{R}^3 \to \mathbb{R}^2$?

Figure 1.10 makes it clear. Building this intuition is all you need at this point of the course – it will come back later.
In this case, it is easy to spot if some column/row is a combination of the others. However, we need a systematic way to find that out. The *elimination* you are learning this week is one such method. The *pivot number tells you this.*

5. Solving Linear Equations
Now, let's extend the idea in section 4b. An n by n invertible matrix attains the full space in the same dimension of n. On the contrary, a matrix that is not invertible will transform vectors only to a *subspace* of the n^{th} dimensional space. This is essentially saying:

There is some parts in the space the transformed vector cannot go if the transformation matrix is singular.

   (a) With this perspective, can you explain Example 1 in Section 2.1 to me in column picture why there is no solution to the equations?
   (b) How about Example 2?