Institute of Mathematical Sciences
ShanghaiTech University
Matthew Randall

Spectral Theorem

Theorem. Suppose S is a symmetric matrix. Then S is orthogonally diagonalizable, i.e. we can find an orthogonal matrix Q such that $\Lambda = Q^T S Q$ is diagonal.

Proof. We will prove by induction on the size of S. When S is a 1×1 matrix, take $Q = [1]$. Assume now that any symmetric $n \times n$ matrix B is orthogonally diagonalizable. We show that this is true for any $(n+1) \times (n+1)$ matrix S as well.

Pick a real eigenvalue λ_1 of S and normalise the eigenvector $\boldsymbol{x_1}$ with the corresponding eigenvalue so that it is of length 1. Find an orthonormal basis $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_{n+1}}$ of $\mathbb{R}^{n+1}$ using the Gram-Schmidt process.

Let $P = \begin{bmatrix} \boldsymbol{x_1} & \boldsymbol{x_2} \ldots & \boldsymbol{x_{n+1}} \end{bmatrix}$. Computing $P^{-1}SP = P^T SP$, we find that the first column of $P^T SP$ is $\lambda_1 \boldsymbol{e_1}$, the vector with λ_1 in the first component and zero everywhere else. By the symmetric property of S, we have $P^{-1}SP = P^T SP = (P^T SP)^T$ is symmetric as well. Hence $P^{-1}SP$ has the block form $\begin{bmatrix} \lambda_1 & 0 \\ 0 & B \end{bmatrix}$ where B is a symmetric $n \times n$ matrix.

By our induction hypothesis, B is orthogonally diagonalizable and $B = QDQ^T = QDQ^{-1}$ for some orthogonal Q. Now take $R = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$. Then we have

$$\begin{aligned} R^{-1} \begin{bmatrix} \lambda_1 & 0 \\ 0 & B \end{bmatrix} R &= \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & Q^{-1}BQ \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & D \end{bmatrix}. \end{aligned}$$

We therefore find $R^{-1}P^{-1}SPR = \begin{bmatrix} \lambda_1 & 0 \\ 0 & D \end{bmatrix}$ is diagonal. Take $\tilde{Q} = PR$. The product of orthogonal matrices is again orthogonal. Then $\tilde{Q}^{-1} = (PR)^{-1} = R^{-1}P^{-1}$. Thus $\tilde{Q}^{-1}S\tilde{Q} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & D \end{bmatrix}$ is also diagonal and S is orthogonally diagonalizable. $\square$