from the German adjective *eigen* ‘own.’

Eigenvector and Eigenvalue
Eigen-decomposition
with change of basis

Shijie Gu

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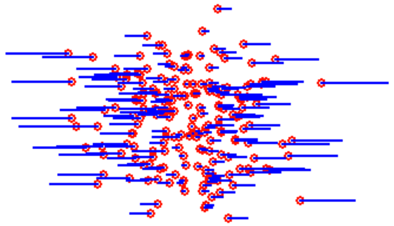
Some slides modified from Prof. Michale Fee’s slides for 9.40 - Intro to Neural Computation, MIT

Preview

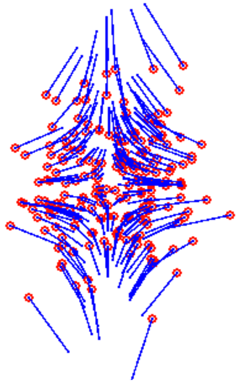
- To facilitate this section, I asked students to watch two youtube videos with animations:
- **Change of Basis:** https://www.youtube.com/watch?v=P2LTAUO1TdA&t=0s&index=13&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab
- **Eigenvectors and Eigenvalues:** https://www.youtube.com/watch?v=PFDu9oVAE-g&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=13

Intro to Eigenvectors and eigenvalues

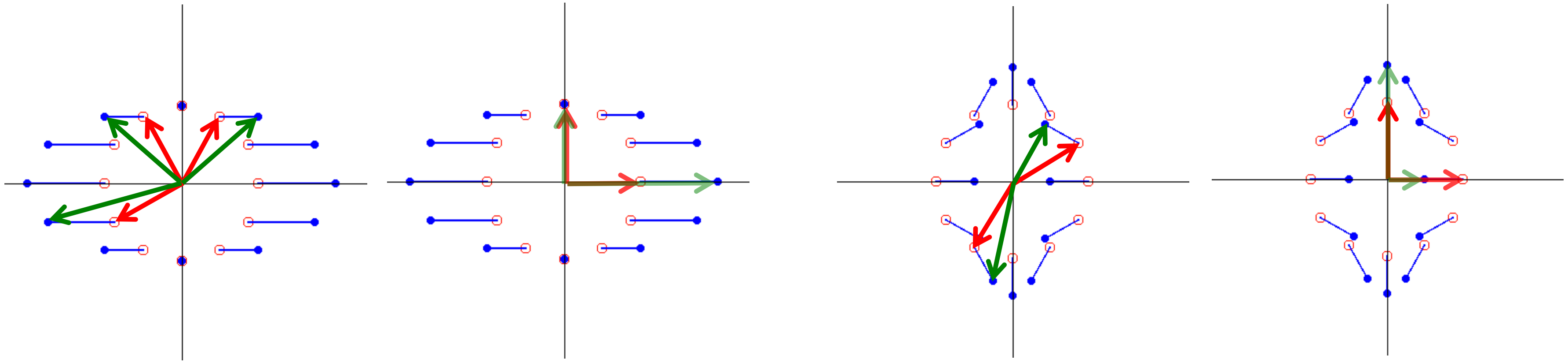
$$A = \begin{pmatrix} 1+\delta & 0 \\ 0 & 1 \end{pmatrix}$$



$$A = \begin{pmatrix} 1-\delta & 0 \\ 0 & 1+\delta \end{pmatrix}$$



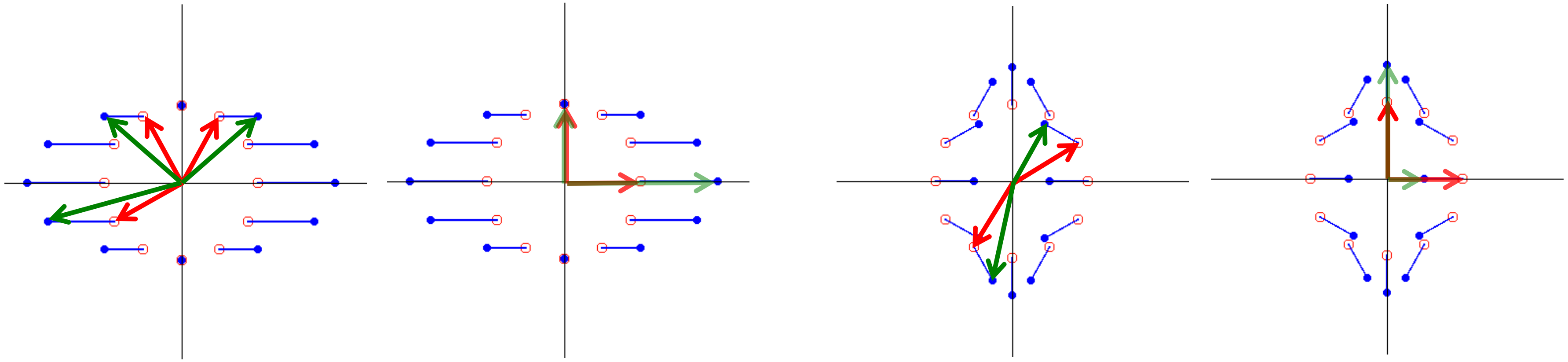
- Some vectors are rotated, some are not.
- Matrix transformations have special directions



These are all diagonal matrices

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

- Some vectors are rotated, some are not.



- For a diagonal matrix, vectors along the axes are scaled, but not rotated.

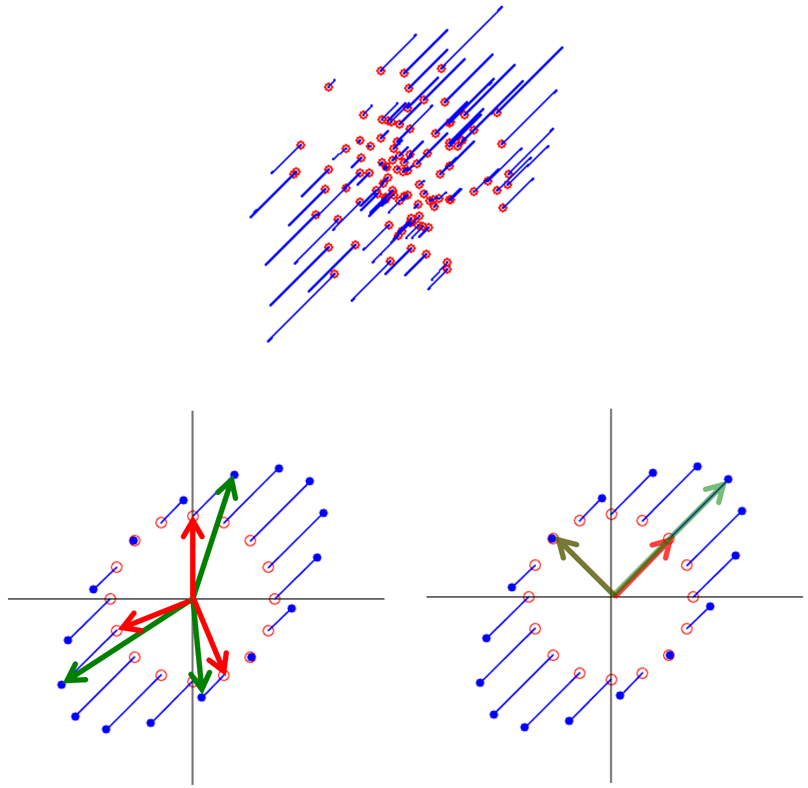
$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Lambda \hat{e}_1 = \lambda_1 \hat{e}_1$$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \Lambda \hat{e}_2 = \lambda_2 \hat{e}_2$$

Intro to Eigenvectors and eigenvalues

- Another example

$$\begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$$



Eigenvectors and eigenvalues

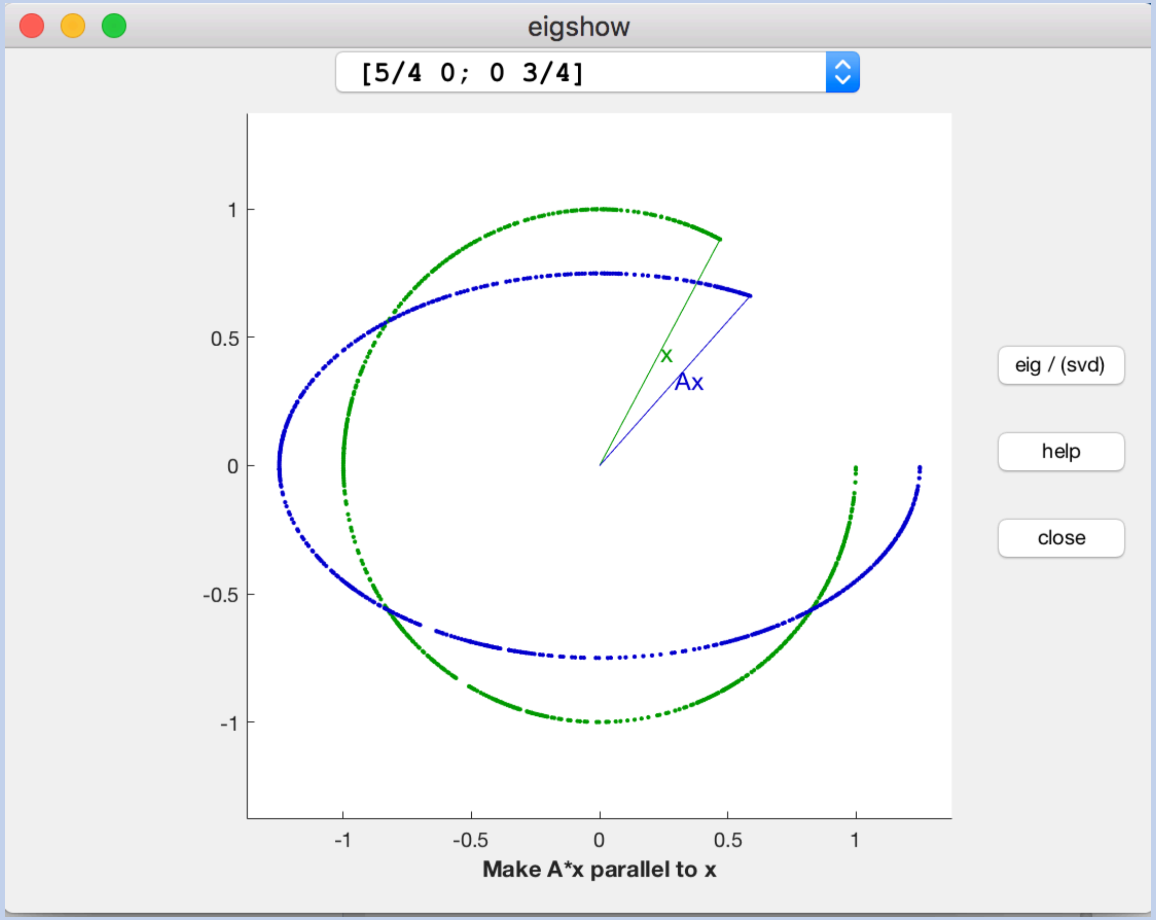
- Any vector $\vec{x}$ that is mapped by matrix A onto a parallel vector $\lambda\vec{x}$ is called an eigenvector of A. The scale factor λ is called the eigenvalue of vector $\vec{x}$.
- $A\vec{x} = \lambda\,\vec{x}$

Why do we need to find out eigenvectors and eigenvalues?

- To simplify calculation:
- $A\vec{x} = \lambda\,\vec{x}$: what is $A^{100}\vec{x}$?

Optional: Some visualization (no need to dig into it)

- You can take a look at https://blogs.mathworks.com/cleve/2013/07/08/eigshow-week-1/
- Code for eigshow() is emailed to you.



求解 Eigenvectors and eigenvalues

- What are the eigenvalues of a general 2-dim matrix A ?

$$A\vec{x} = \lambda\vec{x}$$

we only want solutions where

$$\vec{x} \neq 0$$

$$A\vec{x} = \lambda I\vec{x}$$

Which means $A - \lambda I$ has a non-empty nullspace, thus it is not invertible: $\det(A - \lambda I) = 0$

$$(A - \lambda I)\vec{x} = 0$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \qquad \det(A - \lambda I) = (a-\lambda)(d-\lambda) - bc$$

$$\det(A - \lambda I) = (a-\lambda)(d-\lambda) - bc = 0$$

$$ad - \lambda(a+d) + \lambda^2 - bc = 0$$

Characteristic equation of matrix A

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

$\lambda_1\lambda_2 = \det(\mathrm{A})$; $\lambda_1 + \lambda_2 = trace$

求解 Eigenvalues

- What are the eigenvalues of a general 2-dim matrix?

Characteristic equation of matrix A

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

$\lambda_1 \lambda_2 = \det A; \lambda_1 + \lambda_2 = trace$

- Solutions are given by the quadratic formula

$$\lambda_{\pm} = \frac{1}{2}(a+d) \pm \frac{1}{2}\sqrt{\underbrace{(a-d)^2 + 4bc}}$$

eigenvalues can be real, complex, imaginary

(one way to understand them will be explored in Chapter 6.3)

- For matrix not invertible: there must be one eigenvalue equal to 0.

(zero is an eigenvalue iif A is not invertible)

求解 Eigenvalues

$$\lambda_{\pm} = \frac{1}{2}(a+d) \pm \frac{1}{2}\sqrt{(a-d)^2 + 4bc}$$

- For a symmetric matrix

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

$$\lambda_{\pm} = \frac{1}{2}(a+d) \pm \frac{1}{2}\sqrt{\underbrace{(a-d)^2 + 4b^2}_{}} \quad \geq 0$$

The eigenvalues of a symmetric matrix are always real.

$$A = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\lambda_{\pm} = \frac{1}{2}\left(\frac{3}{2} + \frac{1}{2}\right) \pm \frac{1}{2}\sqrt{\left(\frac{3}{2} - \frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^2}$$

$$\lambda_{\pm} = 1 \pm \frac{\sqrt{2}}{2}$$

求解 Eigenvectors

- 算出eigenvalue后，分别带入原式求出对应的eigenvector

$$(A-\lambda I)\vec{x}=0$$

- Some tricks on problem solving:
  - When dealing with 2 by 2 matrices: both rows of $(A-\lambda I)$ are multiples of a vector (a,b). The eigenvector is any multiple of (b,a).
  - Use the product of eigenvalues = det; the sum of eigenvalues = trace to check your answers.

Eigenvectors

- Note that eigenvectors are not unique…

If $\vec{x}_i$ is an eigenvector of A then so is $a\vec{x}_i$

$$A\vec{x}_i = \lambda_i \vec{x}_i \qquad\qquad A\left(a\vec{x}_i\right) = \lambda_i \left(a\vec{x}_i\right)$$

… so we usually write eigenvectors as unit vectors

- Chapter 6.4: For a symmetric matrix, the eigenvectors of A are orthogonal (and we write them as unit vectors)…

… the eigenvectors of A form a complete orthonormal basis set.

- For eigenvectors of A: $A\vec{x} = \lambda\,\vec{x}$, then nicely we have $A^n\vec{x} = \lambda^n\,\vec{x}$
- What about other arbitrary vectors? Can we simplify?

If we can write $\overrightarrow{u_0}$ as: $\overrightarrow{u_0} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2}$,

then $\mathrm{A}^{\mathrm{n}}\,\overrightarrow{u_0} = \mathrm{A}^{\mathrm{n}}(c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2})$

$$= c_1\mathrm{A}^{\mathrm{n}}\overrightarrow{x_1} + c_2\mathrm{A}^{\mathrm{n}}\overrightarrow{x_2}$$

$$= c_1{\lambda_1}^{\mathrm{n}}\overrightarrow{x_1} + c_2{\lambda_2}^{\mathrm{n}}\overrightarrow{x_2}$$

- Thus, as long as we find c_1 and c_2 , we can simplify as:

$$\mathrm{A}^{\mathrm{n}}\,\overrightarrow{u_0} = c_1{\lambda_1}^{\mathrm{n}}\overrightarrow{x_1} + c_2{\lambda_2}^{\mathrm{n}}\overrightarrow{x_2}$$

- Prerequisite: there are enough independent eigenvectors.
  - Independent $\vec{x}$'s from different λ's (proved in your textbook, read it by yourself)
  - (digression) Question: (True or False) different λ's from Independent $\vec{x}$'s ?
- Finding c_1 and c_2 is essentially a process called 'change of basis'

Change of Basis

Decompose other non-eigenvectors into combinations of eigenvectors (Finding c_1 and c_2)

- Let's look at an example (6.2 Example 3):

  $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ has $\lambda_1 = 2$ and $\overrightarrow{x_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$; $\lambda_2 = -1$ and $\overrightarrow{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

  - Express $\overrightarrow{u_0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by $c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2}$ (find c1 and c2)

    You can do it easily by expressing the above equation as:

$$\overrightarrow{u_0} = [\overrightarrow{x_1}\overrightarrow{x_2}]\ \boldsymbol{c}$$

Then $\boldsymbol{c} = [\overrightarrow{x_1}\overrightarrow{x_2}]^{-1}\ \overrightarrow{u_0}$

$$\boldsymbol{c} = S^{-1}\ \overrightarrow{u_0}\quad,\ S = [\overrightarrow{x_1}\overrightarrow{x_2}]$$

Eigen-decomposition

- Now, find $A^n \overrightarrow{u_0}$
  Substitute it into the equation we derived previously: $A^n \overrightarrow{u_0} = c_1 \lambda_1{}^n \overrightarrow{x_1} + c_2 \lambda_2{}^n \overrightarrow{x_2}$

$$A^n \overrightarrow{u_0} = [\lambda_1{}^n \overrightarrow{x_1},\ \lambda_2{}^n \overrightarrow{x_2}]\ \boldsymbol{c}$$

$$A^n \overrightarrow{u_0} = [\lambda_1{}^n \overrightarrow{x_1},\ \lambda_2{}^n \overrightarrow{x_2}]\ S^{-1} \overrightarrow{u_0} \quad , S = [\overrightarrow{x_1} \overrightarrow{x_2}]$$

$$A^n \overrightarrow{u_0} = [\overrightarrow{x_1},\ \overrightarrow{x_2}] \begin{bmatrix} \lambda_1{}^n & \\ & \lambda_2{}^n \end{bmatrix} S^{-1} \overrightarrow{u_0}$$

$$A^n \overrightarrow{u_0} = S \begin{bmatrix} \lambda_1{}^n & \\ & \lambda_2{}^n \end{bmatrix} S^{-1} \overrightarrow{u_0}$$

Explain intuitively what it means.

Eigen-decomposition

$$A^n \overrightarrow{u_0} = S \begin{bmatrix} \lambda_1{}^n & \\ & \lambda_2{}^n \end{bmatrix} S^{-1} \overrightarrow{u_0}$$

Homework assistant: MATLAB

- Matlab has a function 'eig' to calculate eigenvectors and eigenvalues

```
>> A=[1.5 0.5;0.5 1.5]

A =

    1.5000    0.5000
    0.5000    1.5000
```

```
>> [F,V]=eig(A)

F =

   -0.7071    0.7071
    0.7071    0.7071


V =

     1     0
     0     2
```

$$A = F\,V\,F^T$$

```
>> F*V*F'

ans =

    1.5000    0.5000
    0.5000    1.5000
```

The Transpose here is because A is symmetric. For now, understand it as F^{-1}

Diagonalizing

Eigen-decomposition

$$A^n = S\begin{bmatrix}\lambda_1{}^n & \\ & \lambda_2{}^n\end{bmatrix}S^{-1}$$

Diagonalizing

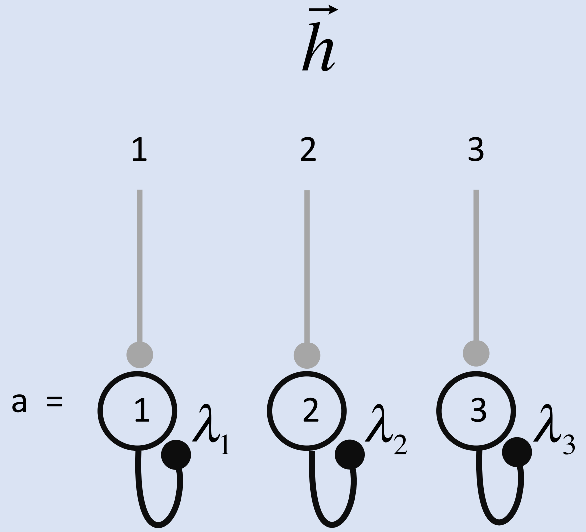
$$S^{-1}A^nS = \begin{bmatrix}\lambda_1{}^n & \\ & \lambda_2{}^n\end{bmatrix}$$

Application for diagonalization Recurrent networks

- M is a diagonal matrix

$$M = \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 & \\ & & & \ddots \end{pmatrix}$$

$$\tau_n \frac{d\vec{v}}{dt} = -\vec{v} + \Lambda \vec{v} + \vec{h}$$



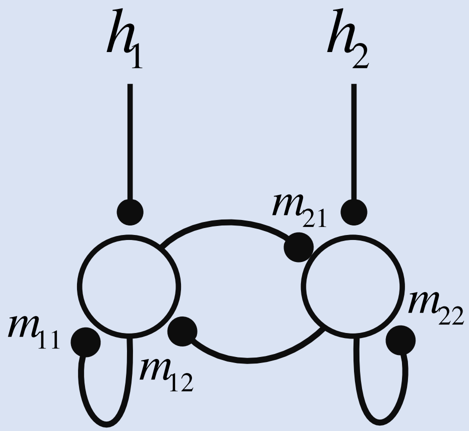
$$\tau_n \frac{dv_a}{dt} = -v_a + \lambda_a v_a + h_a$$

We have n independent equations – each neuron acts independently of all the others

- A simple system to solve

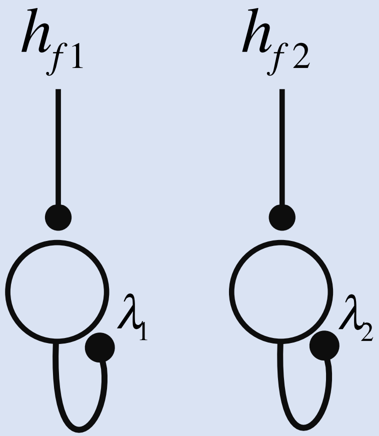
Recurrent networks

- What if it the system is not described by a diagonal matrix?
- We can let's make M diagonal!



$$M = \Phi \Lambda \Phi^T$$

Change basis
into eigenbasis



$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

Eigenvectors of AB

If x is an eigenvector of B $\quad ABx = A\beta x$

$ABx = \beta Ax \quad (1)$

This side of reasoning is flawed. To prove the inverse, some advanced linear algebra is needed, we will skip them.

If x is also an eigenvector of A

$ABx = \beta\lambda x$

Similarly, $BAx = \beta\lambda x$

Therefore, $AB = BA$

If $AB = BA$

$ABx = BAx \quad (2)$

$\beta Ax = BAx$ by (1) and (2)

All Ax's are also eigenvectors of B,
So eigenvectors of A's are eigenvectors of B

Suppose both A and B can be diagonized, they share the same eigenvector matrix S if and only if AB=BA

More exercises

- If A has eigenvalues 1 and 3, and eigenvectors x_1 and x_2 , what is the eigenvalues and eigenvectors of matrix A-1 ? (Strang 6.1-A)

- What are the eigenvalues of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$; $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ (Strang 6.1-29)

- Try it yourself (when you review for quiz):
  - Strang 6.1-21: **The eigenvalues of A equal the eigenvalues of A^T**
  - Strang 6.2-4, 31, 34