LINEAR ALGEBRA: QUIZ 2 (SOLUTION GUIDE)

SHANGHAITECH UNIVERSITY

1. (10 pts) Let

$$\boldsymbol{v}_0 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

be five vectors in $\mathbb{R}^4$. Let P_1, P_2 be two planes in $\mathbb{R}^4$ defined as

$$P_1 = \{\boldsymbol{v} \in \mathbb{R}^4 \mid \boldsymbol{v} = \boldsymbol{v}_0 + x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 \text{ for some } x_1, x_2 \in \mathbb{R}\};$$

$$P_2 = \{\boldsymbol{v} \in \mathbb{R}^4 \mid \boldsymbol{v} = x_3\boldsymbol{v}_3 + x_4\boldsymbol{v}_4 \text{ for some } x_3, x_4 \in \mathbb{R}\}.$$

a) Find the condition on a, b, c, d such that $P_1 \cap P_2$ is non-empty.
b) Suppose

$$\boldsymbol{v}_0 = \begin{bmatrix} -1 \\ -3 \\ -2 \\ -2 \end{bmatrix}.$$

What is $P_1 \cap P_2$?

Solution:

- (a) We want to find the condition on a, b, c, d such that the linear combinations

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

admits a solution. In matrix-vector form, we obtain the system

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -a \\ -b \\ -c \\ -d \end{bmatrix}. \quad \textbf{(3 points)}$$

Gaussian elimination on the augmented matrix gives

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -a \\ 0 & 1 & 0 & -1 & -d \\ 0 & 0 & 1 & 1 & b \\ 0 & 0 & 0 & 0 & a+b-c-d \end{array}\right].$$

This system of equations is consistent when $a+b-c-d=0$. When $a+b-c-d=0$, the system admits a solution so $P_1 \cap P_2 \neq \emptyset$. (**2 points**)

- (b) When $v_0 = \begin{bmatrix} -1 \\ -3 \\ -2 \\ -2 \end{bmatrix}$, we solve for the system of equations to get

$$x_1 = -1 - x_4,$$
$$x_2 = 2 + x_4,$$
$$x_3 = -3 - x_4,$$

with x_4 the free variable. We therefore find

$$P_1 \cap P_2 = \left\{ \begin{bmatrix} 0 \\ -3 \\ -3 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(**5 points**)

2. (10 pts) Let A be a 3×3 real matrix.

a) Is it always true that $rank(I - A) = 3 - rank(A)$? If yes, prove it; if not, give a counterexample.

b) If $A^2 = A$, prove that $rank(I - A) = 3 - rank(A)$. [Hint: You could use the following two facts: 1) $AB = 0$ implies $C(B) \subseteq N(A)$; 2) $C(A + B) \subseteq C(A) + C(B)$.]

Solution:

- (a) False. The matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has rank 1 and $I - A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has rank 3. (**5 points**)

- (b) We have $A(I - A) = 0$, so $C(I - A) \subseteq N(A)$. Let $rank(I - A) = p$ and $rank(A) = r$. Then $p \leq 3 - r$. From the hint, $3 \leq p + r$. Together this gives $p = 3 - r$. (**5 points**)

3. (10 pts) Let A be a 2×2 matrix.

   a) Define $V = \{X \in M_{2\times2}(\mathbb{R}) | AX = 0\}$. Show that V is a subspace of $M_{2\times2(\mathbb{R})}$.

   b) If $rank(A) = 1$, what is *dim V*? If $rank(A) = 2$, what is *dim V*? Explain your result.

Solution:

- (a) Let X, Y be matrices in V. For any real scalar c we have $A(cX) = c(AX) = 0$ since scalar multiplication commutes with matrix multiplication. Moreover, $A(X + Y) = AX + AY = 0$ by the distributive law for matrix multiplication. Since V is closed under scalar multiplication and vector addition, V is a subspace of $M_{2\times2}(\mathbb{R})$.   (**4 points**)

- (b) If $rank(A) = 1$, then $\dim(V) = 2$. We have $C(X) \subseteq N(A) = N(R)$. We may assume wlog $R = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}$, so that $N(R)$ is given by the span of the vector $\begin{bmatrix} -a \\ 1 \end{bmatrix}$. The columns of X are given by scalar multiples of this vector. Hence $\dim(V) = 2$.

   If $rank(A) = 2$, the matrix A is invertible and X is the zero matrix. Therefore $\dim(V) = 0$.   (**6 points**)

4. (20 pts) Find a basis for each of the four subspaces associated with A:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

What is the dimension of each of the four subspaces?

Solution:

- Reduced row echelon form of A gives $R = \begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (**4 points**)

- $C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\}$ and $\dim(C(A)) = 2$ (**4 points**)

- $C(A^T) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ and $\dim(C(A^T)) = 2$ (**4 points**)

- $N(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ and $\dim(N(A)) = 3$ (**4 points**)

- $N(A^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ and $\dim(N(A^T)) = 1$ (**4 points**)