Chapter 6.3 Application of Diagonalizing a Matrix
Solving Constant-Coefficient Differential Equation Systems

1. **Review: Change from standard basis into "eigenvector basis"**
   Recall last time when we were introducing diagonalizing a matrix: $A = S\Lambda S^{-1}$, we understood it from $A\boldsymbol{x} = S\Lambda S^{-1}\boldsymbol{x}$, where $S^{-1}\boldsymbol{x}$ first change your vectors from standard world coordinates into those of eigenvectors', then Λ applies the stretching or compressing in this eigen-coordinate system. Finally S transforms your vectors back into the normal standard basis. Essentially it is taking the advantage of the eigenvectors' directions, where a matrix multiplication is simply a scaling.

   This time, we follow the same spirit of changing our perspective on the world to simplify calculations: we apply to constant-coefficient differential equations.

2. **Some vector notations**
   Before we begin, let's clarify some notations using an example. For the following differential equation system on the left, we can use matrix notation shown on the right.

$$\frac{du_1}{dt} = -u_1 + 2u_2 \qquad \frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$$
$$\frac{du_2}{dt} = u_1 - 2u_2 \qquad \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

   More Exercise 1 (next page)
   We will also be using e^{At}, which can be seen from substituting At into e^x.

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$
$$e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \dots$$
$$\frac{de^{At}}{dt} = Ae^{At} = A + A^2t + \frac{1}{2}A^3t^2 + \dots$$

   Comments: (1) e^{At} is a matrix. (2) e^{At} always has the inverse e^{-At} (3) the eigenvalues of e^{At} are $e^{\lambda t}$. (4) e^Ae^B is different from e^Be^A, both are different from e^{A+B}. See Strang 6.3-23.
   Exercise 1(Strang 6.3-22) If $A^2 = A$, show that $e^{At} = I + (e^t - 1)A$

3. **Solving constant-coefficient differential equations**

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$$
$$S^{-1}\frac{d\boldsymbol{u}}{dt} = S^{-1}A\boldsymbol{u}$$
$$S^{-1}\frac{d\boldsymbol{u}}{dt} = S^{-1}S\Lambda S^{-1}\boldsymbol{u}$$
$$S^{-1}\frac{d\boldsymbol{u}}{dt} = \Lambda S^{-1}\boldsymbol{u}$$
$$\frac{d\boldsymbol{v}}{dt} = \Lambda\boldsymbol{v}, \qquad \boldsymbol{v} = S^{-1}\boldsymbol{u}$$
$$\begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} Ce^{\lambda_1 t} \\ De^{\lambda_2 t} \end{bmatrix}, \tag{1}$$

where C and D are determined by the initial condition

$$\boldsymbol{u} = S\boldsymbol{v} = S \begin{bmatrix} Ce^{\lambda_1 t} \\ De^{\lambda_2 t} \end{bmatrix} \tag{2}$$

$$\boldsymbol{u} = C\boldsymbol{s_1}e^{\lambda_1 t} + D\boldsymbol{s_2}e^{\lambda_2 t} \tag{3}$$

So, to solve for a system in our very first example, we just need to find the eigenvalue and eigenvectors of the coefficient matrix A and plug them into equation 3.

Exercise 2(Strang 6.3-7)
Suppose P is the projection matrix onto the line $y = x$ in $\mathbf{R^2}$. What are its eigenvalues? If $\frac{d\boldsymbol{u}}{dt} = -P\boldsymbol{u}$, what is the limit of $\boldsymbol{u}(t)$ at t=∞ starting from $\boldsymbol{u}(0) = (3, 1)$?

4. **Limiting behavior of the system and Stability**
   Stable: solution approach $\boldsymbol{u} = \boldsymbol{0}$ as t goes to ∞. From our solution, you see that we need **both** eigenvalues to have **real parts negative**.
   This is equivalent to (1) trace of A is negative and (2) determinant of A is positive.

5. **More general way to express the solution to $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$: $e^{At}\boldsymbol{u}(0)$**
   The above derivation is feasible only when an n by n matrix A has n independent eigenvectors. Otherwise, it will fail. So we need a more general way to solve the problem. We propose a solution $\boldsymbol{u} = e^{At}\boldsymbol{u}(0)$

   First, check if this proposed solution satisfies the equation.
   Then check if our previous solution is the same as $e^{At}\boldsymbol{u}(0)$ (See Strang 6.3 "The Exponential of a Matrix" for help)

   Exercise 3(Strang 6.3 Example 4)
   Solve $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$, with $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ (hint: use what we have just found in **Exercise 1**)

6. **Solving the system $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} + \boldsymbol{b}$**
   By now you should be armed with skills to derive solutions on your own. Check with **Strang 6.3-15**

More Exercise 1(Strang 6.3-27) What are the coefficient matrix A for the following systems?

$$\frac{dx}{dt} = 0x - 4y \qquad \frac{dy}{dt} = -2x + 2y \qquad A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$\frac{dy}{dt} = -2x + 2y \qquad \frac{dx}{dt} = 0x - 4y$$

More Exercise 2 (Strang 6.3-14) For anti-symmetric matrix, e^{At} is orthogonal.
(a) With $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$, A is above on the right, show that $\|\boldsymbol{u}(t)\|^2 = u_1^2 + u_2^2 + u_3^2$ does not change.
(b) If $Q = e^{At}$, show $Q^TQ = I$, Q is orthogonal.[1]

More Exercise 3(Strang 6.3-20)
Starting from $\boldsymbol{u}(0)$ the solution at time T is $e^{AT}\boldsymbol{u}(0)$. Go an additional time t to reach $e^{At}e^{AT}\boldsymbol{u}(0)$. This solution at time $t + T$ can also be written as _____. Conclusion: e^{At} times e^{AT} equals _____.

[1] You may also check out this proof for more properties: https://yutsumura.com/eigenvalues-of-real-skew-symmetric-matrix-are-zero-or-purely-imaginary-and-the-rank-is-even/