Math 1441 (Calc I) Sections 2.4, 2.5 & 2.6 — Instructor: Dr. Shijun Zheng

Course website — 01/23-27/2020

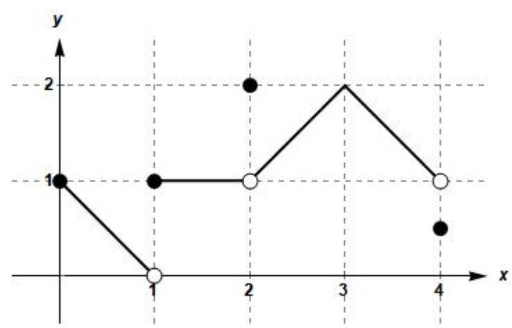
§2.4 One-Sided Limits

Example 1. The domain of $f(x) = \sqrt{4-x^2}$ is $[-2,2]$; its graph is a semicircle centered at origin with radius 2. We have $\lim_{x\to -2^+} \sqrt{4-x^2} = 0$ and $\lim_{x\to 2^-} \sqrt{4-x^2} = 0$.

Theorem 6. Suppose that a function f is defined on an open interval containing c, except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \qquad \Longleftrightarrow \qquad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

Example 2. Discuss the limit of the function $y = f(x)$ graphed below.



Ex. Discuss the existence of the one-sided limit for $f(x) = \frac{x}{|x|}$ as $x \to 0+$ or $x \to 0-$.

Exercise 18. Find the limits: (a) $\lim_{x\to 1^+} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$ (b) $\lim_{x\to 1^-} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$.

§2.5 Continuity

Example 1. At which numbers does the function f in §2.4 Example 2 appear to be not continuous? Explain why. What occurs at other numbers in the domain?

Definition. Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f.
The function f is **continuous at c** if $\lim_{x\to c} f(x) = f(c)$.
The function f is **right-continuous at c (or continuous from the right)** if $\lim_{x\to c^+} f(x) = f(c)$.
The function f is **left-continuous at c (or continuous from the left)** if $\lim_{x\to c^-} f(x) = f(c)$.

Definition. We define a **continuous function** to be one that is continuous at every point in its domain. If a function is discontinuous at one or more points of its domain, we say it is a **discontinuous** function.

Theorem 8. If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1 & 2. *Sums & Differences:*	$f \pm g$	
3. *Constant Multiples:*	$k \cdot f$,	for any number k
4. *Products:*	$f \cdot g$	
5. *Quotients:*	f/g,	provided $g(c) \neq 0$
6. *Powers:*	f^n,	n a positive integer
7. *Roots:*	$\sqrt[n]{f}$,	provided well-defined and n a positive integer

Example 6. (a) Every polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is continuous because $\lim_{x \to c} P(x) = P(c)$.
(b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined ($Q(c) \neq 0$).

Example 7. The function $f(x) = |x|$ is continuous.

Exercise 72. The functions $y = \sin x$ and $y = \cos x$ are continuous at every point $x = c$. All six trigonometric functions are continuous wherever they are defined.

Proposition. When a continuous function defined on an interval has an inverse, the inverse function is itself a continuous function over its own domain.

Theorem 9 (Composition of Continuous Functions). If f is continuous at c and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c.

Example 8. Show that the following functions are continuous on their natural domains.
(c) $y = \left| \dfrac{x-2}{x^2-2} \right|$

Theorem 10. If $\lim_{x \to c} f(x) = b$ and g is continuous at the point b, then $\lim_{x \to c} g(f(x)) = g(b)$.

Example 9. Find (a) $\lim_{x \to \pi/2} \cos(2x + \sin(\frac{3\pi}{2} + x))$ (b) $\lim_{x \to 1} \sin^{-1}\left(\dfrac{1-x}{1-x^2}\right)$.

Example 12. Show that $f(x) = \dfrac{x^2+x-6}{x^2-4}$, $x \neq 2$ has a continuous extension to $x = 2$, and find that extension.

§2.6 Limits Involving Infinity; Asymptotes of Graphs

Example 1. (a) $\lim_{x \to \infty} \frac{1}{x} = 0$ (b) $\lim_{x \to -\infty} \frac{1}{x} = 0$

Exercises 93 & 94. (a) $\lim_{x \to \infty} k = k$ (b) $\lim_{x \to -\infty} k = k$ (k is a constant)