NAME: ______________________________

MARK BOX		
PROBLEM	POINTS	
1	10	
2	10	
3	10	
4	10	
TOTAL	40	

ID (last four digits)________________________

please check the box of your section below

☐

or

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INSTRUCTIONS:

(1) To receive credits you must:
  (a) work in a logical fashion, **show all your work and indicate your reasoning** to support and justify your answer
  (b) when applicable put your answer on/in the line/box; use the back of the page if needed

(2) This exam covers (from *Elementary Linear Algebra* by Larson and Falvo 7^{th} ed.): Sections 3.1 – 3.4* .

(1) Compute the determinant.

$$\begin{vmatrix} 1 & 1 & -2 \\ 0 & 15 & 0 \\ 2 & 2 & -4 \end{vmatrix}$$

(2) Find (i) the characteristic equation, (ii) the eigenvalues, and (iii) the corresponding eigenvectors of the matrix.

(a)

$$\begin{vmatrix} 4 & -5 \\ 2 & -3 \end{vmatrix}$$

(b)

$$\begin{vmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{vmatrix}$$

(3) (*optional*)* Find the adjoint $\mathbf{ad(M)}$ of the matrix $M = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 3 & 2 \\ 3 & 0 & -1 \end{pmatrix}$.

Verify that $M\mathbf{ad}(M) = \mathbf{ad}(M)M = \det(M)I_3$.

(4) **Definition**. A vector $\mathbf{u}$ is said to be in the null space of a matrix A provided

$$A\mathbf{u} = \mathbf{0}.$$

or, equivalently, $\mathbf{u}$ is an eigenvector corresponding to the zero eigenvalue of A.

Which of the following vectors, if any, is in the null space of $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{pmatrix}$?

a) $[-1\ 0\ 1\ 0]^T$ b) $[0\ 2\ 1\ -1]^T$ c) $[0\ 4\ 2\ -2]^T$

(5) Determine which of the following statements are equivalent to the fact that a matrix A of size $n \times n$ is invertible?

a) A is nonsingular

b) The row space of A has dimension n

c) The column space of A has dimension n

d) The determinant of A is nonzero

e) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any given $\mathbf{b}$ in $\mathbf{R}^n$

f) The system $A\mathbf{x} = \mathbf{0}$ has nonzero solution

g) The dimension of the null space of A is zero

h) The rows of A are linear independent

i) The columns of A are linear independent

j) The rank of A is n

k) A is row-equivalent to an identity matrix

l) All eigenvalues of A are nonzero

m) A can be written as the product of elementary matrices.

(6) (optional*) The matrix $A = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$ row reduces to $C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

a) Find the rank and nullity of A.

b) Find a basis of the row space and the column space of A respectively.

c) Find a basis of the null space of A

d) Does the system $A\mathbf{x} = \begin{pmatrix} 109 \\ -217 \\ 66 \end{pmatrix}$ have a solution? (Hint: You can draw a conclusion from the fact that dimension of column space is 3, without having to solve the system. Recall that $rank(A) = dim(Col(A)) = dim(Row(A))$)

e) What is the relation between $rank, dim(null(A))$?(Hint: Theorem 4.17 (pp.196) states that $rank(A) + dim(null(A)) = n$, the number of columns)

(7) Find all the eigenvalues of the given matrix.

a) $\begin{pmatrix} 1 & -2 & 0 \\ -3 & 1 & 0 \\ -4 & -5 & 1 \end{pmatrix}$

b) $\begin{pmatrix}1 & 9\\ 0 & -1\end{pmatrix}$ (b) $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ (c) $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ (d) $\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$
where $i = \sqrt{-1}$ ($i^2 = -1$) is the unit for pure imaginary numbers.

(8) We say a vector $\mathbf{u}$ is a linear combination of a finite set of vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ if there exist constants c_1, c_2, c_3 such that

$$\mathbf{u} = c_1\mathbf{v_1} + \mathbf{v_2} + c_3\mathbf{v_3}.$$

Determine whether one can write $\mathbf{u} = [8\ 3\ 8]^T$ as a linear combination of the vectors in the set S.

$S = \{[4\ 3\ 2]^T, [0\ 3\ 2]^T, [0\ 0\ 2]^T\}$

Solutions (3) $\mathbf{ad}(M)$ = transpose of $\begin{pmatrix}-4 & 3 & -9\\ 3 & -5 & 4\\ -5 & 1 & -3\end{pmatrix}$
A straight forward computation shows $M\mathbf{ad}(M) = -11I_3$.

(8) a) $rank(A) = 3$ (number of leading 1's in C), nullity of $A = 1$

b) A basis of Row(A) consists of $\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$.

A basis of Col(A) consists of $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}$

c) $\begin{pmatrix}1\\1\\-1\\0\end{pmatrix}$

d) Yes.