The syllabus for Exam III is Sections 3.2,3.3,3.6 and 4.1-4.4.

1. Solve each of the following homogeneous linear differential equations.
   (a) $y'' + 3y' + 2y = 0$
   (b) $y'' + 6y' + 13y = 0$
   (c) $8y'' + 4y' + y = 0$
   (d) $2y'' - 7y' + 5y = 0$
   (e) $y'' + .2y' + .01y = 0$
   (f) $y'' + 2y' + 2y = 0$
   (g) $y''' + 2y'' - 8y' = 0$
   (h) $y''' - 2y'' - 3y' = 0$
   (i) $y^{(4)} - 5y'' + 4y = 0$
   (j) $t^2y'' - 7ty' + 15y = 0$
   (k) $t^2y'' - 12y = 0$

2. Find the general solution of the constant coefficient homogeneous linear differential equation with the given characteristic polynomial $p(s)$.
   (a) $p(s) = (s-1)(s+3)(s-5)$
   (b) $p(s) = s^3 - 1$
   (c) $p(s) = s^3 - 3s^2 + s + 5$
   (d) $p(s) = (s^2 + 1)^3$
   (e) $p(s)$ has degree 4 and has roots $\sqrt{2}$ with multiplicity 2 and $2 \pm 3i$ with multiplicity 1.
   (f) $p(s)$ has degree 5 and roots 0 with multiplicity 3 and $1 \pm \sqrt{3}$ with multiplicity 1.

3. Solve each of the following initial value problems. You may (and should) use the work already done in exercise 1.
   (a) $y'' + 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = -3.$
   (b) $y'' + 6y' + 13y = 0, \quad y(0) = 0, \quad y'(0) = -1.$
   (c) $y'' + 2y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 2$
   (d) $4t^2y'' - 7ty' + 6y = 0, \quad y(1) = 1, \quad y'(1) = 2$

4. Find a second order linear homogeneous differential equation with constant real coefficients that has the given function as a solution, or explain why there is no such equation.
   (a) $e^{-3t} + 2e^{-t}$
   (b) $e^{-t}\cos 2t$
   (c) $e^t t^{-2}$

5. Find the general solution to each of the following differential equations.
   (a) $y'' - 2y' + y = t^2 - 1$

(b) $y'' - 2y' + y = 4\cos t$

(c) $y'' - 2y' + y = te^t$

6. Find a particular solution $y_p(t)$ of each of the following differential equations by using the method of variation of parameters. In each case, $\mathcal{S}$ denotes a fundamental set of solutions of the associated homogeneous equation.

   (a) $y'' + 2y' + y = t^{-1}e^{-t}$; $\quad \mathcal{S} = \left\{e^{-t},\, te^{-t}\right\}$

   (b) $y'' + 9y = 9\sec 3t$; $\quad \mathcal{S} = \{\cos 3t,\, \sin 3t\}$

   (c) $t^2y'' + 2ty' - 6y = t^2$; $\quad \mathcal{S} = \left\{t^2,\, t^{-3}\right\}$

   (d) $y'' + 4y' + 4y = t^{1/2}e^{-2t}$; $\quad \mathcal{S} = \left\{e^{-2t},\, te^{-2t}\right\}$

7. Find the Laplace transform of each of the following functions:

   (a) $f(t) = t^2\chi_{[1,\,3)}$

   (b) $g(t) = \begin{cases} 0 & \text{if } 0 \le t < 5, \\ t^2 - 5 & \text{if } t \ge 5. \end{cases}$

8. Find the inverse Laplace transform of each of the following functions

   (a) $F(s) = \dfrac{se^{-2s}}{s^2 - 9}$

   (b) $G(s) = \dfrac{e^{-s} - e^{-2s}}{s^4}$

9. Solve the following initial value problem:

   (a)
   $$y'' + 4y' + 5y = \begin{cases} 1 & \text{if } 0 \le t < \pi \\ 0 & \text{if } t \ge \pi \end{cases} \qquad y(0) = 0,\ y'(0) = 0.$$

   (b)
   $$y'' + 9y = h(t - 2\pi)\sin t, \quad y(0) = 1,\ y'(0) = 0.$$

Answers

1. (a) $y = c_1e^{-t} + c_2e^{-2t}$

   (b) $y = c_1e^{-3t}\cos 2t + c_2e^{-3t}\sin 2t$

   (c) $y = (c_1 + c_2t)e^{-3t}$

   (d) $y = c_1e^{5t/2} + c_2e^t$

   (e) $y = e^{0.1t}(c_1 + tc_2)$

   (f) $y = c_1e^{-4t} + c_2e^{-3t}$

   (g) $y = c_1 + c_2e^{3t} + c_3e^{-5t}$

   (h) $y = c_1 + c_2e^{-t} + c_3e^{3t}$

   (i) $y = c_1e^{2t} + c_2e^{-2t} + c_3e^t + c_4e^{-t}$

(j) $y = c_1t^3 + c_2t^5$

(k) $y = c_1t^4 + c_2t^{-3}$

2. (a) $y = c_1e^t + c_2e^{-3t} + c_3e^{5t}$

   (b) $y = c_1e^t + c_2e^{-t/2}\cos\sqrt{3}t/2 + c_2e^{-t/2}\sin\sqrt{3}t/2$

   (c) $y = c_1e^{-t} + c_2e^{2t}\cos t + c_3e^{2t}\sin t$

   (d) $y = (c_1 + c_2t + c_3t^2)\cos t + (c_4 + c_5t + c_6t^2)\sin t$

   (e) $y = (c_1 + c_2t)e^{\sqrt{2}t} + c_3e^{2t}\cos 3t + c_4e^{2t}\sin 3t$

   (f) $y = (c_1 + c_2t + c_3t^2) + c_4e^{(1+\sqrt{3})t} + c_5e^{(1-\sqrt{3})t}$

3. (a) $y = 2e^{-2t} - e^{-t}$

   (b) $y = -\frac{1}{2}e^{-3t}\sin 2t$

   (c) $y = 2e^{-t}\sin t$

   (d) $y = (1/2)(t^{-1/2} + t^6)$

4. (a) $y'' + 4y' + 3y = 0$

   (b) $y'' + 2y' + 5y = 0$

   (c) Not possible since e^tt^{-2} is not included in the list of functions in Theorem 3.3.1 (Page 159).

5. (a) $y = c_2e^t + c_2te^t + t^2 + 4t + 5$

   (b) $y = c_2e^t + c_2te^t + -2\sin t$

   (c) $y = c_2e^t + c_2te^t + \frac{1}{6}t^3e^t$

6. (a) $y_p(t) = (t\ln t)e^{-t}$

   (b) $y_p(t) = \frac{1}{2}(\ln|\cos 3t|)\cos 3t + \frac{3}{2}t\sin 3t$

   (c) $y_p(t) = (1/5)t^2\ln|t|$

   (d) $y_p(t) = (4/15)t^{5/2}e^{-2t}$

7. Find the Laplace transform of each of the following functions:

   (a) $f(t) = t^2\chi_{[1,\,3)}$ Since $f(t) = t^2\chi_{[1,\,3)} = t^2(h(t-1) - h(t-3)) = t^2h(t-1) - t^2h(t-3)$ apply Corollary 4.2.5 to get

$$\begin{aligned}
\mathcal{L}\{f(t)\} = F(s) &= \mathcal{L}\left\{t^2h(t-1)\right\} - \mathcal{L}\left\{t^2h(t-3)\right\} \\
&= e^{-s}\mathcal{L}\left\{(t+1)^2\right\} - e^{-3s}\mathcal{L}\left\{(t+3)^2\right\} \\
&= e^{-s}\mathcal{L}\left\{t^2+2t+1\right\} - e^{-3s}\mathcal{L}\left\{t^2+6t+9\right\} \\
&= \boxed{e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) - e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)}
\end{aligned}$$

   (b) $g(t) = \begin{cases} 0 & \text{if } 0 \le t < 5, \\ t^2 - 5 & \text{if } t \ge 5. \end{cases}$ Since $g(t) = \begin{cases} 0 & \text{if } 0 \le t < 5, \\ t^2 - 5 & \text{if } t \ge 5. \end{cases} = (t^2-5)h(t-5)$ apply Corollary 4.2.5 to get

$$\begin{aligned}
\mathcal{L}\{g(t)\} = G(s) &= \mathcal{L}\left\{(t^2-5)h(t-5)\right\} \\
&= e^{-5s}\mathcal{L}\left\{(t+5)^2 - 5\right\} \\
&= e^{-5s}\mathcal{L}\left\{t^2+10t+20\right\} \\
&= \boxed{e^{-5s}\left(\frac{2}{s^3} + \frac{10}{s^2} + \frac{20}{s}\right)}
\end{aligned}$$

8. Find the inverse Laplace transform of each of the following functions

   (a) $F(s) = \dfrac{se^{-2s}}{s^2-9}$ Since

   $$\frac{s}{s^2-9} = \frac{1}{2}\left(\frac{1}{s-3} + \frac{1}{s+3}\right),$$

   the second translation principle gives

   $$\boxed{f(t) = \frac{1}{2}\left(e^{-3(t-2)} + e^{3(t-2)}\right) h(t-2).}$$

   (b) $G(s) = \dfrac{e^{-s} - e^{-2s}}{s^4}$ Since $\mathcal{L}^{-1}\left\{\dfrac{1}{s^4}\right\} = \dfrac{t^3}{6}$, the second translation principle gives

   $$\boxed{g(t) = \frac{1}{6}(t-1)^3 h(t-1) - \frac{1}{6}(t-2)^3 h(t-2).}$$

9. Solve the following initial value problem:

   (a)

   $$y'' + 4y' + 5y = \begin{cases} 1 & \text{if } 0 \le t < \pi \\ 0 & \text{if } t \ge \pi \end{cases} \qquad y(0) = 0,\ y'(0) = 0.$$

   The right hand side of the equation is the function

   $$f(t) = \begin{cases} 1 & \text{if } 0 \le t < \pi \\ 0 & \text{if } t \ge \pi \end{cases} = 1 - h(t-\pi),$$

   with Laplace transform $F(s) = \dfrac{1}{s} - \dfrac{1}{s}e^{-\pi s}$. Let $Y(s) = \mathcal{L}\{y(t)\}$ and apply the Laplace transform to the given differential equation to get (using $y(0) = y'(0) = 0$)

   $$s^2Y(s) + 4sY(s) + 5Y(s) = F(s) = \frac{1}{s} - \frac{1}{s}e^{-\pi s}.$$

   Solving for $Y(s)$ gives

   $$(\dagger) \qquad Y(s) = \frac{1}{s(s^2+4s+5)} - \frac{1}{s(s^2+4s+5)}e^{-\pi s}.$$

   Applying partial fractions to $G(s) = \dfrac{1}{s(s^2+4s+5)}$ gives a decomposition

   $$\frac{1}{s(s^2+4s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+5}$$

   and clearing the denominators in the usual manner gives an equation

   $$1 = A(s^2+4s+5) + (Bs+C)s = (A+B)s^2 + (4A+C)s + 5A.$$

   Thus, by comparing coefficients of 1, s and s^2 on the left and right of this equation, we conclude that $A = \frac{1}{5}$, $B = -\frac{1}{5}$ and $C = -\frac{4}{5}$. Hence

   $$\begin{aligned} G(s) = \frac{1}{s(s^2+4s+5)} &= \frac{1}{5}\left(\frac{1}{s} - \frac{s+4}{s^2+4s+5}\right) \\ &= \frac{1}{5}\left(\frac{1}{s} - \frac{s+4}{(s+2)^2+1}\right) \\ &= \frac{1}{5}\left(\frac{1}{s} - \frac{s+2}{(s+2)^2+1} - \frac{2}{(s_2)^2+1}\right), \end{aligned}$$

so that

$$g(t) = \mathcal{L}^{-1}\left\{G(s)\right\} = \frac{1}{5}\left(1 - e^{-2t}\cos t - 2e^{-2t}\sin t\right).$$

The second translation principle applied to this formula and (†) shows

$$\begin{aligned} y(t) &= g(t) - g(t-\pi)h(t-\pi) \\ &= \frac{1}{5}\left(1 - e^{-2t}\cos t - 2e^{-2t}\sin t\right) \\ &\qquad -\frac{1}{5}\left(1 - e^{-2(t-\pi)}\cos(t-\pi) - 2e^{-2(t-\pi)}\sin(t-\pi)\right)h(t-\pi). \end{aligned}$$

(b) $y'' + 9y = h(t-2\pi)\sin t, \quad y(0) = 1,\ y'(0) = 0.$
Ans: $y = \cos 3t + \frac{1}{24}h(t-2\pi)(3\sin t - \sin 3t)$