SOLUTION FOR MATH417 MIDTERM

Problem 1.
(1). Since $f(x) = 2x$ for $0 < x < \pi$,

$$f(x) \sim \sum_{n\geq 1} A_n \sin(nx).$$

Then by definition, we have

$$\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^\pi 2x\sin(nx)dx \\
&= -\frac{4}{n\pi}\cos(nx)x|_0^\pi + \int_0^\pi \frac{4}{n\pi}\cos(nx)dx \\
&= -\frac{4}{n}(-1)^n + \frac{4}{n^2\pi}\sin(nx)|_0^\pi \\
&= (-1)^{n+1}\frac{4}{n}.
\end{aligned}$$

(2). Let $\tilde{f}$ be the odd and 2π-periodic extension of f. By convergence theorem, the Fourier sine series $S[f]$ of f equals to $\tilde{f}$ at the point of continuity, and $\frac{\tilde{f}(x-)+\tilde{f}(x+)}{2}$ at the point of discontinuity. In conclusion, we know that for

$$S[f](x) = \begin{cases} 0 & x = -3\pi \\ 2(x+2\pi) & -3\pi < x < -\pi \\ 0 & x = -\pi \\ 2x & -\pi < x < \pi \\ 0 & x = \pi \\ 2(x-2\pi) & \pi < x < 3\pi \\ 0 & x = 3\pi \end{cases}$$

(3). If we use partial sum of eigenfunctions to approximate a function, we know that we have Gibbs phenomenon (roughly 9% overshoot) at the point of discontinuity.

Here the only points of discontinuity are $x_k = \pi + 2k\pi$, at x_k, $f(x_k+) - f(x_k-) = 4\pi$, the overshoot is roughly $9\% \times 4\pi = 0.36\pi$.

(4). If we studied the Fourier cosine series instead, we need to consider $\hat{f}$ which is the even and 2π-periodic extension of f. We can know that $\hat{f}$ is in fact a continuous function, then by convergence theorem, the Fourier cosine series $C[f](x)$ of f equals to $\hat{f}$.

Problem 2. By the method of separation of variables, we assume first that

$$u(x,t) = \phi(x)h(t).$$

Then by the equation, we know that

$$\phi(x)\frac{\partial h}{\partial t}(t) = \frac{\partial^2 \phi}{\partial x^2}(x)h(t),$$

and hence by dividing ϕh, we can separate the variables (and get two equations for ϕ and h),

$$\frac{dh}{dt}(t)/h(t) = \frac{d^2\phi}{dx^2}(x)/\phi(x) = -\lambda.$$

We study first the eigenvalue problem for ϕ. Recall that we must have

$$\phi(0) = \phi(\pi) = 0$$

from the boundary condition if we consider the nontrivial solution. The solution for this eigenvalue problem is that

$$\lambda_n = n^2, \quad \phi_n(x) = \sin(nx), \quad n \geq 1.$$

For any λ_n, the solution for $h(t)$ is

$$h_n(t) = e^{-n^2 t}.$$

Then by the superposition principle, we claim the general solution to this problem is given by

$$u(x,t) = \sum_{n\geq 1} A_n e^{-n^2 t} \sin(nx).$$

(By continuity and BC, we conclude that we have identity for the solution $u = S[u(t)](x)$.)

To solve this problem, we need to check the initial data $u(x,0) = 2x$, thus by setting $t = 0$, we are reduced to determine the coefficients A_n of the Fourier sine series of $f(x) = 2x$. And now continue as in Problem 1, above, i.e.,

$$u(x,t) = \sum_{n\geq 1} (-1)^{n+1} \frac{4\sin(nx)}{n} e^{-n^2 t}.$$

Problem 3. We use the method of eigenfunction expansion to solve the problem.

First method (combine the method of shifting the data) We have inhomogeneous BC, we first construct one reference function $r(x,t)$ s.t.

$$r_x(0,t) = 0, r_x(\pi,t) = 2\pi$$

we choose $r(x,t) = x^2$.

Let $v(x,t) = u(x,t) - r(x,t)$, the the problem satisfied by v is

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \cos x + 2 & (0 < x < \pi, t > 0), \\ \frac{\partial v}{\partial x}(0,t) = 0,\ \frac{\partial v}{\partial x}(\pi,t) = 0 & (t > 0), \\ v(x,0) = 1 & (0 < x < \pi). \end{cases}$$

Now we use the method of eigenfunction expansion. Recall the BC is of second type and the corresponding eigenfunctions are

$$\lambda_n = n^2, \phi_n(x) = \cos nx$$

If v solves the problem, we have for some $a_n(t)$ (we have identity due to the continuity)

$$v(x,t) = a_0(t) + \sum_{n\geq 1} a_n(t) \cos nx$$

We use the term by term differentiation to conclude that

$$v_t(x,t) = a_0'(t) + \sum_{n\geq 1} a_n'(t) \cos(nx),$$

$$v_x(x,t) = -\sum_{n\geq 1} n a_n(t)\sin(nx),$$

By BC for v, we have

$$v_{xx}(x,t) = -\sum_{n\geq 1} n^2 a_n(t)\cos(nx).$$

Then the PDE tells us that

$$a_0'(t) + \sum_{n\geq 1}(a_n'(t) + n^2 a_n(t))\cos nx = 2 + \cos x$$

that is

$$a_0'(t) = 2,\ a_1'(t) + a_1(t) = 1,\ a_n'(t) + n^2 a_n(t) = 0, n \geq 2$$

Moreover, IC tells us that

$$a_0(0) = 1,\ a_1(0) = 0,\ a_n(0) = 0, n \geq 2$$

Solving the ODEs for a_n, we know that

$$a_0(t) = 1 + 2t,\ a_1(t) = 1 - e^{-t},\ a_n(t) = 0$$

And so

$$v(x,t) = 1 + 2t + \cos x - e^{-t}\cos x$$

Recall definition of v, we have

$$u(x,t) = 1 + 2t + (1 - e^{-t})\cos x + x^2.$$

Second method (direct use of method of eigenfunction expansion)

First, note that if we consider the case that the boundary condition is homogeneous, then the eigenfunction for the eigenvalue problem

$$\phi_{xx} + \lambda^2\phi = 0, \quad \phi_x(0) = \phi_x(\pi) = 0,$$

will be given by

$$\lambda_n = n^2, \quad \phi_n(x) = \cos(nx), \quad n = 0, 1, 2, \cdots.$$

Then we assume that the solution u take the form

$$u(x,t) = \sum_{n\geq 0} A_n(t)\cos(nx).$$

We use the term by term differentiation to conclude that

$$u_t(x,t) = \sum_{n\geq 0} A_n'(t)\cos(nx),$$

$$u_x(x,t) = -\sum_{n\geq 1} nA_n(t)\sin(nx).$$

Moreover, by the boundary condition, we have

$$\begin{aligned} u_{xx}(x,t) &= -\sum_{n\geq 1} n^2 A_n(t)\cos(nx) + \frac{1}{\pi}u_x|_0^\pi + \sum_{n\geq 1}\frac{2}{\pi}u_x(y,t)\cos(ny)|_0^\pi\cos(nx), \\ &= 2 + \sum_{n\geq 1}(4\cos n\pi - n^2 A_n(t))\cos(nx). \end{aligned}$$

Then if u is the solution to the PDE, we must have

$$A_0'(t) = 2,$$

$$A_1'(t) = 1 - 4 - A_1(t),$$

$$A_n'(t) = 4(-1)^n - n^2 A_n(t), \quad n \geq 2.$$

To solve these ODEs, we need also to consider the initial conditions, the initial conditions for the coefficients are

$$A_0(0) = a_0 + 1, \quad A_n(0) = a_n, \quad n \geq 1.$$

(let a_n be the Fourier coefficients of x^2)

Then the solution for the ODEs for the coefficients are

$$A_0(t) = a_0 + 1 + 2t,$$

$$A_1(t) = a_1 e^{-t} + 3e^{-t} - 3 = (a_1 + 3)(e^{-t} - 1) + a_1,$$

$$A_n(t) = a_n e^{-n^2 t} + (-1)^{n+1} \frac{4}{n^2}(e^{-n^2 t} - 1) = (a_n + (-1)^{n+1} \frac{4}{n^2})(e^{-n^2 t} - 1) + a_n, \quad n \geq 2.$$

In conclusion, the solution is

$$u(x,t) = A_0(t) + \sum_{n \geq 1} A_n(t) \cos(nx).$$

Here, we can observe that we have in fact got the same result, since $a_n = (-1)^n \frac{4}{n^2}$ for $n \geq 1$

$$A_0(t) = a_0 + 1 + 2t,$$

$$A_1(t) = a_1 e^{-t} + 3e^{-t} - 3 = -(e^{-t} - 1) + a_1,$$

$$A_n(t) = (a_n + (-1)^{n+1} \frac{4}{n^2})(e^{-n^2 t} - 1) + a_n = a_n, \quad n \geq 2.$$

$$u(x,t) = A_0(t) + \sum_{n \geq 1} A_n(t) \cos(nx) = 1 + 2t + x^2 + (1 - e^{-t}) \cos x.$$

Bonus Problem. Since

$$\frac{\pi}{4} \sim \sum_{n \geq 1, odd} \frac{1}{n} \sin \frac{n\pi x}{L} = \sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} + \cdots,$$

by integration from 0 to x, we have

$$\frac{\pi}{4} x = -\sum_{n \geq 1, odd} \frac{1}{n} \frac{L}{n\pi} (\cos \frac{n\pi x}{L} - 1) = a_0 - \sum_{n \geq 1, odd} \frac{L}{n^2 \pi} \cos \frac{n\pi x}{L},$$

that is

$$x = A_0 - \sum_{n \geq 1, odd} \frac{4L}{(n\pi)^2} \cos \frac{n\pi x}{L}.$$

Here we see that A_0 is the first coefficient of the cosine series, and so

$$A_0 = \frac{1}{L} \int_0^L x dx = L/2$$

Now we conclude that

$$x = C[x](x) = L/2 - \sum_{n \geq 1, odd} \frac{4L}{(n\pi)^2} \cos \frac{n\pi x}{L}.$$

By take $x = 0$, we see that

$$0 = C[x](0) = L/2 - \sum_{n \geq 1, odd} \frac{4L}{(n\pi)^2},$$

that is

$$\sum_{n\geq 1, odd} \frac{1}{n^2} = \pi^2/8.$$

To conclude, let $C = \sum_{n\geq 1} \frac{1}{n^2}$, we see that

$$\sum_{n\geq 1, even} \frac{1}{n^2} = \sum_{n=2k, k\geq 1} \frac{1}{4}\frac{1}{k^2} = \frac{1}{4}C$$

and so

$$C = \sum_{n\geq 1} \frac{1}{n^2} = \sum_{n\geq 1, odd} \frac{1}{n^2} + \sum_{n\geq 1, even} \frac{1}{n^2} = \pi^2/8 + \frac{1}{4}C$$

$$C = \frac{4}{3}\pi^2/8 = \pi^2/6$$