

In 3.2 we discussed using elementary row/column operations to evaluate the determinant of a matrix.

§3.3 Properties of Determinants

Theorem 3.5. If A and B are square matrices of order n , then $\det(AB) = \det(A)\det(B)$.

Theorem 3.6. If A is a square matrix of order n and c is a scalar, then the determinant of cA is $\det(cA) = c^n \det(A)$.

Theorem 3.7. A square matrix A is invertible (nonsingular) if and only if $\det(A) \neq 0$.

Example 3. Determine whether each matrix has an inverse.

$$(a) \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \quad (b) \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Solution. (a) $\det(A) = 0$.

(b) $\det(A) \neq 0$.

By the augmented matrix method $[A \ I_3]$, we have $A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ -1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

$$|A^{-1}| = \begin{vmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ -1 & \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{3} \begin{vmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{12}.$$

□

Theorem 3.8. If A is an $n \times n$ invertible matrix, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

Example 4. Find $|A^{-1}|$ for the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$.

Solution. $|A| = 4 \Rightarrow |A^{-1}| = \frac{1}{4}$.

Check: $(A \ I_3)$ -method $\Rightarrow A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} & \frac{3}{4} \\ 1 & -\frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$.

$$\begin{vmatrix} -\frac{1}{2} & \frac{3}{4} & \frac{3}{4} \\ 1 & -\frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{3}{4} & 0 \\ 1 & -\frac{3}{2} & 1 \\ \frac{1}{2} & -\frac{1}{4} & 0 \end{vmatrix} = (-1)^{2+3} \begin{vmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{vmatrix} = (-1) \left(\frac{1}{8} - \frac{3}{8} \right) = \frac{1}{4}.$$

□

Equivalent Conditions for a Nonsingular Matrix. If A is an $n \times n$ matrix, then the statements below are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix \mathbf{b} .
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. A is row equivalent to I_n .
5. A can be written as the product of elementary matrices.
6. $\det(A) \neq 0$.

Example 5. Which of the systems has a unique solution?

$$\begin{array}{ll} \text{(a)} & \begin{array}{l} 2x_2 - x_3 = -1 \\ 3x_1 - 2x_2 + x_3 = 4 \\ 3x_1 + 2x_2 - x_3 = -4 \end{array} \\ \text{(b)} & \begin{array}{l} 2x_2 - x_3 = -1 \\ 3x_1 - 2x_2 + x_3 = 4 \\ 3x_1 + 2x_2 + x_3 = -4 \end{array} \end{array}$$

Theorem 3.9. If A is a square matrix, then $\det(A) = \det(A^T)$.

APPENDIX A. PROOF OF THEOREM 3.8

Proof of Theorem 3.8. Elementary matrices are of the following three forms (2 by 2):

$$\begin{aligned} E_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & E_2 &= \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \quad (c \neq 0) & \text{and} & E_3 &= \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \\ E_1^{-1} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & E_2^{-1} &= \begin{bmatrix} c^{-1} & 0 \\ 0 & 1 \end{bmatrix} & \text{and} & E_3^{-1} &= \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}. \end{aligned}$$

We see each E_i^{-1} has a determinant $\det(E_i^{-1}) = (\det(E_i))^{-1}$ for $i = 1, \dots, q$. If $A = E_1 \cdots E_q$ are product of elementary matrices, then $A^{-1} = E_q^{-1} \cdots E_1^{-1}$. Hence

$$\begin{aligned} \det(A^{-1}) &= \det(E_q^{-1}) \cdots \det(E_1^{-1}) \\ &= \det(E_q)^{-1} \cdots \det(E_1)^{-1} \\ &= (\det(E_q) \cdots \det(E_1))^{-1} = (\det(A))^{-1}. \end{aligned}$$

□

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