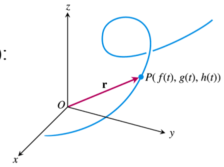
Curves in Space and Their Tangents

The path of a particle moving in space in time interval I is described by a ***vector-valued function*** (***vector function***):

$$\mathbf{r}(t) = \overrightarrow{OP} = \underbrace{f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}}_{\text{component functions}} \qquad t \in I$$



or, equivalently, by parametric equations:

$$x = f(t), \quad y = g(t), \quad z = h(t), \qquad t \in I$$

Example

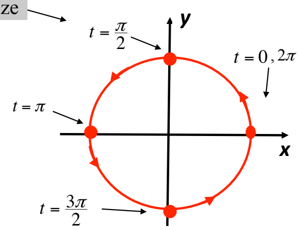
Graph the vector function parametrize

$$\mathbf{r}(t) = (\underset{x=}{\cos t})\mathbf{i} + (\underset{y=}{\sin t})\mathbf{j} + t\mathbf{k}$$

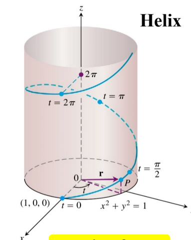
Hint: The components (first two) satisfy the equation

$$x^2 + y^2 = 1$$

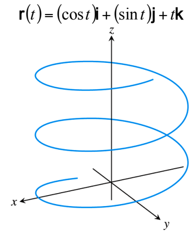
(The particle stays on the surface of a cylinder.)



The particle is spiraling up the surface of the cylinder, completing one full cycle while rising 2π units.

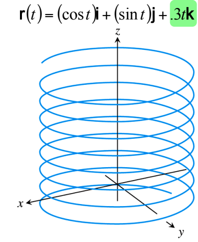


$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$



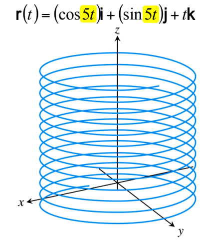
rising slower

$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + .3t\mathbf{k}$



rotating faster

$\mathbf{r}(t) = (\cos 5t)\mathbf{i} + (\sin 5t)\mathbf{j} + t\mathbf{k}$



Can you sketch this one? **the floor function**

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \lfloor t \rfloor\mathbf{k}$$

Discontinuous at every integer.

Recall:
Function f is continuous at x_0 if $\lim_{x \to x_0} f(x)$ exists and equals $f(x_0)$.

If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ then

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$$

means

$$\lim_{t \to t_0} f(t) = L_1, \quad \lim_{t \to t_0} g(t) = L_2, \quad \text{and} \quad \lim_{t \to t_0} h(t) = L_3$$

EXAMPLE 2 If $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, then

$$\lim_{t \to \pi/4} \mathbf{r}(t) = \left(\lim_{t \to \pi/4} \cos t\right)\mathbf{i} + \left(\lim_{t \to \pi/4} \sin t\right)\mathbf{j} + \left(\lim_{t \to \pi/4} t\right)\mathbf{k}$$

$$= \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\pi}{4}\mathbf{k}.$$

1

Definition
A vector function $\mathbf{r}(t)$ is continuous at t_0 if

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$$

It is continuous if it is continuous over its interval domain.

Recall:
Function f is continuous at x_0 if $\lim_{x \to x_0} f(x)$ exists and equals $f(x_0)$.

If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ then

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0) = f(t_0)\mathbf{i} + g(t_0)\mathbf{j} + h(t_0)\mathbf{k}$$

means

$$\lim_{t \to t_0} f(t) = f(t_0) \quad \lim_{t \to t_0} g(t) = g(t_0) \text{ and } \quad \lim_{t \to t_0} h(t) = h(t_0)$$

f is cont. at t_0 $\quad$ g is cont. at t_0 $\quad$ h is cont. at t_0

Note
A vector function is continuous at a point iff each of its component functions is.

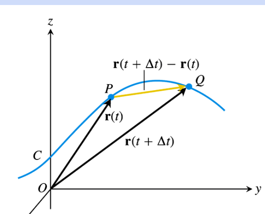
Recall:

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

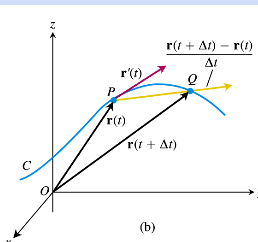
$$\frac{\Delta f}{\Delta x} \xrightarrow{\Delta x \to 0} \frac{df}{dx}$$

Definition
A vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a derivative (is differentiable) at t if $f, g,$ and h have derivatives at t. The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}$$

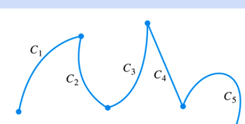


(a)



(b)

Definition
The curve traced by a point with position vector $\mathbf{r}$ is ***smooth*** if $d\mathbf{r}/dt$ is continuous and never $\mathbf{0}$; i.e. if f, g, and h have continuous first derivatives that are not simultaneously 0.



Not smooth; ***piecewise smooth***: smooth pieces connected end to end.

position: $\mathbf{r}(t)$

velocity: $\mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt}$ $\qquad$ speed: $\|\mathbf{v}\|$ $\qquad$ direction of motion: $\mathbf{v}/\|\mathbf{v}\|$

acceleration: $\mathbf{a}(t) = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt}$

$\text{Trig}\left(\frac{7\pi}{2}\right) = \text{Trig}\left(\frac{3\pi}{2}\right)$

Example

Find velocity, speed, and acceleration of a particle whose position is given by

$$\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + (5\cos^2 t)\mathbf{k}$$

Evaluate at $t = \frac{7\pi}{4}$.

$\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\sin\left(\frac{7\pi}{2}\right) = -1$, $\cos\left(\frac{7\pi}{2}\right) = 0$

Solution

$$\mathbf{v}(t) = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + (-10\cos t\sin t)\mathbf{k}$$

$$= -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} - 5\sin 2t\,\mathbf{k} \xrightarrow{t=\frac{7\pi}{4}} \mathbf{v}\left(\tfrac{7\pi}{4}\right) = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j} + 5\mathbf{k}$$

$$\|\mathbf{v}(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (5\sin 2t)^2} = \sqrt{4 + 25\sin^2 2t} \xrightarrow{t=\frac{7\pi}{4}} \left\|\mathbf{v}\left(\tfrac{7\pi}{4}\right)\right\| = \sqrt{29}$$

$$\mathbf{a}(t) = -2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} - 10\cos 2t\,\mathbf{k} \xrightarrow{t=\frac{7\pi}{4}} \mathbf{a}\left(\tfrac{7\pi}{4}\right) = -\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$$

Differentiation Rules for Vector Functions

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of t, $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

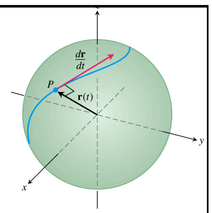
1. *Constant Function Rule:* $\frac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:* $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:* $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:* $\frac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
6. *Cross Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
7. *Chain Rule:* $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Observation (Vector Functions of Constant Magnitude)

If $\mathbf{r}(t)$ is differentiable and $\|\mathbf{r}(t)\|$ is constant then $\mathbf{r}$ and $d\mathbf{r}/dt$ are orthogonal:

$$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0$$



Reason:

Suppose $\|\mathbf{r}(t)\| = c$. Then $\mathbf{r}(t)\bullet\mathbf{r}(t) = c^2$, so $\frac{d}{dt}[\mathbf{r}(t)\bullet\mathbf{r}(t)] = 0$

$$\mathbf{r}'(t)\bullet\mathbf{r}(t) + \mathbf{r}(t)\bullet\mathbf{r}'(t) = 0 \qquad (\mathbf{u}\bullet\mathbf{v} = \mathbf{v}\bullet\mathbf{u})$$

$$2\mathbf{r}(t)\bullet\mathbf{r}'(t) = 0$$

5. *Dot Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$

2

DEFINITION The **indefinite integral** of $\mathbf{r}$ with respect to t is the set of all antiderivatives of $\mathbf{r}$, denoted by $\int \mathbf{r}(t)\,dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}.$$

EXAMPLE 1 To integrate a vector function, we integrate each of its components.

$$\int ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k})\,dt = \left(\int \cos t\,dt\right)\mathbf{i} + \left(\int dt\right)\mathbf{j} - \left(\int 2t\,dt\right)\mathbf{k} \quad (1)$$

$$= (\sin t + C_1)\mathbf{i} + (t + C_2)\mathbf{j} - (t^2 + C_3)\mathbf{k} \quad (2)$$

$$= (\sin t)\mathbf{i} + t\mathbf{j} - t^2\mathbf{k} + \mathbf{C} \qquad \mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} - C_3\mathbf{k}$$

As in the integration of scalar functions, we recommend that you skip the steps in Equations (1) and (2) and go directly to the final form. Find an antiderivative for each component and add a *constant vector* at the end.

DEFINITION The **indefinite integral** of $\mathbf{r}$ with respect to t is the set of all antiderivatives of $\mathbf{r}$, denoted by $\int \mathbf{r}(t)\,dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}.$$

EXAMPLE 2 As in Example 1, we integrate each component.

$$\int_0^{\pi} ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k})\,dt = \left(\int_0^{\pi} \cos t\,dt\right)\mathbf{i} + \left(\int_0^{\pi} dt\right)\mathbf{j} - \left(\int_0^{\pi} 2t\,dt\right)\mathbf{k}$$

$$= \Big[\sin t\Big]_0^{\pi}\,\mathbf{i} + \Big[t\Big]_0^{\pi}\,\mathbf{j} - \Big[t^2\Big]_0^{\pi}\,\mathbf{k}$$

$$= [0 - 0]\mathbf{i} + [\pi - 0]\mathbf{j} - [\pi^2 - 0^2]\mathbf{k}$$

$$= \pi\mathbf{j} - \pi^2\mathbf{k}$$