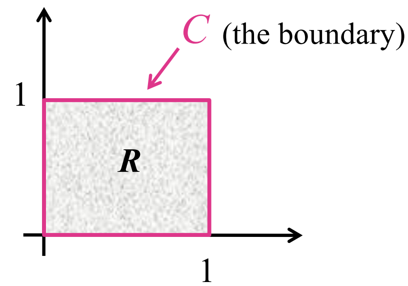
Example

Evaluate $\oint_C xy\,dy - y^2\,dx$ where C is the boundary of the square bounded by:
$x=0,\ x=1,\ y=0,\ y=1.$



Using formula (1):

$$\oint_C \overset{M}{xy}\,dy - \overset{N}{y^2}\,dx = \iint_R \overbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}^{y+2y} dA = 3\iint_R y\,dA = 3\int_0^1\!\!\int_0^1 y\,dx\,dy = 3\int_0^1 y\Big(\overbrace{\int_0^1 dx}^{1}\Big)dy$$

$$= 3\int_0^1 y\,dy = 3\left(\tfrac{1}{2}y^2\right)\Big|_0^1 = \tfrac{3}{2}$$

(1)

$$\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$

OR, using formula (2):

$$\oint_C xy\,dy - y^2\,dx = \oint_C \overset{M}{-y^2}\,dx + \overset{N}{xy}\,dy = \iint_R \overbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}^{y-(-2y)} dA = \boxed{3\iint_R y\,dA} = \tfrac{3}{2}$$

(as before)

(2)

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

Example

Evaluate the outward flux of the vector field $\mathbf{F}(x,y) = x\,\mathbf{i} + y^2\,\mathbf{j}$ across the square bounded by: $x=\pm 1,\ y=\pm 1.$

Green's Thm.

$$\text{Flux} = \int_C \mathbf{F}\bullet\mathbf{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \int_{-1}^{1}\int_{-1}^{1} (1+2y)\,dx\,dy = 4$$

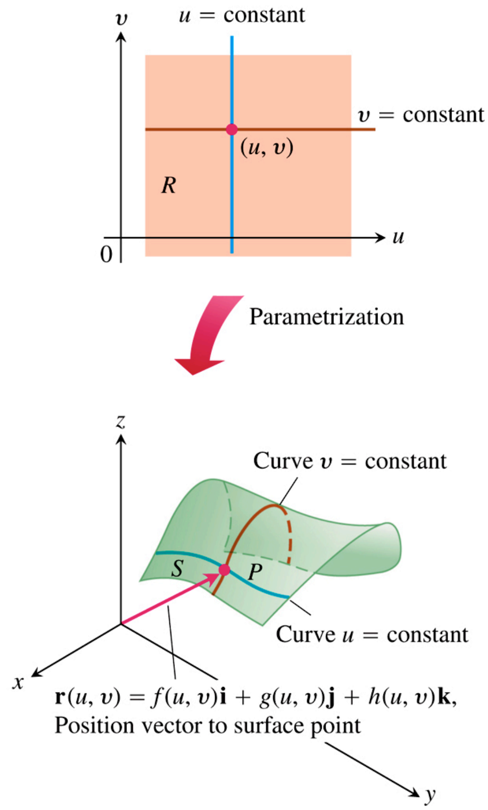
Surfaces and Area

Parametrizations of Surfaces

$$\mathbf{r}(u,v)=\langle f(u,v), g(u,v), h(u,v)\rangle$$

Or, equivalently,

$$x=f(u,v),\quad y=g(u,v),\quad z=h(u,v)$$

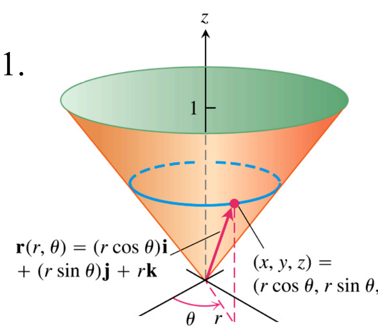


Example
Find a parametrization of the cone $z=\sqrt{x^2+y^2}$, $0\le z\le 1$.

Hint: Use cylindrical coordinates.

$$x=r\cos\theta \qquad \begin{pmatrix} 0\le r\le 1, \\ 0\le\theta\le 2\pi \end{pmatrix}$$
$$y=r\sin\theta$$
$$z=z\ =r$$

$$(u=r,\ v=\theta)$$



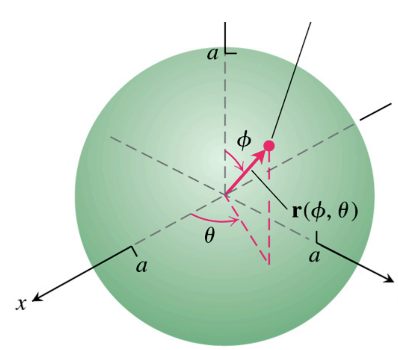
Example
Find a parametrization of the sphere $x^2+y^2+z^2=a^2$.

$$\rho=a$$

Hint: Use spherical coordinates.

$$x=\rho\sin\phi\cos\theta\ =a\sin\phi\cos\theta \qquad \begin{pmatrix} 0\le\phi\le\pi, \\ 0\le\theta\le 2\pi \end{pmatrix}$$
$$y=\rho\sin\phi\sin\theta\ =a\sin\phi\sin\theta$$
$$z=\rho\cos\phi\ =a\cos\phi$$

$$(u=\phi,\ v=\theta)$$



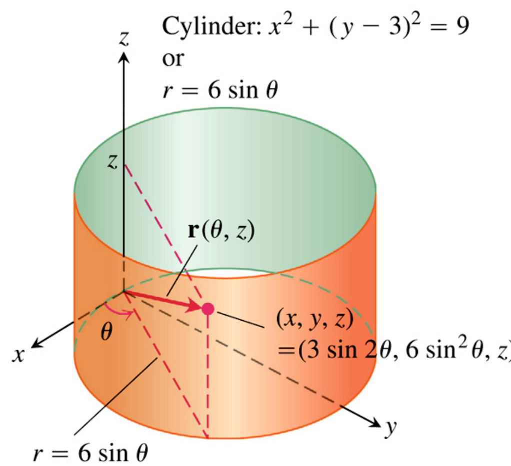
Example

Find a parametrization of the cylinder $x^2+(y-3)^2=9$ $\rightarrow$ $r=6\sin\theta$, $0\le z\le 5.$

Hint: Use cylindrical coordinates.

$$
\begin{aligned}
x &= r\cos\theta = 6\sin\theta\cos\theta = 3\sin 2\theta \\
y &= r\sin\theta = 6\sin\theta\sin\theta = 6\sin^2\theta \\
z &= z
\end{aligned}
$$

$(0\le\theta\le\pi,\ 0\le z\le 5)$



Definition

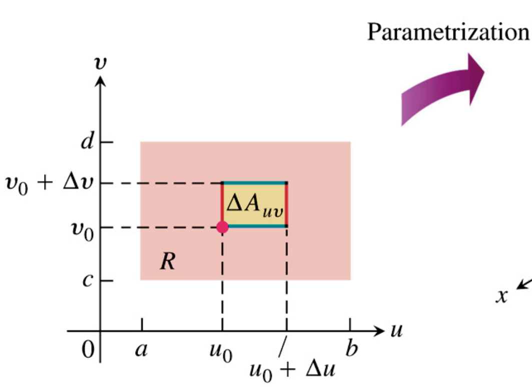
A parametrized surface $\mathbf{r}=\mathbf{r}(u,v)$ is ***smooth*** if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u\times\mathbf{r}_v$ is never zero on the interior of the parameter domain.

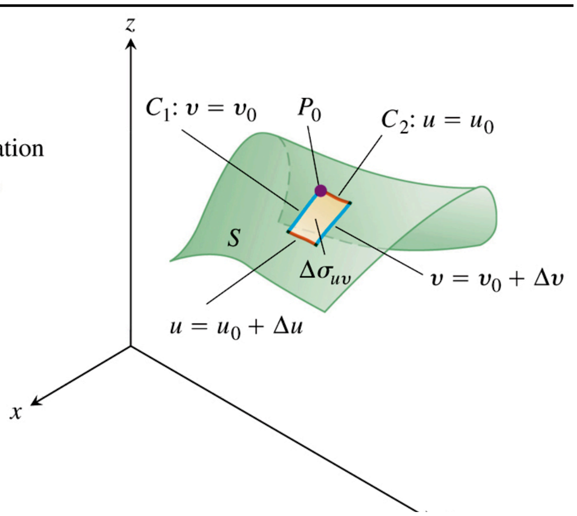
E.g. $\mathbf{r}=\mathbf{r}(\theta,z)=\langle 3\sin 2\theta, 6\sin^2\theta, z\rangle$, $0\le\theta\le\pi$, $0\le z\le 5$, is smooth:

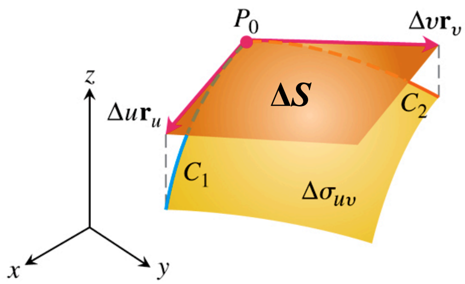
$$\mathbf{r}_\theta=\langle 6\cos 2\theta, 6\sin 2\theta, 0\rangle,\quad \mathbf{r}_z=\langle 0,0,1\rangle \quad \text{are continuous}$$

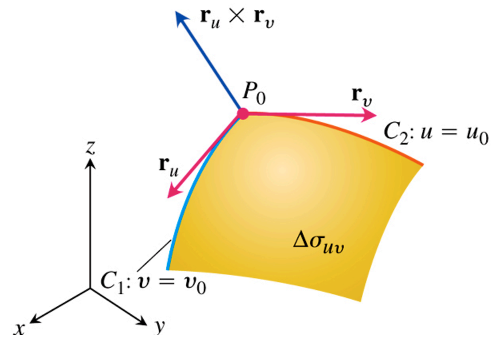
and $\mathbf{r}_\theta\times\mathbf{r}_z=\langle 6\sin 2\theta, -6\cos 2\theta, 0\rangle\neq\mathbf{0}$

The Area of a Smooth Surface









$$\Delta S = \left|\Delta u\mathbf{r}_u \times \Delta v\mathbf{r}_v\right| = \left|\mathbf{r}_u \times \mathbf{r}_v\right| \overbrace{\Delta u\Delta v}^{\Delta A}$$

Definition
The area of the smooth surface S parametrized by $\mathbf{r} = \mathbf{r}(u,v)$ $a \le u \le b,\ c \le v \le d$
is

$$A = \iint_R \left|\mathbf{r}_u \times \mathbf{r}_v\right| dA = \int_c^d \int_a^b \underbrace{\left|\mathbf{r}_u \times \mathbf{r}_v\right| du\, dv}_{d\sigma} = \iint_S d\sigma$$

Surface area differential

Definition
The area of the smooth surface S parametrized by $\mathbf{r} = \mathbf{r}(u,v)$ $a \le u \le b,\ c \le v \le d$
is

$$A = \iint_R \left|\mathbf{r}_u \times \mathbf{r}_v\right| dA = \int_c^d \int_a^b \underbrace{\left|\mathbf{r}_u \times \mathbf{r}_v\right| du\, dv}_{d\sigma} = \iint_S d\sigma$$

Surface area differential

Example
Find the surface area of the cone $z = \sqrt{x^2 + y^2},\ \ 0 \le z \le 1.$

Using the parametrization found before:

$$\mathbf{r} = \mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle \qquad (0 \le r \le 1,\ 0 \le \theta \le 2\pi)$$

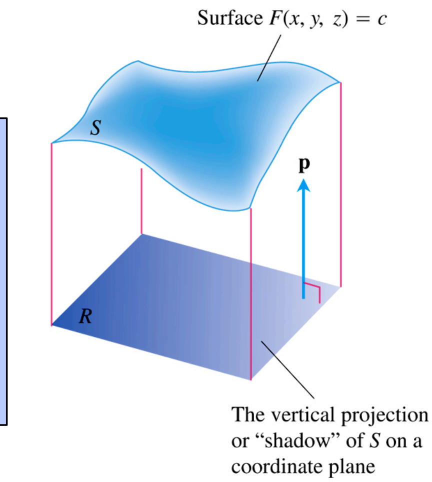
$$\mathbf{r}_r = \langle \cos\theta, \sin\theta, 1\rangle \qquad \mathbf{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$$

$$\mathbf{r}_r \times \mathbf{r}_\theta = \langle -r\cos\theta, -r\sin\theta,\ r\rangle = r\langle -\cos\theta, -\sin\theta,\ 1\rangle \qquad \left|\mathbf{r}_r \times \mathbf{r}_\theta\right| = \sqrt{2}r$$

$$A = \int_0^{2\pi} \underbrace{\int_0^1 \sqrt{2}\, r\, dr}_{\sqrt{2}/2}\, d\theta = \pi\sqrt{2}$$

The Area of a Surface Given Implicitly

(A level surface of a function)



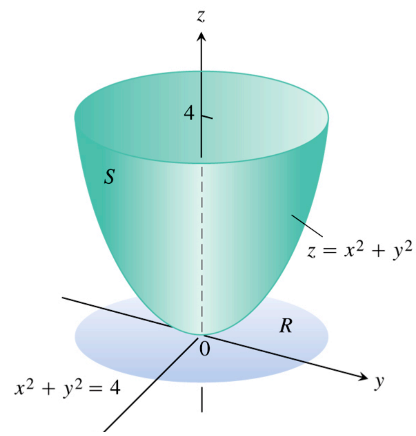
The area of the surface $F(x,y,z)=c$ over a closed, bounded plane region R is

$$A = \iint_R \frac{|\nabla F|}{|\nabla F \bullet p|}\, dA$$

where $p = \mathbf{i}, \mathbf{j},$ or $\mathbf{k}$ is normal to R and $\nabla F \bullet p \neq 0$.

Example

Find the area of the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plain $z = 4$.



$$F(x,y,z) = x^2 + y^2 - z = 0$$

$$\nabla F = \langle 2x, 2y, -1\rangle \qquad |\nabla F| = \sqrt{4x^2 + 4y^2 + 1}$$

$$|\nabla F \bullet p| = |\langle 2x, 2y, -1\rangle \bullet \langle 0,0,1\rangle| = |-1| = 1$$

$$(p = \mathbf{k})$$

So

$$A = \iint_R \frac{|\nabla F|}{|\nabla F \bullet p|}\, dA = \iint_{x^2+y^2\le 4} \sqrt{4x^2+4y^2+1}\ dxdy$$

Switch to polar coordinates

$$= \int_0^{2\pi} \int_0^2 \sqrt{4r^2+1}\ r\, dr\, d\theta = \tfrac{\pi}{6}\left(17\sqrt{17}-1\right)$$

$$u = 4r^2 + 1$$

$$\tfrac{1}{8}\int_1^{17} u^{1/2} du = \tfrac{1}{12}\left(17\sqrt{17}-1\right)$$