Picture the level curves of f:

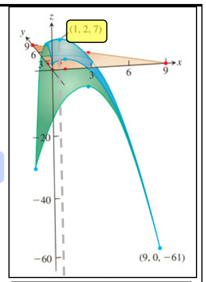
$$L_c = \{(x,y) \mid f(x,y)=c\}$$

e.g. $L_7 = \{(1,2)\}$ (a point),

for $c>7$, $L_c = \varnothing$ (the empty set),

for $c<7$, L_c is the circle centered at $(1,2)$ with radius $\sqrt{7-c}$.

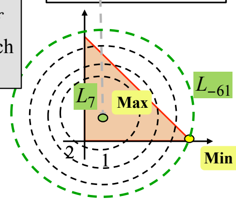
$$f(x,y) = 2+2x+4y-x^2-y^2 = c$$
$$x^2-2x+y^2-4y = 2-c$$
$$(x-1)^2+(y-2)^2 = 7-c$$



Max: $f(1,2)=7$

Min: $f(9,0)=-61$

Observation
The extreme values occur where the two level curves of f corresponding to the two extreme values $(c=7,-61)$ touch region R.

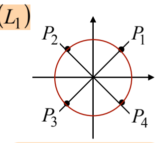


This is true for other functions and regions, including three-variable functions, and regions in space (level ***curves*** replaced with level ***surfaces***).

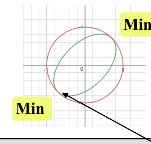
The last observation can be cleverly used if the region R itself is the ***level curve/surface*** of a differentiable function.

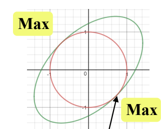
One more look at: f = temperature function

(L_1)



The unit circle is a level curve of $g(x,y)=x^2+y^2$





Observation
The extreme values occur where the two level curves of f corresponding to the two extreme values touch the circle.
i.e. are ***tangent*** to

Recall: $(\nabla f)_P$ is normal to the level curve of f through P.

Hence: Level curves of differentiable functions f and g are tangent at a point P iff their gradients at P are parallel; i.e. $(\nabla f)_P = \lambda(\nabla g)_P$, for some scaler λ.

(two variables)

Same is true for level ***surfaces*** of ***three***-variable functions.

The Method of Lagrange Multipliers

To find extreme values of $f(\bar{x})$ subject to the constraint

(*) $\qquad g(\bar{x})=c$

find all $\bar{x}$ satisfying (*) such that

$$\nabla f(\bar{x}) = \lambda \nabla g(\bar{x})$$

for some λ. ← **Lagrange multiplier**

$(\bar{x} = (x_1, x_2, \ldots, x_n))$

i.e. restricted to $L_c = \{\bar{x} \mid g(\bar{x})=c\}$

Example
Apply the method to $f(x,y)=x^2-xy+y^2$ and the constraint $\underbrace{x^2+y^2}_{g(x,y)}=1$. (*)

$\nabla f(x,y) = \langle 2x-y, 2y-x\rangle \qquad \nabla g(x,y) = \langle 2x,2y\rangle = 2\langle x,y\rangle$

$\langle 2x-y, 2y-x\rangle = \lambda(2\langle x,y\rangle)$ for some λ or, equivalently,

$\langle 2x-y, 2y-x\rangle = \lambda\langle x,y\rangle$ for some (other) λ

fails

$2x-y=\lambda x \longrightarrow y=(2-\lambda)x = (2-\lambda)^2 y$ $\qquad$ $\cancel{y=0}$ or $(\lambda-2)^2=1$ $\boxed{\lambda=1,3}$

$2y-x=\lambda y \longrightarrow x=(2-\lambda)y$ $\qquad$ $\cancel{x=0}$

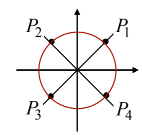
1

$\lambda=1$

$y=(2-1)x = x \xrightarrow{(*)} (x,y)= P_1, P_3$

$\lambda=3$

$y=(2-3)x=-x \xrightarrow{(*)} (x,y)=P_2,P_4$



$f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=\frac{1}{2}$ **Min**

$f\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=\frac{3}{2}$ **Max**

Example
Find the points on the hyperbolic cylinder $\underbrace{x^2-z^2}_{g(x,y)}=1$ (*) that are closest to the origin.

Find (x,y,z) on the surface (*) with the minimum distance to $(0,0,0)$
i.e. minimizing the value of the function

$$d(x,y,z)=\sqrt{x^2+y^2+z^2}$$

or, equivalently, minimizing the value of the function

$$f(x,y,z)=x^2+y^2+z^2$$

$\nabla f = \langle 2x,2y,2z\rangle = 2\langle x,y,z\rangle \qquad \nabla g = \langle 2x,0,-2z\rangle = 2\langle x,0,-z\rangle$

$\langle x,y,z\rangle = \lambda\langle x,0,-z\rangle$

$\langle x,y,z\rangle = \lambda\langle x,0,-z\rangle$ $\qquad$ $x^2-z^2=1$ (*)

$x=\lambda x \longrightarrow x-\lambda x=0 \longrightarrow x(1-\lambda)=0 \longrightarrow \cancel{x=0}$ (*) or $\boxed{\lambda=1}$

$y=0$

$z=-\lambda z$

satisfying (*)

$z=-z \longrightarrow z=0 \xrightarrow[(*)]{} x^2=1 \longrightarrow x=\pm 1$

$\boxed{(x,y,z)=(\pm 1,0,0)}$

$f(1,0,0)=f(-1,0,0)=1$ is an ***extreme*** value of f on the surface (*)
and it is indeed the ***minimum*** value on the surface because

$$x^2-z^2=1 \longrightarrow x^2=z^2+1\ge 1$$

so for every (x,y,z) satisfying (*)

$$f(x,y,z)=x^2+y^2+z^2 \ge x^2 \ge 1$$