Recall:

The Method of Lagrange Multipliers

To find extreme values of $f(\bar{x})$ subject to the constraint

$$(*) \qquad g(\bar{x}) = c$$

find all $\bar{x}$ satisfying $(*)$ such that

$$\nabla f(\bar{x}) = \lambda \nabla g(\bar{x})$$

Example

Find the points on the hyperbolic cylinder $\underbrace{x^2 - z^2}_{g(x,y)} = 1 \quad (*)$ that are closest to the origin.

Find (x, y, z) on the surface $(*)$ with the minimum distance to $(0,0,0)$

i.e. minimizing the value of the function

$$d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

or, equivalently, minimizing the value of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

$$\nabla f = \langle 2x, 2y, 2z \rangle = 2\langle x, y, z \rangle \qquad \nabla g = \langle 2x, 0, -2z \rangle = 2\langle x, 0, -z \rangle$$

$$\langle x, y, z \rangle = \lambda \langle x, 0, -z \rangle$$

$$\langle x, y, z \rangle = \lambda \langle x, 0, -z \rangle \qquad\qquad x^2 - z^2 = 1 \quad (*)$$

$x = \lambda x \longrightarrow x - \lambda x = 0 \longrightarrow x(1 - \lambda) = 0 \xrightarrow{(*)} \cancel{x = 0}$ or $\boxed{\lambda = 1}$

$y = 0$

$z = -\lambda z$

satisfying (*)

$z = -z \longrightarrow z = 0 \xrightarrow[(*)]{} x^2 = 1 \longrightarrow x = \pm 1$

$$\boxed{(x, y, z) = (\pm 1, 0, 0)}$$

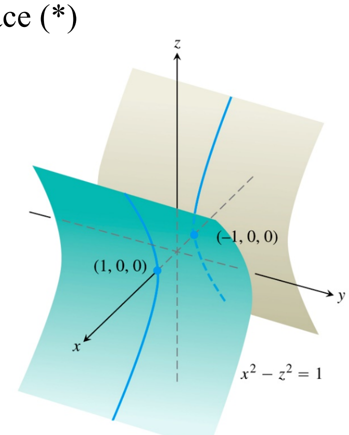
$f(1,0,0) = f(-1,0,0) = 1$ is an ***extreme*** value of f on the surface (*)

and it is indeed the ***minimum*** value on the surface because

$$x^2 - z^2 = 1 \longrightarrow x^2 = z^2 + 1 \geq 1$$

so for every (x,y,z) satisfying $(*)$

$$f(x, y, z) = x^2 + y^2 + z^2 \geq x^2 \geq 1$$



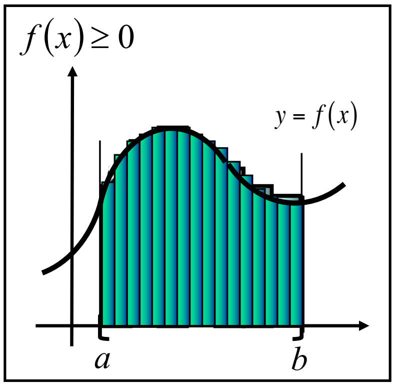
Double Integrals

Recall:

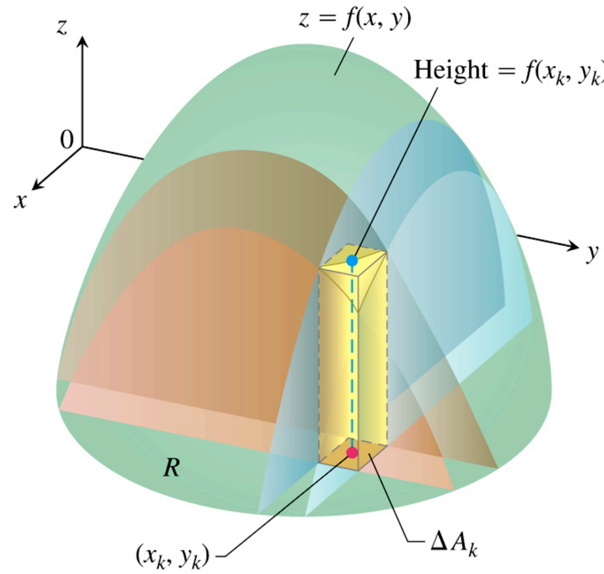
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

If $f(x) \ge 0$ on $[a,b]$ then

$$\int_a^b f(x)\,dx = \text{area below the graph}$$



For $f(x,y) \ge 0$ over a region R in the plane:

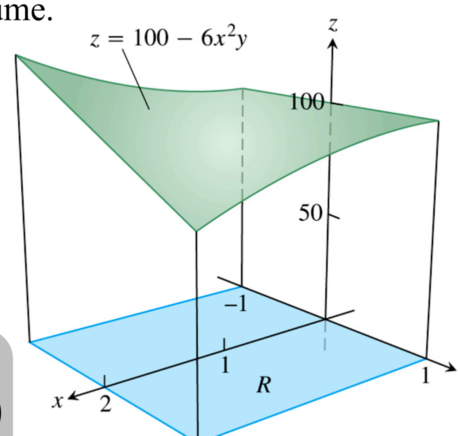


$$\text{Volume} = \lim \sum f(x_k, y_k)\,\Delta A_k = \iint_R f(x, y)\,dA$$

Example

Find $\iint_R f(x,y)\,dA$ for $f(x,y) = 100 - 6x^2 y$ and $R:\ 0 \le x \le 2,\ -1 \le y \le 1$

Since $f(x,y) \ge 0$ over R we can use Calc I to find volume.



A. By cross sections ***perpendicular*** to the **x-axis** :

$$V = \int_0^2 A(x)\,dx$$

where $A(x)$ is $\left\{\begin{array}{l}\text{the area of the slice obtained by} \\ \text{cutting the solid at location } x.\end{array}\right.$

= the area of the region below the curve C_x :

$z = g(y) = f(x,y)$ $\quad (x \text{ is fixed})$

in the (y,z)-plane between $y=-1$ and $y=1$;

i.e.

$$A(x) = \int_{-1}^{1} f(x,y)\,dy = \left(100y - 3x^2y^2\right)\Big|_{y=-1}^{y=1} = \left(100 - 3x^2\right) - \left(-100 - 3x^2\right) = 200$$

so

$$V = \int_0^2 A(x)\,dx = \int_0^2 200\,dx = (200x)\Big|_0^2 = \boxed{400}$$

B. By cross sections ***perpendicular*** to the **y-axis** :

$$V=\int_{-1}^{1}A(y)dy$$

where $A(y)$ is $\left\{\begin{array}{l}\text{the area of the slice obtained by}\\ \text{cutting the solid at location } y.\end{array}\right.$

= the area of the region below the curve C_y :

$$z=h(x)=f(x,y) \qquad (y \text{ is fixed})$$

in the (x,z)-plane between $x=0$ and $x=2$;

i.e.

$$A(y)=\int_{0}^{2}f(x,y)dx=\left(100x-2x^3y\right)\Big|_{x=0}^{x=2}=200-16y$$

so

$$V=\int_{-1}^{1}A(y)dy=\int_{-1}^{1}(200-16y)\,dy=\left(200y-8y^2\right)\Big|_{-1}^{1}=(200-8)-(-200-8)=\boxed{400}$$

Example

Find the volume of the prism in the picture;

i.e. $\displaystyle\iint_R f(x,y)dA$

for $f(x,y)=3-x-y$ and R the triangle below: