Math 2243, Calculus III, Formulae

Vectors and Projection

In $\mathbb{R}^2$, $\mathbf{v} = \langle x, y\rangle = x\mathbf{i} + y\mathbf{j}$ with $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$.
In $\mathbb{R}^3$, $\mathbf{v} = \langle x, y, z\rangle = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ with $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$ and $\mathbf{k} = \langle 0, 0, 1\rangle$.
Displacement vector: $\overline{PQ} = Q - P$, Position vector: $\overline{0Q} = Q$.
Dot Product: $\mathbf{a}\cdot\mathbf{b} = a_x\, b_x + a_y\, b_y + a_z\, b_z$
Length: $|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$, $|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$. Distance: $d(P, Q) = |\overline{PQ}| = |Q - P|$.
In $\mathbb{R}^2$, $|\langle x, y\rangle)| = \sqrt{x^2 + y^2}$. In $\mathbb{R}^3$, $|\langle x, y, z\rangle| = \sqrt{x^2 + y^2 + z^2}$.

Cross Product: $\mathbf{a}\times\mathbf{b} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = \langle a_y b_z - b_y a_z, b_x a_z - a_x b_z, a_x b_y - b_x a_y\rangle$.

Cross Product in $\mathbb{R}^2$: $\mathbf{a}\times\mathbf{b} = a_x b_y - b_x a_y$. (I.e., take $\langle a_x, a_y, 0\rangle \times \langle b_x, b_y, 0\rangle$ in $\mathbb{R}^3$).
Area of parallelogram: $|\mathbf{a}\times\mathbf{b}|$. Area of triangle: $|\mathbf{a}\times\mathbf{b}|/2$.

Volume of parallelepiped: $|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}| = \det\begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{bmatrix}$. Tetrahedron: $|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|/6$.

Angle between vectors $\mathbf{a}$ and $\mathbf{b}$: $\cos(\theta) = \mathbf{a}\cdot\mathbf{b}/|\mathbf{a}||\mathbf{b}|$, $\sin(\theta) = |\mathbf{a}\times\mathbf{b}|/|\mathbf{a}||\mathbf{b}|$.
Projection Theorem: $\mathbf{a} = \mathbf{a}_\| + \mathbf{a}_\perp$, $\mathbf{a}_\| = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$, $\mathbf{a}_\perp = \mathbf{a} - \mathbf{a}_\|$.
Length of projection: $|\mathbf{a}_\||^2 + |\mathbf{a}_\perp|^2 = |\mathbf{a}|^2$, $|\mathbf{a}_\|| = |\mathbf{a}\cdot\mathbf{b}|/|\mathbf{b}|$, $|\mathbf{a}_\perp| = |\mathbf{a}\times\mathbf{b}|/|\mathbf{b}|$.
Parametrized line from point Q in direction $\mathbf{b}$: $\mathbf{r}(t) = Q + t\,\mathbf{b}$.
Distance from a point P to this line is $|\mathbf{a}_\perp|$ using $\mathbf{a} = \overline{QP}$.
The point $S = Q + \mathbf{a}_\|$ is nearest to P from the line $\mathbf{r}(t)$.
Distance between skew lines $P_1 + t\,\mathbf{a}$ and $P_2 + s\,\mathbf{b}$ is $\overline{P_1P_2}\cdot\mathbf{n}/|\mathbf{n}|$ with $\mathbf{n} = \mathbf{a}\times\mathbf{b}$.
Distance from a point P to the plane through Q with normal $\mathbf{n}$ is $|\overline{QP}\cdot\mathbf{n}|/|\mathbf{n}|$.
Normal form of a plane through $\mathbf{x}_0$ with normal $\mathbf{n}$: $(\mathbf{x} - \mathbf{x}_0)\cdot\mathbf{n} = 0$ or $\mathbf{x}\cdot\mathbf{n} = \mathbf{x}_0\cdot\mathbf{n}$.
This reduces to $a\,x + b\,y + c\,z = d$ when $\mathbf{n} = \langle a, b, c\rangle$, $\mathbf{x} = \langle x, y, z\rangle$ and $d = \langle x_0, y_0, z_0\rangle\cdot\mathbf{n}$.
Parametrized plane: $\mathbf{r}(u, v) = \mathbf{r}_0 + u\,\mathbf{a} + v\,\mathbf{b}$. The normal vector is $\mathbf{n} = \mathbf{a}\times\mathbf{b}$.

Parametric Curves

$\mathbf{r}(t) = \langle x(t), y(t))\rangle$ or $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Let $\mathbf{r}' := d\mathbf{r}/\,dt$ and $\dot{\mathbf{r}} := d\mathbf{r}/\,ds$.
Differentiable if $\mathbf{r}'(t)$ exists. Regular if also $|\mathbf{r}'(t)| \neq 0$ for all t in $[a, b]$.
Position: $\mathbf{r}(t)$, Velocity: $\mathbf{v}(t) = \mathbf{r}'(t)$, Speed $v(t) = |\mathbf{v}(t)|$, Acceleration: $\mathbf{a} = \mathbf{r}''(t)$.
Arc length: $s = s(t) = \int_a^t v(t)\,dt$, $ds/dt = v(t)$, $dt/ds = 1/v(t)$.
Arc length parametrized curves have unit speed, i.e., $|\dot{\mathbf{r}}(s)| = 1$ for all $s \in [0, L]$.
Unit Tangent: $\mathbf{T} = \dot{\mathbf{r}}(s) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$. Note: $\mathbf{T}\cdot\mathbf{T} = 1 \Rightarrow \mathbf{T}'\cdot\mathbf{T} = \dot{\mathbf{T}}\cdot\mathbf{T} = 0$.
Unit Normal Vector: $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'| = \dot{\mathbf{T}}/|\dot{\mathbf{T}}| = \mathbf{B}\times\mathbf{T} = (\mathbf{a} - a_T\,\mathbf{T})/a_N$. In $\mathbb{R}^2$ take $\mathbf{N}\cdot\mathbf{r}''>0$.
Unit Binormal Vector: $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \mathbf{r}'\times\mathbf{r}''/|\mathbf{r}'\times\mathbf{r}''|$.
Acceleration: $\mathbf{a} = a_T\,\mathbf{T} + a_N\,\mathbf{N}$ with $a_T = v' = \mathbf{r}'\cdot\mathbf{r}''/|\mathbf{r}'|$ and $a_N = \kappa\, v^2 = |\mathbf{r}'\times\mathbf{r}''|/|\mathbf{r}'|$.
Curvature: $\kappa = |\dot{\mathbf{T}}| = |\mathbf{T}'|/v = |\mathbf{r}'\times\mathbf{r}''|/|\mathbf{r}'|^3$. Torsion: $\tau = (\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''/|\mathbf{r}'\times\mathbf{r}''|^2$, $|\tau| = |\dot{\mathbf{B}}|$.
Signed curvature of planar curves $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ is $\kappa = \dot{\phi} = (x'y'' - x''y')/|\mathbf{r}'|^3$.
Signed curvature of functions $y = f(x)$ is $\kappa = y''/(1 + (y')^2)^{3/2}$.
Osculating circle: Radius $\rho = 1/\kappa$, Center $\mathbf{c} = \mathbf{r} + \rho\mathbf{N}$.

Conic Sections

$$f(x,y) = a\,x^2 + 2\,b\,x\,y + c\,y^2 + 2\,d\,x + 2\,e\,y + f = \mathbf{x}^T Q \mathbf{x} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

Nondegenerate if $D := \det(Q) \neq 0$. Let $\Delta = a\,c - b^2$.
Ellipse: $D \neq 0$ and $\Delta > 0$; Parabola: $D \neq 0$ and $\Delta = 0$; Hyperbola: $D \neq 0$ and $\Delta < 0$.

Quadric Surfaces

Plane: $a\,x + b\,y + c\,z = d$. Sphere: $x^2 + y^2 + z^2 = \rho^2$. Cylinder: $f(x,y) = 0$ in $\mathbb{R}^3$.
Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Elliptic Paraboloid: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. Hyperbolic Paraboloid: $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$.
Elliptic Cone: $\frac{z^2}{c^2} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$. Hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. Two sheets: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

Gradient, Directional Derivative

Normal Vector to the surface $F(x,y,z) = 0$ is the gradient $\nabla F = \langle f_x, f_y, f_z \rangle$.
The vector $\mathbf{u}$ is a "direction" if $|\mathbf{u}| = 1$.
Directional Derivative of $F(\mathbf{x})$ at $\mathbf{x}_0$ in the direction $\mathbf{u}$: $D_\mathbf{u} F(\mathbf{x}_0) = \nabla F(\mathbf{x}_0) \cdot \mathbf{u}$.
$D_\mathbf{u} F(\mathbf{x})$ is maximized when $\mathbf{u} = \nabla F(\mathbf{x}) / |\nabla F(\mathbf{x})|$, which is the direction of greatest increase (steepest ascent) and perp to the contours $F(\mathbf{x}) = c$ (i.e., normal to the curve or surface $F(\mathbf{x}) = c$).
Chain Rule for $w = f(\mathbf{r}(t))$ with $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = \nabla F(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$$

Chain Rule for $w = f(x(s,t), y(s,t), z(s,t))$:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

Tangent Planes

TP to $z = f(x,y)$ at (a,b): $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$.
TP to $F(x,y,z) = 0$ at $\mathbf{x}_0$: $\nabla F(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$.
TP to $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ at (a,b):

$$TP(u,v) = \mathbf{r}(a,b) + (u-a)\,\mathbf{r}_u(a,b) + (v-b)\,\mathbf{r}_v(a,b)$$

Linear and Quadratic Approximations to $z = f(x,y)$ at (a,b)

Linear:
$$\begin{aligned} L(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &= f(a,b) + \nabla f(a,b) \cdot (x-a, y-b) \end{aligned}$$

Quadratic:
$$\begin{aligned} Q(x,y) &= L(x,y) + \tfrac{1}{2}\left(f_{xx}(a,b)(x-a)^2 + 2\,f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right) \\ &= L(x,y) + \tfrac{1}{2}\,[x-a,\; y-b] \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} \end{aligned}$$

Local Max/Min/Saddle for $z = f(x,y)$

Critical points: (a,b) such that $f_x(a,b) = f_y(a,b) = 0$.
Let $\Delta(a,b) := \det \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix} = f_{xx}(a,b) f_{yy}(a,b) - (f_{xy}(a,b))^2$.
If $\Delta > 0$ and $f_{xx}(a,b) > 0$ then (a,b) is a local minimizer and $Q(x,y)$ is an elliptic paraboloid opening up.
If $\Delta > 0$ and $f_{xx}(a,b) < 0$ then (a,b) is a local maximizer and $Q(x,y)$ is an elliptic paraboloid down.
If $\Delta < 0$ then (a,b) is a saddle point and $Q(x,y)$ is a hyperbolic paraboloid.

Constrained Max/Min

Extremize $f(\mathbf{x})$ subject to the constraint $g(\mathbf{x}) = 0$,
Lagrange Multipliers: Solve $\nabla f = \lambda\,\nabla g$ and $g = 0$ simultaneously.

Vector Fields

Vector Fields: $\mathbf{F}(\mathbf{x}) = \langle P(x,y), Q(x,y)\rangle$ or $\mathbf{F}(\mathbf{x}) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$

Del Operator: $\nabla := \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle$ or $\nabla := \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$

Gradient of $f(x,y,z)$: $\text{grad}(f) := \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$

Divergence: $\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

Curl: $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\rangle$

Identity: $\text{div}(\text{curl}(\mathbf{F})) = \nabla \cdot (\nabla \times \mathbf{F}) = 0$

Identity: $\text{curl}(\nabla f) = 0$ (i.e., gradient vector fields are conservative)

Conservative Vector Fields

Definition: The vector field $\mathbf{F}$ is *conservative* if $\mathbf{F} = \nabla f$ for some (potential) function f

Corollary: $\mathbf{F}$ is conservative iff $\nabla \times \mathbf{F} = 0$

Irrotational Vector Field: $\nabla \times \mathbf{F} = 0$

Conservation of Mass: $\nabla \cdot \mathbf{F} = 0$

Source: $\nabla \cdot \mathbf{F} > 0$; Sink: $\nabla \cdot \mathbf{F} < 0$

Path Independence: The value of $\int_C \mathbf{F} \cdot \, d\mathbf{r}$ is independent of the path C

Fundamental Theorem of line integrals: $\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$

Corollary: $\mathbf{F}$ is conservative $\Longrightarrow \int_C \mathbf{F} \cdot \, d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ and $\oint_C \mathbf{F} \cdot \, d\mathbf{r} = 0$

Theorem: $\mathbf{F}$ conservative iff path independent iff circulation $= 0$ iff curl $\mathbf{F}$=0 (irrotational)

Line Integrals of curves $\mathbf{r}(t) = \langle \mathbf{x}(t), \mathbf{y}(t), \mathbf{z}(t)\rangle$

Unit Tangent: $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$; Unit Normal: $\mathbf{N}$

Scalar line integral: $\int_c f(x,y)\, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt$ (Arc length when $f \equiv 1$)

Differentials: $d\mathbf{r} = \langle\, dx, \, dy\rangle$ or $d\mathbf{r} = \langle\, dx, \, dy, \, dz\rangle$

Vector line integrals: $W = \int_c \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_c \mathbf{F} \cdot \, d\mathbf{r}$

Green's Theorem (circulation): $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_{\partial R} P\, dx + Q\, dy = \int\int_R (Q_x - P_y)\, dA = \int\int_R \nabla \times \mathbf{F}\, dA$

Green's Theorem (flux): $\oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \int_{\partial R} P\, dy - Q\, dx = \int\int_R (P_x + Q_y)\, dA = \int\int_R \nabla \cdot \mathbf{F}\, dA$

Stoke's Theorem: $\oint_{\partial S} \mathbf{F} \cdot \mathbf{T}\, ds = \int\int_S \text{curl}(\mathbf{F}) \cdot \mathbf{N}\, dS$

Surface Integrals of $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

Unit Normal Vector (outward): $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v / |\mathbf{r}_u \times \mathbf{r}_v|$

Scalar Surface integrals: $\int\int_S f(\mathbf{r}(u,v))\, dS = \int\int_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$

Surface area when $f \equiv 1$

Rectangular: $\int\int_S f(x,y)\sqrt{1 + z_x^2 + z_y^2}\, dx\, dy$

Cylindrical: $\int\int_S f(r,\theta)\sqrt{r^2 + r^2 z_r^2 + z_\theta^2}\, dr\, d\theta$

Vector Surface integrals (Flux):

$$\int\int_S \mathbf{F} \cdot \mathbf{N}\, dS = \int\int_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv = \int\int_S P\, dy\, dz + Q\, dz\, dx + R\, dx\, dy$$

Divergence (Gauss') Theorem (for closed surfaces): $\int\int_{\partial V} \mathbf{F} \cdot \mathbf{N}\, dS = \int\int\int_V \nabla \cdot \mathbf{F}\, dV$

Coordinate Transformations

	From Cartesian	From Cylindrical	From Spherical
To Cartesian (x, y, z)		$x = r\cos(\theta)$ $y = r\sin(\theta)$ $z = z$	$x = \rho\sin(\varphi)\cos(\theta)$ $y = \rho\sin(\varphi)\sin(\theta)$ $z = \rho\cos(\varphi)$
To Cylindrical (r, θ, z)	$r^2 = x^2 + y^2$ $\tan(\theta) = y/x$ $z = z$		$r = \rho\sin(\varphi)$ $\theta = \theta$ $z = \rho\cos(\varphi)$
To Spherical (ρ, θ, ϕ)	$\rho^2 = x^2 + y^2 + z^2$ $\tan(\varphi) = \sqrt{x^2 + y^2}/z$ $\tan(\theta) = y/x$	$\rho^2 = r^2 + z^2$ $\tan(\varphi) = r/z$ $\theta = \theta$	

Mass

Density function in units mass per arc length, area, volume: $\delta(s)$, $\delta(x, y)$, $\delta(x, y, z)$.

When $\delta \equiv 1$, mass = length, area or volume.

	Curves	Areas	Volumes
Mass m =	$\int \delta(s)\,\mathrm{ds}$	$\int\int \delta(x, y)\,\mathrm{d}A$	$\int\int\int \delta(x, y, z)\,\mathrm{d}V$

First Moments

Center of mass: $(\overline{x}, \overline{y})$ or $(\overline{x}, \overline{y}, \overline{z})$.

When density is constant, center of mass = centroid.

	Curves	Areas	Volumes
$m\,\overline{x} =$	$\int x\,\delta(s)\,\mathrm{ds}$	$\int\int x\,\delta(x, y)\,\mathrm{dA}$	$\int\int\int x\,\delta(x, y, z)\,\mathrm{d}V$
$m\,\overline{y} =$	$\int y\,\delta(s)\,\mathrm{ds}$	$\int\int y\,\delta(x, y)\,\mathrm{dA}$	$\int\int\int y\,\delta(x, y, z)\,\mathrm{d}V$
$m\,\overline{z} =$	$\int z\,\delta(s)\,\mathrm{ds}$		$\int\int\int z\,\delta(x, y, z)\,\mathrm{d}V$

Second Moments (Moments of Inertia)

Moments of Inertia about axes: I_L, I_x, I_y, I_z.

Polar moment of inertia of plane lamina: $I_o = I_x + I_y$.

Radius of Gyration: R, R_x, R_y, R_z. $(R = \sqrt{I_L/m})$

	Curves	Areas	Volumes
$I_L = m\,R^2 =$	$\int r^2\,\delta(s)\,\mathrm{ds}$	$\int\int r^2\,\delta(x, y)\,\mathrm{d}A$	$\int\int\int r^2\,\delta(x, y, z)\,\mathrm{d}V$
$I_x = m\,{R_x}^2 =$	$\int(y^2 + z^2)\,\delta(s)\,\mathrm{ds}$	$\int\int y^2\,\delta(x, y)\,\mathrm{dA}$	$\int\int\int(y^2 + z^2)\,\delta(x, y, z)\,\mathrm{d}V$
$I_y = m\,{R_y}^2 =$	$\int(x^2 + z^2)\,\delta(s)\,\mathrm{ds}$	$\int\int x^2\,\delta(x, y)\,\mathrm{dA}$	$\int\int\int(x^2 + z^2)\,\delta(x, y, z)\,\mathrm{d}V$
$I_z = m\,{R_z}^2 =$	$\int(x^2 + y^2)\,\delta(s)\,\mathrm{ds}$		$\int\int\int(x^2 + y^2)\,\delta(x, y, z)\,\mathrm{d}V$