§*5.3 The Definite Integral* (Continued)

Exercise 20. Evaluate $\displaystyle\int_{-1}^{1}(1-|x|)\,dx$.

Definition. If f is integrable on $[a,b]$, then its **average value on $[\boldsymbol{a},\boldsymbol{b}]$**, which is also called its **mean**, is $\operatorname{av}(f)=\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$.

Example 5. Find the average value of $f(x)=\sqrt{4-x^2}$ on $[-2,2]$.

Video The Integral (28 min) Definition of the integral. Signed area. Interval additivity property.

§*5.4 The Fundamental Theorem of Calculus*

Theorem 3 (The Mean Value Theorem for Definite Integrals). If f is continuous on $[a,b]$, then at some point c in $[a,b]$, $f(c)=\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$.

Theorem 4 (The Fundamental Theorem of Calculus, Part 1). If f is continuous on $[a,b]$, then $F(x)=\displaystyle\int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b) and its derivative is $f(x)$: $F'(x)=\dfrac{d}{dx}\displaystyle\int_a^x f(t)dt=f(x)$.

Example 2. Find dy/dx if

(a) $y=\displaystyle\int_a^x(t^3+1)dt$ (b) $y=\displaystyle\int_x^5 3t\sin t dt$ (c) $y=\displaystyle\int_1^{x^2}\cos t dt$ (d) $y=\displaystyle\int_{1+3x^2}^{4}\frac{1}{2+e^t}dt$.

Exercise 80. Find $f(4)$ if $\displaystyle\int_0^x f(t)dt=x\cos(\pi x)$.

Theorem 4 Continued (The Fundamental Theorem of Calculus, Part 2). If f is continuous over $[a,b]$ and F is any antiderivative of f on $[a,b]$, then $\displaystyle\int_a^b f(x)dx=F(b)-F(a)$.

Example 3. Calculate (a) $\displaystyle\int_{-\pi/4}^{0}\sec x\tan x dx$ (c) $\displaystyle\int_1^4\left(\frac{3}{2}\sqrt{x}-\frac{4}{x^2}\right)dx$ (d) $\displaystyle\int_0^1\frac{dx}{x^2+1}$.

Theorem 5 (The Net Change Theorem). The net change in a differentiable function $F(x)$ over an interval $[a,b]$ is the integral of its rate of change: $F(b)-F(a)=\displaystyle\int_a^b F'(x)dx$.

Example 8. Find the area of the region between the x-axis and the graph of $f(x)=x^3-x^2-2x$, $-1\le x\le 2$.

Video The Fundamental Theorem of Calculus (26 min) Average value theorem. The function $F(x) = \int_a^x f(s)\ ds$. The fundamental theorem of calculus.

§5.5 Indefinite Integrals and the Substitution Method

Example 1. Find the integral $\displaystyle\int (x^3 + x)^5(3x^2 + 1)dx$.

Theorem 6 (The Substitution Rule). If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then $\displaystyle\int f(g(x)) \cdot g'(x)dx = \int f(u)du$.

Example 4. Find $\displaystyle\int \cos(7\theta + 3)d\theta$.

Example 5. Find $\displaystyle\int x^2 e^{x^3} dx$.

Example 6. Evaluate $\displaystyle\int x\sqrt{2x + 1}\ dx$.

Integrals of the Tangent, Cotangent, Secant, and Cosecant Functions.

$$\int \tan x dx = \ln|\sec x| + C \qquad \int \sec x dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x dx = \ln|\sin x| + C \qquad \int \csc x dx = -\ln|\csc x + \cot x| + C$$

Video Change of Variables (Substitution) (21 minutes)
Differentials. Using basic "u-substitutions" to find indefinite integrals and compute definite integrals.

§5.6 Definite Integral Substitutions and the Area Between Curves

Theorem 7 (Substitution in Definite Integrals). If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then $\displaystyle\int_a^b f(g(x)) \cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

Example 1. Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3 + 1}dx$.

Example 2. Evaluate (a) $\displaystyle\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta d\theta$.

Definition. If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b: $A = \displaystyle\int_a^b [f(x) - g(x)]dx$.

Example 4. Find the area of the region bounded above by the curve $y = 2e^{-x} + x$, below by the curve $y = e^x/2$, on the left by $x = 0$, and on the right by $x = 1$.

Example 5. Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Examples 6 & 7. Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Video Areas Between Curves (19 minutes)