*§4.1 Extreme Values of Functions on Closed Intervals* (Continued)

**Definition.** An interior point of the domain of a function $f$ where $f'$ is zero or undefined is a **critical point** of $f$.

Ex. Find all critical points of $f(x) = x^3 - 3x^2 + 2$.

*Solution.* $f'(x) = 3x^2 - 6x$. To find the critical point(s), we solve $3x^2 - 6x = 0 \Rightarrow x = 0$ and $x = 2$. Therefore, $x = 0,\ 2$ are the critical points of the function $y = f(x)$ on $\mathbb{R} = (-\infty, \infty)$. □

**Exercise 52.** Determine all critical points for $g(x) = \sqrt{2x - x^2}$.

*Solution.* $g'(x) = \frac{1}{2}(2x - x^2)^{-1/2}(2 - 2x)$. Solve $\frac{1}{2}(2x - x^2)^{-1/2}(2 - 2x) = 0$ to obtain $2 - 2x = 0$, that is, $x = 1$, which is the critical point in the interior of the domain $[0, 2]$. □

**Example 3.** Find the absolute maximum and minimum values of $f(x) = 10x(2 - \ln x)$ on the interval $[1, e^2]$.

**Example 4.** Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

[Answer: $x = 0$ where the derivative d.n.e.]
Ex. MML Determine the critical point(s) of $y = x^2 + \frac{54}{x}$.
[Answer: $x = 3$]

*§4.2 The Mean Value Theorem*

**Theorem 3 (Rolle's Theorem).** Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior $(a, b)$. If $f(a) = f(b)$, then there is at least one number $c$ in $(a, b)$ at which $f'(c) = 0$.

**Theorem 4 (The Mean Value Theorem).** Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior $(a, b)$. Then there is at least one point $c$ in $(a, b)$ at which $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

**Corollary 1.** If $f'(x) = 0$ at each point $x$ of an open interval $(a, b)$, then $f(x) = C$ for all $x$ in $(a, b)$, where $C$ is a constant.

**Corollary 2.** If $f'(x) = g'(x)$ at each point $x$ in an open interval $(a, b)$, then there exists a constant $C$ such that $f(x) = g(x) + C$ for all $x$ in $(a, b)$. That is, $f - g$ is a constant function on $(a, b)$.

## *§4.3 Monotonic Functions and the First Derivative Test*

**Definition.** Let $f$ be a function defined on an interval $I$ and let $x_1$ and $x_2$ be two distinct points in $I$.
1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then $f$ is said to be **increasing** on $I$.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then $f$ is said to be **decreasing** on $I$.

Ex. (a) $y = x^3$, $x \in \mathbb{R}$
(b) $y = \sqrt{x}$, $x \geq 0$
**Corollary 3.** Suppose that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. If $f'(x) > 0$ at each point $x$ in $(a, b)$, then $f$ is increasing on $[a, b]$. If $f'(x) < 0$ at each point $x$ in $(a, b)$, then $f$ is decreasing on $[a, b]$.

**Example 1.** Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which $f$ is increasing and on which $f$ is decreasing.

*Solution.* $f'(x) = 3x^2 - 12 = 3(x + 2)(x - 2)$. Solve $3(x + 2)(x - 2) = 0 \Rightarrow x = -2$ and $x = 2$ critical points. Making a table and using testing point method show that $f' > 0$ on $(-\infty, -2)$, increasing; $f' < 0$ on $(-2, 2)$, decreasing; $f' > 0$ on $(2, \infty)$, increasing. $\square$

**Example 2.** Find the critical points of $f(x) = x^{1/3}(x - 4)$. Identify the open intervals on which $f$ is increasing and decreasing. Find the function's local and absolute extreme values.

**Example 3.** Find the critical points of $f(x) = (x^2 - 3)e^x$. Identify the open intervals on which $f$ is increasing and decreasing. Find the function's local and absolute extreme values.

## *§4.4 Concavity and Curve Sketching*

**Definition.** The graph of a differentiable function $y = f(x)$ is
(a) **concave up** on an open interval $I$ if $f'$ is increasing on $I$;
(b) **concave down** on an open interval $I$ if $f'$ is decreasing on $I$.

**The Second Derivative Test for Concavity.** Let $y = f(x)$ be twice-differentiable on an interval $I$. If $f'' > 0$ on $I$, the graph of $f$ over $I$ is concave up. If $f'' < 0$ on $I$, the graph of $f$ over $I$ is concave down.

**Example 1.** (a) The curve $y = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.
(b) The curve $y = x^2$ is concave up on $(-\infty, \infty)$.

**Example 2.** Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

**Definition.** A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

**Example 3.** Determine the concavity and find the inflection points of the function $f(x) = x^3 - 3x^2 + 2$. See also Ex. in §4.1.

**Example 4.** Determine the concavity and find the inflection points of $f(x) = x^{5/3}$.

**Theorem 5 (Second Derivative Test for Local Extrema).** Suppose $f''$ is continuous on an open interval that contains $x = c$.
1. If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function $f$ may have a local maximum, local minimum, or neither.

**Example 8.** Sketch a graph of the function $f(x) = x^4 - 4x^3 + 10$.