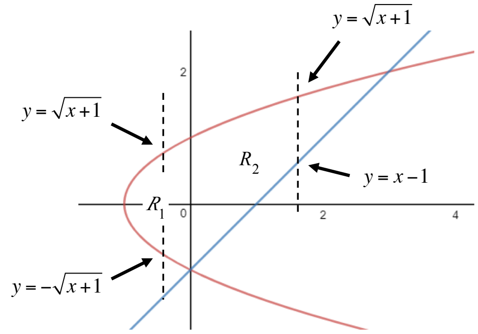
Example

Find the area of the region R bounded by $y^2 = x+1$ and $y = x-1$.



Solution 1

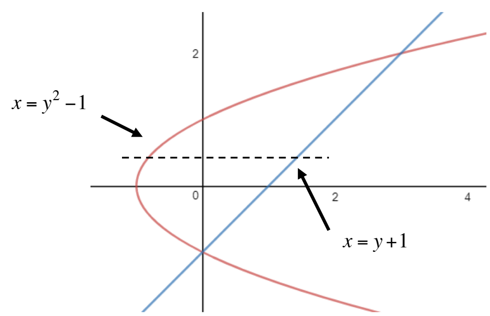
$$A_1 = \iint\limits_{R_1} dA = \int_{-1}^{0}\int_{-\sqrt{x+1}}^{\sqrt{x+1}} dydx = \int_{-1}^{0}\left(\sqrt{x+1}-\left(-\sqrt{x+1}\right)\right)dx = 2\int_{-1}^{0}\sqrt{x+1}dx = 2\left(\tfrac{2}{3}\left(x+1\right)^{3/2}\right)\Big|_{-1}^{0} = \tfrac{4}{3}$$

$$A_2 = \iint\limits_{R_2} dA = \int_{0}^{3}\int_{x-1}^{\sqrt{x+1}} dydx = \int_{0}^{3}\left(\sqrt{x+1}-\left(x-1\right)\right)dx = \left(\tfrac{2}{3}\left(x+1\right)^{3/2} - \tfrac{1}{2}x^2 + x\right)\Big|_{0}^{3}$$

$$= \left(\tfrac{2}{3}\left(4\right)^{3/2} - \tfrac{1}{2}3^2 + 3\right) - \tfrac{2}{3}\left(1\right)^{3/2} = \tfrac{19}{6}$$

$$A = A_1 + A_2 = \tfrac{4}{3} + \tfrac{19}{6} = \tfrac{9}{2}$$

Solution 2



$$A = \int_{-1}^{2}\int_{y^2-1}^{y+1} dxdy = \int_{-1}^{2}\left(y+1-\left(y^2-1\right)\right)dy$$

$$= \left(\tfrac{1}{2}y^2 + 2y - \tfrac{1}{3}y^3\right)\Big|_{-1}^{2}$$

$$= \left(2+4-\tfrac{8}{3}\right) - \left(\tfrac{1}{2} - 2 + \tfrac{1}{3}\right) = \tfrac{9}{2}$$

Definition

$$\textbf{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f \, dA.$$

EXAMPLE 3 Find the average value of $f(x, y) = x \cos xy$ over the rectangle R: $0 \le x \le \pi$, $0 \le y \le 1$.

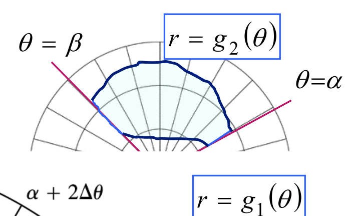
Solution

$$\int_0^\pi \int_0^1 x \cos xy \, dy \, dx = \int_0^\pi \left[\sin xy \right]_{y=0}^{y=1} dx \qquad \int x \cos xy \, dy = \sin xy + C$$

$$= \int_0^\pi (\sin x - 0) \, dx = -\cos x \Big]_0^\pi = 1 + 1 = 2.$$

The area of R is π. The average value of f over R is $2/\pi$.

Double Integrals in Polar Form



$$\iint_R f(r,\theta) dA = \lim_{n \to \infty} S_n$$

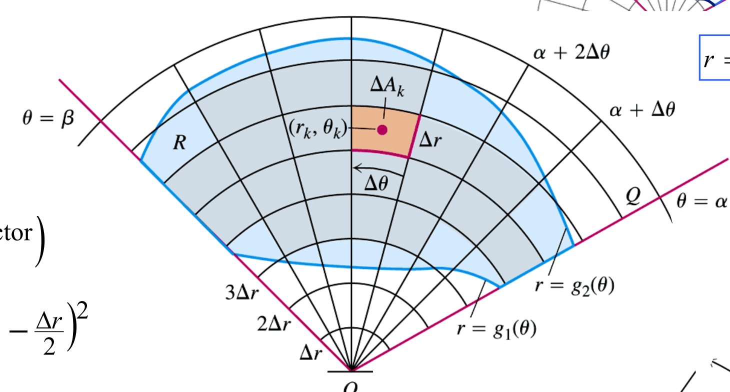
$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k) \boxed{\Delta A_k}$$

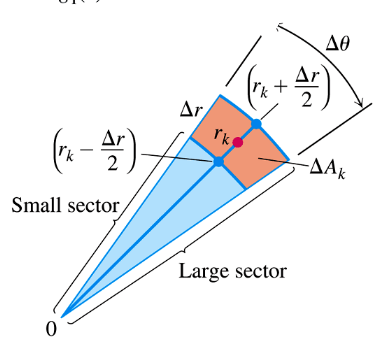
$$\Delta A_k = \begin{pmatrix}\text{large sector} \\ \text{area}\end{pmatrix} - \begin{pmatrix}\text{small sector} \\ \text{area}\end{pmatrix}$$

$$= \frac{\Delta\theta}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 - \frac{\Delta\theta}{2}\left(r_k - \frac{\Delta r}{2}\right)^2$$

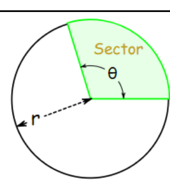
$$= \frac{\Delta\theta}{2} \underbrace{\left(\left(r_k + \frac{\Delta r}{2}\right) + \left(r_k - \frac{\Delta r}{2}\right)\right)}_{2r_k} \underbrace{\left(\left(r_k + \frac{\Delta r}{2}\right) - \left(r_k - \frac{\Delta r}{2}\right)\right)}_{\Delta r}$$

$$= \boxed{r_k \, \Delta r \, \Delta\theta}$$





$$A = \frac{\theta}{2\pi} \pi r^2 = \frac{\theta}{2} r^2$$



$$\iint_R f(r,\theta)\,dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

$$S_n = \sum_{k=1}^{n} f(r_k,\theta_k)\,\Delta A_k = \sum_{k=1}^{n} f(r_k,\theta_k)\, r_k\, \Delta r\, \Delta\theta \xrightarrow{n\to\infty}$$