Math 5530 Review Test I

Read each question carefully. Avoid making simple mistakes. Use the back of the page if necessary. You must show your work in order to receive full credits.

(1) A differential equation is an equation involving derivatives or differentials. Determine the type of the following equations by indicating the order, ODE/PDE, linear/nonlinear. If linear, tell if it is homogeneous or inhomogeneous.

a $(y'')^2 - 6x = (y')^3$
b $y' = \frac{a\cos x + b\sin y}{a\sin x + b\cos y}$ $\quad$ (a, b are constants)
c $\frac{d^2y}{dt^2} + 13\frac{dy}{dt} + 36y = 4e^t$
d $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})U = 0$
e $y = xy' - y'^2$
f $(x^2 - x)dy = (2x-1)ydx$
g $y' = x^2 + y^2$
h $u_t = \Delta u + f(t,x)$ $\quad$ (heat equation)
i $u_t + u_{xxx} + 6uu_x = 0$
j $u_t = \frac{\partial}{\partial x}F(u, u_x)$
k $u_t + uu_x = 0$ $\quad$ (Burgers equation)
l $\nabla \cdot (\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = 0$ $\quad$ (minimal surface equation)

(2) Find the solution to the initial value problem

$$x' = x\sin t + 2te^{-cost}, \quad x(0) = 1$$

(3) Determine if the equation is exact and solve it if it is.

$$(2x\sin y + 3x^2y)dx + (x^3 + x^2\cos y + y^2)dy = 0$$

(4) The ODE $-ydx + xdy = 0$ is not exact. Multiply by $1/x^2$ will make it exact. some other integrating factors are $1/y^2, 1/(xy), 1/(x^2+y^2)$. In general, given $Mdx + Ndy = 0$, by Theorem 1.4 and 1.5 in Section 1.4 (E. Kreyszig):

(a) If $R(x) := \frac{1}{N}(M_y - N_x)$ depends on x only, then the integrating factor

$$\mu(x) = e^{\int Rdx}$$

(b) If $R^*(x) := \frac{1}{M}(N_x - M_y)$ depends on y only, then the integrating factor

$$\mu(y) = e^{\int R^*dy}$$

Solve $(x^2+y^2)dx - 2xydy = 0$ [Hint: #5 in [Kreyszig Section 1.4]]

(5) Solve the equations. Determine if the differential equations are homogeneous. If so, determine its degree.

(a)

$$ydx + (y-x)dy = 0$$

(b) $xy' = y + 3x^4\cos^2(y/x)$, $y(1) = 0$. [Clue: substitution $y = xu$, [Kreyszig, Section 1.3, # 17]]

(6) * Find a general solution of
$$\frac{dy}{dx} = 6\frac{y}{x} - xy^2$$

(7) Find the solutions of $y^{(4)} + 8y'' + 16y = 0$ (answer: $y = c_1 \sin 2t + c_2 \cos 2t + c_3 t \sin 2t + c_4 t \cos 2t$)

(8) * Find the orthogonal trajectories of the family of curves
  (a) $xy = c$
  (b) $x^2 + y^2 = cx$
  (c) $y^2 = cx^2 - 2y$

(9) Let $D = d/dx$. Solve the Cauchy-Euler equation $(x^2D^2 + xD - 4)y = x^3$ [Clue I: change of variable $x = e^t$; clue II: let $y = x^k$]

(10) The operator $L := a_0(x)D^2 + a_1(x)D + a_2(x)$ is exact $\iff$ $a_0'' - a_1' + a_2 = 0$, in which case
$$Ly = (a_0D^2 + a_1D + a_2)y = D(a_0D + a_1 - a_0')y$$
Find the solution of $(1 - x^2)y'' - 3xy' - y = 1$.

(11) Figure 1 (Page 4) is the direction field for the differential equation $y' = y(y-1)(y+1)$.
  (a) Draw on the direction field the solutions of the differential equation satisfying each of the following initial values.
    (i) $y(0) = 0.0$
    (ii) $y(0) = 0.5$
    (iii) $y(0) = -1.5$
  (b) For the solution $y(t)$ with initial condition $y(0) = 0.5$, what is $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?
  (c) For the solution $y(t)$ with initial condition $y(0) = -1.5$, what is $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

(12) Figure 2 (Page 4) is the direction field for the differential equation $y' = y - t$.
  (a) Draw on the direction field the solutions of the differential equation satisfying each of the following initial values.
    (i) $y(0) = 0.0$
    (ii) $y(0) = 1.0$
    (iii) $y(0) = -1.0$
    (iv) $y(0) = 2.0$
  (b) Are there any constant solutions $y = c$ to this differential equation? If so, show them on the direction field.
  (c) Are there any straight line solutions $y = mt + b$? If so indicate them on the direction field.
  (d) There is a number c such that all solutions with initial condition $y(0) > c$ satisfy $\lim_{t\to\infty} = \infty$ and all solutions with initial condition $y(0) < c$ satisfy $\lim_{t\to\infty} = -\infty$. Find this number c by inspecting the direction field.

Figure 1. Direction Field for Exercise 11

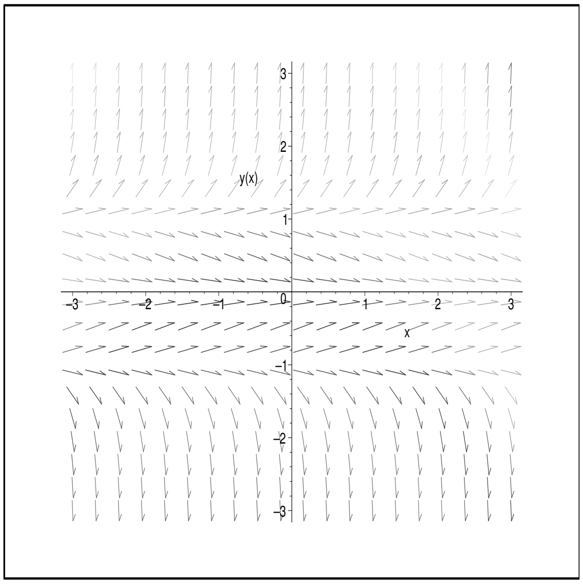
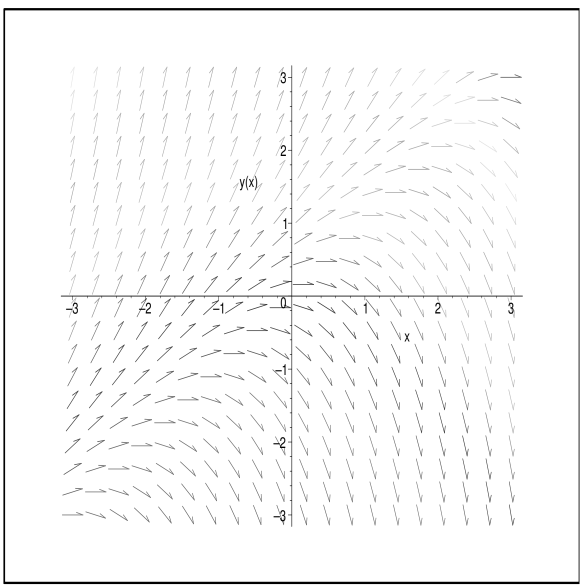


Figure 2. Direction Field for Exercise 12



(13) Solve each of the following initial value problems. You **must** show your work to tell if they are unique?
(a) $y' = \pm y^{-3}, \quad y(1) = -1.$
(b) $y' = |y|^{2/3}, \quad y(t_0) = y_0$
(c) $y' + \dfrac{3}{t}y = 7t^3, \quad y(1) = -1.$

(14) [# 5, Kreyszig, Section 2.4] What are the frequencies of vibration of a body of mass $m = 5$ kg

(a) on a spring of modulus $k_1 = 20nt/m$
(b) on a spring of modulus $k_2 = 45nt/m$
(c) on the two springs in parallel?

(15) [# 7, Kreyszig, Section 2.4] Find the frequency of oscillation of a pendulum of mass m and of length L, neglecting air resistance and the weight of the rod, and assuming the angle θ to be so small that $\sin\theta$ practically equals θ.

(16) [Ex.2, Sec. 2.4, Kreyszig] Consider the damped system $my" + cy' + ky = 0$ with IC $y(0) = 0.16m, y'(0) = 0$ where $m = 10, k = 90$ under the following conditions
(a) $c = 100$kg/sec,
(b) $c = 60$kg/sec,
(c) $c = 10$kg/sec.
[Clue: a) $y = -0.02e^{-9t} + 0.18e^{-t}$ (overdamping)
b) $y = (0.16 + 0.48t)e^{-3t}$ (critical damping)
c) $y = e^{-t/2}(0.16\cos 2.96t + 0.027\sin 2.96t)$ (underdamping)]

Solutions

2. This is first order linear ODE $x' + Px = Q$, where $P = -\sin t, Q = 2te^{-cost}$. The general formula gives

$$
\begin{aligned}
x(t) = e^{-\int P}\int e^{\int P}Qdt &= e^{\int(\sin t)}\int e^{\int(-\sin t)}2te^{-cost}dt \\
=& e^{\int(\sin t)}\int e^{\cos t}2te^{-cost}dt = e^{\int(\sin t)}\int 2tdt \\
=& e^{-cost}(t^2 + C)
\end{aligned}
$$

Now plugging in $t = 0, x = 1$ to obtain $C = e$.

3. It is Exact by the following test: The differential form $Mdx + Ndy = 0$ is exact $\iff$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Since it is exact, there exists $f(x, y)$ such that $df = f_x dx + f_y dy = Mdx + Ndy$. We will solve f to obtain the equation $f(x, y) = C$ which implicitly defines the solution of

$$(2x\sin y + 3x^2y)dx + (x^3 + x^2\cos y + y^2)dy = 0$$

From $\frac{\partial f}{\partial x} = 2x\sin y + 3x^2 y$ we get

$$f(x,y) = \int (2x\sin y + 3x^2y)dx = x^2 \sin y + x^3 y + C(y)$$

Taking derivative in y of the above yields

$$\begin{aligned}\partial_y f(x,y) &= \partial_y (x^2 \sin y + x^3 y + C(y)) \\ =& x^2\cos y + x^3 + C'(y) = N = x^3 + x^2 \cos y + y^2\end{aligned}$$

which suggests $C'(y) = y^2 \to C(y) = y^3/3$. Hence we arrive at the equation

$$f(x,y) = x^2\sin y + x^3 y + y^3/3 = C.$$

6*. This is first-order quadratic equation (Bernoulli type). $n = 2$ Substitution $w = y^{1-n} = y^{-1} \to y = w^{-1}$. We have $\frac{dy}{dx} = -w^{-2}\frac{dw}{dx}$ and so

$$\begin{aligned}& -w^{-2}\frac{dw}{dx} = 6\frac{w^{-1}}{x} - xw^{-2} \\ & \text{(multiplying } -w^2 \text{ both sides} \to) \quad \frac{dw}{dx} = -6\frac{w}{x} + x\end{aligned}$$

This is a 1st-order ODE, you can solve to get $w = w(x)$ and then replace w by y^{-1} and then simplify to obtain the solution $y = y(x)$.

Indeed,

$$\begin{aligned}w = w(x) &= e^{-\int \frac{6}{x}} \left(\int e^{\int \frac{6}{x}} x dx \right) \\ =& e^{-6\ln|x|}\left(\int x^6 x dx\right) = x^{-6}(x^8/8 + C) \\ =& x^2/8 + Cx^{-6}.\end{aligned}$$

From this we obtain $y = \frac{1}{x^2/8 + Cx^{-6}}$.

7 Solve the characteristic equation

$$\begin{aligned}& r^4 + 8r^2 + 16 = 0 \\ & (r^2+4)^2 = 0 \\ & r_{1,2} = \pm 2i, \quad r_{3,4} = \pm 2i.\end{aligned}$$

10 $a_0 = 1 - x^2$, $a_1 = -3x, a_2 = -1$, we find $a_0'' - a_1' + a_2 = -2 - (-3) + (-1) = 0 \Longrightarrow$ $D((1-x^2)D - 3x + 2x)y = D\left(((1-x^2)D - x)y\right) = 1$

$$\begin{aligned}& ((1-x^2)D - x)y = \int 1dx = x + C_1 \\ & (1-x^2)Dy - xy = x + C_1 \\ & Dy - \frac{x}{1-x^2}y = \frac{x+C_1}{1-x^2}\end{aligned}$$

$\square$