Review Final Exam **Name**

Math 2160 **Id**

Read carefully each problem. Show all your work in order to justify and support your answer. Credits will be given mainly depending on your work, not just an answer. Put a box around the final answer to a question. Use the back of the page if necessary.

(1) Which of the following equations are linear?
a) $xy + 3y = 1$
b) $3x - y + z = 9w$
c) $x \cos 15° + (2 - y) \sin 15° = \sqrt{2}$
d) $e^5 x - e^{11} y = 0$

(2) Write the system of linear equations in the form $A\mathbf{x} = \mathbf{b}$ and solve the matrix equation for $\mathbf{x}$ using either Gaussian elimination with back-substitution or Gauss-Jordan elimination.

(a)

$$\begin{cases} 3x_1 & & +12x_3 = -6 \\ -9x_1 & & -35x_3 = 2 \\ 18x_1 & +x_2 & +70x_3 = 8 \end{cases}$$

(b) $$\begin{cases} x_1 & -x_2 & -x_3 + x_4 & = 0 \\ x_1 & +x_2 & & = 5 \\ 2x_1 & +2x_2 & & = 10 \end{cases}$$

(3) Find the inverse of the matrix (if it exists).

(a) $\begin{pmatrix} 10 & -5 \\ 5 & -3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & -2 & -1 \\ 3 & -5 & -2 \\ 2 & -5 & -2 \end{pmatrix}$

(4) Compute the determinants.

$$(a) \quad \begin{vmatrix} 1 & 1 & 2 \\ 4 & 5 & 6 \\ 2 & 3 & -1 \end{vmatrix} \qquad (b^*) \quad \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \lambda & 0 \\ 0 & \lambda & 1 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix}$$

(5) Which of the following sets of vectors $x = [x_1, x_2, x_3, x_4]^T$ are subspace of $\mathbf{R}^4$?
a) All x such that $x_1 + x_2 = 7x_3$
b) All x such that $x_3 = 0$
c) All x such that $x_1 + x_4 = -12$

d) All x such that each x_i component is positive, that is, the first "I-quadrant" set $= \{x_i \geq 0, i = 1, 2, 3, 4\}$.

(6) Which of the following vectors, if any, is in the null space of $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{pmatrix}$?

a) $[-1\ 0\ 1\ 0]^T$ b) $[0\ 2\ 1\ -1]^T$ c) $[0\ 4\ 2\ -2]^T$

(7) [Testing for liner independence] Determine whether the following set S of vectors is linearly independent or linearly dependent?

(a) $S = \{(-2, 2), (3, 5)\}$ in $\mathbf{R}^2$ (4.4, #29)

(b) $S = \{(-4, -3, 4), (1, -2, 3), (6, 0, 0)\}$ in $\mathbf{R}^3$ (4.4, #37)

(c) $S = \{9, x^2, x^2 + 1\}$ in P_2.

(8) [True or False]

(a) A set of vectors $S = \{v_1, v_2, \ldots, v_k\}$ in a vector space is called linearly dependent when the vector equation $c_1v_2 + c_2v_2 + \cdots + c_kv_k = 0$ has only the trivial solution.
(4.4, #59 (a))

(b) The set $S = \{(1, 0, 0, 0), (0, -2, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ spans $\mathbf{R}^4$.

(c) A set $S = \{v_1, v_2, \ldots, v_k\}$, $k \geq 2$, is linearly independent if and only if at least one of the vectors v_j can be written as a linear combination of the other vectors.
(4.4, #60 (a))

(d) If a subset S spans a vector space V, then every vector in V can be written as a linear combination of the vectors in S.

(9) Determine which of the following statements are equivalent to the fact that a matrix A of size $n \times n$ is invertible?

a) A is nonsingular
b) The row space of A has dimension n
c) The determinant of A is nonzero
d) $A\mathbf{x} = \mathbf{b}$ has a unique solution for any given $\mathbf{b}$ in $\mathbf{R}^n$
e) The system $A\mathbf{x} = \mathbf{0}$ has nonzero solution
f) The dimension of the null space of A is zero
g) The rows of A are linear independent
h) The columns of A are linear independent
i) The rank of A is n
j) A is row-equivalent to an identity matrix
k) All eigenvalues of A are nonzero
l) A has n linear independent eigenvectors

m) A is similar to an diagonal matrix

n) A can be written as the product of elementary matrices.

(10) Find all the eigenvalues and eigenvectors of the given matrix.

a) $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(11) [optional*] The matrix $A = \begin{pmatrix} 6 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{pmatrix}$ has eigenvalues 5 and 8.

a) Find the eigenspaces E_5 and E_8 by solving $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

b*) By the theorem in Section 7.3 we know that a symmetric matrix of size n by n is always diagonalizable, equivalently speaking, always has n linear independent eigenvectors. Find an ordered basis consisting of eigenvectors of A.

c*) Specify the matrices P and D in the diagonalization $P^{-1}AP = D$

d*) Find an orthogonal matrix U such that $U^{-1}AU = D$ (Hint: An (real) orthogonal matrix means $U^{-1} = U^T$ or equivalently $U^TU = UU^T = I_n$).

Solutions

(2) $[62, 12, -16]$

(3) (b) Form the matrix $[A|I_3] = \left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 3 & -5 & -2 & 0 & 1 & 0 \\ 2 & -5 & -2 & 0 & 0 & 1 \end{array}\right)$ Then use row operation to reduce to $[I_3|B]$. Hence the inverse equals $B = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & -1 \\ -5 & 1 & 1 \end{pmatrix}$

(4) (a) -3 (b) $15(\lambda^2 - 1)$

(5) (a), (b) are vector subspaces;
(c), (d) are not subspaces.

(6) (b), (c)

(8) (a) False. (b) True. (c) False (d) True.

(9) (a), (b), (c), (d), (f), (g), (h), (i), (j), (k), (n)

(10) (a)

Solution. The eigenvalues are repeated, $\lambda_1 = \lambda_2 = 1$.

To solve for the corresponding eigenvectors, we plug $\lambda = 1$ in the linear equation $(I - A)X = 0$:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We then obtain the eigenvector $X = (x_1, x_2) = (t, 0) = t(1, 0)$, t is any real number. There we find that only one linearly independent vector in $E_1 = span\,\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$. $\square$

(b) $|\lambda I - A| = (\lambda + 1)^2(\lambda - 2)$

(11) (a)&(b) $E_5 = \text{span}\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\}$, $E_8 = \text{span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}$

(c*) $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}$

(d*) The orthogonal matrix U consists of three eigenvectors that are orthogonal in $\mathbf{R}^3$. So we need to orthogonalise the base $u = [-1, 1, 0]^T$, $v = [-1, 0, 1]^T$, $w = [1, 1, 1]$. Since the third vector in E_8 is orthogonal the any vectors in E_5. We only need to orthogonalise the two vectors in E_5 by Grant-Schmidt method. Let

$$u = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \tilde{v} = v - \frac{\langle v, u \rangle}{\langle u, u \rangle} u = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix},$$

Normalise the three vectors $u, \tilde{v}, w$ to obtain

$$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$