$$\left(8-x^2-y^2\right)-\left(x^2+3y^2\right)=8-2x^2-4y^2=2\left(4-x^2-2y^2\right)$$

$$V=\iiint_D dV=\int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx=2\int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\left(4-x^2-2y^2\right)dy\,dx$$

an even function of y

$$\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\left(4-x^2-2y^2\right)dy=2\int_{0}^{\sqrt{(4-x^2)/2}}\left(4-x^2-2y^2\right)dy=2\left(\left(4-x^2\right)y-\frac{2}{3}y^3\right)\Bigg|_0^{\sqrt{(4-x^2)/2}}$$

$$=\frac{2}{3}y\left(12-3x^2-2y^2\right)\Bigg|_0^{\sqrt{(4-x^2)/2}}$$

$$=\frac{2}{3}\sqrt{\left(4-x^2\right)/2}\left(12-3x^2-2\left(4-x^2\right)/2\right)=\boxed{\frac{2\sqrt{2}}{3}\left(4-x^2\right)^{3/2}}$$

$$\frac{\sqrt{2}}{3}\left(4-x^2\right)^{1/2} \qquad 8-2x^2=2\left(4-x^2\right)$$

$$\frac{4\sqrt{2}}{3}\int_{-2}^{2}\left(4-x^2\right)^{3/2}dx$$

So

an even function of x

$$V=\frac{4\sqrt{2}}{3}\int_{-2}^{2}\left(4-x^2\right)^{3/2}dx=\frac{8\sqrt{2}}{3}\int_{0}^{2}\left(4-x^2\right)^{3/2}dx=8\pi\sqrt{2}$$

$$\boxed{x=\sin 2u}$$

$$16\int_0^{\pi/2}\cos^4 u\,du=4\int_0^{\pi/2}\left(\tfrac{3}{2}+2\cos 2u+\tfrac{1}{2}\cos 4u\right)du=3\pi$$

$$\cos^2\theta=\tfrac{1}{2}\left(1+\cos 2\theta\right)$$

(apply twice)

Example

Set up the limits of integration for evaluating the triple integral over the tetrahedron with vertices (0, 0, 0), (1, 1, 0), (0, 1, 0), and (0, 1, 1). Use the order of integration *dy dz dx*

The ***y*-limits** *of* integration:

$$y = f_1(x,z) = x + z$$

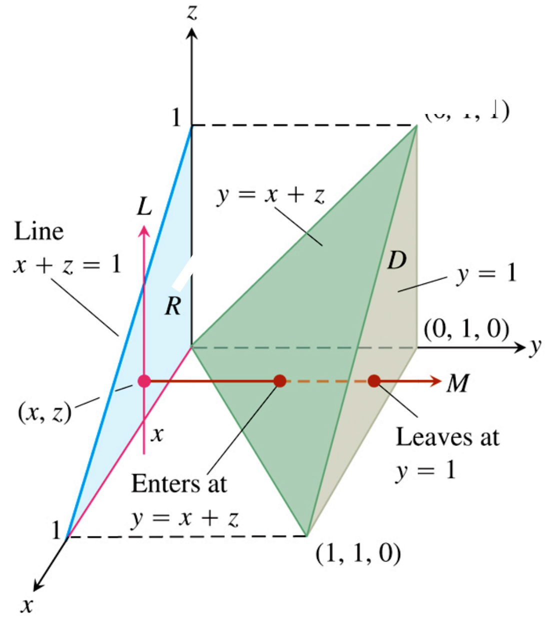
$$y = f_2(x,z) = 1$$

The ***z*-limits** *of* integration:

$$g_1(x) = 0$$

$$g_2(x) = 1 - x$$

$$\int_0^1 \int_0^{1-x} \int_{x+z}^1 F(x,y,z)\, dy\, dz\, dx$$



Example

Find the volume of the tetrahedron above.

$$V = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy\, dz\, dx \quad \ldots = \frac{1}{6}$$

Recalculate integrating in the order *dz dy dx*.

The ***z*-limits** *of* integration:

$$z = f_1(x,y) = 0$$

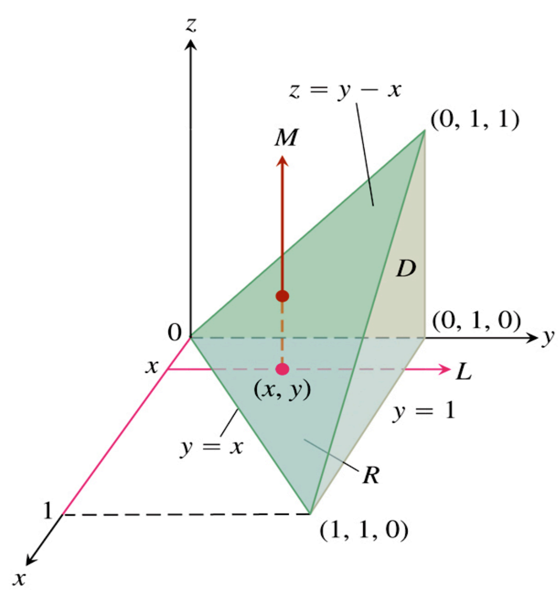
$$z = f_2(x,y) = y - x$$

The ***y*-limits** *of* integration:

$$g_1(x) = x$$

$$g_2(x) = 1$$

$$V = \int_0^1 \int_x^1 \int_0^{y-x} dz\, dy\, dx \quad \ldots = \frac{1}{6}$$



Definition

The ***average value*** of a function F over a region D in space is defined by

$$\frac{1}{\text{volume of } D} \iiint_D F \, dV$$

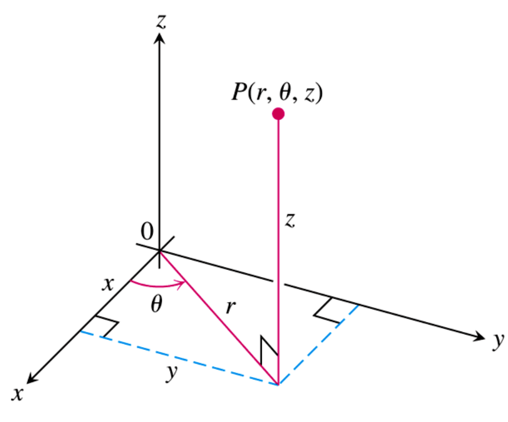
Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical Coordinates

$$P = (\underbrace{r, \theta}, z)$$

polar coordinates of the projection of P on the xy-plane

rectangular vertical coordinate



Conversion Formulas

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$x^2 + y^2 = r^2$

$\tan\theta = y/x$

Example

Describe the set given by the equation:

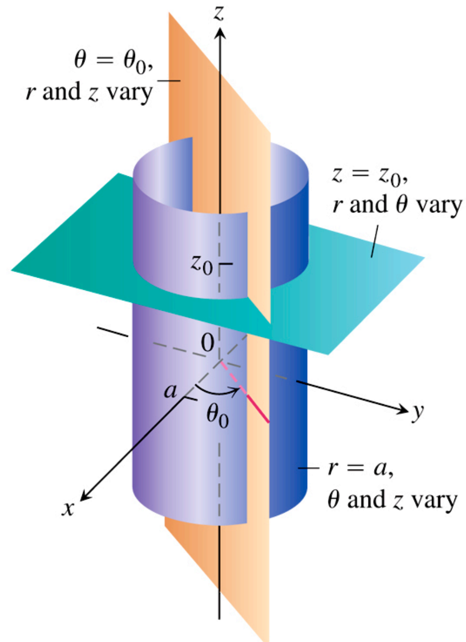
a) $r = a \quad (a \geq 0)$

The vertical cylinder around the z-axis with radius a $(z\text{-axis, if } a = 0)$

b) $\theta = \theta_0$

The plane that contains the z-axis and makes an angle θ_0 with the positive x-axis

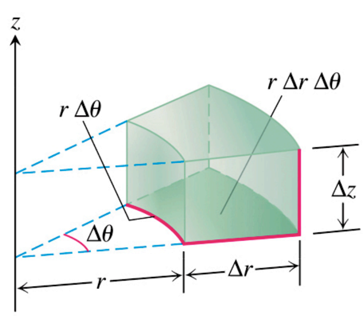
c) $z = z_0$ A plane perpendicular to the z-axis



The Definite Integral in Cylindrical Coordinates

$$\boxed{\iiint_D f\, dV = \int_{\cdots}^{\cdots}\int_{\cdots}^{\cdots}\int_{\cdots}^{\cdots} f(r,\theta,z)\; dz\; r\, dr\; d\theta}$$

$$\Delta V = \Delta z \cdot r\,\Delta r\,\Delta\theta$$



Example

Find the limits of integration in cylindrical coordinates for integrating a function $f(r,\theta,z)$ over the region D bounded below by the plane z=0, laterally by the cylinder $x^2 + (y-1)^2 = 1$ and above by the paraboloid $z = x^2 + y^2$.

$$\boxed{\iiint_D f(r,\theta,z)\, dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} f(r,\theta,z)\; dz\; r dr\; d\theta}$$