§1.1 Introduction to Systems of Linear Equations

Definition. A **linear equation in n variables** $x_1, x_2, x_3, \cdots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b.$$

The **coefficients** $a_1, a_2, a_3, \cdots, a_n$ are real numbers, and the **constant term** b is a real number. The number a_1 is the **leading coefficient**, and x_1 is the **leading variable**.

Example 1. Linear or nonlinear?

(a) $3x + 2y = 7$ (b) $\frac{1}{2}x + y - \pi z = \sqrt{2}$ (c) $(\sin \pi)x_1 - 4x_2 = e^2$

(d) $xy + z = 2$ (e) $e^x - 2y = 4$ (f) $\sin x_1 + 2x_2 - 3x_3 = 0$

Example 3. Solve the linear equation $3x + 2y - z = 3$.

Definition. A **system of m linear equations in n variables** is a set of m equations, each of which is linear in the same n variables:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1,\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2,\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3,\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}$$

A system of linear equations is also called a **linear system**.

Example 4. Solve and graph each system of linear equations.

(a) $\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$ (b) $\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$ (c) $\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$

Theorem. For a system of linear equations, precisely one of the statements below is true.
1. The system has exactly one solution (consistent system).
2. The system has infinitely many solutions (consistent system).
3. The system has no solution (inconsistent system).

Example 6. Sovle the system.

$$\begin{aligned}
x - 2y + 3z &= 9\\
y + 3z &= 5\\
z &= 2
\end{aligned}$$

Operations That Produce Equivalent Systems. Each of these operations on a system of linear equations produces an *equivalent* system.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

Example 7. Solve the system. Then check your answer.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Example 8. Solve the system.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 2x_1 - x_2 - 2x_3 &= 2 \\ x_1 + 2x_2 - 3x_3 &= -1 \end{aligned}$$

Example 9. Solve the system.

$$\begin{aligned} x_2 - x_3 &= 0 \\ x_1 - 3x_3 &= -1 \\ -x_1 + 3x_2 &= 1 \end{aligned}$$

§1.2 Gaussian Elimination and Gauss-Jordan Elimination

Definition. If m and n are positive integers, then an $m \times n$ matrix is a rectangular array

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

in which each **entry**, a_{ij}, of the matrix is a number. An $m \times n$ matrix has m rows and n columns. Matrices are usually denoted by capital letters.

Elementary Row Operations.
1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.