Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A copy of the table of Laplace transforms and convolution products from the text will be supplied, and these tables can be used for all problems.

1. [10 Points]

   (a) Complete the following definition: Suppose f and g are continuous functions defined for all $t \geq 0$. The **convolution** of f and g is the function $f * g$ defined as follows:

   $$f * g(t) = \boxed{\int_0^t f(\tau)g(t-\tau)\,d\tau}$$

   (b) Compute the convolution of t^2 and t^2 (you can use either the formula in part (a) or the table). By Table C, with $m = n = 2$,

   $$t^2 * t^2 = \frac{m!n!t^{m+n+1}}{(m+n+1)!} = \boxed{\frac{t^5}{30}}$$

2. [30 Points] Compute the Laplace transform of each of the following functions.

   (a) $f_1(t) = te^t$
   Using Table C.2:

   $$\boxed{F_1(s) = \frac{1}{(s-1)^2}}$$

   (b) $f_2(t) = te^{-t}\sin 3t$
   Using Formula 12 from Table C.2 with $n = 1$, $a = -1$, and $b = 3$:

   $$\begin{aligned} F_2(s) &= \mathrm{Im}\left(\frac{1}{(s-(-1+3i))^2}\right) \\ &= \mathrm{Im}\left(\frac{1}{((s+1)-3i))^2}\right) \\ &= \mathrm{Im}\left(\frac{1}{((s+1)-3i))^2}\frac{((s+1)+3i))^2}{((s+1)+3i))^2}\right) \\ &= \mathrm{Im}\left(\frac{(s+1)^2-9+6(s+1)i}{((s+1)^2+9)^2}\right) \\ &= \frac{6(s+1)}{((s+1)^2+9)^2}. \end{aligned}$$

   Alternatively, one can use Formula 8 from Table C.1 with $f(t) = e^{-t}\sin 3t$ so that $F(s) = \dfrac{3}{(s+1)^2+9}$ from Formula 9 of Table C.2. Then, since $f_2(t) = tf(t)$, Formula 8 (Table C.1) gives

$$\begin{aligned} F_2(s) &= -F'(s) = -\left(3((s+1)^2+9)^{-1}\right)' \\ &= -\left(3(-1)((s+1)^2+9)^{-2}2(s+1)\right) \\ &= \frac{6(s+1)}{((s+1)^2+9)^2}. \end{aligned}$$

(c) The solutions $y(t)$ of the initial value problem

$$2y'' - 3y' + 2y = te^t, \qquad y(0) = -1,\ y'(0) = 2.$$

Note that you are only asked to find the Laplace transform $Y(s)$ of $y(t)$, not $y(t)$ itself.

Taking the Laplace transform of each side of the equation gives:

$$2(s^2Y - (-1)s - 2) - 3(sY - (-1)) + 2Y = \frac{1}{(s-1)^2}.$$

Collecting the Y terms together gives

$$(2s^2 - 3s + 2)Y = -2s + 7 + \frac{1}{(s-1)^2},$$

and solving for Y:

$$\boxed{Y = \frac{-2s+7}{2s^2-3s+2} + \frac{1}{(s-1)^2(2s^2-3s+2)}.}$$

3. [30 Points] Compute the inverse Laplace transform of each of the following functions.

(a) $F(s) = \dfrac{2s^2-3s+1}{s^3}$

$F(s) = \dfrac{2}{s} - \dfrac{3}{s^2} + \dfrac{1}{s^3}$ so using Formula 3 (Table C.2) gives

$$\boxed{f(t) = 2 - 3t + \frac{1}{2}t^2.}$$

(b) $G(s) = \dfrac{s^2+4}{(s+1)(s^2-4)}$

Since $s^2 - 4 = (s+2)(s-2)$, the partial fraction expansion of $G(s)$ is

$$G(s) = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s+2}.$$

The constants are computed from the Heaviside expansion formula (Page 98):

$$\begin{aligned} A &= \frac{(-1)^2+4}{(-1-2)(-1+2)} = -\frac{5}{3} \\ B &= \frac{2^2+4}{((2+1)(2+2)} = \frac{2}{3} \\ C &= \frac{(-2)^2+4}{(-2+1)(-2-2)} = 2. \end{aligned}$$

Hence,

$$\boxed{g(t) = -\frac{5}{3}e^{-t} + \frac{2}{3}e^{2t} + 2e^{-2t}.}$$

(c) $H(s) = \dfrac{10}{s^2 + 2s + 5}$

Completing the square gives $s^2 + 2s + 5 = (s+1)^2 + 4$ and we can write $H(s)$ as

$$H(s) = \frac{10}{(s+1)^2 + 4} = \frac{10}{(s+1)^2 + 4} = 5\frac{2}{(s+1)^2 + 2^2}.$$

Then Formulas 9 (Table C.2) gives

$$\boxed{h(t) = 5e^{-t}\sin 2t.}$$

4. [20 Points] Solve the following initial value problem using Laplace transforms:

$$y'' - 3y' + 2y = 6, \quad y(0) = 1,\ y'(0) = 0$$

Letting $Y = \mathcal{L}\{y(t)\}$ and substituting in the differential equation gives

$$s^2Y - s - 0 - 3(sY - 1) + 2Y = \frac{6}{s}.$$

Then

$$(s^2 - 3s + 2)Y = s - 3 + \frac{6}{s}$$

and solving for Y gives (since $s^2 - 3s + 2 = (s-2)(s-1)$)

$$Y = \frac{s-3}{(s-2)(s-1)} + \frac{6}{s(s-2)(s-1)} = \frac{s^2 - 3s + 6}{s(s-2)(s-1)}$$

But

$$\frac{s^2 - 3s + 6}{s(s-2)(s-1)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-1}$$

where $A = 3$, $B = 2$ and $C = -4$. Then

$$y(t) = \mathcal{L}^{-1}\{Y\} = 3 + 2e^{2t} - 4e^{t}.$$

5. [10 Points]

(a) Verify that $y_p(t) = 2t - 3$ is a particular solution to the nonhomogeneous equation

$$y'' + 3y' + 2y = 4t.$$

Since $y_p'(t) = 2$ and $y_p''(t) = 0$, substitution into the equation gives

$$y_p'' + 3y_p' + 2y_p = 0 + 3\cdot 2 + 2(2t - 3) = 4t.$$

Hence $y_p(t) = 2t - 3$ is a solution.

(b) Verify that e^{-t} and e^{-2t} are two solutions to the associated homogeneous equation $y'' + 3y' + 2y = 0$

Substitute e^{-t} and e^{-2t} resp. into the homogeneous equation to check the left hand side should be equal to the right hand side, as in (a).

(c) What is the general solution of the equation in Part (a)?

The associated homogeneous equation is $y'' + 3y' + 2y = 0$ which has the general solution $y_h = c_1e^{-2t}+c_1e^{-t}$. The general solution of the nonhomogeneous equation is then

$$\boxed{y_{\text{gen}} = y_h + y_p = c_1e^{-2t} + c_1e^{-t} + 2t - 3.}$$

(d) Find the solution of the initial value problem

$$y'' + 3y' + 2y = 4t, \quad y(0) = y'(0) = 0.$$

It is necessary to choose c_1 and c_2 in the solution to part (b) so that $y(0) = y'(0) = 0$. This gives two equations

$$\begin{aligned} 0 = y(0) &= c_1 + c_2 - 3 \\ 0 = y'(0) &= -2c_1 - c_2 + 2, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} c_1 + c_2 &= 3 \\ 2c_2 + c_2 &= 2. \end{aligned}$$

This system is solved to give $c_1 = -1$ and $c_2 = 4$. Thus the solution of the initial value problem is

$$\boxed{y = 4e^{-t} - e^{-2t} + 2t - 3.}$$