Example

Identify the set of points whose coordinates satisfy the given condition:

a) $z = 1$ The horizontal plane 1 unit above the xy-plane (**parallel to** the xy-plane)

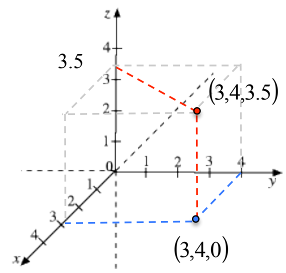
b) $z^2 = 1$ The pair of horizontal planes: 1 unit above, and 1 unit below the xy-plane

c) $z^2 \le 1$ The two planes from part b) together with all the points in between

d) $x = 3$ The plane parallel to the yz-plane, 3 units in front of it (see the picture)

e) $x = 3,\ z = 1$

The intersection of the two planes from a) and d);
i.e. a straight line parallel to the y-axis that intersects
the xz-plane at the point $(3, 0, 1)$



f) $y = 0$ The xz-plane

g) $x^2 + y^2 = 9,\ z = 1$ The circle of radius 3, in the plane from part a),
with the center at the point $(0, 0, 1)$

In the xy-plane: the circle of radius 3,
with the center at the origin.

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h) $x^2 + y^2 = 9,\ z^2 = 1$ The pair of circles of radius 3 in the planes form part b)
with the centers at $(0, 0, 1)$ and $(0, 0, -1)$

i) $x^2 + y^2 = 9,\ z^2 \le 1$ The cylinder extending between the two circles form part h)

j) $x^2 + y^2 = 9$ A cylinder like in part i) extending up and down infinitely

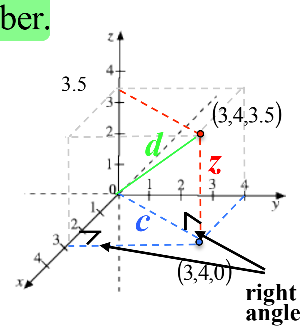
k) $x^2 + y^2 = z^2$ Two vertical cones extending infinitely up and down from the common vertex at the origin.

Hint: Same as

$x^2 + y^2 = a$ and $z^2 = a$, where a is any nonnegative real number.

Pair of horizontal circles of radius $\sqrt{a}$
with the centers at $\left(0, 0, \pm\sqrt{a}\right)$

Distance in Space



Example

How far is the point $(3, 4, 3.5)$ from the origin? (x, y, z)

By Pythagorean Theorem: $\boxed{d = \sqrt{x^2 + y^2 + z^2}} = \sqrt{3^2 + 4^2 + 3.5^2} \approx 6.1$

$$d^2 = c^2 + z^2 = x^2 + y^2 + z^2$$

and:

$$c^2 = x^2 + y^2$$

In general: The distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

$$|P_1P_2| = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Equation of a Sphere

A point $P(x, y, z)$ lies on the sphere of radius r centered at $P_0(x_0, y_0, z_0)$ iff $|P_0P| = r$

i.e.

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2 \qquad \textbf{(standard form)}$$

Example

For the sphere of radius 2 centered at $(3, -1, 0)$ we get

$$(x-3)^2 + (y+1)^2 + (z-0)^2 = 4$$

or, equivalently,

$$x^2 + y^2 + z^2 - 6x + 2y + 6 = 0 \qquad \text{(neither radius nor center are clear)}$$

Example

Find the radius and the center of the sphere

$$x^2 + y^2 + z^2 + 10x - 3z - 15 = 0 \qquad \textbf{(need standard form)}$$

Hint: Complete the squares on the x-, y-, z- terms as necessary to write as squared sum or differences.

$$x^2 + 10x = (x + \boxed{5})^2 - 25 \qquad y^2 = (y-0)^2 \qquad z^2 - 3z = \left(z - \tfrac{3}{2}\right)^2 - \frac{9}{4}$$

$$(x+5)^2 - 25 + (y-0)^2 + \left(z - \frac{3}{2}\right)^2 - \frac{9}{4} - 15 = 0$$

$$(x+5)^2 + (y-0)^2 + \left(z - \frac{3}{2}\right)^2 = 25 + \frac{9}{4} + 15 = \frac{169}{4} = \left(\frac{13}{2}\right)^2$$

center at $\left(-5, 0, \frac{3}{2}\right)$

$$\boxed{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2}$$

radius $= \frac{13}{2}$

Example

Identify the set of points whose coordinates satisfy the given condition:

a) $x^2 + y^2 + z^2 < 9$ The interior of the sphere of radius 3 centered at the origin.

b) $x^2 + y^2 + z^2 \le 9$ The sphere of radius 3 centered at the origin with its interior.

c) $x^2 + y^2 + z^2 > 9$ The exterior of the sphere from a).

d) $x^2 + y^2 + z^2 = 9$, $x^2 + y^2 = 4$ The intersection of the sphere as above with the infinite vertical cylinder of radius 2 centered at the origin:

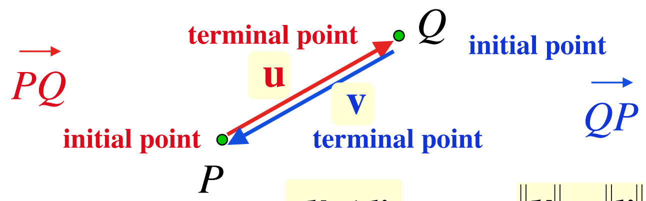
$z^2 = 5$ A pair of horizontal circles of radius 2 centered at $(0, 0, \pm\sqrt{5})$

Vectors

Quantities that require **direction** as well as **magnitude** in their description (force, displacement, velocity). Represented by directed line segments(arrows).

$$\boxed{\overrightarrow{PQ} \neq \overrightarrow{QP}}$$

$$\|\overrightarrow{PQ}\| = \|\overrightarrow{QP}\|$$



$\boldsymbol{u} \neq \boldsymbol{v}$ $\|\boldsymbol{u}\| = \|\boldsymbol{v}\|$

The length of the arrow signifies the magnitude of the vector.

Notation: $\|\overrightarrow{PQ}\|$ = the magnitude of $\overrightarrow{PQ}$.

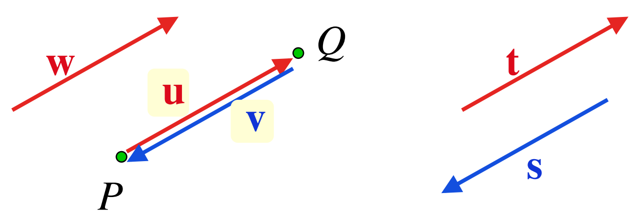
Often, single boldface letters are used to denote vectors.

Vectors with the same magnitude and direction are considered ***equal***.

Note
If the magnitude of a vector equals zero, its direction is undefined.
All such vectors are considered equal: the ***zero vector***, denoted by **0**.

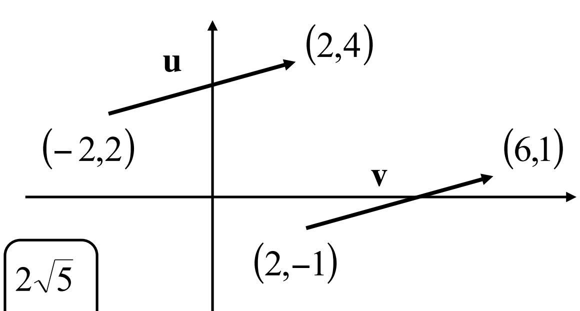
Example

$\mathbf{w} = \mathbf{u} = \mathbf{t}$
$\mathbf{s} = \mathbf{v}$



Example (in 2D Cartesian system)

Show that u=v



magnitudes:

$$\|\mathbf{u}\| = \sqrt{(2-(-2))^2 + (4-2)^2} = \sqrt{16+4} = \boxed{2\sqrt{5}}$$

$$\|\mathbf{v}\| = \sqrt{(6-2)^2 + (1-(-1))^2} = \sqrt{16+4} = \boxed{2\sqrt{5}}$$

direction: (compare the slopes)

$$m_{\boldsymbol{u}} = \frac{4-2}{2-(-2)} = \boxed{\frac{1}{2}} \qquad m_{\boldsymbol{v}} = \frac{1-(-1)}{6-2} = \boxed{\frac{1}{2}}$$

Conclusion:
u = v