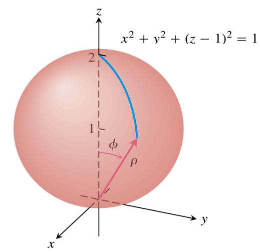
Conversion Formulas

$r = \rho\sin\phi$ $x = r\cos\theta = \rho\sin\phi\cos\theta$

$z = \rho\cos\phi$ $y = r\sin\theta = \rho\sin\phi\sin\theta$

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$$



Example

Convert to a spherical coordinate equation

a) $x^2+y^2+(z-1)^2 = 1$ (a sphere)

$x^2+y^2+z^2-2z+\cancel{1}=\cancel{1}$

$\rho^2 - 2\rho\cos\phi = 0$

$\rho(\rho - 2\cos\phi) = 0$

$\rho = 0$ or $\boxed{\rho = 2\cos\phi}$ (includes $\rho = 0$)

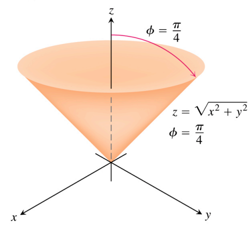
($\phi = \pi/2$)

b) $z = \sqrt{x^2+y^2} = \sqrt{r^2} = r$

$\rho\cos\phi = \rho\sin\phi$ (a cone)

$\rho = 0$ or $\cos\phi = \sin\phi \Rightarrow \boxed{\phi = \pi/4}$ $(0 \le \phi \le \pi)$

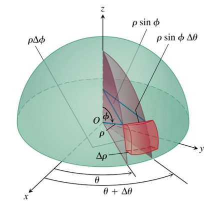
(included) $\longleftarrow$ (includes all $\rho \ge 0$)



The Definite Integral in Spherical Coordinates

$$\iiint_D f\,dV = \iiint_D f(\rho,\phi,\theta)\,\rho^2\sin\phi\;d\phi\,dr\,d\theta$$

$$\Delta V \approx \Delta\rho\cdot\rho\Delta\phi\cdot\rho\sin\phi\,\Delta\theta$$
$$= \rho^2\sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$$



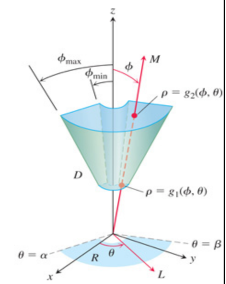
Finding Limits of Integration $(d\phi\,dr\,d\theta)$

1. Sketch D along with its projection R on the xy-plane. Label the bounding surfaces.
2. Draw a ray M from the origin through D, along with its projection L on the xy-plane.

ρ-limits: as ρ increases, M enters D at $\rho = g_1(\phi,\theta)$ and leaves at $\rho = g_2(\phi,\theta)$

ϕ-limits: For any given θ, the min and max values of the angle ϕ that M can make with the positive z-axis.

θ-limits: The min and max values of θ as ray L sweeps over R.

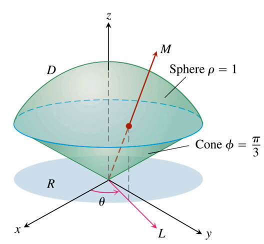


Example

Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \le 1$ by the cone $\phi = \pi/3$.

$$V = \iiint_D dV = \int_0^{2\pi}\int_0^{\pi/3}\underbrace{\int_0^1 \rho^2\sin\phi\,d\rho}_{\frac{1}{3}\sin\phi}\,d\phi\,d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\underbrace{\int_0^{\pi/3}\sin\phi\,d\phi}_{\left(-\cos\phi|\right)_0^{\pi/3} = \frac{1}{2}}\,d\theta = \frac{1}{6}\int_0^{2\pi}d\theta = \frac{\pi}{3}$$



Recalculate in the cylindrical coordinates.