Math 2065 Section 1
Final Exam Review Sheet

The final exam will be on Wednesday, Dec. 8 from 10:00 to 12:00 A.M. in the normal classroom. The exam is closed book, but you can bring the usual table of Laplace transforms.
The final exam is comprehensive, and thus the material we covered from Chapters 1 to 4, and Chapter 6 is a valid source for questions.
A good strategy for study is to do the review sheets, and some of the old exams and homework (if you have extra time) *without* looking at the answers. If you then compare with the answer sheets, you can identify the areas in which you need additional work. Some additional review exercises are included here, of exactly the same type as found in your text and on the previous review sheets.
I will be at office for the exam week. I will be happy to answer your questions with special appointments.

Review Exercises

Solve each of the following differential equations.

1. $y' = t - 2y$
2. $y' = t - 4ty$
3. $y' + \frac{4}{t}y = t^4$
4. $yy' = (t-1)^2$
5. $y' = 1 + t + y^2 + ty^2$
6. For the equation $y'(1-t) = y$,
   (a) Find the general solution.
   (b) Find the particular solution with $y(2) = 1$, and give its interval of existence.
7. Consider the initial value problem $ty' = e^t - y$, $y(1) = e$.
   (a) Without solving the equation, given the domain of existence of the solution, as guaranteed by the existence and uniqueness theorem.
   (b) Now solve the equation and see if your answer is indeed defined on the interval you found in Part (a).
8. $y'' - 3y' + 2y = 0$
9. $y'' + 2y' + 2y = 0$

10. $y'' + 4y' + 4y = 0$

11. $y'' + 6y' + 9y = 0$

12. $y'' - 6y' + 13y = 0$

13. $y'' + 16y = 0$

14. $y'' - 2y' - 3y = 0, \quad y(0) = 0, \quad y'(0) = 1$

15. $y'' + 6y' + 13y = 0, \quad y(0) = 1, \quad y'(0) = -1$

16. $y'' - 2y' - y = 0$

17. $y'' + 2y' - 15y = 0$

18. $y'' - 2y' + y = 3e^{2t}$

19. $y'' + 2y' + y = 2e^{-t}$

20. $y'' - y' - 2y = -9e^{-t}$

21. $y'' - 2y' + y = \dfrac{e^t}{t^5}$

22. $y'' + \dfrac{1}{t}y' - \dfrac{1}{t^2}y = \ln t$, $(t > 0)$. You may assume that a fundamental set for the associated homogeneous equation is $\left\{\varphi_1(t) = t,\, \varphi_2(t) = t^{-1}\right\}$.

Find the Laplace transform of each of the following functions.

23. $t^2 e^{-9t}$

24. $e^{2t} - t^3 + t^2 - \sin 5t$

25. $t \cos 6t$

26. $2 \sin t + 3 \cos 2t$

27. $e^{-5t} \sin 6t$

28. $t^2 \cos at$ where a is a constant

29. $f(t) = \begin{cases} 1 & \text{if } 0 \le 0 < 2, \\ -1 & \text{if } 2 \le t < 4, \text{ and} \\ 0 & \text{if } t \ge 4. \end{cases}$

30. $f(t) = (t^2 - 100)h(t - 10)$

Find the inverse Laplace transform of each of the following functions.

31. $\dfrac{1}{s^2 - 10s + 9}$

32. $\dfrac{2s - 18}{s^2 + 9}$

33. $\dfrac{2s + 18}{s^2 + 25}$

34. $\dfrac{s + 3}{s^2 + 5}$

35. $\dfrac{s - 3}{s^2 - 6s + 25}$

36. $\dfrac{1}{x(s^2 + 4)}$

37. $\dfrac{1}{s^2(s + 1)^2}$

38. $\dfrac{1 - e^{-s}}{s}$

39. $\dfrac{1 + e^{-\pi s}}{s^2 + 1}$

40. Solve the matrix differential equation $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$.

41. Consider a pond with 1000 cubic meters of water. There is a stream flowing out from the pond at a rate of 10 cubic meters a day. Nearby is a field which is regularly irrigated and fertilized. Each day, 10 cubic meters of water from the field enters the pond, and this is contaminated with 3 kilograms of ammonium nitrate per cubic meter. Write down a differential equation for the amount of ammonium nitrate in the pond at time t. Assume the ammonium nitrate is perfectly mixed and ignore the effect of rain and evaporation. Do not solve the equation.

42. (a) Verify that $\mathbf{\Phi}(t) = \begin{bmatrix} e^{2t} & 3e^{3t} \\ e^{2t} & 2e^{3t} \end{bmatrix}$ is a fundamental matrix of the system

    $$(*) \qquad \mathbf{y}' = A\mathbf{y}$$

    where $A = \begin{bmatrix} 1 & -4 \\ 1 & 5 \end{bmatrix}$.

    (b) Find a fundamental matrix $\mathbf{\Psi}(t)$ for $(*)$ such that $\mathbf{\Psi}(0) = I_2$ (where I_2 is the 2×2 identity matrix).

(c) Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

(d) Find e^{At}.

43. What is the largest interval $I = (a, b)$ for which the existence and uniqueness theorem guarantees that the following initial value problem has a solution?

$$\mathbf{y}' = \begin{bmatrix} \frac{t^2+2}{t+4} & t+2 \\ t-3 & \frac{t+4}{t-8} \end{bmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{bmatrix} 7 \\ -3 \end{bmatrix}.$$

44. Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

(a) Compute $(sI - A)^{-1}$.

(b) Find $\mathcal{L}^{-1}((sI - A)^{-1})$.

(c) What is e^{At}?

(d) Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$.

(e) Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Answers

1. $y = \frac{1}{2}t - \frac{1}{4} + ce^{-2t}$
2. $4y = 1 + ce^{-2x^2}$
3. $y = \frac{c}{t^4} + \frac{1}{9}t^5$
4. $3y^2 - 2(t-1)^3 = c$
5. $\arctan y - t - \frac{t^2}{2} = c$
6. (a) $y = \frac{c}{1-t}$

   (b) $y = \frac{1}{t-1}$. The interval of existence is $(1, \infty)$
7. (a) $(0, \infty)$ (b) $y(t) = \frac{e^t}{t}$
8. $y = c_1e^t + c_2e^{2t}$
9. $y = e^{-t}(c_1 \cos t + c_2 \sin t)$
10. $y = c_1e^{-2t} + c_2te^{-2t}$
11. $y = c_1e^{-3t} + c_2te^{-3t}$
12. $y = c_1e^{3t} \cos 2t + c_2e^{3t} \sin 2t$
13. $y = c_1 \cos 4t + c_2 \sin 4t$
14. $y = \frac{1}{4}(e^{3t} - e^{-t})$
15. $y = e^{-3t}(\cos 2t + \sin 2t)$
16. $y = c_1e^{(1+\sqrt{2})t} + c_2e^{(1-\sqrt{2})t}$

17. $y = c_1e^{3t} + c_2e^{-5t}$
18. $y = c_1e^t + c_2te^t + 3e^{2t}$
19. $y = c_1e^{-t} + c_2te^{-t} + t^2e^{-t}$
20. $y = c_1e^{-t} + c_2e^{2t} + 3te^{-t}$
21. $y = c_1e^t + c_2te^t + \frac{1}{12}t^{-3}e^t$
22. $y = c_1t + c_2t^{-1} + \frac{t^2}{3}\ln t - \frac{4}{9}t^2$
23. $\frac{2}{(s+9)^3}$
24. $\frac{1}{s-2} - \frac{6}{s^4} + \frac{2}{s^3} - \frac{5}{s^2+25}$
25. $\frac{s^2-36}{(s^2+36)^2}$
26. $\frac{2}{s^2+1} + \frac{3s}{s^2+4}$
27. $\frac{6}{(s+5)^2+36}$
28. $\frac{2s^3-6sa^2}{(s^2+a^2)^2}$
29. $\frac{1-2e^{-2s}+e^{-4s}}{s}$
30. $\frac{2e^{-10s}}{s^3} + \frac{20e^{-10s}}{s^2}$
31. $\frac{1}{8}(e^{9t} - e^t)$
32. $2\cos 3t - 6\sin 3t$
33. $2\cos 5t + \frac{18}{5}\sin 5t$
34. $\cos\sqrt{5}t + \frac{3}{\sqrt{5}}\sin\sqrt{5}t$
35. $e^{3t}\cos 4t$
36. $\frac{1}{4}(1-\cos 2t)$
37. $te^{-t} + 2e^{-t} + t - 2$
38. $1 - h(t-1)$
39. $\sin t\,(1 - h(t-\pi))$
40. $\mathbf{y} = \frac{1}{6}\begin{bmatrix} (5c_1 - c_2)e^{4t} + (c_1 + c_2)e^{-2t} \\ (-5c_1 + c_2)e^{4t} + (5c_1 + 5c_2)e^{-2t} \end{bmatrix}$
41. If $y(t)$ denotes the number of kilogram of ammonium nitrate at time t, then $y'(t) = 30 - \frac{y(t)}{100}$
42. The solution is similar to that to Exercise 3 or one Example in 6.3.
43. $(-4, 8)$
44. Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

    (a) Compute $(sI - A)^{-1}$. $sI - A = \begin{bmatrix} s-1 & -2 \\ 2 & s-1 \end{bmatrix}$ so $p(s) = \det(sI - A) = (s-1)^2 + 4$ and

$$(sI - A)^{-1} = \frac{1}{p(s)}\begin{bmatrix} s-1 & 2 \\ -2 & s-1 \end{bmatrix} = \begin{bmatrix} \dfrac{s-1}{(s-1)^2+4} & \dfrac{2}{(s-1)^2+4} \\ \dfrac{-2}{(s-1)^2+4} & \dfrac{s-1}{(s-1)^2+4} \end{bmatrix}.$$

(b) Find $\mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}$.

$$\begin{aligned}\mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} &= \mathcal{L}^{-1}\left\{\begin{bmatrix}\dfrac{s-1}{(s-1)^2+4} & \dfrac{2}{(s-1)^2+4}\\ \dfrac{-2}{(s-1)^2+4} & \dfrac{s-1}{(s-1)^2+4}\end{bmatrix}\right\}\\ &= \begin{bmatrix}e^t\cos 2t & e^t\sin 2t\\ -e^t\sin 2t & e^t\cos 2t\end{bmatrix}.\end{aligned}$$

(c) What is e^{At}?

$$e^{At} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = \begin{bmatrix}e^t\cos 2t & e^t\sin 2t\\ -e^t\sin 2t & e^t\cos 2t\end{bmatrix}.$$

(d) Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$. The general solution is given by $\mathbf{y}(t) = e^{At}\mathbf{y}(0)$, so if we write $\mathbf{y}(0) = \begin{bmatrix}c_1\\ c_2\end{bmatrix}$ where c_1 and c_2 are arbitrary constants, then the general solution is given by

$$\mathbf{y}(t) = e^{At}\mathbf{y}(0) = \begin{bmatrix}e^t\cos 2t & e^t\sin 2t\\ -e^t\sin 2t & e^t\cos 2t\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}c_1e^t\cos 2t + c_2e^t\sin 2t\\ -c_1e^t\sin 2t + c_2e^t\cos 2t\end{bmatrix}.$$

(e) Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix}2\\ 1\end{bmatrix}$. Take $c_1 = 2$ and $c_2 = 1$ in the previous step to get

$$\mathbf{y}(t) = \begin{bmatrix}2e^t\cos 2t + e^t\sin 2t\\ -2e^t\sin 2t + e^t\cos 2t\end{bmatrix}.$$