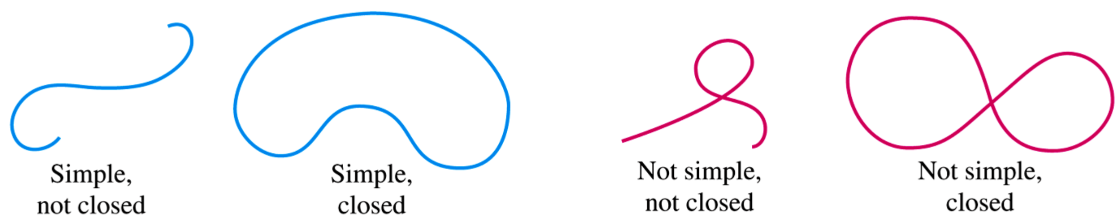
Flux Across a Simple Closed Plane Curve

A plane curve is

- ***simple*** if it does not cross itself
- ***closed**,* or a ***loop***, if it starts and ends at the same point.



Flux Across a Plane Curve

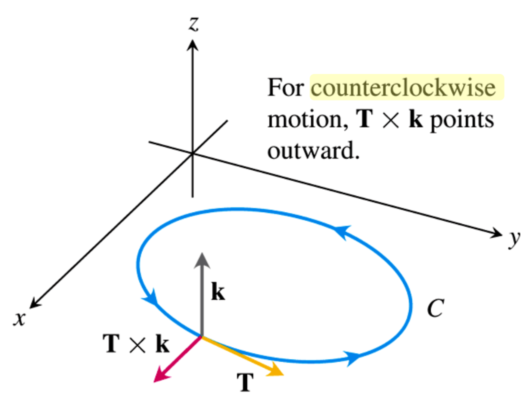
If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ then the ***flux*** of $\mathbf{F}$ across C is

$$\int_C \mathbf{F} \bullet \mathbf{n}\, ds$$

where $\boxed{\mathbf{n}}$ is the outward-pointing unit normal vector on C.

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}$$

$$\boxed{\mathbf{n} = \mathbf{T}\times\mathbf{k}} = \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\right)\times\mathbf{k} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}$$



$$\mathbf{F}\bullet\mathbf{n} = \langle M(x,y), N(x,y)\rangle \bullet \left\langle \frac{dy}{ds}, -\frac{dx}{ds} \right\rangle$$

$$= M(x,y)\frac{dy}{ds} - N(x,y)\frac{dx}{ds}$$

counterclockwise parametrization

$$\int_C \mathbf{F}\bullet\mathbf{n}\, ds = \int_C \left(M(x,y)\frac{dy}{ds} - N(x,y)\frac{dx}{ds}\right) ds = \oint_C M(x,y)\,dy - N(x,y)\,dx$$

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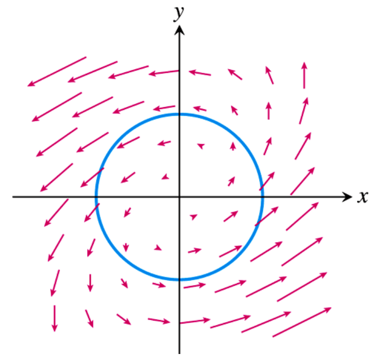
$$\int_C \mathbf{F}\bullet\mathbf{n}\, ds = \oint_C M\,dy - N\,dx$$

Evaluate with any smooth parametrization that traces C ***counterclockwise*** exactly once.

where $\boxed{\mathbf{n}}$ is the outward-pointing unit normal vector on C.

Example

Find the flux of $\mathbf{F}(x,y)=\overset{M}{(x-y)}\mathbf{i}+\overset{N}{(x)}\mathbf{j}$ across the circle $x^2+y^2=1$.



$$\text{Flux} = \oint_C (x-y)\,dy - x\,dx = \int_a^b \left((x-y)\frac{dy}{dt} - x\frac{dx}{dt}\right)dt$$

where *a, b*, $x(t)$ and $y(t)$ come from any smooth parametrization that traces the circle counterclockwise exactly once; e.g. with $x=\cos t, y=\sin t,\ 0\le t\le 2\pi$ we get

$$= \int_0^{2\pi}\left((\cos t-\sin t)(\cos t)-(\cos t)(-\sin t)\right)dt$$

$$= \int_0^{2\pi}\cos^2 t\,dt = \tfrac{1}{2}\int_0^{2\pi}(1+\cos 2t)\,dt = \tfrac{1}{2}\left(t+\tfrac{1}{2}\sin 2t\right)\Big|_0^{2\pi} = \pi$$

A positive answer indicates that the net flow across the curve is outward.
(A net inward flow would have given a negative flux.)

$$\text{Flux} = \int_C \mathbf{F}\bullet\mathbf{n}\,ds = \oint_C M\,dy - N\,dx$$

where $\mathbf{F} = M(x,y)\mathbf{i}+N(x,y)\mathbf{j}$

Evaluate with any smooth parametrization that traces C ***counterclockwise*** exactly once.

Path Independence, Conservative Fields, and Potential Functions

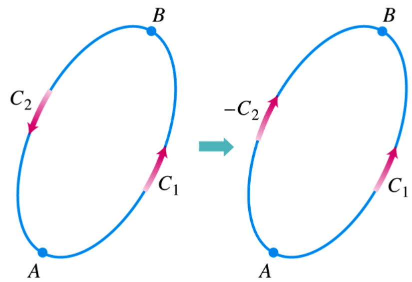
Definition

A vector field $\mathbf{F}$ is said to be ***path independent*** in a region D, or ***conservative*** on D, if for any two points A and B in D the line integral of $\mathbf{F}$ along a path C from A to B is the same for all paths from A to B. $\left(\text{Can use } \int_A^B \text{ to denote } \int_C .\right)$

Note

F is path independent in D iff $\int_C \mathbf{F}\bullet d\mathbf{r}=0$ around any **loop** C in D.

$\Rightarrow$

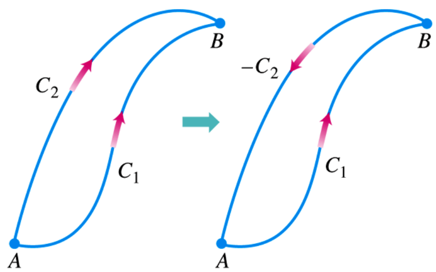


$$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_{C_1} \mathbf{F}\bullet d\mathbf{r} + \int_{C_2} \mathbf{F}\bullet d\mathbf{r}$$

$$= \int_{C_1} \mathbf{F}\bullet d\mathbf{r} - \int_{-C_2} \mathbf{F}\bullet d\mathbf{r} =$$

$$= \int_A^B \mathbf{F}\bullet d\mathbf{r} - \int_A^B \mathbf{F}\bullet d\mathbf{r} = 0$$

$\Leftarrow$



$$\int_{C_1} \mathbf{F}\bullet d\mathbf{r} - \int_{C_2} \mathbf{F}\bullet d\mathbf{r} =$$

$$= \int_{C_1} \mathbf{F}\bullet d\mathbf{r} + \int_{-C_2} \mathbf{F}\bullet d\mathbf{r} =$$

$$= \int_{C_1\cup -C_2} \mathbf{F}\bullet d\mathbf{r} = 0$$

Theorem

F is path independent iff $\mathbf{F}=\nabla f$ for some function f (called a ***potential*** for **F**).

Fundamental Theorem of Line Integrals

Suppose C is a curve from A to B parametrized by $\mathbf{r}(t)$. If $\mathbf{F}=\nabla f$ then

$$\int_C \mathbf{F}\bullet d\mathbf{r} = f(B)-f(A)$$

Reason:

$$\int_C \nabla f \bullet d\mathbf{r} = \int_a^b \underbrace{\nabla f(\mathbf{r}(t))\bullet \frac{d\mathbf{r}}{dt}}_{\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}} dt = \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\,dt \underset{\text{Fund. Thm. Calc}}{=} f(\mathbf{r}(t))\Big|_a^b = f(B)-f(A)$$

Example

Find work done by the conservative field $\mathbf{F}= yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k} =\nabla f$, where $f=xyz$ along any smooth curve joining $A=(-1,3,9)$ to $B=(1,6,-4)$.

$$W=\int_C \mathbf{F}\bullet d\mathbf{r} = f(B)-f(A) = xyz\big|_{(1,6,-4)} - xyz\big|_{(-1,3,9)} = (-24)-(-27) = 3$$

Component Test for Conservative Fields

$\mathbf{F} = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$ is conservative iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \qquad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}, \qquad \text{and} \qquad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}.$$

To find f such that $\mathbf{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$ i.e.

$$\frac{\partial f}{\partial x} = M, \qquad \frac{\partial f}{\partial y} = N, \qquad \text{and} \qquad \frac{\partial f}{\partial z} = P,$$

integrate any one of these equations and "adjust" for the other two.

Example

Show that $\mathbf{F} = (\underbrace{e^x \cos y + yz}_{M})\mathbf{i} + (\underbrace{xz - e^x \sin y}_{N})\mathbf{j} + (\underbrace{xy + z}_{P})\mathbf{k}$ is conservative and find a potential function for it.

$$\frac{\partial M}{\partial y} = -e^x \sin y + z \qquad \frac{\partial N}{\partial z} = x \qquad \frac{\partial P}{\partial x} = y$$

$$\frac{\partial N}{\partial x} = z - e^x \sin y \qquad \frac{\partial P}{\partial y} = x \qquad \frac{\partial M}{\partial z} = y$$

A potential f must satisfy:

$$(1)\ \frac{\partial f}{\partial x} = e^x \cos y + yz, \qquad (2)\ \frac{\partial f}{\partial y} = xz - e^x \sin y, \qquad \text{and} \qquad (3)\ \frac{\partial f}{\partial z} = xy + z$$

Integrating (1) w.r.t. x while holding y and z fixed, we get

$$(1) \Rightarrow f(x,y,z) = e^x \cos y + xyz + C(y,z)$$

$$\xrightarrow{\partial/\partial y} \quad -e^x \sin y + xz + C_y(y,z) \overset{(2)}{=} xz - e^x \sin y \quad \Rightarrow C_y(y,z) = 0$$

$$\Rightarrow C(y,z) = D(z)$$

So $f(x,y,z) = e^x \cos y + xyz + D(z)$

$$\xrightarrow{\partial/\partial z} \quad xy + D'(z) \overset{(3)}{=} xy + z \quad \Rightarrow D'(z) = z \quad \Rightarrow D(z) = \tfrac{1}{2}z^2 + E$$

So

$$\boxed{f(x,y,z) = e^x \cos y + xyz + \tfrac{1}{2}z^2 + E, \quad \text{where } E \text{ is any constant.}}$$