§4.1 Extreme Values of Functions on Closed Intervals (Continued)

Definition. An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

Ex. Find all critical points of $f(x) = x^3 - 3x^2 + 2$.

Solution. $f'(x) = 3x^2 - 6x$. To find the critical point(s), we solve $3x^2 - 6x = 0 \Rightarrow x = 0$ and $x = 2$. Therefore, $x = 0,\ 2$ are the critical points of the function $y = f(x)$ on $\mathbb{R} = (-\infty, \infty)$. □

Exercise 52. Determine all critical points for $g(x) = \sqrt{2x - x^2}$.

Solution. $g'(x) = \frac{1}{2}(2x - x^2)^{-1/2}(2 - 2x)$. Solve $\frac{1}{2}(2x - x^2)^{-1/2}(2 - 2x) = 0$ to obtain $2 - 2x = 0$, that is, $x = 1$, which is the critical point in the interior of the domain $[0, 2]$. □

Example 3. Find the absolute maximum and minimum values of $f(x) = 10x(2 - \ln x)$ on the interval $[1, e^2]$.

Example 4. Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

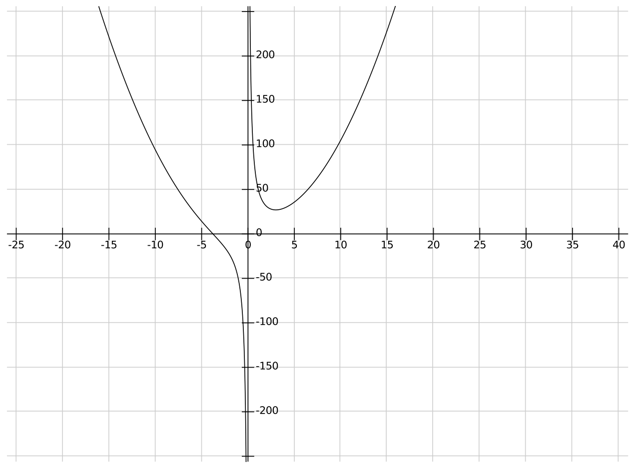
[Answer: $x = 0$ where the derivative d.n.e.]

Ex. MML Determine the critical point(s) of $y = x^2 + \frac{54}{x}$.

[Answer: $x = 3$]

§4.2 The Mean Value Theorem

Theorem 3 (Rolle's Theorem). Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.



Theorem 4 (The Mean Value Theorem). Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

Corollary 1. If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all x in (a, b), where C is a constant.

Corollary 2. If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all x in (a, b). That is, $f - g$ is a constant function on (a, b).

Ex. From MML §4.2: Let $f(x) = \cos^{-1} x$. Find value(s) of $c \in [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$ s.t. $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

[Solution] By the derivative formula for the inverse of cosine function, see §3.9 on inverse trigonometric functions. $\dfrac{d}{dx}\left(\cos^{-1}(x)\right) = -\dfrac{1}{\sqrt{1 - x^2}}$. Identify $a = -\frac{\sqrt{3}}{2}$, $b = \frac{\sqrt{3}}{2}$, and $f(a) = \frac{5\pi}{6}$, $f(b) = \frac{\pi}{6}$. We substitute into the mean value theorem formula to solve for $c = \pm\sqrt{1 - \frac{27}{4\pi^2}}$.

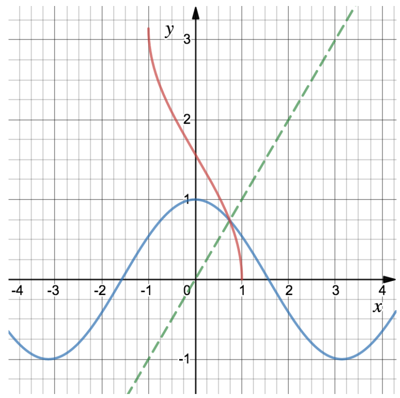


FIGURE 1. $\arccos x$ vs $\cos x$

§4.3 Monotonic Functions and the First Derivative Test

Definition. Let f be a function defined on an interval I and let x_1 and x_2 be two distinct points in I.
1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

Ex. (a) $y = x^3$, $x \in \mathbb{R}$
(b) $y = \sqrt{x}$, $x \geq 0$

Corollary 3. Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) > 0$ at each point x in (a, b), then f is increasing on $[a, b]$. If $f'(x) < 0$ at each point x in (a, b), then f is decreasing on $[a, b]$.

Example 1. Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

Solution. $f'(x) = 3x^2 - 12 = 3(x + 2)(x - 2)$. Solve $3(x + 2)(x - 2) = 0 \Rightarrow x = -2$ and $x = 2$ critical points. Making a table and using testing point method show that $f' > 0$ on $(-\infty, -2)$, increasing; $f' < 0$ on $(-2, 2)$, decreasing; $f' > 0$ on $(2, \infty)$, increasing. $\square$

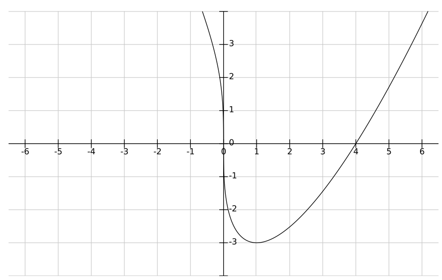
Example 2. Find the critical points of $f(x) = x^{1/3}(x - 4)$. Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution. $f'(x) = (x^{4/3} - 4x^{2/3})' = \frac{4}{3}\frac{x-1}{\sqrt[3]{x^2}}$. Solve $\frac{4}{3}\frac{x-1}{\sqrt[3]{x^2}} = 0$ to obtain $x = 1$, a critical point.
From the table we see f attains its local minimum $f(1) = -3$ at $x = 1$.

TABLE 1. $y = f(x) = x^{1/3}(x-4)$

x	$-\infty < x < 1$	$1 < x < \infty$
f'	< 0	> 0
f	$\searrow$	$\nearrow$

The following figure plots the graph of the function $y = f(x)$ over $(-6.5, 6.5)$.



$\square$

Example 3. Find the critical points of $f(x) = (x^2 - 3)e^x$. Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Ex. (a) $y = x + \sin x$ (b) $y = \frac{\sin x}{x}$ (c) $y = e^{-x}\cos x$

§4.4 Concavity and Curve Sketching

Definition. The graph of a differentiable function $y = f(x)$ is
(a) **concave up** on an open interval I if f' is increasing on I;
(b) **concave down** on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity. Let $y = f(x)$ be twice-differentiable on an interval I. If $f'' > 0$ on I, the graph of f over I is concave up. If $f'' < 0$ on I, the graph of f over I is concave down.

Example 1. (a) The curve $y = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.
(b) The curve $y = x^2$ is concave up on $(-\infty, \infty)$.

Example 2. Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Definition. A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

Example 3. Determine the concavity and find the inflection points of the function $f(x) = x^3 - 3x^2 + 2$. See also Ex. in §4.1.

[Solution] $f'(x) = 3x^2 - 6x = 3x(x-2)$. Solve $f'(x) = 0$ and we obtain $x = 0, 2$ are critical points in $\mathbb{R}$.
$f''(x) = 6x - 6 = 6(x-1)$. Solve $6(x-1) = 0$ we obtain $x = 1$ is a reflection point.

TABLE 2. $f(x) = x^3 - 3x^2 + 2$

Function \ x	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, \infty)$
f'	$+$	$-$	$-$	$+$
f	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$
f''	$-$	$-$	$+$	$+$
f	concave down	concave down	concave up	concave up

The function $y = f(x)$ attains its local maximum $y_{max} = f(0) = 2$ at $x = 0$ and local minimum $y_{min} = f(2) = -2$ at $x = 2$; and has a refection point at $(1, 0)$ as Figure 2 shows. Note that $f''(0) = -6 < 0$ (local max), and $f''(2) = 6 > 0$ (local min.). See Figure 2.

Example 4. Determine the concavity and find the inflection points of $f(x) = x^{5/3}$.

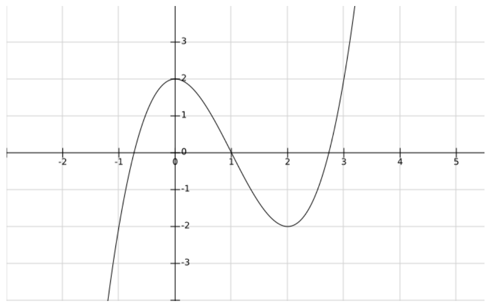
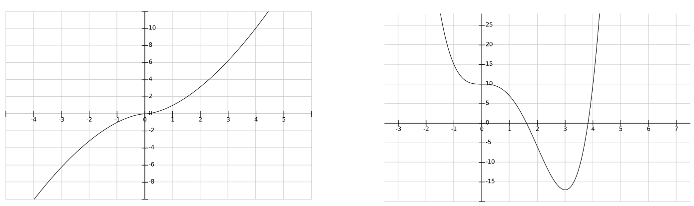


FIGURE 2. Plot of the cubic function $f(x) = x^3 - 3x^2 + 2$

Theorem 5 (Second Derivative Test for Local Extrema). Suppose f'' is continuous on an open interval that contains $x = c$.
1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, local minimum, or neither.

Example 8. Sketch a graph of the function $f(x) = x^4 - 4x^3 + 10$.



(A) Plot of $x^{5/3}$ (B) Plot of $x^4 - 4x^3 + 10$

FIGURE 3. graphs for Ex.4 and Ex.8