NAME: ______________________________

MARK BOX		
PROBLEM	POINTS	
1	10	
2	10	
3	10	
4	10	
TOTAL	40	

ID (last four digits)______________________

please check the box of your section below

☐

or

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INSTRUCTIONS:

(1) To receive credits you must:
   (a) work in a logical fashion, **show all your work and indicate your reasoning** to support and justify your answer
   (b) when applicable put your answer on/in the line/box; use the back of the page if needed

(2) This exam covers (from *Elementary Linear Algebra* by Larson 8^{th} ed.): Sections 3.1 – 3.3; 4.1– 4.4 .

(1) Compute the determinant.

$$\begin{vmatrix} 1 & 1 & -2 \\ 0 & 15 & 0 \\ 2 & 2 & -4 \end{vmatrix}$$

(2) Find (i) the characteristic equation, (ii) the eigenvalues, and (iii) the corresponding eigenvectors of the matrix.

(a)

$$\begin{vmatrix} 4 & -5 \\ 2 & -3 \end{vmatrix}$$

(b)

$$\begin{vmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{vmatrix}$$

(3) (*optional*)* Find the adjoint $\mathbf{ad}(M)$ of the matrix $M = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 3 & 2 \\ 3 & 0 & -1 \end{pmatrix}$.

Verify that $M\mathbf{ad}(M) = \mathbf{ad}(M)M = \det(M)I_3$.

(4) **Definition**. A vector $\mathbf{u}$ is said to be in the null space of a matrix A provided

$$A\mathbf{u} = \mathbf{0}.$$

or, equivalently, $\mathbf{u}$ is an eigenvector corresponding to the zero eigenvalue of A.

Which of the following vectors, if any, is in the null space of $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{pmatrix}$?

a) $[-1\ 0\ 1\ 0]^T$ b) $[0\ 2\ 1\ -1]^T$ c) $[0\ 4\ 2\ -2]^T$

(5) Determine which of the following statements are equivalent to the fact that a matrix A of size $n \times n$ is invertible?

a) A is nonsingular
b) The row space of A has dimension n
c) The column space of A has dimension n
d) The determinant of A is nonzero
e) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any given $\mathbf{b}$ in $\mathbf{R}^n$
f) The system $A\mathbf{x} = \mathbf{0}$ has nonzero solution
g) The dimension of the null space of A is zero
h) The rows of A are linear independent
i) The columns of A are linear independent
j) The rank of A is n
k) A is row-equivalent to an identity matrix
l) All eigenvalues of A are nonzero
m) A can be written as the product of elementary matrices.

(6) (optional*) The matrix $A = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$ row reduces to $C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

a) Find the rank and nullity of A.
b) Find a basis of the row space and the column space of A respectively.
c) Find a basis of the null space of A
d) Does the system $A\mathbf{x} = \begin{pmatrix} 109 \\ -217 \\ 66 \end{pmatrix}$ have a solution? (Hint: You can draw a conclusion from the fact that dimension of column space is 3, without having to solve the system. Recall that $rank(A) = dim(Col(A)) = dim(Row(A))$)
e) What is the relation between $rank, dim(null(A))$?(Hint: Theorem 4.17 (pp.196) states that $rank(A) + dim(null(A)) = n$, the number of columns)

(7) Find all the eigenvalues of the given matrix.

a) $\begin{pmatrix} 1 & -2 & 0 \\ -3 & 1 & 0 \\ -4 & -5 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 9 \\ 0 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$
where $i = \sqrt{-1}$ ($i^2 = -1$) is the unit for pure imaginary numbers.

(8) We say a vector $\mathbf{u}$ is a linear combination of a finite set of vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ if there exist constants c_1, c_2, c_3 such that

$$\mathbf{u} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}.$$

Determine whether one can write $\mathbf{u} = [8\ 3\ 8]^T$ as a linear combination of the vectors in the set S.

$S = \{[4\ 3\ 2]^T, [0\ 3\ 2]^T, [0\ 0\ 2]^T\}$

Solutions 2 (a). (i) The characteristic equation is $|\lambda I - A| = 0$, that is,

$$\begin{vmatrix} \lambda - 4 & 5 \\ -2 & \lambda + 3 \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2) = 0$$

(ii) The eigenvalues are solutions to the characteristic equation:

$$\lambda_1 = -1,\ \lambda_2 = 2.$$

(iii) The eigenvectors corresponding to $\lambda = -1$ is the set of nonzero solutions to $(\lambda I - A)\mathbf{x} = \mathbf{0}$

$$\begin{pmatrix} -5 & 5 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving it yields

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad t \neq 0$$

Similarly the eigenvectors corresponding to $\lambda = 2$ are

$$\begin{pmatrix} -2 & 5 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving it yields

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad t \neq 0$$

2 (b). (i) The characteristic equation reads

$$\begin{vmatrix} \lambda - 1 & 1 & 1 \\ -1 & \lambda - 3 & -1 \\ 3 & -1 & \lambda + 1 \end{vmatrix} = 0$$

(ii) The eigenvalues are obtained by solving the above equation. We start with simplifying

$$\begin{aligned}
&\begin{vmatrix} \lambda-1 & 1 & 1 \\ -1 & \lambda-3 & -1 \\ 3 & -1 & \lambda+1 \end{vmatrix} = \begin{vmatrix} \lambda-2 & \lambda-2 & 0 \\ -1 & \lambda-3 & -1 \\ 3 & -1 & \lambda+1 \end{vmatrix} \\
=&(\lambda-2)\begin{vmatrix} 1 & 1 & 0 \\ -1 & \lambda-3 & -1 \\ 3 & -1 & \lambda+1 \end{vmatrix} = (\lambda-2)\begin{vmatrix} 1 & 0 & 0 \\ -1 & \lambda-2 & -1 \\ 3 & -4 & \lambda+1 \end{vmatrix} \\
=&(\lambda-2)\begin{vmatrix} \lambda-2 & -1 \\ -4 & \lambda+1 \end{vmatrix} \\
=&(\lambda-2)(\lambda+2)(\lambda-3).
\end{aligned}$$

Hence $\lambda_1 = -2$, $\lambda_2 = 2$ and $\lambda_3 = 3$.

2 (b) (iii) To find the eigenvectors for λ, we solve the linear homogeneous equation

$$\begin{bmatrix} \lambda-1 & 1 & 1 \\ -1 & \lambda-3 & -1 \\ 3 & -1 & \lambda+1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If $\lambda_1 = -2$, row reduction yields

$$\begin{bmatrix} \lambda_1-1 & 1 & 1 \\ -1 & \lambda_1-3 & -1 \\ 3 & -1 & \lambda_1+1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{pmatrix} \qquad t \neq 0.$$

The eigenvectors for λ_2 and λ_3 can be found in a similar way. If $\lambda_3 = 3$, say, row reduction yields

$$\begin{bmatrix} \lambda_3-1 & 1 & 1 \\ -1 & \lambda_3-3 & -1 \\ 3 & -1 & \lambda_3+1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \qquad t \neq 0.$$

3*. By definition the adjoint matrix of a matrix $A = (C_{ij})_{n\times n}$ is given by

$$\mathbf{ad}(A) = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

where $C_{ij} = (-1)^{i+j}M_{ij}$ are cofactors of A.

$$\begin{pmatrix} -3 & 0 & -6 \\ 6 & -5 & 2 \\ -9 & 0 & -3 \end{pmatrix}$$

A straight forward computation shows $M\mathbf{ad}(M) = \mathbf{ad}(M)M = -15I_3$.

4. Answer: (b) and (c). If multiplying A and the vector in (b), we will have $A\mathbf{u} = 0$. The same occurs for (c).

(Here is some more details. Given a matrix A, the null space $Null(A)$ is a vector space consisting of all those vectors $\mathbf{u}$ satisfying the equation $A\mathbf{x} = 0$.

So if you want to check if certain vector u is in the null space, all you need to do is to substitute $\mathbf{x} = \mathbf{u}$ into the linear equation $A\mathbf{x} = 0$.

If you find $A\mathbf{u} = 0$ then $\mathbf{u}$ belongs to $Null(A)$; otherwise it does not belong to $Null(A)$.)

6*. (a) $Rank\,(A) = 3$. $nullity(A) = 1$ (nullity is the dimension for the null space of A)

(b) A basis for $Row(A)$ is given by $\{[2\ 1\ 3\ 1]^T, [1\ \ -1\ 0\ 1]^T, [1\ 1\ 2\ 1]^T\}$. A basis for $Col(A)$ is given by $\{[2\ 1\ 1]^T, [3\ 0\ 2]^T, [1\ 1\ 1]^T\}$.

(c) The solutions to $A\mathbf{x} = \mathbf{0}$ consist vectors of the form $\{t \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, t \neq 0\}$. So a basis can be chosen as $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$.

(d) Yes. Because the dimension of the column space of A equals to 3, and, the dimension of the column space of the augmented matrix $[Ab]$ is also 3. We see that the column space and the augmented space are consistent in the case. Therefore the system $A\mathbf{x} = \mathbf{b}$ is consistent or solvable.

(e) $Rank(A) + \dim(null(A)) = 3 + 1 = 4$ which should be the number of columns.

7. (a) The eigenvalues are solutions of

$$\begin{vmatrix} \lambda - 1 & 2 & 0 \\ 3 & \lambda - 1 & 0 \\ 4 & 5 & \lambda - 1 \end{vmatrix} = 0$$

$$(\lambda - 1) \begin{vmatrix} \lambda - 1 & 2 \\ 3 & \lambda - 1 \end{vmatrix} = (\lambda - 1)(\lambda^2 - 2\lambda - 5) = 0.$$

Hence $\lambda_1 = 1$, $\lambda_{2,3} = 1 \pm \sqrt{6}$.

7 (c). $\lambda = \pm i$.

7 (d) $\lambda = \pm 1$.

7. (e) Solving

$$\begin{vmatrix} \lambda & -i \\ -i & \lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

we obtain $\lambda_1 = i$, $\lambda_2 = -i$.

(8) We can rewrite $\mathbf{u} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$ in the form

$$\begin{pmatrix} 4 & 0 & 0 \\ 3 & 3 & 0 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \\ 8 \end{pmatrix}.$$

Solve this equation using either row reduction or in the traditional way as follows.

$$\begin{cases} 4c_1 = 8 \\ 3c_1 + 3c_2 = 3 \\ 2c_1 + 2c_2 + 2c_3 = 8 \end{cases} \Rightarrow \begin{cases} c_1 = 2 \\ c_1 + c_2 = 1 \\ c_1 + c_2 + c_3 = 4 \end{cases} \Rightarrow$$

$$\boxed{\therefore\ \mathbf{c} = [c_1, c_2, c_3]^T = [2\ \ -1\ \ 3]^T}$$