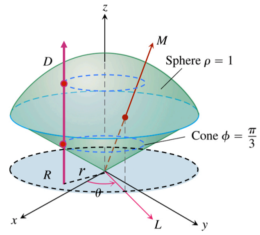
**Example**

Find the volume of the "ice cream cone" $D$ cut from the solid sphere $\rho\le 1$ by the cone $\phi = \pi/3$.

$$V = \iiint_D dV = \int_0^{2\pi}\int_0^{\pi/3}\underbrace{\int_0^{1} \rho^2 \sin\phi \, d\rho}_{\frac{1}{3}\sin\phi}\, d\phi\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3} \sin\phi\, d\phi\, d\theta = \frac{1}{6}\int_0^{2\pi} d\theta = \frac{\pi}{3}$$

$$\left(-\cos\phi\right)\Big|_0^{\pi/3} = \frac{1}{2}$$




**Recalculate in the cylindrical coordinates.**

$$V = \int\int\int_{r/\sqrt{3}}^{\sqrt{1-r^2}} dz\, r\, dr\, d\theta$$

$z$-limits: Need cylindrical equations for the bounding surfaces.

top: $r^2 + z^2 = 1 \;\Rightarrow\; z = \sqrt{1-r^2}$

bottom: $z = \frac{1}{\sqrt{3}} r$

$r$-limits: Need the radius of circle along which the bounding surfaces intersect.

$$r^2 + \tfrac{1}{3}r^2 = 1 \qquad r^2 = \tfrac{3}{4}$$

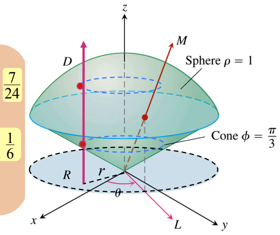
$$\tfrac{4}{3}r^2 = 1 \qquad r = \tfrac{\sqrt{3}}{2}$$

$$\int_0^{\sqrt{3}/2} r\sqrt{1-r^2}\, dr = -\tfrac{1}{2}\int_1^{1/4}\sqrt{u}\, du = \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2}\right)\Big|_{1/4}^{1} = \tfrac{1}{3}\left(1-\tfrac{1}{8}\right) = \tfrac{7}{24}$$

$$\boxed{u = 1 - r^2}$$

$$\tfrac{1}{\sqrt{3}}\int_0^{\sqrt{3}/2} r^2\, dr = \tfrac{1}{3\sqrt{3}}\left(r^3\right)\Big|_0^{\sqrt{3}/2} = \tfrac{1}{3\sqrt{3}}\,\tfrac{3\sqrt{3}}{8} = \tfrac{1}{8}$$

$$\tfrac{7}{24} - \tfrac{1}{8} = \tfrac{1}{6}$$




$$r\sqrt{1-r^2} - \tfrac{1}{\sqrt{3}}r^2$$

$$V = \int_0^{2\pi}\int_0^{\sqrt{3}/2}\int_{r/\sqrt{3}}^{\sqrt{1-r^2}} dz\, r\, dr\, d\theta = \tfrac{1}{6}\int_0^{2\pi} d\theta = \tfrac{\pi}{3}$$

**OR**

$$V = \int_0^{2\pi}\int_0^{1}\int_0^{g(z)} r\, dr\, dz\, d\theta \qquad \text{where} \quad g(z) = \begin{cases} \sqrt{3}\, z & 0 \le z \le \frac{1}{2} \\ \sqrt{1-z^2} & \le z \le 1 \end{cases}$$

$$\int_0^{1/2}\int_0^{g(z)} r\, dr\, dz + \int_{1/2}^{1}\int_0^{g(z)} r\, dr\, dz = \int_0^{1/2}\int_0^{z\sqrt{3}} r\, dr\, dz + \int_{1/2}^{1}\int_0^{\sqrt{1-z^2}} r\, dr\, dz$$

$$\int_0^{1/2}\int_0^{z\sqrt{3}} r\, dr\, dz = \tfrac{3}{2}\int_0^{1/2} z^2 dz = \tfrac{1}{2}\left(z^3\right)\Big|_0^{1/2} = \tfrac{1}{16} \longrightarrow \tfrac{1}{16} + \tfrac{5}{48} = \tfrac{1}{6} \text{ (like before)}$$

$$\left(\tfrac{1}{2}r^2\right)\Big|_0^{z\sqrt{3}} = \tfrac{3}{2}z^2$$

$$\int_{1/2}^{1}\int_0^{\sqrt{1-z^2}} r\, dr\, dz = \tfrac{1}{2}\int_{1/2}^{1}\left(1-z^2\right)dz = \tfrac{1}{2}\left(z - \tfrac{1}{3}z^3\right)\Big|_{1/2}^{1} = \tfrac{1}{6}z\left(3-z^2\right)\Big|_{1/2}^{1}$$

$$= \tfrac{1}{6}\underbrace{\left(1\cdot 2 - \tfrac{1}{2}\cdot\tfrac{11}{4}\right)}_{\frac{5}{8}} = \tfrac{5}{48}$$

$$\left(\tfrac{1}{2}r^2\right)\Big|_0^{\sqrt{1-z^2}} = \tfrac{1}{2}\left(1-z^2\right)$$

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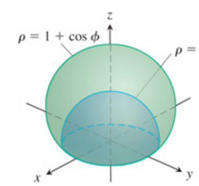
**Example**

Find the volume of the solid bounded below by the hemisphere $\rho = 1,\ z \ge 0$ and above by the cardioid of revolution $\rho = 1 + \cos\phi$ .

$$V = \iiint_D dV = \int_0^{2\pi}\int_0^{\pi/2}\int_1^{1+\cos\phi} \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

$$\left(\tfrac{1}{3}\sin\phi\, \rho^3\right)\Big|_1^{1+\cos\phi} = \tfrac{1}{3}\sin\phi\left((1+\cos\phi)^3 - 1\right)$$




$$\tfrac{1}{3}\int_0^{\pi/2}\left((1+\cos\phi)^3 - 1\right)\sin\phi\, d\phi = -\tfrac{1}{3}\int_2^{1}\left(u^3-1\right)du = \tfrac{1}{3}\left(\tfrac{1}{4}u^4 - u\right)\Big|_1^2 = \tfrac{1}{3}u\left(\tfrac{1}{4}u^3 - 1\right)\Big|_1^2$$

$$\boxed{u = 1 + \cos\phi}$$

$$= \tfrac{1}{12}u\left(u^3 - 4\right)\Big|_1^2$$

$$= \tfrac{1}{12}\left(2\cdot 4 - 1\cdot(-3)\right) = \tfrac{11}{12}$$

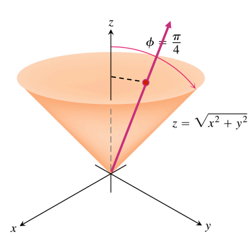
$$V = \tfrac{11}{12}\int_0^{2\pi} d\theta = \tfrac{11}{6}\pi$$

**Example**

Let $D$ be the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the plane $z = 1$. Set up the triple integrals in spherical coordinates that give the volume of $D$ using the following orders of integration.

a) $d\rho\, d\phi\, d\theta$

b) $d\phi\, d\rho\, d\theta$




a) $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

b)