

§2.1 Rates of Change and Tangent Lines to Curves

Example 1. A rock breaks loose from the top of a tall cliff. What is its average speed

(a) during the first 2 sec of fall?

(b) during the 1-sec interval between second 1 and second 2?

Example 2. Find the speed of the falling rock in Example 1 at $t = 1$ and $t = 2$ sec.

Definition. The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0.$$

§2.2 Limit of a Function and Limit Laws

Example 1. How does the function $f(x) = \frac{x^2 - 1}{x - 1}$ behave near $x = 1$?

Definition (informal). Suppose that $f(x)$ is defined on an open interval about c , except possibly at c itself. If $f(x)$ is arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to c , other than c itself, then we say that f approaches the **limit** L as x approaches c , and write $\lim_{x \rightarrow c} f(x) = L$.

Example 2. The limit of a function does not depend on how the function is defined at the point being approached. Consider the three functions

$$f(x) = \frac{x^2 - 1}{x - 1}, \quad g(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}, \quad h(x) = x + 1.$$

Example 3. (a) If f is the **identity function** $f(x) = x$, then for any value of c , $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} x = c$.

(b) If f is the **constant function** $f(x) = k$ (function with constant value k), then for any value of c , $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} k = k$.

Example 4. Discuss the behavior of the following functions, explaining why they have no limit as $x \rightarrow 0$.

$$(a) U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \quad (b) g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad (c) f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$$

Theorem 1 (Limit Laws). If L , M , c , and k are real numbers and $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$, then

- | | |
|--|---|
| 1 & 2. <i>Sum and Difference Rule:</i> | $\lim_{x \rightarrow c} (f(x) \pm g(x)) = L \pm M$ |
| 3. <i>Constant Multiple Rule:</i> | $\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot L$ |
| 4. <i>Product Rule:</i> | $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$ |
| 5. <i>Quotient Rule:</i> | $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$ |
| 6. <i>Power Rule:</i> | $\lim_{x \rightarrow c} [f(x)]^n = L^n, \quad n \text{ a positive integer}$ |
| 7. <i>Root Rule:</i> | $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \quad n \text{ a positive integer}$ |
- (If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c .)

Example 5. Use Example 3 and Theorem 1 to find the following limits.

(a) $\lim_{x \rightarrow c} (x^3 + 4x^2 - 3)$ (b) $\lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5}$ (c) $\lim_{x \rightarrow -2} \sqrt{4x^2 - 3}$

Theorem 2. If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then $\lim_{x \rightarrow c} P(x) = P(c)$.

Theorem 3. If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then $\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$.

Exercise 79. If $\lim_{x \rightarrow 4} \frac{f(x) - 5}{x - 2} = 1$, find $\lim_{x \rightarrow 4} f(x)$.

Example 7. Evaluate $\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}$.

Example 9. Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$.

Theorem 4 (The Sandwich Theorem). Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c , except possibly at $x = c$ itself. Suppose also that $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$. Then $\lim_{x \rightarrow c} f(x) = L$.

Example 11. (a) $\lim_{\theta \rightarrow 0} \sin \theta = 0$ (b) $\lim_{\theta \rightarrow 0} \cos \theta = 1$

(c) For any function f , $\lim_{x \rightarrow c} |f(x)| = 0$ implies $\lim_{x \rightarrow c} f(x) = 0$.