NAME: ____________________________

MARK BOX		
PROBLEM	POINTS	
1	10	
2	10	
3	10	
4	10	
Bonus Credits	5	
TOTAL	40	

ID (last four digits)______________________

please check the box of your section below

☐

or

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INSTRUCTIONS:

(1) To receive credits you must:
  (a) work in a logical fashion, **show all your work and indicate your reasoning** to support and justify your answer
  (b) when applicable put your answer on/in the line/box; use the back of the paper if needed

(2) This exam covers (from *Elementary Linear Algebra* by Larson and Falvo 7^{th} ed.):
Section 3.1 – 3.4, 4.1 – 4.6, 5.1, 5.2, 5.3* .

Problem Inspiration:

- homework problem § 3.4 # 5 and 11
- homework problem § 4.1 # 21 and 23
- homework problem § 4.4 # 43
- homework problem § 4.5 # 49
- the other type of problems in this exam are either discussed in class or appear in the examples in the text or the exercises

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(1) Compute the determinant.

$$\begin{vmatrix} 1 & 1 & -2 \\ 0 & 15 & 0 \\ 2 & 2 & -4 \end{vmatrix}$$

(2) [§3.4 #5, #11] Find (i) the characteristic equation, (ii) the eigenvalues, and (iii) the corresponding eigenvectors of the matrix.

(a)

$$\begin{vmatrix} 4 & -5 \\ 2 & -3 \end{vmatrix}$$

(b)

$$\begin{vmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{vmatrix}$$

(3) [§4.1 #21, #23] Let $\mathbf{u} = \langle 1\ 2\ 3 \rangle$, $\mathbf{v} = \langle 2\ 2\ -1 \rangle$ and $\mathbf{w} = \langle 4\ 0\ -4 \rangle$.

a) Find $2\mathbf{u} - \mathbf{4v} - \mathbf{w}$, b) Find $\mathbf{z}$ such that $2\mathbf{z} - \mathbf{3u} = \mathbf{w}$.

(4) Find the adjoint $\mathbf{ad(M)}$ of the matrix $M = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 3 & 2 \\ 3 & 0 & -1 \end{pmatrix}$.

Verify that $M\mathbf{ad}(M) = \mathbf{ad}(M)M = \det(M)I_3$.

(5) (i) Which of the following sets of vectors $x = [x_1, x_2, x_3]^T$ are subspace of $\mathbf{R}^3$?

a) All x such that $x_1 + x_2 = 7x_3$

b) All x such that $x_2 = 0$

c) All x such that $x_1 + x_3 = 10$

(ii) We know that $P_2 = \{f : f \text{ is a polynomial of degree} \le 2\}$ is a vector space. Which set of functions satisfying the following properties constitutes a subspace of P_2 ?

a) $f(-x) = -f(x)$ b) $f(0) + f(1) = 5$ c) $f'(0) = 0$

(6) Which of the following vectors, if any, is in the null space of $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{pmatrix}$?

a) $[-1\ 0\ 1\ 0]^T$ b) $[0\ 2\ 1\ -1]^T$ c) $[0\ 4\ 2\ -2]^T$

(7) Determine which of the following statements are equivalent to the fact that a matrix A of size $n \times n$ is invertible?

a) A is nonsingular

b) The row space of A has dimension n

c) The column space of A has dimension n

d) The determinant of A is nonzero

e) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any given $\mathbf{b}$ in $\mathbf{R}^n$

f) The system $A\mathbf{x} = \mathbf{0}$ has nonzero solution

g) The dimension of the null space of A is zero

h) The rows of A are linear independent

i) The columns of A are linear independent

j) The rank of A is n

k) A is row-equivalent to an identity matrix

l) All eigenvalues of A are nonzero

m) A can be written as the product of elementary matrices.

(8) The matrix $A = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$ row reduces to $C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

a) Find the rank and nullity of A.

b) Find a basis of the row space and the column space of A respectively.

c) Find a basis of the null space of A

d) Does the system $A\mathbf{x} = \begin{pmatrix} 109 \\ -217 \\ 66 \end{pmatrix}$ have a solution? (Hint: You can draw a conclusion from the fact that dimension of column space is 3, without having to solve the system. Recall that $rank(A) = dim(Col(A)) = dim(Row(A))$)

e) What is the relation between $rank, dim(null(A))$?(Hint: The theorem states that $rank(A) + dim(null(A)) = n$, the number of columns)

(9) Find all the eigenvalues of the given matrix.

a) $\begin{pmatrix} 1 & -2 & 0 \\ -3 & 1 & 0 \\ -4 & -5 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(10) [§4.4 # 43] Determine whether each set in P_2 is linear independent.

$$S = \{2 - x, 2x - x^2, 6 - 5x + x^2\}$$

(11) [§4.5 # 49] Determine whether S is a basis for $\mathbb{R}^3$. If it is, write $\mathbf{u} = [8\ 3\ 8]^T$ as a linear combination of the vectors in S.

$$S = \{[4\ 3\ 2]^T, [0\ 3\ 2]^T, [0\ 0\ 2]^T\}$$

(12) [Bonus §5.3, #22] Find the coordinate of x relative to the orthonormal basis B in $\mathbf{R}^2$.

$$B = \{(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}), (\frac{-2\sqrt{5}}{5}, \frac{\sqrt{5}}{5})\}, \quad \mathbf{x} = (-3, 4)$$

(Hint: If $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is an orthonormal basis in V, then any vector $\mathbf{w}$ in V can be written as $\mathbf{w} = \langle \mathbf{w}, \mathbf{u_1} \rangle \mathbf{u_1} + \cdots + \langle \mathbf{w}, \mathbf{u_k} \rangle \mathbf{u_k}$, where $\langle \mathbf{w}, \mathbf{u} \rangle$ means the inner product which agrees with $\mathbf{Proj_u w} = \frac{\langle \mathbf{w}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$, the projection of $\mathbf{w}$ onto $\mathbf{u}$)

Solutions (4) $\mathbf{ad}(M) =$ transpose of $\begin{pmatrix} -4 & 3 & -9 \\ 3 & -5 & 4 \\ -5 & 1 & -3 \end{pmatrix}$

A straight forward computation shows $M\mathbf{ad}(M) = -11I_3$.

(8) a) $rank(A) = 3$ (number of leading 1's in C), nullity of $A = 1$

b) A basis of Row(A) consists of $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

A basis of Col(A) consists of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$

d) Yes.