Math 2160 (Section A) **Instructor Shijun Zheng**
12/4/2019

§*7.2 Diagonalization* (Continued)

Theorem 7.6. If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

§*7.3* Diagonalization*

Definition. A matrix $A = (a_{ij})_{n\times n}$ is orthogonal if $A^T A = AA^T = I$. That is, the inverse of A is the transpose of that matrix.

proposition 1. *Let $A = [v_1, v_2, \ldots, v_n]$ be a matrix with n columns v_j, $1 \leq j \leq n$ Then A is orthogonal* $\iff$

$$\langle v_j, v_k \rangle = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

Definition. [Orthogonal diagonalization] A square matrix A is orthogonally diagonalizable provided there exists an orthogonal matrix P such that

$$P^T A P = D \tag{1}$$

where D is a diagonal matrix.
†If A is orthogonally diagonalizable then $AP = PD$, which means the j-th column of P is an eigenvector corresponding to the eigenvalue λ_j.

This example illustrate how to find the eigenvalues and orthogonal eigenvectors of a given matrix.

Example. Given matrix $A = \begin{pmatrix} 6 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{pmatrix}$ has eigenvalues 5 and 8.

(a) Find the eigenspaces E_5 and E_8 by solving $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

(b) By the theorem in Section 7.3 we know that a symmetric matrix of size n by n is always diagonalizable, equivalently speaking, always has n linear independent eigenvectors. Find an ordered basis consisting of eigenvectors of A.

(c) Specify the matrices P and D in the diagonalization
$P^{-1}AP = D$

(d) Find an orthogonal matrix U such that $U^{-1}AU = D$. Recall an (real) orthogonal matrix means $U^{-1} = U^T$ or equivalently $U^T U = UU^T = I_n$).

[Solution] (a) and (b). $E_5 = \text{span}\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\}$, $E_8 = \text{span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}$

(c*) $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}$

(d*) The orthogonal matrix U consists of three eigenvectors that are orthogonal in $\mathbb{R}^3$. So we need to orthogonalise the base $u = [-1, 1, 0]^T$, $v = [-1, 0, 1]^T$, $w = [1, 1, 1]$. Since the third vector in E_8 is orthogonal the any vectors in E_5. We only need to orthogonalise the two vectors in E_5 by Grant-Schmidt method. Let

$$u = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \tilde{v} = v - \frac{\langle v, u\rangle}{\langle u, u\rangle} u = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix},$$

Normalise the three vectors $u, \tilde{v}, w$ to obtain

$$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$