Review Test 1 **Name**

Math 2331 **Id**

Read carefully each problem. Show all your work. Credits will be given mainly depending on your work, not just an answer. Put a box around the final answer to a question. Use the back of the page if necessary.

1 [**10**] *Solve the system using either Gaussian elimination with back-substiution or Gauss-Jordan elimination.*

a)

$$\begin{aligned} -x + 2y &= 1.5 \\ 2x - 4y &= 3 \end{aligned}$$

b)

$$\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 \qquad - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$$

2 [**10**] *Solve the homogeneous linear system corresponding to the coefficient matrix provided:*

a) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

3 [**10**] *a) Write the system of linear equations in the form $A\mathbf{x} = \mathbf{b}$ and solve the matrix equation for* $\mathbf{x}$.

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ x_1 + 4x_2 &= 10 \end{aligned}$$

(b) Solve the matrix equation for a, b, c, d

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 17 \\ 4 & -1 \end{pmatrix}$$

4 [**10**] *a) If $AB = 0$, is it necessarily $A = 0$ or $B = 0$? Consider the example* $\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{pmatrix}$

b)Show that if $AB = 0$ and A is invertible, then $B = 0$.

5 [**10**] *Find the inverse of the matrix (if it exists).*

a) $\begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ 7 & 16 & -21 \end{pmatrix}$

6 [**10**] *Prove that if $A^2 = A$, then $I - 2A = (I - 2A)^{-1}$.*

7 [**10**] *Let A be an n by n matrix. Which of the following statements are equivalent?*

(1) *A is invertible*
(2) *There exists a matrix B such that $BA = I_n$*
(3) *$A\mathbf{x} = \mathbf{b}$ has a unique solution for every n by 1 column matrix $\mathbf{b}$*
(4) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution*
(5) *A is row-equivalent to I_n*
(6) *A is column-equivalent to I_n*
(7) *A can written as the product of elementary matrices.*
(8) *Determinant of A is nonzero*
(9) *$A\mathbf{x} = \mathbf{0}$ has infinitely many solutions*

8 [**10**] *Factor the matrix into a product of elementary matrices.*

$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

$B = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$C = \begin{pmatrix} 4 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$

9 [**10**] *(optional) Solve the system $Ax = b$ by*

1) *finding the LU-factorization of the coefficient matrix A*
2) *solving the lower triangular system $Ly = b$, and*
3) *solving the upper triangular system $Ux = y$.*

a)

$$\begin{aligned} &2x + y = 1 \\ &y - z = 2 \\ &-2x + y + z = -2 \end{aligned}$$

b)

$$\begin{aligned} &2x_1 = 4 \\ &-2x_1 + x_2 - x_3 = -4 \\ &6x_1 + 2x_2 + x_3 = 15 \\ &-x_4 = -1 \end{aligned}$$

c)

$$\begin{aligned} &x_1 - 3x_2 = -5 \\ &x_2 + 3x_3 = -1 \\ &2x_1 - 10x_2 + 2x_3 = -20 \end{aligned}$$

10 [**10**] *Let A be a nonsingular matrix. Prove that a) if B is row-equivalent to A, then B is also nonsingular. b) Use $(AB)^T = B^T A^T$ and $(AB)^{-1} = B^{-1}A^{-1}$ to show that A^T is also invertible.*

11[**10**] *(optional) Using a system of equations to write the partial fraction decomposition of the rational expression. Then solve the system using matrices.*

$$\frac{4x^2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

where A, B, C are constants.

12 [**10**] *Give three distinct examples of elementary matrices and explain how they correspond to row operations for a given matrix of 3 by 3.*

Solutions

2 (a) The homogeneous system $Ax = 0$ means

$$\begin{aligned} x_1 + x_4 &= 0 \\ x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Hence the solution is $x_1 = -t, x_2 = s, x_3 = 0, x_4 = t$, where $-\infty < s, t < \infty$.

2 (b) The system is equivalent to $0x_1 + 0x_2 + 0x_3 = 0$ Therefore, we have the freedom of choosing values of the unknown variables. $x_1 = s, x_2 = t, x_3 = r$, where the parameters s, t, r can be any real numbers.

3 b) Multiplying the equation both sides by $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}^{-1}$ on the right, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 17 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}^{-1}$$

Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 17 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}^{-1} = \cdots$$

4 a) No. b) We need to show if A^{-1} exists and $AB = 0$, then $B = 0$. Multiplying A^{-1} on the left on both sides of the equation $AB = 0$, we have

$$A^{-1}AB = A^{-1} \cdot 0,$$

that is, $B = 0$ (since $A^{-1}A = I$ and $I \cdot B = 0$)

5. b) Row operation $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{pmatrix} \xrightarrow{(-3)R1+R2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 5 & 5 \end{pmatrix}$ which shows that one of the rows has all zeors, thus it is Not row equivalent to identity matrix. Hence the matrix is Not invertible.

c) Perform row operation on the adjoining matrix $\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 7 & -10 & 0 & 1 & 0 \\ 7 & 16 & -21 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{(-3)R1+R2}$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -3 & 1 & 0 \\ 7 & 16 & -21 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{(-7)R1+R3} \begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -3 & 1 & 0 \\ 0 & 2 & -14 & -7 & 0 & 1 \end{pmatrix} \xrightarrow{(-2)R2+R3}$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -3 & 1 & 0 \\ 0 & 0 & -2 & -1 & -2 & 1 \end{pmatrix} \xrightarrow{(-3)R3+R2} \begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 7 & -3 \\ 0 & 0 & -2 & -1 & -2 & 1 \end{pmatrix} \xrightarrow{(-1/2)*R3}$$

$$\begin{pmatrix}1&2&-1&1&0&0\\0&1&0&0&7&-3\\0&0&1&\frac{1}{2}&1&-\frac{1}{2}\end{pmatrix}\xrightarrow{R3+R1}\begin{pmatrix}1&2&0&\frac{3}{2}&1&-\frac{1}{2}\\0&1&0&0&7&-3\\0&0&1&\frac{1}{2}&1&-\frac{1}{2}\end{pmatrix}\xrightarrow{(-2)*R2+R1}\begin{pmatrix}1&0&0&\frac{3}{2}&-13&\frac{11}{2}\\0&1&0&0&7&-3\\0&0&1&\frac{1}{2}&1&-\frac{1}{2}\end{pmatrix}$$

Hence $A^{-1}=\begin{pmatrix}\frac{3}{2}&-13&\frac{11}{2}\\0&7&-3\\\frac{1}{2}&1&-\frac{1}{2}\end{pmatrix}$

6. Proof. Since $A^2=A$, *we have*

$$\begin{aligned}(I-2A)^2&=I-4A+4A^2\\&=I-4A+4A=I,\end{aligned}$$

which shows that $(1-2A)^{-1}=I-2A$ *by definition of an inverse matrix.*

□

7. The fact that (1) A is invertible is equivalent to $(3)\Leftrightarrow(2)\Leftrightarrow(4)\Leftrightarrow(5)\Leftrightarrow(6)\Leftrightarrow(7)\Leftrightarrow(8)$

8. Do a finite sequence of row operations to convert A *to* I. *This corresponds to* $E_k\cdots E_1A=I$ *for certain elementary matrices* E_i. *Record the* E_i *corresponding to each row operation. Then* $A=E_1^{-1}\cdots E_k^{-1}$ *is the factorization.*

8(c). Row operation $C=\begin{pmatrix}4&0&0&2\\0&1&0&1\\0&0&-1&2\\1&0&0&-2\end{pmatrix}\xrightarrow{R1\leftrightarrow R4}\begin{pmatrix}1&0&0&-2\\0&1&0&1\\0&0&-1&2\\4&0&0&2\end{pmatrix}\xrightarrow{(-4)R1+R4}$

$$\begin{pmatrix}1&0&0&-2\\0&1&0&1\\0&0&-1&2\\0&0&0&10\end{pmatrix}\xrightarrow{\frac{1}{10}R4}\begin{pmatrix}1&0&0&-2\\0&1&0&1\\0&0&-1&2\\0&0&0&1\end{pmatrix}\xrightarrow{(-2)R4+R3}\begin{pmatrix}1&0&0&-2\\0&1&0&1\\0&0&-1&0\\0&0&0&1\end{pmatrix}\xrightarrow{(-1)R4+R2}$$

$$\begin{pmatrix}1&0&0&-2\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}\xrightarrow{2R4+R1}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}\xrightarrow{(-1)*R3}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

From the above sequence we keep track the corresponding elementary matrices at each step $E_1=\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix}$, $E_2=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-4&0&0&1\end{pmatrix}$; $E_3=$

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\frac{1}{10}\end{pmatrix}\ E_4=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-2\\0&0&0&1\end{pmatrix}\ E_5=\begin{pmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&0\\0&0&0&1\end{pmatrix}\ E_6=$$

$$\begin{pmatrix}1&0&0&2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \quad E_7 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}$$

Then $A = E_1^{-1} \cdots E_7^{-1}$ *where* $E_1^{-1} = \begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix}$, $E_2^{-1} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\4&0&0&1\end{pmatrix}$; $E_3^{-1} =$

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&10\end{pmatrix} \quad E_4^{-1} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&2\\0&0&0&1\end{pmatrix} \quad E_5^{-1} = \begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix} \quad E_6^{-1} =$$

$$\begin{pmatrix}1&0&0&-2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \quad E_7^{-1} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}$$

9. $A = LU$. *First solve* $Ly = b$ *then solve* $Ux = y$, *this process can be expressed via matrix notion*

$$x = U^{-1}(L^{-1}b)$$

if both U and L are invertible. If not, then we just solve the two equations directly (and separately) by row-echelon.

10. a) Proof. B is row equivalent to A means that one can do a finite sequence of elementary row operation to convert A to B (or B to A). Since each row operation amounts to multiplication on the left by an elementary matrix E_i, *we know that B can be written as* $B = E_k \cdots E_1 A$. *Now B is invertible (or nonsingular) because of the existence of* $B^{-1} = A^{-1}E_1^{-1} \cdots E_k^{-1}$.

b) The problem asks to prove that if A is invertible, then A^T *is also invertible. Hence we need to show the existence of the inverse of* A^T.

Claim. The inverse of A^T *is* $(A^{-1})^T$. *In fact,*

$$(A^{-1})^T(A^T) = (AA^{-1})^T = I^T = I$$

also

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

Therefore the claim is proved true.

11. First find the common denominator $(x-1)(x+1)^2$. *The equation then becomes*

$$\frac{4x^2}{(x-1)(x+1)^2} = \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$$

Compare the coefficients of the x^2, x *and constant terms for the numerator, we obtain three linear equations with three unknowns* A, B, C. *Solve the linear equations for* A, B, C.

12. For example $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Multiplied by E_1, E_2, E_3 *on the left of a given matrix* A *correspond to row operations: exchanging the first and second rows, multiplying the third by 3 and Row 2 add* $-2\times$ *Row 1*$\rightarrow$ *Row 2, respectively for* A.