Review Test 3 **Name**

Math 142 **Section** **Id**

Use exactly one page for each of the five numbered questions (use the back of the page if necessary).
Put your name and the question number on each page.
Put a box around the final answer to a question.
You must *show* your work in order to get possible credits.

1. Find the limit of the sequence

a) $\lim_{n\to\infty} n \tan \frac{1}{n}$

Hint: Substitution $x = 1/n$. Then as $n \to \infty$, $x \to 0$. Apply L'hopital rule to $\frac{\tan x}{x}$ as $x \to 0$.

b) $\lim_{n\to\infty} \frac{\ln(3n+5)}{n}$

Hint: $a_n \sim \frac{\ln(3n)}{n}$, $n \to \infty$, so a first guess is the limit should be zero. To show this, you can apply L'hopital to

$$\lim_{x\to\infty} \frac{\ln(3x+5)}{x}$$

c) $\lim_{n\to\infty} \frac{2n^2}{(n+1)^2}$

Hint:

$$\frac{2n^2}{(n+1)^2} = \frac{2n^2 \frac{1}{n^2}}{(n+1)^2 \frac{1}{n^2}}$$

$$= \frac{2}{(1+\frac{1}{n})^2}$$

d) $\lim_{n\to\infty} \frac{e^n}{e^{2n}-1}$

e) $\lim_{n\to-\infty} \frac{e^n}{e^{2n}-1}$

2. Determine whether the limit of the following sequence exists as $n \to +\infty$, if so, find the limit:

a) $\frac{(-1)^n}{n!}$

Hint: Notice

$$-\frac{1}{n!} \leq \frac{(-1)^n}{n!} \leq \frac{1}{n!}$$

Obviously $\frac{1}{n!} \leq \frac{1}{n} \to 0$. By squeezing theorem, the original sequence goes to zero.

b) $\frac{\sin(3n)}{\sqrt{n\pi}}$

Hint: Notice

$$0 \leq \left|\frac{\sin(3n)}{\sqrt{n\pi}}\right| \leq \frac{1}{\sqrt{n\pi}}$$

Apply squeezing theorem.

c) $(-1)^n + 100$

Hint: The sequence is (starting with $n = 1$) : 99, 101, 99, 101, 99, 101, ...

So limit D.N.E.

d^*) $\tan(n - \pi)$

Hint: $\tan(n - \pi) = \tan n$, which does not have a limit

3. Evaluate the limit

a. $\displaystyle\lim_{n\to\infty} \frac{n^{5/2} + 7n^2 + 9}{-n^{5/2} + 3n^2 - 3n - 11} =$

Hint: multiplying $1/n^{5/2}$ both top and bottom

b. $\displaystyle\lim_{n\to\infty} \frac{5n^5 - 7n^3 + 10}{5n^4 + 6n^2 + 9} =$

Hint: multiplying $1/n^4$ both top and bottom

Fill in the blanks or parenthesis in Problems **4** to **8**.

4. n^{th}**-term test:** Let $\{a_n\}$ be an arbitrary sequence.

(a) If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ __________

(b) If $\sum_n a_n$ converges, then $\lim a_n =$ __________

5. **Integral Test:** $a_n > 0$. Let $f : [1, \infty) \to \mathbf{R}$ be so that

- $a_n = f(\ )$ for each $n \in \mathbf{N}$
- f is a ______________________________ function
- f is a ______________________________ function

- f is a ______________________________ function .

Then $\sum a_n$ converges if and only if ______________________________ converges.

6. (a) **Comparison Test:** $a_n > 0$

- If $0 \leq a_n \leq b_n$ for all $n \in \mathbf{N}$ and $\sum b_n$ _______________, then $\sum a_n$ _______________.
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbf{N}$ and $\sum b_n$ _______________, then $\sum a_n$ _______________.

(b). **Limit Comparison Test:** $a_n > 0$
Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.
If ________ $< L <$ ________, then $\sum a_n$ converges if and only if $\sum b_n$ _______________ .

7. (a) **Ratio Test:** $a_n > 0$
Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

- If $\rho <$ ________ then $\sum a_n$ converges.
- If $\rho >$ ________ then $\sum a_n$ diverges.
- If $\rho =$ ________ then the test is inconclusive.

(b) **Root Test:** $a_n > 0$
Let $\rho = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$.

- If $\rho <$ ________ then $\sum a_n$ converges.
- If $\rho >$ ________ then $\sum a_n$ diverges.
- If $\rho =$ ________ then the test is inconclusive.

8*. (Optional) **Alternating Series Test:** $a_n > 0$. If

- a_n ________$\geq a_{n+1}$ for n large (enventually decreasing)
- $\lim_{n\to\infty} a_n =$ ________0

then $\sum(-1)^n a_n$ ________ converges.

9. Determine whether the series converges. If it does, find the value of the sum.

$$(a) \quad \sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^n$$

Hint:

$$\sum a_n = \frac{\frac{-2}{3}}{1-(-2/3)}$$

$$(b) \quad \sum_{k=1}^{\infty} \frac{1}{k^2+2k-3}$$

Hint:

$$\frac{1}{k^2+2k-3} = \frac{1}{4}\left(\frac{1}{k-1} - \frac{1}{k+3}\right)$$

10. Determine the convergence/divergence of the series below. A correctly checked box without appropriate explanation will receive 0 or 1 point.

$$\sum_{n=1}^{\infty} (-1)^n \; \frac{\ln\ n}{n}$$

☐ absolutely convergent

☐ conditionally convergent

☐ divergent

(Hint: Conditional convergent means $\sum a_n$ is convergent but $\sum |a_n|$ divergent. Since

$$|a_n| = \frac{\ln n}{n} \geq \frac{1}{n} \qquad n \geq 3$$

and $\sum \frac{1}{n} = \infty$, it follows by direct comparison test $\sum |a_n| = \infty$. Since $a_n = \frac{\ln n}{n}$ is eventually positive, decreasing to zero, by the alternating series test, $\sum a_n$ converges, hence converges conditionally.

11. Let $a_n = \frac{n^3\ (n!)}{(2n)!}$ Find a_{n+1}/a_n. Simplify your answer so that no factorial sign (i.e., !) appears.

answer: $\dfrac{a_{n+1}}{a_n} \quad =$

$$\sum_{n=2}^{\infty} (-1)^n \ \frac{n^3\ (n!)}{(2n)!}$$

- ☐ absolutely convergent
- ☐ conditionally convergent
- ☐ divergent

Answer: since $|a_{n+1}/a_n| \to 0 < 1$, by ratio test converge absolutely

12. Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(x+7)^n}{n^2} \ .$$

In the box below draw a diagram indicating for which x's this series is: absolutely convergent, conditionally convergent, and divergent. Of course, indicate your reasoning.

(Hint: use ratio/root test to determine the radius of convergence, interval of convergence.
We find radius of convergence $R = 1$.

At the endpoints $x = -8$ and $x = -6$, the series converges absolutely because $\sum \frac{1}{n^2}$ does.

Hence, interval of convergence is $[-7 - R, -7 + R] = [-8, -6]$.

13. Find the Taylor or Maclaurin series of $y = f(x)$

(a)

$$f(x) = e^{2x}$$

about $x = 1$

Hint: Know

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$$

Sub $y = 2(x - 1)$ gives

$$e^{2(x-1)} = \sum_{n=0}^{\infty} \frac{2^n (x-1)^n}{n!}$$

So

$$e^{2x} = e^2 e^{2(x-1)} = e^2 \sum_{n=0}^{\infty} \frac{2^n (x-1)^n}{n!}$$

$$= \sum_{n=0}^{\infty} e^2 \frac{2^n (x-1)^n}{n!}$$

(b)

$$f(x) = \frac{1}{1+x}$$

about $x = 0$.

Hint: Method I. Compute the $n - th$ derivatives of $f(x) = (1 + x)^{-1}$ at $x = 0$ and use Taylor expansion:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} x^n$$

Method II. From summation formula for geometric series we know

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$$

with radius of convergence 1. Interval of convergence $(-1, 1)$.

14. **Geometric Series**

a. If $|r| < 1$, then

$$\sum_{n=0}^{\infty} r^n = \boxed{}$$

Answer for (a): $\frac{1}{1-r}$

(Hint for part (b), if $|r| < 1$, then $\sum_{\mathbf{n=0}}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + \ldots$.)

b. Find the sum of the below series. (Note that the sum begins at $n = 10$ instead of $n = 0$.)

$$\sum_{n=10}^{\infty} 2 \left(\frac{1}{5}\right)^{n-2} = \boxed{}$$

c. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{\pi}{e}\right)^n$

- ☐ absolutely convergent
- ☐ conditionally convergent
- ☐ divergent

Hint: $\frac{\pi}{e} \approx \frac{3.14159}{2.71828} \approx 1.16$.

Ans:

$$|a_n| = \left(\frac{\pi}{e}\right)^n \to \infty, \; n \to \infty$$

So a_n does not tend to 0. Divergent by $n-th$ term test.