Review Test 1 **Name**
Math 5339 **Id**

Read each question carefully. Avoid simple mistakes. (Use the back of the page if necessary). **You must show your work in order to get credits or partial credits**.

1. [10pts] For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons.

   (a) $u_{tt} - c^2 u_{xx} + u^3 = 0$

   (b) $u_t - k u_{xx} + 6u_{xt} = 0$

   (c) $u_{xx} + u_{yy} + 2xy u_{xy} = 0$

   (d) $u_{tt} - u_{xx} = \sin u$

   (e) $u_t - u_{xxx} + 6uu_x = 0$

   (f) $\frac{du}{dt} = G(u)$

2. [15pts] Solve the first order $2u_t + 3u_x = 0$ with auxiliary condition $u = \sin x$ when $t = 0$. (Hint: Express the solution as, according to characteristic curve method, $u(t,x) = f(bt - ax)$)

3. [15pts] Solve $u_{tt} = c^2 u_{xx}$, $u(x,0) = e^x$, $u_t(x,0) = \sin x$.

4. [10pts] Find the solution of the Laplace equation $u_{xx} + u_{yy} = 0$ for $x, y \in [0,1]$. Use separation of variables.

5. [10pts] Solve the wave equation $u_{tt} - u_{xx} = 0$ for $x \in [0, \ell], t \in \mathbb{R}$.

6. [10pts] Use change of variables $t' = at + bx, x' = bt - ax$ to solve

$$au_t + bu_x = f(x,t), (t,x) \in R \times \mathbb{R} \tag{1}$$

7. [20pts] The solution to $u_t - ku_{xx} = 0, u(0,x) = f(x)$ is given by

$$u(t,x) = e^{-kt\Delta} f(x) = \frac{1}{(4k\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4kt}} f(y)dy := \int_{\mathbb{R}^n} p_t(x,y) f(y)dy,$$

where $p_t(x,y)$ denote the fundamental solution or heat kernel. Show that

$$a)\int_{\mathbb{R}^n} e^{-|x|^2}dx = \pi^{n/2}$$
$$b)\int p_t(x,y)dx = 1 \qquad \forall t>0$$
$$c)\text{ Given any } \delta>0, \lim_{t\to 0}\int_{|x-y|\geq\delta} p_t(x,y)dy = 0$$
$$d)\text{ If } f\in L^p, 1\leq p\leq\infty, \text{ then } u(t,x)\to f(x) \text{ as } t\to 0.$$

8. [20pts] a) Consider the Schrödinger equation

$$i\partial_t u + \Delta u + \varepsilon|u|^p u = 0$$

Show that the conserved quantities (constants of motion) are

$$Q(t) = \int |u|^2 dx$$
$$E(t) = \frac{1}{2}\int(|\nabla_x u|^2 + \frac{\varepsilon}{p+1}|u|^{p+1})dx$$

(Hint: Use integration by parts to prove $dE(t)/dt = 0$, thus $E(t)$ must be a constant)
b) How about wave equation

$$\partial_{tt}u - u_{xx} + \varepsilon|u|^p u = 0$$

Is $E(t) = \frac{1}{2}\int_{\mathbb{R}^n}(u_t^2 + |\nabla u|^2 + \frac{\varepsilon}{p+1}|u|^{p+1})dx$ conserved in time?

9. [15pts] Solve the heat equation with convection

$$u_t - ku_{xx} + Vu_x = 0, \qquad -\infty < x < \infty$$
$$u(x,0) = \phi(x)$$

where V is a constant. (Hint: Sub $y = x - Vt$)

10. [10pts] Solve $u_{tt} = 9u_{xx}$ in $x\in(0,\frac{\pi}{2})$, $u(x,0)=\cos x$, $u_t(x,0)=0$, $u_x(0,t)=0, u(\frac{\pi}{2},t)=0$.

11. [Bonus 10pts] Solve the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition

$$u_t - ku_{xx} = f(x,t) \qquad 0 < x, t < \infty$$
$$u(x,0) = \phi(x), u(0,t) = 0$$

using the method of reflection.

Solutions 2. The general solution for the transport equation is given by $u(x,t) = f(bt - ax)$; see Lecture notes or Section 1.2 in Strauss. Sub the initial condition with $t = 0$ into this, we get $f(-ax) = \sin x$. Therefore $f(x) = -\sin(x/a)$.

3. The general solution for the wave equation on the line $(-\infty, \infty)$ is given by $u(x,t) = f(x+ct) + g(x-ct)$. Substituting the initial conditions into this expression yields

$$\begin{cases} f(x) + g(x) = e^x \\ cf'(x) - cg'(x) = \sin x \end{cases}$$

Now taking derivative on the first equation we obtain $f'(x) + g'(x) = e^x$; this, together with the second equation will lead to a solution of f' and g'. From there antiderivative will recover f and g.

4. Write $u(x,y) = X(x)Y(y)$ and sub into $\Delta u = 0$. We then have

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

where $\lambda \geq 0$ is a constant (or eigenvalue of d^2/dx^2).

5. Write $u(x,y) = T(t)X(x), \ldots$.

6. This is a transport equation with inhomogeneous term f. A detailed solution was given in the handout Monday Feb 22.

7. a)

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-x_1^2} \cdots e^{-x_n^2} dx_1 \cdots dx_n$$
$$= \int_{-\infty}^{\infty} e^{-x_1^2} dx_1 \cdots \int_{-\infty}^{\infty} e^{-x_n^2} dx_n$$

There are a few ways to evaluate $\int_{-\infty}^{\infty} e^{-x^2} dx$. For instance using polar coordinate (r, θ) in 2 dimensions for $x = (x_1, x_2)$.

$$\int_{\mathbb{R}^2} e^{-|x|^2} dx_1 dx_2 = \int_0^{2\pi} d\theta \int_0^\infty e^{-r^2} r dr$$
$$= 2\pi \int_0^\infty e^{-r^2} r dr = -\pi \int_0^\infty d(e^{-r^2}) = \pi$$

Meanwhile,

$$\int_{\mathbb{R}^2} e^{-|x|^2} dx_1 dx_2 = \int_{\mathbb{R}^2} e^{-x_1^2} e^{-x_2^2} dx_1 dx_2$$
$$= \int_{\mathbb{R}} e^{-x_1^2} dx_1 \int_{\mathbb{R}} e^{-x_2^2} dx_2$$

which shows $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$
b) $\int p_t(x, y) dx = 1$ follows from a) by scaling or change of variables.
c) For fixed $\delta > 0$, a change of variables $u = (y - x)/\sqrt{4kt}$ gives

$$\int_{|x-y| \geq \delta} \frac{1}{(4k\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4kt}} dy = \int_{|u| \geq \delta/\sqrt{4kt}} e^{-|u|^2} du \to 0 \qquad as\ t \to 0$$

d) Let $f \in L^p$, $1 \leq p \leq \infty$. We may assume $f \in C_b(\mathbb{R}^n)$, that is, f is bounded continuous function. Then

$$u(t, x) - f(x) =$$
$$\int_{|x-y| < \delta} p_t(x, y)[f(y) - f(x)] dy + \int_{|x-y| \geq \delta} p_t(x, y)[f(y) - f(x)] dy := I_1 + I_2.$$

Given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ whenever $|y - x| < \delta$. Since

$$|I_1| \leq \varepsilon \int_{|x-y| < \delta} p_t(x, y) dy \leq \varepsilon \int_{\mathbb{R}^n} p_t(x, y) dy = \varepsilon$$
$$and\ |I_2| \leq 2C \int_{|x-y| \geq \delta} p_t(x, y) dy \to 0\ as\ t \to 0,$$

we see immediately that $u(t, x) - f(x) \to 0$ as $t \to 0$.

8. a) Show $Q'(t) = 0$ using the Schrödinger equation and integration by parts.
b) Similar method.

9. Sub $y = x - Vt$ and we have $u(x,t) = u(y+Vt,t) := \tilde{u}(y,t)$, where $x = x(y,t) = y + Vt$. By chain rule,

$$\frac{\partial}{\partial t}\tilde{u}(y,t) = \frac{\partial}{\partial t}[u(y+Vt,t)] = \frac{\partial}{\partial x}u(x,t)\cdot V + \frac{\partial}{\partial t}u(x,t)$$
$$\frac{\partial}{\partial y}\tilde{u}(y,t) = \frac{\partial}{\partial y}[u(y+Vt,t)] = \frac{\partial}{\partial x}u(x,t)\cdot 1.$$

Sub the above into the $u_t - ku_{xx} + Vu_x = 0$, $u(x,0) = \phi(x)$. We find that $\tilde{u}_t - k\tilde{u}_{yy} = 0$, $\tilde{u}(y,0) = \phi(y)$. Thus $\tilde{u}(y,t) = \int p_t(y,z)\phi(z)dz$ which gives that

$$\begin{aligned} u(x,t) &= \int p_t(x - Vt, z)\phi(z)dz \\ &= \frac{1}{(4k\pi t)^{n/2}}\int_{\mathbb{R}} e^{-\frac{|x-Vt-z|^2}{4kt}}\phi(z)dz. \end{aligned}$$

10. By reflection method as described in Section 3.3, extend $u(x,t)$ as odd function on $(-\frac{\pi}{2}, \frac{\pi}{2})$, and then periodically on $\mathbb{R}$, which we call $v(x,t)$. Also do the same for ϕ and ψ. Then v satisfies the wave equation on $\mathbb{R}$ with initial condition $v(x,0) = \tilde{\phi}$, $v_t(x,0) = \tilde{\psi}$, with a little abuse of notation we will continue to call them ϕ, psi. Thus the solution is given by

$$v(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \int_{x-ct}^{x+ct}\psi(s)ds$$

Note that the boundary condition is automatically satisfied because of the odd symmetries on **both** $x = 0, x = \pi/2$. As in the text, the strip $(0, \pi/2)\times\mathbb{R}$ divided into diamond regions. Inside each of these diamond, the wave propagates through a chain of reflections against the boundaries, $\phi(x \pm ct)$ has different sign depending on the number of reflections, similarly for $\pm\psi$. After some simple calculations, the express of $v(x,t)$ has a domain of dependence resulting from those reflections.

11. The inhomogeneous problem on the line has the solution

$$\begin{aligned} v(x,t) &= e^{t\Delta}\phi + \int_0^t e^{(t-s)\Delta}f(x,s)ds \\ &= \int p_t(x,y)\phi(y)dy + \int_0^t ds\int p_{t-s}(x,y)f(y,s)dy \end{aligned}$$

Make odd extension of u to v with $\phi(-x) = -\phi(x)$, then restrict $u = v|_{\mathbb{R}\times\mathbb{R}_+}$ to obtain the solution.