## Surface Integrals

Suppose $G(x,y,z)$ is continuous on a smooth surface $S$.
The ***surface integral*** of $G$ over $S$ is given as follows:

**1.** If $S$ is given parametrically by $\mathbf{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}, \quad (u,v) \in R,$ then

$$\iint_S G(x,y,z)\, d\sigma = \iint_R G\big(f(u,v), g(u,v), h(u,v)\big)\, |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$$

**2.** If $S$ is given implicitly by $F(x,y,z) = c$ where $F$ continuously differentiable, and $S$ lies above its closed bounded shadow $R$ in the coordinate plane beneath it, then

$$\iint_S G(x,y,z)\, d\sigma = \iint_R G(x,y,z)\, \frac{|\nabla F|}{|\nabla F \bullet p|}\, dA$$

where $p = \mathbf{i}, \mathbf{j},$ or $\mathbf{k}$ is normal to $R$ and $\nabla F \bullet p \neq 0$.

**Note**

If $G(x,y,z) = 1$, for all $(x,y,z) \in S$, then $\iint_S G(x,y,z)\, d\sigma =$ the area of $S$.

**Example** $G(x,y,z)$

Evaluate $\iint_S x^2 d\sigma$ where $S$ is the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$.

$f \quad g \quad h$

We already evaluated $\iint_S d\sigma$ using the parametrization $\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle$, $(r,\theta) \in [0,1]\times[0,2\pi]$. We got $|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{2}r$. Since $G(f,g,h) = r^2\cos^2\theta$,

we get $\qquad (\cos^2\theta)(\tfrac{1}{4})$

$$\iint_S x^2 d\sigma = \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta)(\sqrt{2}\,r)\,dr\,d\theta = \sqrt{2}\int_0^{2\pi}\int_0^1 r^3\cos^2\theta\,dr\,d\theta$$

$$= \frac{\sqrt{2}}{4}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{\sqrt{2}}{8}\int_0^{2\pi}(1+\cos 2\theta)\,d\theta = \frac{\sqrt{2}}{8}\left(\theta + \tfrac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = \frac{\pi\sqrt{2}}{4}$$

$\tfrac{1}{2}(1+\cos 2\theta)$

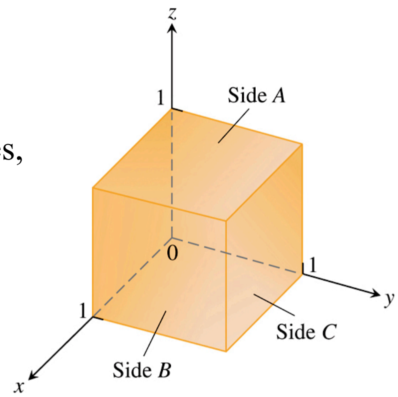
**Additivity of Surface Integrals**

If $S = S_1 \cup S_2 \cup \ldots \cup S_n$ where $S_1, S_2, \ldots, S_n$ are smooth and overlap along smooth curves then

$$\iint_S G\, d\sigma = \iint_{S_1} G\, d\sigma + \iint_{S_2} G\, d\sigma + \ldots + \iint_{S_n} G\, d\sigma$$

**Example**

Integrate $G(x,y,z)=xyz$ over the surface of the cube cut from the first octant by the planes $x=1$, $y=1$, and $z=1$.




We integrate $xyz$ over the six sides and add the results.

Since $xyz=0$ on the sides that lie in the coordinate planes, we get

$$\iint\limits_{\substack{\text{Cube}\\ \text{surface}}} xyz\,d\sigma = \iint\limits_{\text{Side }A} xyz\,d\sigma + \iint\limits_{\text{Side }B} xyz\,d\sigma + \iint\limits_{\text{Side }C} xyz\,d\sigma$$

Side $A$: $z=1$

Can be regarded as a level surface of the function $F(x,y,z)=z$ (namely $F(x,y,z)=1$)

$$\left.\begin{aligned} |\nabla F| &= |\langle 0,0,1\rangle| = 1 \\ |\nabla F \bullet p| &= |\,1\,| = 1 \end{aligned}\right\}\ d\sigma = 1$$

$$\iint\limits_{\text{Side }A} \overset{xy}{\overbrace{xyz}}\,d\sigma = \iint\limits_{[0,1]\times[0,1]} xy\,dA = \int_0^1 \underbrace{\int_0^1 xy\,dx}_{\frac{1}{2}y}\,dy = \left(\tfrac{1}{4}y^2\right)_0^1 = \frac{1}{4}$$

$(p=\mathbf{k})$

$R=[0,1]\times[0,1]$

$$\iint\limits_{S} G(x,y,z)\,d\sigma = \iint\limits_{R} G(x,y,z)\overbrace{\frac{|\nabla F|}{|\nabla F \bullet p|}}^{d\sigma}\,dA$$

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The other two can be done the same way. Actually, by symmetry it follows they have the same value, i.e. the final answer is $\boxed{\frac{3}{4}}$.

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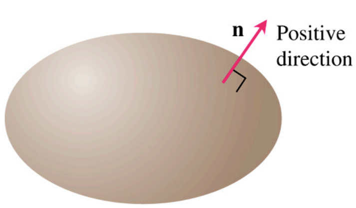
**Definition**

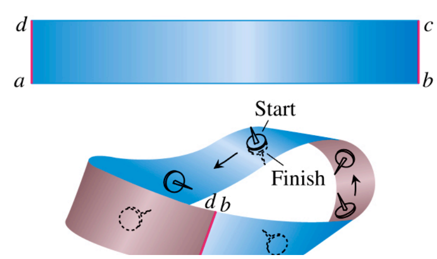
A smooth surface $S$ is ***orientable*** (or ***two-sided***) if it is possible to define a field of unit normal vectors $\mathbf{n}$ on $S$ which varies continuously with position.

Spheres and other smooth $\overset{\textbf{enclose solids}}{\text{closed surfaces}}$ in space are orientable.

By convention, we usually choose $\mathbf{n}$ on a closed surface to point outward.




A surface that is **not** orientable:




**Möbius band**

## Surface Integrals of Vector Fields

**Definition**

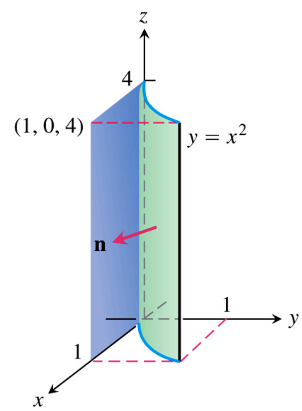
Let $\mathbf{F}$ be a 3D vector field defined over a smooth surface $S$ having a chosen field of normal unit vectors $\mathbf{n}$ orienting $S$. Then the **surface integral of F over *S*** is

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, d\sigma \qquad \text{(the } \textbf{flux} \text{ of } \mathbf{F} \text{ across } S\text{)}$$

**Recall:** For a 2D vector field $\mathbf{F}$ and a closed curve $C$, flux of $\mathbf{F} = \iint_C \mathbf{F} \bullet \mathbf{n}\, ds$.

**Example**

Find the flux of $\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}$ through the parabolic cylinder $y = x^2$ in the direction $\mathbf{n}$ indicated in the picture.




**Solution 1** (parametrizing the cylinder)

$$\mathbf{r} = \mathbf{r}(x,z) = \langle x, x^2, z\rangle \qquad \mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_z}{|\mathbf{r}_x \times \mathbf{r}_z|} \qquad d\sigma = |\mathbf{r}_x \times \mathbf{r}_z|\, dA$$

$$\mathbf{r}_x \times \mathbf{r}_z = \langle 2x, -1, 0\rangle$$

$$|\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2+1} \qquad \mathbf{F} \bullet \mathbf{n}\, d\sigma = \mathbf{F} \bullet (\mathbf{r}_x \times \mathbf{r}_z)\, dA$$

On the cylinder,

$$\mathbf{F} = \langle yz, x, -z^2\rangle = \langle x^2 z, x, -z^2\rangle \qquad (y = x^2)$$

and so

$$\mathbf{F} \bullet (\mathbf{r}_x \times \mathbf{r}_z) = \langle x^2 z, x, -z^2\rangle \bullet \langle 2x, -1, 0\rangle = 2x^3 z - x$$

Therefore

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, d\sigma = \iint_R \mathbf{F} \bullet (\mathbf{r}_x \times \mathbf{r}_z) dA = \int_0^4 \int_0^1 (2x^3 z - x)\, dx\, dz = \tfrac{1}{2}\int_0^4 (z-1)\, dz = \tfrac{1}{2}\left(\tfrac{1}{2}z^2 - z\right)\Big|_0^4$$

$$= \boxed{2}$$

$$\left(\tfrac{1}{2}x^4 z - \tfrac{1}{2}x^2\right)\Big|_0^1 = \tfrac{1}{2}z - \tfrac{1}{2}$$

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cylinder $y = x^2$ in the direction $\mathbf{n}$ indicated in the picture.

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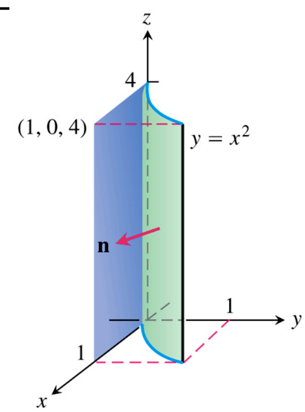


**Solution 2** (treating the cylinder as a level surface)

$$g(x,y,z) = x^2 - y = 0 \qquad \mathbf{n} = \frac{\nabla g}{|\nabla g|} \qquad d\sigma = \frac{|\nabla g|}{|\nabla g \bullet \mathbf{p}|}\, dA$$

$$\nabla g = \langle 2x, -1, 0\rangle$$

$$|\nabla g| = \sqrt{4x^2+1} \qquad \mathbf{F} \bullet \mathbf{n}\, d\sigma = \mathbf{F} \bullet \frac{\nabla g}{|\nabla g \bullet \mathbf{p}|}\, dA$$

On the cylinder,

$$\mathbf{F} = \left\langle yz, x, -z^2 \right\rangle = \left\langle x^2 z, x, -z^2 \right\rangle \qquad \left(y = x^2\right)$$

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$$\nabla g \bullet \mathbf{p} = -1 \qquad (\mathbf{p} = \mathbf{j})$$

So

$$\mathbf{F} \bullet \frac{\nabla g}{|\nabla g \bullet \mathbf{p}|} = \left\langle x^2 z, x, -z^2 \right\rangle \bullet \langle 2x, -1, 0 \rangle = 2x^3 z - x$$

and

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, d\sigma = \iint_R \mathbf{F} \bullet \frac{\nabla g}{|\nabla g \bullet p|}\, dA = \int_0^4 \int_0^1 \left(2x^3 z - x\right) dx\, dz \qquad \text{(same as before)}$$

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