§2.1 Rates of Change and Tangent Lines to Curves

Example 1. A rock breaks loose from the top of a tall cliff. What is its average speed
(a) during the first 2 sec of fall?
(b) during the 1-sec interval between second 1 and second 2?

Example 2. Find the speed of the falling rock in Example 1 at $t = 1$ and $t = 2$ sec.

Definition. The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

§2.2 Limit of a Function and Limit Laws

Example 1. How does the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ behave near $x = 1$?

Definition (informal). Suppose that $f(x)$ is defined on an open interval about c, except possibly at c itself. If $f(x)$ is arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to c, other than c itself, then we say that f approaches the **limit** L as x approaches c, and write $\lim\limits_{x\to c} f(x) = L$.

†Typically we can observe the limit by checking the behavior of the function near $x = c$ using graphing or numerical table methods.

Example 2. The limit of a function does not depend on how the function is defined at the point being approached. Consider the three functions

$$f(x) = \frac{x^2 - 1}{x - 1}, \qquad g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}, \qquad h(x) = x + 1.$$

Example 3. (a) If f is the **identity function** $f(x) = x$, then for any value of c, $\lim_{x\to c} f(x) = \lim_{x\to c} x = c$.
(b) If f is the **constant function** $f(x) = k$ (function with constant value k), then for any value of c, $\lim_{x\to c} f(x) = \lim_{x\to c} k = k$.

Example 4. Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$ (b) $g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ (c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$

Theorem 1 (Limit Laws). If L, M, c, and k are real numbers and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

1 & 2. *Sum and Difference Rule:* $\lim_{x\to c}(f(x) \pm g(x)) = L \pm M$

3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$

4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$

5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$

6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n, \quad n$ a positive integer

7. *Root Rule:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \quad n$ a positive integer

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c.)

Example 5. Use Example 3 and Theorem 1 to find the following limits.

(a) $\lim_{x\to c}(x^3 + 4x^2 - 3)$ (b) $\lim_{x\to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$ (c) $\lim_{x\to -2} \sqrt{4x^2 + 3}$

Theorem 2. If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then $\lim_{x\to c} P(x) = P(c)$.

Theorem 3. If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then $\lim_{x\to c} \dfrac{P(x)}{Q(x)} = \dfrac{P(c)}{Q(c)}$.

Example 6 $\lim_{x\to -1} \dfrac{x^3 + 4x^2 - 3}{x^2 + 5} = \dfrac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = 0$

Exercise 79.* If $\lim_{x\to 4} \dfrac{f(x) - 5}{x - 2} = 1$, find $\lim_{x\to 4} f(x)$.

Example 7. Evaluate $\lim_{x\to 1} \dfrac{x^2 + x - 2}{x^2 - x}$.

Example 9. Evaluate $\lim_{x\to 0} \dfrac{\sqrt{x^2 + 100} - 10}{x^2}$.

Theorem 4 (The Sandwich Theorem). Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. Then $\lim_{x\to c} f(x) = L$.

Example 11. (a) $\lim_{\theta\to 0} \sin\theta = 0$ **(b)** $\lim_{\theta\to 0} \cos\theta = 1$

(c) For any function f, $\lim_{x\to c} |f(x)| = 0$ implies $\lim_{x\to c} f(x) = 0$.