§5.3 The Definite Integral (Continued)

Exercise 20. Evaluate $\int_{-1}^{1} (1 - |x|)\ dx$.

Definition. If f is integrable on $[a, b]$, then its **average value on $[\boldsymbol{a}, \boldsymbol{b}]$**, which is also called its **mean**, is $\operatorname{av}(f) = \dfrac{1}{b-a} \displaystyle\int_a^b f(x)dx$.

Example 5. Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

§5.4 The Fundamental Theorem of Calculus

Theorem 3 (The Mean Value Theorem for Definite Integrals). If f is continuous on $[a, b]$, then at some point c in $[a, b]$, $f(c) = \dfrac{1}{b-a} \displaystyle\int_a^b f(x)dx$.

Theorem 4 (The Fundamental Theorem of Calculus, Part 1). If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\ dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$: $F'(x) = \dfrac{d}{dx} \displaystyle\int_a^x f(t)dt = f(x)$.

Example 2. Find dy/dx if

(a) $y = \displaystyle\int_a^x (t^3 + 1)dt$ (b) $y = \displaystyle\int_x^5 3t \sin t dt$ (c) $y = \displaystyle\int_1^{x^2} \cos t dt$ (d) $y = \displaystyle\int_{1+3x^2}^4 \frac{1}{2 + e^t} dt$.

Exercise 80. Find $f(4)$ if $\int_0^x f(t)dt = x \cos(\pi x)$.

Theorem 4 Continued (The Fundamental Theorem of Calculus, Part 2). If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Example 3. Calculate (a) $\displaystyle\int_{-\pi/4}^0 \sec x \tan x dx$ (c) $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx$ (d) $\displaystyle\int_0^1 \frac{dx}{x^2 + 1}$.

Theorem 5 (The Net Change Theorem). The net change in a differentiable function $F(x)$ over an interval $[a, b]$ is the integral of its rate of change: $F(b) - F(a) = \int_a^b F'(x)dx$.

Example 8. Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.

§5.5 Indefinite Integrals and the Substitution Method

Example 1. Find the integral $\int (x^3+x)^5(3x^2+1)dx$.

Theorem 6 (The Substitution Rule). If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then $\int f(g(x)) \cdot g'(x)dx = \int f(u)du$.

Example 4. Find $\int \cos(7\theta+3)d\theta$.

Example 5. Find $\int x^2 e^{x^3} dx$.

Example 6. Evaluate $\int x\sqrt{2x+1}\, dx$.

Integrals of the Tangent, Cotangent, Secant, and Cosecant Functions.

$\int \tan x dx = \ln|\sec x| + C$

$\int \cot x dx = \ln|\sin x| + C$

$\int \sec x dx = \ln|\sec x + \tan x| + C$

$\int \csc x dx = -\ln|\csc x + \cot x| + C$

§5.6 Definite Integral Substitutions and the Area Between Curves

Theorem 7 (Substitution in Definite Integrals). If g' is continuous on the interval $[a,b]$ and f is continuous on the range of $g(x) = u$, then $\int_a^b f(g(x)) \cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

Example 1. Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx$.

Example 2. Evaluate (a) $\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta d\theta$.

Definition. If f and g are continuous with $f(x) \geq g(x)$ throughout $[a,b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f-g)$ from a to b: $A = \int_a^b [f(x) - g(x)]dx$.

Example 4. Find the area of the region bounded above by the curve $y = 2e^{-x} + x$, below by the curve $y = e^x/2$, on the left by $x = 0$, and on the right by $x = 1$.

Example 5. Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Examples 6 & 7. Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.