§4.4 *Spanning Sets and Linear Independence* (Continued)

Definition. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the **span of S** is the set of all linear combinations of the vectors in S,

$$\text{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k : c_1, c_2, \cdots, c_k \text{ are real numbers}\}.$$

The span of S is denoted by span(S) or span$\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$. When span$(S) = V$, it is said that V is **spanned** by $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$, or that S **spans** V.

Theorem 4.7. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then span(S) is a subspace of V. Moreover, span(S) is the smallest subspace of V that contains S, in the sense that every other subspace of V that contains S must contain span(S).

Definition. A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ in a vector space V is **linearly independent** when the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ has only the trivial solution $c_1 = 0, c_2 = 0, \cdots, c_k = 0$. If there are also nontrivial solutions, then S is **linearly dependent**.

Example 7. The followings are examples of linearly dependent sets.
(a) $S = \{(1, 2), (2, 4)\}$ (b) $S = \{(1, 0), (0, 1), (-2, 5)\}$

Example 8. Determine whether the set of vectors in R^3 is linearly independent.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

Example 9. Determine whether the set of vectors in P_2 is linearly independent.

$$S = \{1 + x - 2x^2, 2 + 5x - x^2, x + x^2\}$$

Theorem 4.8. A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\mathbf{v}_i$ can be written as a linear combination of the other vectors in S.

§4.5 *Basis and Dimension*

Definition. A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ in a vector space V is a **basis** for V when the conditions below are true.
1. S spans V. 2. S is linearly independent.

Example 1. Show that the set $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis for R^3.

Example 2. Show that the set $S = \{(1, 1), (1, -1)\}$ is a basis for R^2.

Example 4. Show that the vector space P_3 has the basis $S = \{1, x, x^2, x^3\}$.

Theorem 4.9. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

Theorem 4.10. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.

Example 7. (b) P_3 has a basis consisting of four vectors, so the set

$$S = \{1, 1+x, 1-x, 1+x+x^2, 1-x+x^2\}$$

must be linearly dependent.

Theorem 4.11. If a vector space V has one basis with n vectors, then every basis for V has n vectors.

Definition. If a vector space V has a basis consisting of n vectors, then the number n is the **dimension** of V, denoted by $\dim(V) = n$. When V consists of the zero vector alone, the dimension of V is defined as zero.

Example 9. Find the dimension of the subspace of R^3.
(a) $W = \{(d, c-d, c) : c \text{ and } d \text{ are real numbers}\}$

Example 11. Let W be the subspace of all symmetric matrices in $M_{2,2}$. What is the dimension of W?

Solution. Each vector in $V \subset M_{2,2}$ consisting of all symmetric matrices has the form

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1}$$

□

Ex. (§4.5, # 17) Determine if $S = \{(7,0,3),\ (8,-4,1)\}$ is a basis in $\mathbb{R}^3$.
[Solution] Consider $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ as a basis in $\mathbb{R}^3$. We see the dimension of $V = \mathbb{R}^3$ is $d = 3$. However, S has only two vectors, and so it is not a basis.
Ex. (§4.5, # 27) Determine if the set S is a basis in M_{22}:

$$S = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 8 & -4 \\ -4 & 3 \end{bmatrix}\}.$$

[Solution] Recall that a set S is a basis in V provided
  (1) S spans V;
  (2) S is linearly independent.
We can see the fourth matrix is a linear combination of the other three.

$$\begin{bmatrix} 8 & -4 \\ -4 & 3 \end{bmatrix} = c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} c_1 + c_3 & c_2 \\ c_2 & c_3 \end{bmatrix}. \tag{3}$$

Now comparing the corresponding components (entry values) of the matrices in both sides of the above equation, we obtain

$$c_1 + c_3 = 8 \Rightarrow c_1 = 5 \tag{4}$$

$$c_2 = -4 \tag{5}$$

$$c_3 = 3. \tag{6}$$

So, the set S is linearly dependent. Thus we can infer S is not a basis for $M_{2,2}$.
Method II. (growth mindset)
[Solution] Input the matrices into a matrix. Since the determinant of the matrix equals zero, it is linearly dependent and does not span $M_{2,2}$.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 8 & -4 & -4 & 3 \end{vmatrix} = 0. \tag{7}$$

*Summary.	In the above we give the definitions of **basis** and **dimension** of a given vector space. We show how to determine a set of vectors to be a basis or not a basis in the context of $\mathbb{R}^n$, P_n, M_{mn}.

§4.6* *Rank of a Matrix and Systems of Linear Equations*

Definition. The dimension of the row (or column) space of a matrix A is the **rank** of A and is denoted by rank(A).

Example 6. Find the rank of the matrix $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix}$.

Theorem 4.16. If A is an $m \times n$ matrix, then the set of all solutions of the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$ is a subspace of R^n called the **nullspace** of A and is denoted by $N(A)$. So, $N(A) = \{\mathbf{x} \in R^n : A\mathbf{x} = \mathbf{0}\}$. The dimension of the nullspace of A is the **nullity** of A.

Example 7. Find the nullspace of the matrix $A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$.