Answers for Review Ex.II

1. Compute the Laplace transform of each of the following functions using the Laplace transform Tables in Appendix C.

   (a) $f(t) = 3t^3 - 2t^2 + 7$

   $$\boxed{F(s) = 3\frac{3!}{s^4} - 2\frac{2!}{s^3} + \frac{7}{s} = \frac{18}{s^4} - \frac{4}{s^3} + \frac{7}{s}.}$$

   (b) $g(t) = e^{-3t} + \sin\sqrt{2}t$

   $$\boxed{G(s) = \frac{1}{s+3} + \frac{\sqrt{2}}{s^2+2}.}$$

   (c) $h(t) = -8 + \cos(t/2)$

   $$\boxed{H(s) = -\frac{8}{s} + \frac{s}{s^2+1/4} = -\frac{8}{s} + \frac{4s}{4s^2+1}.}$$

2. Compute the Laplace transform of each of the following functions. You may use the Laplace Transform Tables.

   (a) $f(t) = 7e^{2t}\cos 3t - 2e^{7t}\sin 5t$

   $$\boxed{F(s) = \frac{7s}{(s-2)^2+9} - \frac{10}{(s-7)^2+25}.}$$

   (b) $g(t) = 3t\sin 2t$ Use formula 8 in Table C.1: $\mathcal{L}\{tf(t)\}(s) = -F'(s)$. Apply this formula to the function $f(t) = 3\sin 2t$ so that $F(s) = 6/(s^2+4)$. Since $g(t) = tf(t)$, formula 21) gives:

   $$G(s) = -F'(s) = -\frac{-12s}{(s^2+4)^2} = \frac{12s}{(s^2+4)^2}.$$

   (c) $h(t) = (2-t^2)e^{-5t}$ Use the shifting theorem (Formula 3 in Table C.1). Then

   $$\boxed{H(s) = \frac{2}{s+5} - \frac{2}{(s+5)^3}.}$$

3. Find the inverse Laplace transform of each of the following functions. You may use the Laplace Transform Tables.

(a) $F(s) = \dfrac{7}{(s+3)^3}$

$$\boxed{f(t) = \frac{7}{2}t^2 e^{-3t}.}$$

(b) $G(s) = \dfrac{s+2}{s^2-3s-4}$ Use partial fractions to write

$$G(s) = \frac{s+2}{s^2-3s-4} = \frac{1}{5}\left(\frac{6}{s-4} - \frac{1}{s+1}\right).$$

Thus $\boxed{g(t) = \left(6e^{4t} - e^{-t}\right)/5}$.

(c) $H(s) = \dfrac{s}{(s+4)^2+4}$ Since

$$H(s) = \frac{s}{(s+4)^2+4} = \frac{(s+4)-4}{(s+4)^2+4} = \frac{s+4}{(s+4)^2+4} - 2\frac{2}{(s+4)^2+4},$$

it follows that $\boxed{h(t) = e^{-4t}\cos 2t - 2e^{-4t}\sin 2t}$.

4. Find the Laplace transform of each of the following functions.

(a) $t^2 e^{-9t}$

$$\boxed{\frac{2}{(s+9)^3}}$$

(b) $e^{2t} - t^3 + t^2 - \sin 5t$

$$\boxed{\frac{1}{s-2} - \frac{6}{s^4} + \frac{2}{s^3} - \frac{5}{s^2+25}}$$

(c) $t\cos 6t$

$$\boxed{-\frac{d}{ds}\left(\frac{s}{s^2+36}\right) = \frac{s^2-36}{(s^2+36)^2}}$$

(d) $2\sin t + 3\cos 2t$

$$\boxed{\frac{2}{s^2+1} + \frac{3s}{s^2+4}}$$

(e) $e^{-5t}\sin 6t$

$$\boxed{\frac{6}{(s+5)^2+36}}$$

(f) $t^2\cos at$ where a is a constant Use Formula 9 in Table C.1, applied to $f(t) = \cos at$. Then, $F(s) = s/(s^2+a^2)$ and $\mathcal{L}\{t^2\cos at\}(s) = F''(s)$. Since $F'(s) = (a^2-s^2)/(s^2+a^2)^2$, the Laplace transform of $t^2\cos at$ is

$$\boxed{F''(s) = \frac{2s^2-6sa^2}{(s^2+a^2)^3}.}$$

5. Find the inverse Laplace transform of each of the following functions.

(a) $\dfrac{1}{s^2-10s+9}$ Since $s^2-10s-9 = (s-9)(s-1)$, use partial fractions:

$$\frac{1}{s^2-10s+9} = \frac{1}{8}\left(\frac{1}{s-9} - \frac{1}{s-1}\right) \Rightarrow \boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2-10s+9}\right\} = \frac{1}{8}(e^{9t}-e^t).}$$

(b) $\dfrac{2s-18}{s^2+9}$ $\boxed{2\cos 3t - 6\sin 3t}$

(c) $\dfrac{2s+18}{s^2+25}$ $\boxed{2\cos 5t + (18/5)\sin 5t}$

(d) $\dfrac{s+3}{s^2+5}$ $\boxed{\cos\sqrt{5}t + (3/\sqrt{5})\sin\sqrt{5}t}$

(e) $\dfrac{s-3}{s^2-6s+25}$ Since $s^2-6s+25 = (s-3)^2+4^2$, we conclude:

$$\boxed{\mathcal{L}^{-1}\left\{\frac{s-3}{s^2-6s+25}\right\} = e^{3t}\cos 4t.}$$

(f) $\dfrac{1}{s(s^2+4)}$ Use Formula 12 in Table C.1 with $F(s) = 1/(s^2+4)$ so that $f(t) = \sin 2t$. Then

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \int_0^t \sin 2v\,dv = \frac{1}{2}(1-\cos 2t).}$$

(g) $\dfrac{1}{s^2(s+1)^2}$ Use Formula 12 in Table C.1 twice, starting with $F(s) = 1/(s+1)^2$ (so $f(t) = te^{-t}$). Using the integral formula $\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C$ (which can be found in a table of integrals or derived by integration by parts, we find using Formula 12:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)^2}\right\} = \int_0^t ve^{-v}\,dv = (-v-1)e^{-v}\Big|_0^t = -te^{-t} - e^{-t} + 1.$$

Now integrating the right hand side a second time from 0 to t gives (after some algebraic simplification:

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = -te^{-t} + 2e^{-t} + t - 2.}$$

This exercise can also be solved by partial fraction expansion and by use of the convolution product formula (Formula 13 in Table C.1).

6. Solve each of the following differential equations by means of the Laplace transform:

(a) $y' + 3y = t^2e^{-3t} + te^{-2t} + t, \qquad y(0) = 1$ If $Y(s) = \mathcal{L}(y(t))$ then applying $\mathcal{L}$ to both sides of the equation gives:

$$sY(s) - 1 + 3Y(s) = \frac{2}{(s+3)^3} + \frac{1}{(s+2)^2} + \frac{1}{s^2};$$

and solving for $Y(s)$:

$$Y(s) = \frac{1}{s+3} + \frac{2}{(s+3)^4} + \frac{1}{(s+3)(s+2)^2} + \frac{1}{s^2(s+3)}.$$

Using partial fractions:

$$\frac{1}{(s+3)(s+2)^2} = \frac{1}{s+3} - \frac{1}{s+2} + \frac{1}{(s+2)^2}$$
$$\frac{1}{s^2(s+3)} = \frac{1}{9}\left(\frac{1}{s+3} - \frac{1}{s} + \frac{3}{s^2}\right)$$

Therefore, combining like terms in $Y(s)$ gives

$$Y(s) = \frac{18}{9(s+3)} + \frac{2}{(s+3)^4} - \frac{1}{s+2} + \frac{1}{(s+2)^2} - \frac{1}{9s} + \frac{1}{3s^2},$$

which gives

$$\boxed{y(t) = \frac{18}{9}e^{-3t} + \frac{1}{3}t^3e^{-3t} - e^{-2t} + te^{-2t} - \frac{1}{9} + \frac{t}{3}.}$$

(b) $y'' - 3y' + 2y = 4, \qquad y(0) = 2, \quad y'(0) = 3$ As usual, $Y = \mathcal{L}(y)$. Applying $\mathcal{L}$ to both sides of the equation gives

$$s^2Y(s) - 2s - 3 - 3(Y(s) - 2) + 2Y(s) = \frac{4}{s}$$

and solving for $Y(s)$ gives:

$$\begin{aligned} Y(s) &= \frac{2s^2 - 3s + 4}{s(s-2)(s-1)} \\ &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} \\ &= \frac{2}{s} - \frac{3}{s-1} + \frac{3}{s-2}, \end{aligned}$$

where the last two lines represent the decomposition of $Y(s)$ into partial fractions. Taking the inverse Laplace transform gives

$$\boxed{y(t) = 2 - 3e^t + 3e^{2t}.}$$

(c) $y'' + 4y = 6\sin t, \qquad y(0) = 6, \quad y'(0) = 0$ As usual, $Y = \mathcal{L}(y)$. Applying $\mathcal{L}$ to both sides of the equation and solving for Y gives:

$$Y(s) = \frac{6s}{s^2+4} + \frac{6}{(s^2+4)(s^2+1)}.$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y(t) &= 6\cos 2t + 3\sin 2t * \sin t \\ &= 6\cos 2t + 3\frac{1}{3}\left(2\sin t - \sin 2t\right), \end{aligned}$$

where the second equality comes from the 12th formula in Table 2.7. Hence

$$\boxed{y(t) = 6\cos 2t + 2\sin t - \sin 2t.}$$

(d) $y''' - y' = 2, \qquad y(0) = y'(0) = y''(0) = 4$ Letting $Y(s) = \mathcal{L}(y(t))$ and taking the Laplace transform of both sides of the equation gives:

$$s^3Y(s) - 4s^2 - 4s - 4 - (sY(s) - 4) = \frac{2}{s};$$

solving for $Y(s)$ and expanding in partial fractions gives

$$\begin{aligned} Y(s) &= \frac{4s^3+4s^2+2}{s^4-s^2} \\ &= \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+1}+\frac{D}{s-1} \\ &= \frac{-2}{s^2}-\frac{1}{s+1}+\frac{5}{s-1}. \end{aligned}$$

Now taking the inverse Laplace transform gives

$$\boxed{y(t) = 5e^t - e^{-t} - 2t.}$$

(e) $y''' - y' = 6 - 3t^2, \qquad y(0) = y'(0) = y''(0) = 1 \qquad \boxed{y(t) = t^3 + e^t}$

7. Using the Laplace transform, find the solution of the following differential equations with initial conditions $y(0) = 0$, $y'(0) = 0$:

(a) $y'' - y = 2\sin t \qquad \boxed{y(t) = (1/2)(e^t - e^{-t}) - \sin t}$

(b) $y'' + 2y' = 5y \qquad \boxed{y(t) = 0}$

(c) $y'' + y = \sin 4t \qquad \boxed{y(t) = (1/15)(4\sin t - \sin 4t)}$

(d) $y'' + y' = 1 + 2t \qquad \boxed{y(t) = 1 - e^{-t} + t^2 - t}$

(e) $y'' + 4y' + 3y = 6 \qquad \boxed{y(t) = e^{-3t} - 3e^{-t} + 2}$

(f) $y'' - 2y' = 3(t + e^{2t}) \qquad \boxed{y(t) = (3/8)(1 - 2t - 2t^2 - e^{2t} + 4te^{2t}}$

(g) $y'' - 2y' = 20e^{-t}\cos t \qquad \boxed{y(t) = 3e^{2t} - 5 + 2e^{-t}(\cos t - 2\sin t)}$

(h) $y'' + y = 2 + 2\cos t \qquad \boxed{y(t) = 2 - 2\cos t + t\sin t}$

(i) $y'' - y' = 30\cos 3t \qquad \boxed{y(t) = 3e^t - 3\cos 3t - \sin 3t}$

8. Compute the convolution $t * t^3$ directly from the definition.

$$\begin{aligned} t * t^3 &= \int_0^t \tau(t-\tau)^3\,d\tau \\ &= \int_0^t \tau(t^3 - 3t^2\tau + 3t\tau^2 - \tau^3)\,d\tau \\ &= \int_0^t \left(t^3\tau - 3t^2\tau^2 + 3t\tau^3 - \tau^4\right)\,d\tau \\ &= \left.\left(t^3\frac{\tau^2}{2} - t^2\tau^3 + \frac{3}{4}t\tau^4 - \frac{\tau^5}{5}\right)\right|_0^t \\ &= t^5\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right) \\ &= \frac{t^5}{20}. \end{aligned}$$

9. Using the table of convolutions, compute each of the following convolutions:

(a) $(1+3t) * e^{5t}$

$$\begin{aligned} (1+3t) * e^{5t} &= 1 * e^{5t} + 3t * e^{5t} \\ &= \int_0^t e^{5\tau}\,d\tau + 3\left(\frac{e^{5t} - (1+5t)}{25}\right) \\ &= \frac{1}{5}(e^{5t} - 1) + 3\left(\frac{e^{5t} - (1+5t)}{25}\right) \\ &= \frac{8e^{5t} - 8 - 15t}{25}. \end{aligned}$$

(b) $(1/2 + 2t^2) * \cos\sqrt{2}t$ $\boxed{2t - \frac{3\sqrt{2}}{4}\sin\sqrt{2}t}$

(c) $(e^{2t} - 3e^{4t}) * (e^{2t} + 4e^{3t})$ $\boxed{te^{2t} - \frac{5}{2}e^{2t} + 16e^{3t} - \frac{27}{2}e^{4t}}$

10. (c) $y = 2t^2 + c_1 t + c_2\frac{1}{t}$

(d) $y = 2t^2 - 3t$

(e) $(0, \infty)$. The actual solution is defined on all of $\mathbb{R}$, which certainly includes $(0, \infty)$.