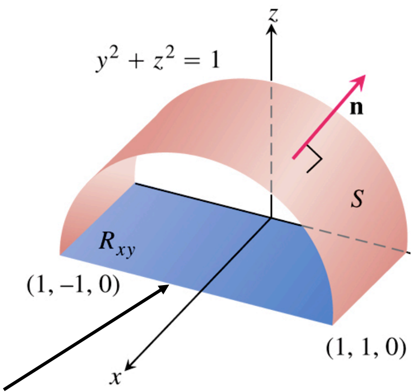
Example

Find the flux of $\mathbf{F} = yz\mathbf{j} + z^2\mathbf{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1,\ z \ge 0$ by the planes $x = 0$ and $x = 1$.



$$\iint_S \mathbf{F}\bullet\mathbf{n}\,d\sigma = \iint_R \mathbf{F}\bullet\frac{\nabla g}{|\nabla g\bullet p|}\,dA = \iint_R dA = 2$$

the area of the rectangle

S: $g(x,y,z) = y^2 + z^2 = 1,\ z \ge 0$

$$\frac{\nabla g}{|\nabla g\bullet\mathbf{k}|} = \frac{2\langle 0,y,z\rangle}{2z} = \left\langle 0,\frac{y}{z},1\right\rangle$$

$$\mathbf{F}(x,y,z) = \langle 0, yz, z^2\rangle$$

$$\mathbf{F}\bullet\frac{\nabla g}{|\nabla g\bullet\mathbf{k}|} = y^2 + z^2 = 1$$

Generalizing Green's Theorem

Recall:

C $\int_C \mathbf{F}\bullet d\mathbf{r}$ **orem** $\nabla\times\mathbf{F}$ **ulation-Curl)** **(2)**

$$\oint_C \mathbf{F}\bullet\mathbf{T}\,ds = \iint_R (\text{curl}\,\mathbf{F})\bullet\mathbf{k}\,dA$$

Or, equivalently:

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

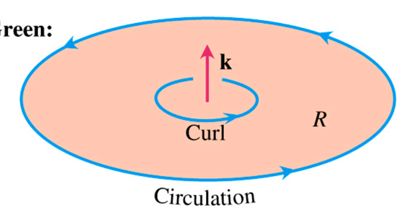
Stoke's Theorem (a field of unit normal vectors $\mathbf{n}$ on S has been chosen)

Suppose S is an oriented surface with a smooth boundary C given the ***induced*** orientation. Then (by the right-hand rule)

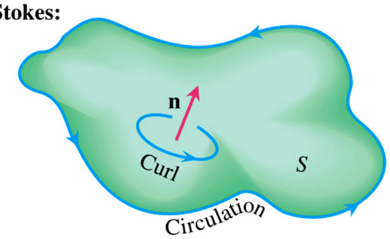
$$\int_C \mathbf{F}\bullet d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\bullet\mathbf{n}\,d\sigma$$

circulation integral **curl integral**

Green:

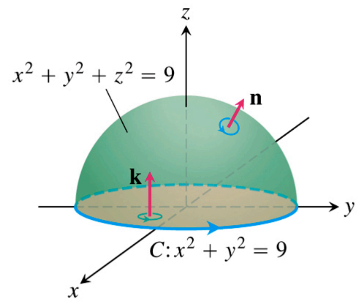


Stokes:



Example

Verify Stoke's theorem for the hemisphere in the picture and the field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$.



$$\mathbf{F}(x,y,z) = \langle y,-x,0 \rangle$$

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt}\,dt = -9\int_0^{2\pi} dt = \boxed{-18\pi}$$

$$C: \quad \mathbf{r}(t) = \langle 3\cos t, 3\sin t, 0 \rangle, \qquad 0 \le t \le 2\pi$$

$$\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}(3\cos t, 3\sin t, 0) = \langle 3\sin t, -3\cos t, 0 \rangle = 3\langle \sin t, -\cos t, 0 \rangle$$

$$\frac{d\mathbf{r}}{dt} = \langle -3\sin t, 3\cos t, 0 \rangle = 3\langle -\sin t, \cos t, 0 \rangle$$

$$\mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt} = 9\langle \sin t, -\cos t, 0 \rangle \bullet \langle -\sin t, \cos t, 0 \rangle = 9\left(-\sin^2 t - \cos^2 t\right) = -9$$

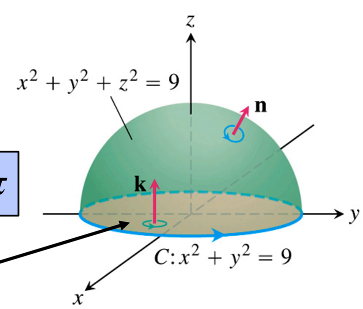
Stoke's Theorem

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\,d\sigma$$

$$\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\,d\sigma = \iint_R (\nabla \times \mathbf{F}) \bullet \frac{\nabla g}{|\nabla g \bullet k|}\,dA =$$

$$= \iint_R (-2)\,dA = -2\iint_R dA = -2\pi(3)^2 = \boxed{-18\pi}$$

the area of the circle



$$S: \quad g(x,y,z) = x^2 + y^2 + z^2 = 9$$

$$\mathbf{n} = \frac{\nabla g}{|\nabla g|} \qquad d\sigma = \frac{|\nabla g|}{|\nabla g \bullet \mathbf{k}|}\,dA$$

$$\frac{\nabla g}{|\nabla g \bullet \mathbf{k}|} = \frac{2\langle x,y,z \rangle}{|2z|} = \frac{\langle x,y,z \rangle}{z} = \left\langle \frac{x}{z}, \frac{y}{z}, 1 \right\rangle$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & 0 \end{vmatrix} = \langle 0,0,-2 \rangle$$

$$(\nabla \times \mathbf{F}) \bullet \frac{\nabla g}{|\nabla g \bullet \mathbf{k}|} = -2$$

$$\mathbf{F}(x,y,z) = \langle y,-x,0 \rangle$$

Stoke's Theorem

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\,d\sigma$$

Stoke's Theorem

Suppose S is an oriented surface with a smooth boundary C given the ***induced*** orientation. Then

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, d\sigma$$

circulation integral **curl integral**

Corollary

If two oriented surfaces S_1 and S_2 have the same boundary C (with the same induced orientation) then their curl integrals are equal (both equal the circulation integral).

(So fields of the form $\nabla \times \mathbf{F}$ are "surface independent", just like fields of the form ∇f are path independent.)

Generalizing Green's Theorem (cont.)

Recall:

Green's Theorem (Flux-Divergence Form) **(1)**

$$\int_C \mathbf{F} \bullet \mathbf{n}\, ds = \iint_R \underbrace{\operatorname{div} \mathbf{F}}_{\nabla \bullet \mathbf{F}}\, dA$$

Or, equivalently:

$$\oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$$

Divergence Theorem

Suppose S is an oriented surface that encloses a solid D (S is said to be closed). Then

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, d\sigma = \iiint_D \nabla \bullet \mathbf{F}\, dV$$

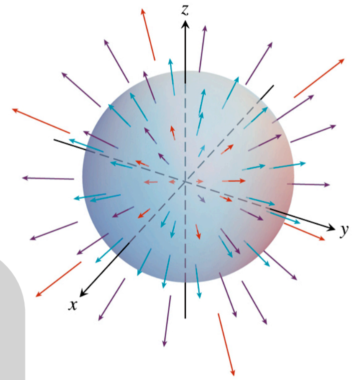
outward flux **divergence integral**

Example

Verify the Divergence Theorem for the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the sphere of radius a centered at the origin.

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, d\sigma = a\iint_S d\sigma = a\left(4\pi a^2\right) = \boxed{4\pi a^3}$$

the area of the sphere



$$S:\ g(x,y,z) = x^2 + y^2 + z^2 = a^2$$

$$\mathbf{n} = \frac{\nabla g}{|\nabla g|} = \frac{2\langle x,y,z\rangle}{2\sqrt{x^2+y^2+z^2}} = \frac{\langle x,y,z\rangle}{\sqrt{a^2}} = \frac{1}{a}\langle x,y,z\rangle$$

$$\mathbf{F} \bullet \mathbf{n} = \frac{1}{a}\langle x,y,z\rangle \bullet \langle x,y,z\rangle = \frac{1}{a}\left(x^2+y^2+z^2\right) = \frac{1}{a}a^2 = a$$

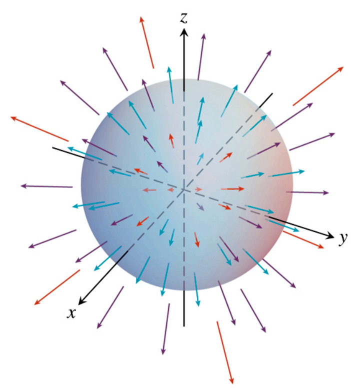
$$\mathbf{F}(x,y,z) = \langle x,y,z\rangle$$

Divergence Theorem

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, d\sigma = \iiint_D \nabla \bullet \mathbf{F}\, dV$$

$$\iiint_D \nabla \bullet \mathrm{F}\, dV = \iiint_D (3)\, dV = 3\iiint_D dV = 3\left(\tfrac{4}{3}\pi a^3\right) = \boxed{4\pi a^3}$$

the volume of the sphere



$$\nabla \bullet \mathrm{F} = \left\langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \right\rangle \bullet \langle x,y,z\rangle = \tfrac{\partial x}{\partial x} + \tfrac{\partial y}{\partial y} + \tfrac{\partial z}{\partial z} = 1+1+1 = 3$$

Divergence Theorem

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, d\sigma = \iiint_D \nabla \bullet \mathbf{F}\, dV$$