Line Integrals

Definition

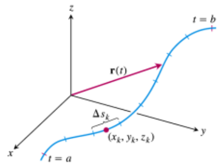
If f is defined on a curve C given parametrically by

$$\mathbf{r} = \mathbf{r}(t), \quad a \le t \le b$$

then the ***line integral*** of f over C is

(*) $$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k$$

(if it exists.)



Recall (arc length parameter):

The (directed) distance along the curve from the base point $\mathbf{r}(a)$ to $\mathbf{r}(t)$

$$s(t) = \int_a^t |\mathbf{v}|\,d\tau \qquad (\mathbf{v} = d\mathbf{r}/dt)$$

So $\frac{ds}{dt} = |\mathbf{v}(t)|$ i.e. $ds = |\mathbf{v}(t)|\,dt.$ If f is continuous and C is ***smooth*** ($\mathbf{v}$ is continuous and never $\mathbf{0}$) then (*) exists and can be expressed as an ordinary one-variable integral: t

$$\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{v}(t)|\,dt$$

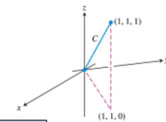
If f has the constant value 1 then this is the length of C.

Note

To use this formula you need a smooth parametrization of C $\;(\mathbf{r} = \mathbf{r}(t), \;\; a \le t \le b).$

Example

Integrate $f(x,y,z) = x - 3y^2 + z$ over the line segment C joining the origin to the point (1, 1, 1).



A natural parametrization: $\mathbf{r}(t) = t\langle 1,1,1\rangle, \; 0 \le t \le 1$

Line through P_0 parallel to $\mathbf{u}$: $\mathbf{r}(t) = \overrightarrow{OP_0} + t\mathbf{u}, \quad -\infty < t < \infty$

$\mathbf{v} = d\mathbf{r}/dt = \langle 1,1,1\rangle$ (continuous, never $\mathbf{0}$) $\quad |\mathbf{v}| = \sqrt{3}$

$f(\mathbf{r}(t)) = f(t,t,t) = t - 3t^2 + t = 2t - 3t^2$

$$\int_C f(x,y,z)\,ds = \int_0^1 \left(2t - 3t^2\right)\sqrt{3}\,dt = \sqrt{3}\left(t^2 - t^3\right)\Big|_0^1 = 0$$

Additivity of Line Integrals

If C is a ***piecewise smooth curve*** made by joining a finite number of smooth curves:

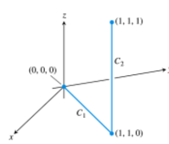
$$C = C_1 \cup C_2 \cup \ldots \cup C_n$$

then

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \ldots + \int_{C_n} f\,ds$$

Example

Integrate $f(x,y,z) = x - 3y^2 + z$ over the path $C_1 \cup C_2$.



Parametrizations

$C_1 : \mathbf{r}_1(t) = t\langle 1,1,0\rangle = \langle t,t,0\rangle, \;\; 0 \le t \le 1 \qquad |\mathbf{v}_1| = \sqrt{2}$

$C_2 : \mathbf{r}_2(t) = \langle 1,1,0\rangle + t\langle 0,0,1\rangle = \langle 1,1,t\rangle, 0 \le t \le 1 \qquad |\mathbf{v}_2| = 1$

$f(\mathbf{r}_1(t)) = f(t,t,0) = t - 3t^2 \qquad f(\mathbf{r}_2(t)) = f(1,1,t) = -2 + t$

$$\int_{C_1\cup C_2} f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds = \int_0^1 \left(t - 3t^2\right)\sqrt{2}\,dt + \int_0^1 (-2+t)\,dt$$

$$= \sqrt{2}\left(\tfrac{1}{2}t^2 - t^3\right)\Big|_0^1 + \left(-2t + \tfrac{1}{2}t^2\right)\Big|_0^1 = \left(-\tfrac{\sqrt{2}}{2}\right) + \left(-\tfrac{3}{2}\right) = \boxed{-\tfrac{3+\sqrt{2}}{2}}$$

different from last example

1

Note

The value of the line integral along a path joining two points can change if you change the path between them.

Example (Integrating over a plane curve)

Integrate $f(x,y) = \sqrt{y}/x$ along the curve C: $\mathbf{r}(t) = t^3\mathbf{i} + t^4\mathbf{j}, \;\; 1/2 \le t \le 1.$

$\langle t^3, t^4\rangle$

$\mathbf{v} = \langle 3t^2, 4t^3\rangle = t^2\langle 3, 4t\rangle \qquad |\mathbf{v}| = t^2\sqrt{9 + 16t^2} \qquad f(\mathbf{r}(t)) = f(t^3, t^4) = \sqrt{t^4}/t^3 = 1/t$

$$\int_C f\,ds = \int_{1/2}^1 t\sqrt{9+16t^2}\,dt = \frac{1}{32}\int_{13}^{25}\sqrt{u}\,du = \frac{1}{32}\left(\frac{2}{3}u^{3/2}\right)\Big|_{13}^{25} = \frac{1}{48}\left(125 - 13\sqrt{13}\right)$$

$u = 9 + 16t^2$

Vector Fields and Line Integrals

Definition

A ***vector field*** over a region D is an assignment of a vector to each point in D.

2D: $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$

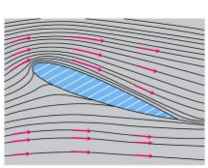


FIGURE 16.6 Velocity vectors of a flow around an airfoil in a wind tunnel.

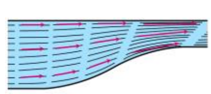


FIGURE 16.7 Streamlines in a contracting channel. The water speeds up as the channel narrows and the velocity vectors increase in length.

3D: $\mathbf{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$

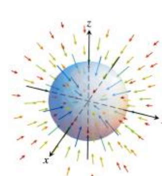


FIGURE 16.8 Vectors in a gravitational field point toward the center of mass that gives the source of the field.

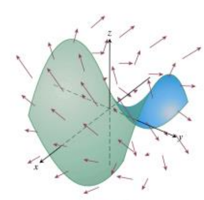
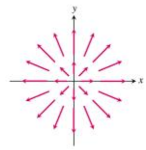


FIGURE 16.9 A surface, like a mesh net or parachute, in a vector field representing water or wind flow velocity vectors. The arrows show the direction and their lengths indicate speed.

Example

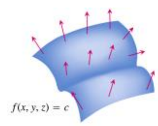
Sketch the vector field given by $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$.



Important Example

The ***gradient field*** of a differentiable function $f(x, y, z)$

$$\mathbf{F}(x, y, z) = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$



Line Integral of a Vector Field

Suppose $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$, is a smooth parametrization of the curve C.
Recall, the unit tangent vector: $\mathbf{T} = d\mathbf{r}/ds = \mathbf{v}/|\mathbf{v}|$.

The ***line integral of a vector field*** **F** over C is the line integral (as defined before) of the ***scalar-valued*** function $\mathbf{F} \bullet \mathbf{T}$ (the scalar component of **F** in the direction **T**):

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds = \int_C \mathbf{F} \bullet \frac{d\mathbf{r}}{ds}\, ds = \int_C \mathbf{F} \bullet d\mathbf{r} = \boxed{\int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt}\, dt}$$

Example

Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ for $\mathbf{F}(x, y, z) = z\mathbf{i} + xy\mathbf{j} - y^2\mathbf{k}$, $\mathbf{r}(t) = t^2\mathbf{i} + t\,\mathbf{j} + \sqrt{t}\,\mathbf{k}$, $0 \le t \le 1$.

$\langle z, xy, -y^2 \rangle$ $\quad\quad$ $\langle t^2, t, \sqrt{t} \rangle$

$$\left.\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= \mathbf{F}(t^2, t, \sqrt{t}) = \langle \sqrt{t}, t^3, -t^2 \rangle \\ \frac{d\mathbf{r}}{dt} &= \langle 2t, 1, 1/2\sqrt{t} \rangle \end{aligned}\right\} \mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt} = 2t\sqrt{t} + t^3 - t^2/2\sqrt{t} = \tfrac{3}{2}t\sqrt{t} + t^3$$

So

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^1 \left(\tfrac{3}{2}t^{3/2} + t^3\right) dt = \ldots = \tfrac{17}{20}$$