Review Test 1 **Name**
Math 2331 **Id**

Read carefully each problem. Show all your work. Credits will be given mainly depending on your work, not just an answer. Put a box around the final answer to a question. Use the back of the page if necessary.

1 [**10**] *Solve the system using either Gaussian elimination with back-substiution or Gauss-Jordan elimination.*

a)

$$\begin{aligned} -x + 2y &= 1.5 \\ 2x - 4y &= 3 \end{aligned}$$

b)

$$\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 \qquad - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$$

2 [**10**] *Solve the homogeneous linear system corresponding to the coefficient matrix provided:*

a) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

3 [**10**] *a) Write the system of linear equations in the form $A\mathbf{x} = \mathbf{b}$ and solve the matrix equation for $\mathbf{x}$.*

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ x_1 + 4x_2 &= 10 \end{aligned}$$

(b) Solve the matrix equation for a, b, c, d

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 17 \\ 4 & -1 \end{pmatrix}$$

4 [**10**] *a) If $AB = 0$, is it necessarily $A = 0$ or $B = 0$? Consider the example* $\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{pmatrix}$

b)Show that if $AB = 0$ and A is invertible, then $B = 0$.

5 [**10**] *Find the inverse of the matrix (if it exists).*

a) $\begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 7 & -9 \\ 7 & 16 & -21 \end{pmatrix}$ *d)* $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ 7 & 16 & -21 \end{pmatrix}$

6 [**10**] *Prove that if $A^2 = A$, then $I - 2A = (I - 2A)^{-1}$.*

7 [**10**] *Let A be an n by n matrix. Which of the following statements are equivalent to the statement that A is invertible?*

(1) *A is singular*
(2) *There exists a matrix B such that $BA = I_n$*
(3) *$A\mathbf{x} = \mathbf{b}$ has a unique solution for every n by 1 column matrix $\mathbf{b}$*
(4) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution*
(5) *A is row-equivalent to I_n*
(6) *A is column-equivalent to I_n*
(7) *A can written as the product of elementary matrices.*
(8) *Determinant of A is nonzero*
(9) *$A\mathbf{x} = \mathbf{0}$ has infinitely many solutions*
(10) *$A\mathbf{x} = \mathbf{b}$ has one and only one solution for one given (constant) vector $\mathbf{b}_0$*

8 [**10**] *Factor the matrix into a product of elementary matrices.*

$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

$B = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$C = \begin{pmatrix} 4 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & -2 \end{pmatrix}$

9 [**10**] *(optional) Solve the system $Ax = b$ by*

1) *finding the LU-factorization of the coefficient matrix A*
2) *solving the lower triangular system $Ly = b$, and*
3) *solving the upper triangular system $Ux = y$.*

a)

$$\begin{aligned} &2x + y = 1 \\ &y - z = 2 \\ &-2x + y + z = -2 \end{aligned}$$

b)

$$\begin{aligned} &2x_1 = 4 \\ &-2x_1 + x_2 - x_3 = -4 \\ &6x_1 + 2x_2 + x_3 = 15 \\ &-x_4 = -1 \end{aligned}$$

c)

$$\begin{aligned} &x_1 - 3x_2 = -5 \\ &x_2 + 3x_3 = -1 \\ &2x_1 - 10x_2 + 2x_3 = -20 \end{aligned}$$

10 [**10**] *Let A be a nonsingular matrix. Prove that a) if B is row-equivalent to A, then B is also nonsingular. b) Use $(AB)^T = B^T A^T$ and $(AB)^{-1} = B^{-1}A^{-1}$ to show that A^T is also invertible.*

11[**10**] *(optional) Using a system of equations to write the partial fraction decomposition of the rational expression. Then solve the system using matrices.*

$$\frac{4x^2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

where A, B, C are constants.

12 [**10**] *Give three distinct examples of elementary matrices and explain how they correspond to row operations for a given matrix of 3 by 3.*

Solutions

1 (a) No solutions.

1 (b) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, *where t is a parameter running through* $(-\infty, \infty)$.

2 (a) The homogeneous system $Ax = 0$ means

$$\begin{aligned} x_1 + x_4 &= 0 \\ x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Hence the solution is $x_1 = -t, x_2 = s, x_3 = 0, x_4 = t$, where $-\infty < s, t < \infty$.

2 (b) The system is equivalent to $0x_1 + 0x_2 + 0x_3 = 0$ Therefore, we have the freedom of choosing values of the unknown variables. $x_1 = s, x_2 = t, x_3 = r$, where the parameters s, t, r can be any real numbers.

3. (a) $(x_1, x_2) = (-2, 3)$.

(b) Multiplying the equation both sides by $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$ *on the right, we have*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 17 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}^{-1}$$

Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 17 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 48 & -31 \\ -7 & 6 \end{pmatrix}$$

4 a) No. b) We need to show if A^{-1} exists and $AB = 0$, then $B = 0$. Multiplying A^{-1} on the left on both sides of the equation $AB = 0$, we have

$$A^{-1}AB = A^{-1} \cdot 0,$$

that is, $B = 0$ (since $A^{-1}A = I$ and $I \cdot B = 0$)

5. b) Row operation $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{pmatrix} \xrightarrow{(-3)R1+R2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 5 & 5 \end{pmatrix}$ *which shows that one of the rows has all zeors, thus it is Not row equivalent to identity matrix. Hence the matrix is Not invertible.*

c) *Perform row operation on the adjoining matrix* $\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 7 & -9 & 0 & 1 & 0 \\ 7 & 16 & -21 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{(-3)R1+R2}$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -3 & 1 & 0 \\ 7 & 16 & -21 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{(-7)R1+R3} \begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -3 & 1 & 0 \\ 0 & 2 & -14 & -7 & 0 & 1 \end{pmatrix} \xrightarrow{(-2)R2+R3}$$
$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -3 & 1 & 0 \\ 0 & 0 & -2 & -1 & -2 & 1 \end{pmatrix} \xrightarrow{(-3)R3+R2} \begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 7 & -3 \\ 0 & 0 & -2 & -1 & -2 & 1 \end{pmatrix} \xrightarrow{(-1/2)*R3}$$
$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 7 & -3 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix} \xrightarrow{R3+R1} \begin{pmatrix} 1 & 2 & 0 & \frac{3}{2} & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 7 & -3 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix} \xrightarrow{(-2)*R2+R1} \begin{pmatrix} 1 & 0 & 0 & \frac{3}{2} & -13 & \frac{11}{2} \\ 0 & 1 & 0 & 0 & 7 & -3 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}$$

Hence $A^{-1} = \begin{pmatrix} \frac{3}{2} & -13 & \frac{11}{2} \\ 0 & 7 & -3 \\ \frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}$

d) *Not invertible.*

6. *Proof. Since* $A^2 = A$, *we have*

$$\begin{aligned} &(I - 2A)^2 = I - 4A + 4A^2 \\ =&I - 4A + 4A = I, \end{aligned}$$

which shows that $(1-2A)^{-1} = I-2A$ *by definition of an inverse matrix.*
□

7. *The fact that* A *is invertible is equivalent to* $(2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7) \Leftrightarrow (8)$.

Basically this question asks the student to describe an invertible matrix A (using alternative statements). This is a logic-related type of question to help the understanding and getting more familiar with certain concepts in LA.

For instance, (2) is an equivalent statement to suggest A *is invertible, because: if* $BA = I$, *then* $AB = I$ *by the fact* A *factors into product of elementary matrices. Then by definition, this means* A *is invertible, whose inverse is given by* B.

(3) $Ax = b$ *has a unique solution for each vector* b. *This statement also means that* A *is invertible, because if* A *is not invertible, then one can row reduces* A *to a matrix having an echelon form that is NOT the identity matrix. More specifically the echelon form will have at least on the bottom line all zeros.*

At this point you can take b *vector to have on the last entry value equal to 1, say. Then you will find immediately the system* $Ax = b$ *with such* b *will have no solution.*

The above argument suggests that A must be invertible (or equivalently, A can be row-reduced to identity matrix).
8. Do a finite sequence of row operations to convert A to I. This corresponds to $E_k \cdots E_1 A = I$ for certain elementary matrices E_i. Record the E_i corresponding to each row operation. Then $A = E_1^{-1} \cdots E_k^{-1}$ is the factorization.
8. (a)

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

(c). Row operation $C = \begin{pmatrix} 4 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & -2 \end{pmatrix} \overset{R1 \leftrightarrow R4}{\rightarrow} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 4 & 0 & 0 & 2 \end{pmatrix} \overset{(-4)R1+R4}{\rightarrow}$
$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 10 \end{pmatrix} \overset{\frac{1}{10}R4}{\rightarrow} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{(-2)R4+R3}{\rightarrow} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{(-1)R4+R2}{\rightarrow}$
$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{2R4+R1}{\rightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{(-1)*R3}{\rightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
From the above sequence we keep track the corresponding elementary matrices at each step $E_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{pmatrix}$; $E_3 =$
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{10} \end{pmatrix}$ $E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $E_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $E_6 =$
$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $E_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Then $A = E_1^{-1} \cdots E_7^{-1}$ *where* $E_1^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}$; $E_3^{-1} =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix} \quad E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad E_5^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad E_6^{-1} =$$

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad E_7^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

9. $A = LU$. First solve $Ly = b$ then solve $Ux = y$, this process can be expressed via matrix notion

$$x = U^{-1}(L^{-1}b)$$

if both U and L are invertible. If not, then we just solve the two equations directly (and separately) by row-echelon.

10. a) Proof. B is row equivalent to A means that one can do a finite sequence of elementary row operation to convert A to B (or B to A). Since each row operation amounts to multiplication on the left by an elementary matrix E_i, we know that B can be written as $B = E_k \cdots E_1 A$. Now B is invertible (or nonsingular) because of the existence of $B^{-1} = A^{-1}E_1^{-1} \cdots E_k^{-1}$.

b) The problem asks to prove that if A is invertible, then A^T is also invertible. Hence we need to show the existence of the inverse of A^T.

Claim. The inverse of A^T is $(A^{-1})^T$. In fact,

$$(A^{-1})^T(A^T) = (AA^{-1})^T = I^T = I$$

also

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

Therefore the claim is proved true.

11. First find the common denominator $(x-1)(x+1)^2$. The equation then becomes

$$\frac{4x^2}{(x-1)(x+1)^2} = \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$$

Compare the coefficients of the x^2, x and constant terms for the numerator, we obtain three linear equations with three unknowns A, B, C. Solve the linear equations for A, B, C.

12. For example $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Multiplied by E_1, E_2, E_3 on the left of a given matrix A correspond to row operations: exchanging the first and second rows, multiplying the third by 3 and Row 2 add $(-2)\times$ Row 1$\rightarrow$ Row 2, respectively for A.