**Review Test 2** **Name**

**Math 5339** **Id**

Read each question carefully. Avoid simple mistakes. **You must show your work in order to possibly get full credits**.

1. Solve the wave equation with Neumann boundary condition on half line

$$\begin{aligned}&\partial_{tt}u - c^2 u_{xx} = 0 \quad (t,x) \in \mathbb{R} \times (0,\infty)\\ &u_x(t,0) = 0 \qquad t \in \mathbb{R}\\ &u(0,x) = \phi(x),\ u_t(0,x) = \psi(x), \quad x \in (0,\infty)\end{aligned}$$

2. Prove the uniqueness of the diffusion problem with Neumann boundary condition by energy method:

$$\begin{aligned}&u_t - k u_{xx} = f(t,x), \qquad (t,x) \in (0,\infty) \times (0,\ell)\\ &u_x(t,0) = g(t),\ u_x(t,\ell) = h(t)\\ &u(0,x) = \phi(x)\end{aligned}$$

(Clue: This is #15 in 2.4 [Strauss])

3. Let $a$ be a constant. Solve the Dirichlet problem
$u_{tt} - 9u_{xx} = e^{ax}$ in $(t,x) \in (-\infty,\infty) \times (0,\infty)$,
$u(0,x) = \cos x,\ u_t(0,x) = 0$
$u(t,0) = 0$.

4. Solve the heat equation with convection on the half line.

$$\begin{aligned}&u_t - k u_{xx} + V u_x = 0, \qquad 0 < t, x < \infty\\ &u(0,x) = \phi(x)\end{aligned}$$

where $V$ is a constant. (Hint: Sub $y = x - Vt$ then reflection)

5. Solve the inhomogeneous diffusion equation on the half-line:

$$\begin{aligned}&u_t - k u_{xx} = f(t,x) \qquad 0 < x, t < \infty\\ &u(0,x) = \phi(x)\\ &u(t,0) = 0\end{aligned}$$

using the method of reflection.

6. (a) Prove the **comparison principle** for the diffusion equation: If $u$ and $v$ are two solutions, and if $u \leq v$ for $t = 0$, for $x = 0$ and $x = \ell$, then $u(t,x) \leq v(t,x)$ for $(t,x) \in [0,\infty) \times [0,\ell]$. (this is 2.3, #6 from [Strauss])

   (b) [Bonus] If $v_t - v_{xx} \geq \sin x$ for $t > 0, 0 \leq x \leq \pi$, and if $v(t,0) \geq 0, v(t,\pi) \geq 0$ and $v(0,x) \geq \sin x$, show that $v(t,x) \geq (1-e^{-t})\sin x$.

7. **[optional] Solve on the interval $(0, \frac{\pi}{2})$ the wave equation $u_{tt} - c^2 u_{xx} = 0$ with IC: $v(0,x) = \tilde{\phi}$, $v_t(0,x) = \tilde{\psi}$ and Dirichlet boundary conditions at $x = 0$ and $x = \pi/2$.

8. Verify that the heat equation $u_t = u_{xx}$ on $(0,\infty) \times [0,\pi]$ with IC $u(0,x) = \phi(x) \in L^2([0,\pi])$ has the solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx) e^{-n^2 t}$$

   where $c_n = \frac{2}{\pi} \int_0^\pi \phi(x) \sin nx dx$. Do you see some connection to the same problem on $(0,\infty) \times \mathbb{R}$?

9. Solve the wave equation on the half line with Dirichlet condition:
   $u_{tt} - c^2 \Delta_x u = 5x, \quad (t,x) \in \mathbb{R} \times (0,\infty)$
   $u(t,0) = 0$
   $u(0,x) = e^{-x}$, $u_t(0,x) = 0$.

10. Solve the mixed boundary condition problem for wave equation

$$\begin{aligned} &u_{tt} = c^2 u_{xx} \qquad x \in (0,\ell), t \in \mathbb{R} \\ &u(t,0) = 0, u_x(t,\ell) = 0 \\ &u(0,x) = 0, u_t(0,x) = \beta \sin((2N+1)\pi x/2\ell), \end{aligned}$$

   here $\beta$ is real and $N$ an integral constant.

11. Can the eigenvalue problem

$$\begin{aligned} -X''(x) = \lambda X(x) \quad 0 < x < 1 \\ X'(0) = 0,\ X(1) = 0 \end{aligned}$$

   have nonpositive eigenvalues? Prove your statements. Write down all the eigenvalues and the associated eigenfunctions.

12. Applying the Parseval equality for an $L^2$ function $f = \mathbf{1}_{[0,\pi]}$ to compute

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

13. Find the Fourier cosine series of $h(x) = x$ on $[0, \pi]$. Then evaluate the sum using Parseval's equality

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}.$$

14. [Strauss, 5.4, #11] (Term wise Integration)

    a If $f(x)$ is piecewise continuous on $[-L, L]$, show that its antiderivative $\int_{-\ell}^{x} f dt$ has a full Fourier series that converges pointwise.

    b Write this convergent series explicitly in terns of the Fourier coefficients $a_n, b_n$ of $f$ if $a_0 = 0$.

    (Clue: after solving part (b), you will find that you just have proved the term by term integration theorem for Fourier series!)

15. Is the following identity true for $0 < x < 2\ell$ or $0 < x < \ell$ or neither? What you can say about the convergence at $x = 0$, $x = 2\ell$ and $x = \ell$?

$$1 = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin\left(\frac{(2n+1)\pi}{2\ell} x\right) \quad ?$$

16. Let $k$ be real and $i = \sqrt{-1}$.
    (a) Use separation of variable method to solve the Schrödinger equation on an interval $iu_t = -ku_{xx}$ on $\mathbb{R} \times (0, \ell)$ with (mixed) boundary condition $u(t, 0) = u_x(t, \ell)\,(= c)$.
    (b) [bonus] What is the energy conservation law here?

    [Hint: In oder to have a well posed solution $c$ must be zero]

17. If both $\phi$ and $\psi$ are odd functions of $x$, show that the solution $u(t, x)$ of the wave equation is also odd in $x$ for all $t$.

18. Solve $u_{xx} + u_{xt} - 20u_{tt} = 0$, $u(0, x) = \phi(x)$, $u_t(0, x) = \psi(x)$.

19. A *spherical wave* is a solution of the three-dimensional wave equation of the form $u(t, r)$, where $r = |x|$ is the distance to the origin (spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2(u_{rr} + \frac{2}{r}u_r) \qquad \text{(spherical wave equation)}$$

   (a) Change variables $w = ru$ to obtain the equation for $w$: $w_{tt} = c^2 w_{rr}$

   (b) Solve for $w$ and thereby solve the spherical wave equation.

   (c) Solve the wave equation with initial condition $u(0, r) = \phi(r)$, $u_t(0, r) = \psi(r)$ by taking both $\phi, \psi$ to be even functions of $r$.

   [Hint: d'Alembert formula for 1d wave equation]

20. (a) Sate the definition of a well-posed PDE problem.

   (b) Is the following problem well-posed?

$$\Delta u = 0 \quad (x, y) \in D := \{x^2 + y^2 < 1\}$$
$$\frac{\partial}{\partial \mathbf{n}} u(x, y) = 0 \quad (x, y) \in \partial D$$

   where $\mathbf{n}$ is the unit outward normal on the boundary of $D$.

21. Verifying that for all $n$, $u_n(t, x) = \frac{1}{n} \sin nx e^{-n^2 t}$ solves

$$u_t = u_{xx} \quad (t, x) \in \mathbb{R} \times (0, \pi)$$
$$u(t, 0) = u(t, \pi) = 0 \quad t \in \mathbb{R}$$
$$u(0, x) = \frac{1}{n} \sin nx \quad x \in [0, \pi]$$

   How does the energy change if $t \to \pm\infty$?

22. Is the following problem well-posed?

$$u_t = k u_{xx} \quad (t, x) \in (-\infty, 0) \times (0, \pi)$$
$$u(t, 0) = u(t, \pi) = 0 \quad t < 0$$
$$u(0, x) = 0 \qquad x \in (0, \pi)$$

**Solutions**

4. Sub $y = x - Vt$ and we have $u(x,t) = u(y+Vt,t) := \tilde{u}(y,t)$, where $x = x(y,t) = y + Vt$. By chain rule,

$$\frac{\partial}{\partial t}\tilde{u}(y,t) = \frac{\partial}{\partial t}[u(y+Vt,t)] = \frac{\partial}{\partial x}u(x,t)\cdot V + \frac{\partial}{\partial t}u(x,t)$$
$$\frac{\partial}{\partial y}\tilde{u}(y,t) = \frac{\partial}{\partial y}[u(y+Vt,t)] = \frac{\partial}{\partial x}u(x,t)\cdot 1.$$

Sub the above into the $u_t - ku_{xx} + Vu_x = 0$, $u(x,0) = \phi(x)$. We find that $\tilde{u}_t - k\tilde{u}_{yy} = 0$, $\tilde{u}(y,0) = \phi(y)$. Thus $\tilde{u}(y,t) = \int p_t(y,z)\phi(z)dz$ which gives that

$$u(x,t) = \int p_t(x - Vt, z)\phi(z)dz$$
$$= \frac{1}{(4k\pi t)^{n/2}}\int_{\mathbb{R}} e^{-\frac{|x-Vt-z|^2}{4kt}}\phi(z)dz.$$

5. The inhomogeneous problem on the line has the solution

$$v(x,t) = e^{t\Delta}\phi + \int_0^t e^{(t-s)\Delta}f(x,s)ds$$
$$= \int p_t(x,y)\phi(y)dy + \int_0^t ds\int p_{t-s}(x,y)f(y,s)dy$$

Make odd extension of u to v with $\phi(-x) = -\phi(x)$, then restrict $u = v|_{\mathbb{R}\times\mathbb{R}_+}$ to obtain the solution.

6. a) Fix any $T > 0$. Let $u$ be the solution to the heat equation $w_t - kw_{xx} = 0$ on the rectangle $R := [0,T]\times[a,b]$. The (weak) maximal principle in [Strauss, Section 2.3] states that $u(t,x) \le M = u(t_0,x_0)$ for some $(t_0,x_0)$ in $\Gamma_T := \{(t,x) : t = 0 \; or \; x = a \; or x = b\}$ (bottom and lateral edges of $R$). Given that u, v satisfy $u_t - ku_{xx} = f(t,x)$, $u(0,x) = \phi(x)$, $u(t,a) = g(t), u(t,b) = h(t)$. Ansatz $w = u - v$ and we see that w satisfies

$$w_t - kw_{xx} = 0,$$
$$w(0,x) = 0,$$
$$w(t,a) = 0, w(t,b) = 0$$

Apply the maximal principle we obtain $w(t,x) \le 0$ for all $(t,x) \in [0,T]\times[a,b]$. Since $T$ is arbitrary, this also valid for all $t > 0$ and $x$.

b*) This part is quite deeper. Observe from the same proof in the text or the lecture note in class, the MP is valid for on $R = [0,T]\times[a,b]$ as long as the solution $w$ satisfies general diffusion inequality

$$w_t \leq kw_{xx} \, .$$

Let $u = (1 - e^{-t}) \sin x$, we have $u_t - u_{xx} = \sin x$ which tells that $v_t - v_{xx} \geq u_t - u_{xx}$. Ansatz $w = u - v$ and we obtain

$$w_t \leq w_{xx} \, .$$

It follows from the MP that $w$ attains its maximal value on the bottom and two lateral sides. When $t = 0$, $w(0, x) = u(0, x) - v(0, x) = 0 - v(0, x) \leq -\sin x \leq 0$ if $0 \leq x \leq \pi$.
When $x = 0$, $w(t, 0) = u(t, 0) - v(t, 0) = 0 - v(t, 0) \leq 0$,
When $x = \pi$, $w(t, \pi) = u(t, \pi) - v(t, \pi) = 0 - v(t, \pi) \leq 0$. Hence, for all $t, x$ it holds that

$$w(t, x) \leq 0$$

This proves that $(1 - e^{-t}) \sin x \leq v(t, x)$. $\square$

7. **Method I**. By reflection method as described in Section 3.3, extend $u(t, x)$ as odd function on $(-\frac{\pi}{2}, \frac{\pi}{2})$, and then periodically on $\mathbb{R}$, which we call $v(t, x)$. Also do the same for $\phi$ and $\psi$. Then $v$ satisfies the wave equation on $\mathbb{R}$ with initial condition $v(0, x) = \tilde{\phi}$, $v_t(0, x) = \tilde{\psi}$, with a little abuse of notation we will continue to call them $\phi, \psi$. Thus the solution is given by

$$v(t, x) = \frac{1}{2}(\phi(x + ct) + \phi(x - ct)) + \int_{x-ct}^{x+ct} \psi(s) ds$$

Note that the boundary condition is automatically satisfied because of the odd symmetries on **both** $x = 0, x = \pi/2$. As in the text, the strip $(0, \pi/2) \times \mathbb{R}$ divided into diamond regions. Inside each of these diamond, the wave propagates through a chain of reflections against the boundaries, $\phi(x \pm ct)$ has different sign depending on the number of reflections, similarly for $\pm\psi$. After some simple calculations, the express of $v(x, t)$ has a domain of dependence resulting from those reflections.

**Method II.** Alternatively, we can also use Fourier series to find the solution

$$\begin{aligned} u(t, x) = &\sum_n T_n(t) X_n(t) \\ = &\frac{A_0}{2} + \sum_{n=1} (A_n \cos(2nct) + B_n \sin(2nct)) \sin(2nx) \end{aligned}$$

where

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = -\lambda$$