Review Test 1	**Name**
Math 5339	**Id**

Read each question carefully. Avoid simple mistakes. (Use the back of the page if necessary). **You must show your work in order to get credits or partial credits**.

1. [10pts] For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons.

   (a) $u_{tt} - c^2 u_{xx} + u^3 = 0$

   (b) $u_t - k u_{xx} + 6u_{xt} = 0$

   (c) $u_{xx} + u_{yy} + 2xy u_{xy} = 0$

   (d) $u_{tt} - u_{xx} = \sin u$

   (e) $u_t - u_{xxx} + 6u u_x = 0$

   (f) $\frac{du}{dt} = G(u)$

2. [15pts] Solve the first order $2u_t + 3u_x = 0$ with auxiliary condition $u = \sin x$ when $t = 0$. (Hint: Express the solution as, according to characteristic curve method, $u(t,x) = f(bt - ax)$)

3. [15pts] Solve $u_{tt} = c^2 u_{xx}$, $u(x,0) = e^x$, $u_t(x,0) = \sin x$.

4. [10pts] Find the solution of the Laplace equation $u_{xx} + u_{yy} = 0$ for $x, y \in [0,1]$. Use separation of variables.

5. [10pts] Solve the wave equation $u_{tt} - u_{xx} = 0$ for $x \in [0, \ell], t \in \mathbb{R}$.

6. [10pts] Use change of variables $t' = at + bx, x' = bt - ax$ to solve

$$a u_t + b u_x = f(x,t), (t,x) \in \mathbb{R} \times \mathbb{R} \tag{1}$$

7. [20pts] The solution to $u_t - k u_{xx} = 0, u(0,x) = f(x)$ is given by

$$u(t,x) = e^{-kt\Delta} f(x) = \frac{1}{(4k\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4kt}} f(y) dy := \int_{\mathbb{R}^n} p_t(x,y) f(y) dy,$$

where $p_t(x, y)$ denote the fundamental solution or heat kernel. Show that

$a)$ $\displaystyle\int_{\mathbb{R}^n} e^{-|x|^2} dx = \pi^{n/2}$

$b)$ $\displaystyle\int p_t(x, y) dx = 1 \qquad \forall t > 0$

$c)$ Given any $\delta > 0$, $\displaystyle\lim_{t \to 0} \int_{|x-y| \geq \delta} p_t(x, y) dy = 0$

$d)$ If $f \in L^p, 1 \leq p \leq \infty,$ *then* $u(t, x) \to f(x)$ *as* $t \to 0$.

8. [20pts] a) Consider the Schrödinger equation

$$i\partial_t u + \Delta u + \varepsilon|u|^p u = 0$$

Show that the conserved quantities (constants of motion) are

$$Q(t) = \int |u|^2 dx$$
$$E(t) = \frac{1}{2} \int (|\nabla_x u|^2 + \frac{\varepsilon}{p+1}|u|^{p+1}) dx$$

(Hint: Use integration by parts to prove $dE(t)/dt = 0$, thus $E(t)$ must be a constant)
b) How about wave equation

$$\partial_{tt} u - u_{xx} + \varepsilon|u|^p u = 0$$

Is $E(t) = \frac{1}{2}\int_{\mathbb{R}^n}(u_t^2 + |\nabla u|^2 + \frac{\varepsilon}{p+1}|u|^{p+1})dx$ conserved in time?

9. [15pts] Solve the heat equation with convection

$$u_t - ku_{xx} + Vu_x = 0, \qquad -\infty < x < \infty$$
$$u(x, 0) = \phi(x)$$

where V is a constant. (Hint: Sub $y = x - Vt$)

10. [10pts] Solve $u_{tt} = 9u_{xx}$ in $x \in (0, \frac{\pi}{2})$, $u(x, 0) = \cos x$, $u_t(x, 0) = 0$, $u_x(0, t) = 0, u(\frac{\pi}{2}, t) = 0$.

11. [Bonus 10pts] Solve the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition

$$
\begin{aligned}
&u_t - ku_{xx} = f(x,t) \qquad 0 < x, t < \infty \\
&u(x,0) = \phi(x), u(0,t) = 0
\end{aligned}
$$

using the method of reflection.

Solutions

2. The general solution for the transport equation is given by $u(x,t) = f(bt - ax)$; see Lecture notes or Section 1.2 in Strauss. Sub the initial condition with $t = 0$ into this, we get $f(-ax) = \sin x$. Therefore $f(x) = -\sin(x/a)$.

3. The general solution for the wave equation on the line $(-\infty, \infty)$ is given by $u(x,t) = f(x+ct) + g(x-ct)$. Substituting the initial conditions into this expression yields

$$
\begin{cases}
f(x) + g(x) = e^x \\
cf'(x) - cg'(x) = \sin x
\end{cases}
$$

Now taking derivative on the first equation we obtain $f'(x) + g'(x) = e^x$; this, together with the second equation will lead to a solution of f' and g'. From there antiderivative will recover f and g.

4. Write $u(x,y) = X(x)Y(y)$ and sub into $\Delta u = 0$. We then have

$$
\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda
$$

where $\lambda \geq 0$ is a constant (or eigenvalue of d^2/dx^2).

5. Write $u(x,y) = T(t)X(x), \ldots$.

6*. (I) **Method I. Change of variables**. †This is a transport equation with inhomogeneous term f. As we will see that the solution is the u_{homog} plus a line integral along the characteristics. Let $t' = at + bx, x' = bt - ax$. Then, substituting into (1) yields

$$
(a^2 + b^2)\frac{\partial}{\partial t'}\tilde{u}(t', x') = \tilde{f}(t', x'), \tag{2}
$$

where we write $\tilde{u}(t',x') = u(t,x)$, $t = t(t',x'), x = x(t',x')$, and $\tilde{f}(t',x') = f(t,x)$ and we have

$$\frac{\partial}{\partial t}u(t,x) = a\frac{\partial \tilde{u}}{\partial t'}(t',x') + b\frac{\partial \tilde{u}}{\partial x'}(t',x')$$
$$\frac{\partial}{\partial x}u(t,x) = b\frac{\partial \tilde{u}}{\partial t'}(t',x') - a\frac{\partial \tilde{u}}{\partial x'}(t',x').$$

Integrating (2) we obtain

$$(a^2+b^2)\tilde{u}(t',x') = \int^{t'} \tilde{f}(s',x')ds' + C(x')$$
$$i.e., \quad (a^2+b^2)u(t,x) = \int^{t'} \tilde{f}(s',x')ds' + C(bt-ax)$$

where $t' = t'(t,x) = at+bx,\ x' = bt-ax$; $t(t',x') = (a^2+b^2)^{-1}(at'+bx'), x(t',x') = (a^2+b^2)^{-1}(bt'-ax')$; We can choose the lower limit to be $-(b/a)x'$ or $-\infty$, the latter choice would give an integral $\int_{-bx'/a}^{t'} + \int_{-\infty}^{-bx'/a}$ which can be absorbed into $C(x')$, (x' being a constant independent of s'). So let us simply write, by change of variable $s' = t'(s,x) = as+bx$ (x a constant and $x' = bt-ax$)

$$(a^2+b^2)u(t,x) = \int_{-bx'/a}^{t'} \tilde{f}(t'(s,x),x')ds' + C(x')$$
$$= \int_{-(x'/a+x)b/a}^{(t'-bx)/a} f(t(as+bx,x'),x(as+bx,x'))d(as+bx) + C(x')$$
$$=a\int_{-b^2t/a^2}^{t} f(\frac{a(as+bx)+bx'}{a^2+b^2}, \frac{b(as+bx)-ax'}{a^2+b^2})ds + C(x')$$
$$=a\int_{-b^2t/a^2}^{t} f(\frac{a^2s+b^2t}{a^2+b^2}, \frac{ab(s-t)}{a^2+b^2}+x)ds + C(bt-ax).$$

Note that the integral is a line integral starting from $(0,-bt/a+x)$ to (t,x) with parametric equation $t = t(s), x = x(s)$ and length element $dL(s) = \sqrt{\dot{t}^2+\dot{x}^2}ds = \frac{a}{\sqrt{a^2+b^2}}ds$; each point on the curve satisfying $bt(s)-ax(s) = constant = bt-ax$. Therefore,

$$(a^2+b^2)u(t,x) = \sqrt{a^2+b^2}\int_L f(L(s))dL(s) + C(bt-ax)$$

which proves that $u_{inhomog}$ is given by "Duhamel" (the $\pm$ for $s-t$ and $s+t$ reflect the effect of forward and backward waves while in 3D it has reflection waves from all angles)

$$u(t,x) = \underbrace{C(bt-ax)}_{u_{homog}} + \underbrace{\frac{1}{\sqrt{a^2+b^2}} \int_L f(L(s))dL(s)}_{u_{particular}}, \qquad (**)$$

where L is the characteristic line from $(0, -bt/a)$ to (t,x). $\square$

Remark The ingredient of the whole idea is to use

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$
$$\begin{pmatrix} t \\ x \end{pmatrix} = \frac{-1}{a^2+b^2} \begin{pmatrix} -a & -b \\ -b & a \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

to transform a few times so the final integral can be realized as a line integral with a new (but more complicated) parametric equation for the characteristic line.

(II) **Second method: Characteristics** (without change of variables)

†We can also prove the solution formula via characteristic curve (ODE) method, as I discussed in the class. The result $u = u_{homog} + u_p$ is the same as in (**). In fact, from

$$a\partial_t u + b\partial_x u = f(t,x) \qquad (*)$$

we have the characteristic curve C in the xt plane given by the following parametric equation

$$\begin{cases} x'(s) = b \\ t'(s) = a \end{cases} \tag{3}$$

Then the PDE reduces to an ODE along L for $U(s) := u(t(s), x(s))$:

$$x = bs + x_0 \tag{4}$$
$$t = as + t_0, \qquad s \in (-\infty, \infty) \tag{5}$$

(note that the parameter s here is not the one for arc-length).

If L starts at $(x_0, t_0) = (x - bt/a, 0)$ to (x, t), then

$$\frac{d}{ds}[U(s)] = f(t(s), x(s))$$

$\Rightarrow$ for all s or $(t, x) = (t(s), x(s)) \in L$ along the curve

$$u(t,x)|_{s=s} = U(s) = \int f(as + t_0, bs + x_0)ds$$

$$= \int_0^s f(t(s), x(s))ds + \tilde{C}(x_0) \qquad \because\ U(s) \text{ depends on } x_0 \text{ only, the } x\text{-intercept}$$

$$= \int_0^s f(t(s), x(s))ds + C(ax - bt).$$

Note that $s = 0$ corresponds to $(x_0, 0)$ and $s = t/a$ corresponds to (x, t). Substituting $s = t/a$ we obtain

$$u(t,x) = \int_0^{t/a} f(t(s), x(s))ds + C(ax - bt). \tag{6}$$

†Here we can also apply fundamental theorem of Calculus for all s

$$u(t(s), x(s)) = \int_0^s f(t(\tau), x(\tau))d\tau + u(t(0), x(0))$$

$$= \int_0^s f(t(\tau), x(\tau))d\tau + u(0, x_0).$$

Evaluate the above at $s = t/a$ to obtain

$$u(t,x) = \int_0^{t/a} f(t(\tau), x(\tau))d\tau + u_0(x - bt/a). \tag{7}$$

where u_0 denotes the initial condition at $t = 0$.

On the other hand we look at the line integral in (**) using parametric equations to obtain

$$\int_L f(L(s))dL(s) = \int_{s=0}^{s=t/a} f(t(s), x(s))d\ell$$

$$= \sqrt{a^2 + b^2} \int_0^{t/a} f(t(s), x(s))ds \tag{8}$$

where if $\mathbf{r}(s) = x(s)\mathbf{i} + t(s)\mathbf{j}$, then $\mathbf{v} = \mathbf{r}'(s)$ and the arc length element $d\ell = |\mathbf{v}(s)|ds = \sqrt{a^2 + b^2}ds$.

Now compare (6) and (8) we conclude that

$$u(t,x) = \underbrace{\frac{1}{\sqrt{a^2+b^2}}\int_L f(L(s))dL}_{u_{particular}} + \underbrace{C(ax-bt)}_{u_{homog}} \tag{9}$$

(III) †The third method would be Duhamel principle from ODE for

$$u_t + Au = f$$

where $A = m\partial_x$, where $m = b/a$.

$$u(t,\cdot) = e^{-tA}u_0(\cdot) + \int_0^t e^{-(t-s)A}f(s,\cdot)ds$$
$$= e^{-m\,t\partial_x}u_0(\cdot) + \int_0^t e^{-m(t-s)\partial_x}f(s,\cdot)ds$$

We arrive at (9) once noticing

$$e^{-mt\partial_x}\phi(x) = \phi(x-mt) = \phi(x-bt/a).$$

This can be verified by Taylor series expansion

$$e^{h\partial_x} = 1 + h\partial_x + \frac{h^2}{2!}\partial_x^2 + \cdots + \frac{h^n}{n!}\partial_x^n + \cdots$$
$$\Rightarrow \phi(x+h) = \sum_{n=0}^{\infty}\frac{h^n}{n!}\phi^{(n)}(x).$$

□

7. a)

$$\int_{\mathbb{R}^n} e^{-|x|^2}dx = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-x_1^2}\cdots e^{-x_n^2}dx_1\cdots dx_n$$
$$= \int_{-\infty}^{\infty} e^{-x_1^2}dx_1 \cdots \int_{-\infty}^{\infty} e^{-x_n^2}dx_n$$

There are a few ways to evaluate $\int_{-\infty}^{\infty} e^{-x^2}dx$. For instance using polar coordinate (r,θ) in 2 dimensions for $x = (x_1, x_2)$.

$$\int_{\mathbb{R}^2} e^{-|x|^2}dx_1dx_2 = \int_0^{2\pi}d\theta\int_0^{\infty} e^{-r^2}rdr$$
$$= 2\pi\int_0^{\infty} e^{-r^2}rdr = -\pi\int_0^{\infty} d(e^{-r^2}) = \pi$$

Meanwhile,

$$\int_{\mathbb{R}^2} e^{-|x|^2} dx_1 dx_2 = \int_{\mathbb{R}^2} e^{-x_1^2} e^{-x_2^2} dx_1 dx_2$$
$$= \int_{\mathbb{R}} e^{-x_1^2} dx_1 \int_{\mathbb{R}} e^{-x_2^2} dx_2$$

which shows $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

b) $\int p_t(x,y)dx = 1$ follows from a) by scaling or change of variables.

c) For fixed $\delta > 0$, a change of variables $u = (y-x)/\sqrt{4kt}$ gives

$$\int_{|x-y|\geq\delta} \frac{1}{(4k\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4kt}} dy = \int_{|u|\geq\delta/\sqrt{4kt}} e^{-|u|^2} du \to 0 \quad as\ t \to 0$$

d) Let $f \in L^p$, $1 \leq p \leq \infty$. We may assume $f \in C_b(\mathbb{R}^n)$, that is, f is bounded continuous function. Then

$$u(t,x) - f(x) =$$
$$\int_{|x-y|<\delta} p_t(x,y)[f(y)-f(x)]dy + \int_{|x-y|\geq\delta} p_t(x,y)[f(y)-f(x)]dy := I_1 + I_2.$$

Given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ whenever $|y - x| < \delta$. Since

$$|I_1| \leq \varepsilon \int_{|x-y|<\delta} p_t(x,y)dy \leq \varepsilon \int_{\mathbb{R}^n} p_t(x,y)dy = \varepsilon$$
$$and\ |I_2| \leq 2C \int_{|x-y|\geq\delta} p_t(x,y)dy \to 0\ as\ t \to 0,$$

we see immediately that $u(t,x) - f(x) \to 0$ as $t \to 0$.

8. a) Show $Q'(t) = 0$ using the Schrödinger equation and integration by parts.

b) Similar method.

9. Sub $y = x - Vt$ and we have $u(x,t) = u(y+Vt,t) := \tilde{u}(y,t)$, where $x = x(y,t) = y + Vt$. By chain rule,

$$\frac{\partial}{\partial t}\tilde{u}(y,t) = \frac{\partial}{\partial t}[u(y+Vt,t)] = \frac{\partial}{\partial x}u(x,t)\cdot V + \frac{\partial}{\partial t}u(x,t)$$
$$\frac{\partial}{\partial y}\tilde{u}(y,t) = \frac{\partial}{\partial y}[u(y+Vt,t)] = \frac{\partial}{\partial x}u(x,t)\cdot 1.$$

Sub the above into the $u_t - ku_{xx} + Vu_x = 0$, $u(x,0) = \phi(x)$. We find that $\tilde{u}_t - k\tilde{u}_{yy} = 0$, $\tilde{u}(y,0) = \phi(y)$. Thus $\tilde{u}(y,t) = \int p_t(y,z)\phi(z)dz$ which gives that

$$\begin{aligned} u(x,t) &= \int p_t(x - Vt, z)\phi(z)dz \\ =& \frac{1}{(4k\pi t)^{n/2}} \int_{\mathbb{R}} e^{-\frac{|x-Vt-z|^2}{4kt}} \phi(z)dz. \end{aligned}$$

10. By reflection method as described in Section 3.3, extend $u(x,t)$ as odd function on $(-\frac{\pi}{2}, \frac{\pi}{2})$, and then periodically on $\mathbb{R}$, which we call $v(x,t)$. Also do the same for ϕ and ψ. Then v satisfies the wave equation on $\mathbb{R}$ with initial condition $v(x,0) = \tilde{\phi}$, $v_t(x,0) = \tilde{\psi}$, with a little abuse of notation we will continue to call them ϕ, psi. Thus the solution is given by

$$v(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \int_{x-ct}^{x+ct} \psi(s)ds$$

Note that the boundary condition is automatically satisfied because of the odd symmetries on **both** $x = 0, x = \pi/2$. As in the text, the strip $(0, \pi/2) \times \mathbb{R}$ divided into diamond regions. Inside each of these diamond, the wave propagates through a chain of reflections against the boundaries, $\phi(x \pm ct)$ has different sign depending on the number of reflections, similarly for $\pm\psi$. After some simple calculations, the express of $v(x,t)$ has a domain of dependence resulting from those reflections.

11. The inhomogeneous problem on the line has the solution

$$\begin{aligned} v(x,t) &= e^{t\Delta}\phi + \int_0^t e^{(t-s)\Delta} f(x,s)ds \\ &= \int p_t(x,y)\phi(y)dy + \int_0^t ds \int p_{t-s}(x,y)f(y,s)dy \end{aligned}$$

Make odd extension of u to v with $\phi(-x) = -\phi(x)$, then restrict $u = v|_{\mathbb{R}_+ \times \mathbb{R}_+}$ to obtain the solution.