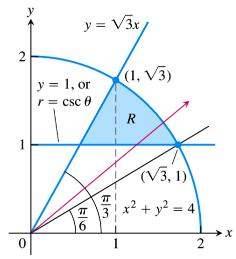
**Example**

Using polar integration, find the area of the region enclosed by the circle $x^2+y^2=4$, above the line $y=1$, and below the line $y=\sqrt{3}x$.

$$A=\int_{\pi/6}^{\pi/3}\int_{\csc\theta}^{2} r\,dr\,d\theta=\frac{1}{2}\int_{\pi/6}^{\pi/3}\left(4-\csc^2\theta\right)d\theta=\frac{1}{2}\left(4\theta+\cot\theta\right)_{\pi/6}^{\pi/3}=\cdots=\frac{\pi-\sqrt{3}}{3}$$

$$\left(\frac{1}{2}r^2\right)_{\csc\theta}^{2}=\frac{1}{2}\left(4-\csc^2\theta\right)$$

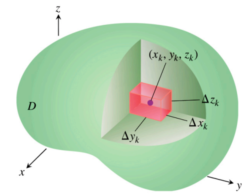



**Triple Integrals in Rectangular Coordinates**

$$\iiint_D F(x,y,z)\,dV=\lim_{n\to\infty}S_n$$

$$S_n=\sum_{k=1}^{n}F(x_k,y_k,z_k)\,\Delta V_k$$

$$\Delta V_k=\Delta x_k\Delta y_k\Delta z_k$$




**Definition**

The ***volume*** of a closed, bounded region $D$ in space is

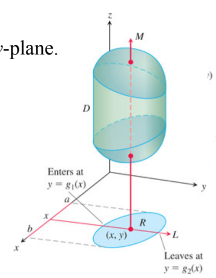
$$V=\iiint_D dV$$

Fubini's Theorem still holds, e.g.: (any other order can be used)

$$\iiint_D F(x,y,z)\,dV=\int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)}F(x,y,z)\,dz\,dy\,dx$$

**Finding Limits of Integration in the Order *dz dy dx***

1. **Sketch** the region $D$ along with its "shadow" $R$ in the $xy$-plane. Label the upper and lower bounding surfaces of $D$ and the upper and lower bounding curves of $R$.
2. **Find the $z$-limits** of integration. Draw a line $M$ passing through a typical point $(x, y)$ in $R$ parallel to the $z$-axis. As $z$ increases, $M$ enters $D$ at $z=f_1(x,y)$ and leaves at $z=f_2(x,y)$.
3. **Find the $y$-limits** *of* integration. Draw a line $L$ through $(x, y)$ parallel to the $y$-axis. As $y$ increases, $L$ enters $R$ at $y=g_1(x)$ and leaves at $y=g_2(x)$.




**Example**

Find the volume of the region $D$ enclosed by the surfaces $z=x^2+3y^2$ and $z=8-x^2-y^2$.

1

**the *z*-limits** of integration

The surfaces' intersection: $x^2+3y^2=8-x^2-y^2$. $\longrightarrow$ $x^2+2y^2=4$

i.e. they intersect on the cylinder $x^2+2y^2=4$ so the projection $R$ of $D$ onto the $xy$ plane is the ellipse $x^2+2y^2=4$ together with its interior: $x^2+2y^2\le 4$.

For every point in $R$ we have

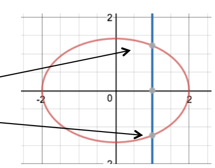
$$\underbrace{x^2+3y^2}_{f_1(x,y)}=\left(x^2+2y^2\right)+y^2\le 4+y^2=8-4+y^2\le 8-\left(x^2+2y^2\right)+y^2=\underbrace{8-x^2-y^2}_{f_2(x,y)}$$

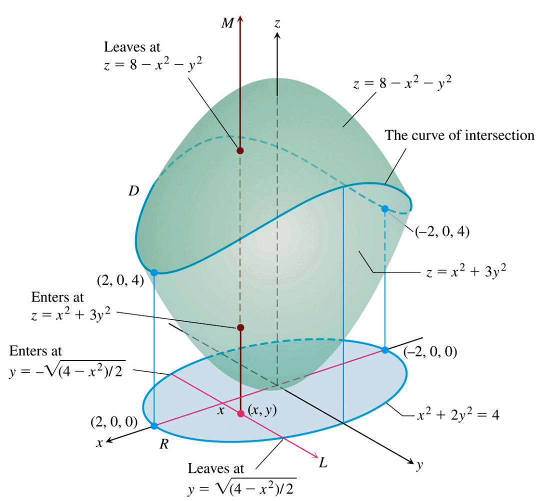
**the *y*-limits** of integration

$$x^2+2y^2=4$$

$$2y^2=4-x^2$$

$$y=\pm\sqrt{\left(4-x^2\right)/2}=g_1(x),g_2(x)$$




$$V=\iiint_D dV=\int_{-2}^{2}\int_{-\sqrt{\left(4-x^2\right)/2}}^{\sqrt{\left(4-x^2\right)/2}}\int_{x^2+3y^2}^{8-x^2-y^2}dz\,dy\,dx=2\int_{-2}^{2}\int_{-\sqrt{\left(4-x^2\right)/2}}^{\sqrt{\left(4-x^2\right)/2}}\left(4-x^2-2y^2\right)dy\,dx$$

$$\left(8-x^2-y^2\right)-\left(x^2+3y\right)^2=8-2x^2-4y^2=2\left(4-x^2-2y^2\right)$$

an even function of y

$$\int_{-\sqrt{\left(4-x^2\right)/2}}^{\sqrt{\left(4-x^2\right)/2}}\left(4-x^2-2y^2\right)dy=2\int_{0}^{\sqrt{\left(4-x^2\right)/2}}\left(4-x^2-2y^2\right)dy=2\left(\left(4-x^2\right)y-\frac{2}{3}y^3\right)\Big|_0^{\sqrt{\left(4-x^2\right)/2}}$$

$$=\frac{2}{3}y\left(12-3x^2-2y^2\right)\Big|_0^{\sqrt{\left(4-x^2\right)/2}}$$

$$=\frac{2}{3}\sqrt{\left(4-x^2\right)/2}\left(12-3x^2-2\left(4-x^2\right)/2\right)=\frac{2\sqrt{2}}{3}\left(4-x^2\right)^{3/2}$$

$$\frac{\sqrt{2}}{3}\left(4-x^2\right)^{1/2}\qquad 8-2x^2=2\left(4-x^2\right)$$

an even function of $x$

$$=\frac{4\sqrt{2}}{3}\int_{-2}^{2}\left(4-x^2\right)^{3/2}dx=\frac{8\sqrt{2}}{3}\int_{0}^{2}\left(4-x^2\right)^{3/2}dx\quad\begin{pmatrix}\text{substitution}\\ x=2\sin u\end{pmatrix}\quad\ldots=8\pi\sqrt{2}$$