Changing Cartesian Integrals into Polar Integrals

$$\boxed{\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

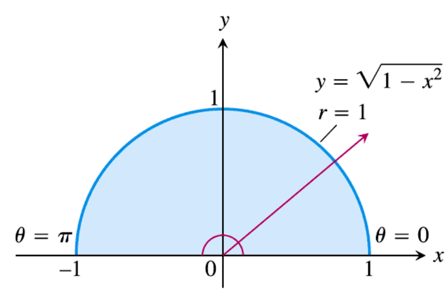
where R and G denote the same region described in Cartesian and polar coordinates, respectively.

Example

Evaluate

$$\iint_R e^{x^2+y^2}\,dx\,dy$$

where R is the upper half of the unit circle



$$\iint_R e^{x^2+y^2}\,dx\,dy = \int_0^\pi \int_0^1 e^{r^2}\,r\,dr\,d\theta = \int_0^\pi \tfrac{1}{2}(e-1)\,d\theta = \frac{\pi}{2}(e-1)$$

$u = e^{r^2}$

$du = 2re^{r^2}\,dr$

$re^{r^2}\,dr = du/2$

$$\frac{1}{2}\int du = \frac{1}{2}u$$

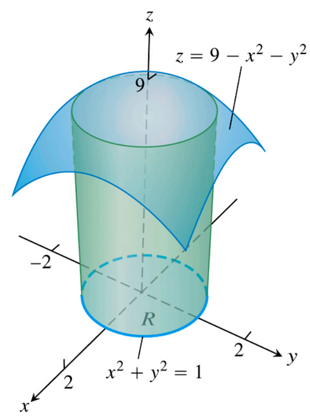
$$\left(\tfrac{1}{2}e^{r^2}\right)\Big|_0^1 = \tfrac{1}{2}(e-1)$$

Example

Find the volume of the solid bounded above by the paraboloid

$$z = 9 - x^2 - y^2$$

and below by the unit circle in the *xy*-plane.



Let $f(x,y) = 9 - x^2 - y^2$. Then $f(r\cos\theta, r\sin\theta) = 9 - r^2$

$$V = \int_0^{2\pi} \int_0^1 \left(9 - r^2\right) r\, dr\, d\theta = \int_0^{2\pi} \left(\frac{17}{4}\right) d\theta = \frac{17\pi}{2}$$

$u = 9 - r^2$

$du = -2r dr$

$rdr = -du/2$

$$-\frac{1}{2}\int u\, du = -\frac{1}{4}u^2$$

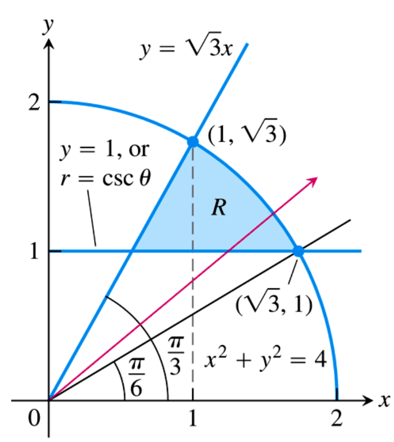
$$\left(-\tfrac{1}{4}\left(9 - r^2\right)^2\right)\Big|_0^1 = \left(-\tfrac{1}{4}(8)^2 + \tfrac{1}{4}(9)^2\right) = \frac{17}{4}$$

Example

Using polar integration, find the area of the region enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

$$A = \int_{\pi/6}^{\pi/3} \int_{\csc\theta}^{2} r\, dr\, d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/3} \left(4 - \csc^2\theta\right) d\theta = \frac{1}{2}\left(4\theta + \cot\theta\right)_{\pi/6}^{\pi/3} = \cdots = \frac{\pi - \sqrt{3}}{3}$$

$$\left(\frac{1}{2}r^2\right)_{\csc\theta}^{2} = \frac{1}{2}\left(4 - \csc^2\theta\right)$$

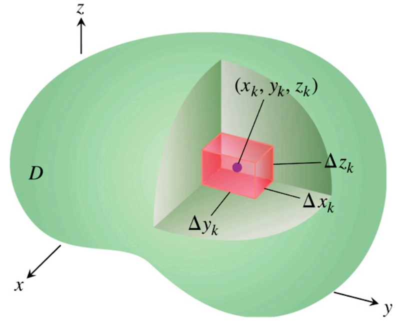


Triple Integrals in Rectangular Coordinates

$$\iiint\limits_D F(x,y,z)\,dV = \lim_{n\to\infty} S_n$$

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\,\Delta V_k$$

$$\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$$



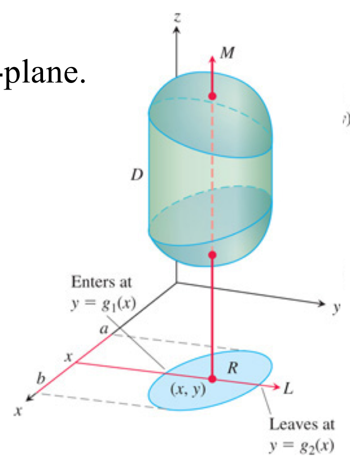
Definition
The ***volume*** of a closed, bounded region D in space is

$$V = \iiint\limits_D dV$$

Fubini's Theorem still holds, e.g.: (any other order can be used)

$$\iiint\limits_D F(x,y,z)\,dV = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x,y,z)\,dz\,dy\,dx$$

Finding Limits of Integration in the Order *dz dy dx*



1. **Sketch** the region D along with its "shadow" R in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.

2. **Find the *z*-limits** of integration. Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x,y)$ and leaves at $z = f_2(x,y)$.

3. **Find the *y*-limits** *of* integration. Draw a line L through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$.

Example
Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

the *z*-limits of integration

The surfaces' intersection: $x^2+3y^2 = 8-x^2-y^2.$ $\longrightarrow$ $x^2+2y^2=4$

i.e. they intersect on the cylinder $x^2+2y^2=4$ so the projection R of D onto the

xy plane is the ellipse $x^2+2y^2 = 4$ together with its interior: $x^2+2y^2 \le 4.$

For every point in R we have

$$\underbrace{x^2+3y^2}_{f_1(x,y)} = \left(x^2+2y^2\right)+y^2 \le 4+y^2 = 8-4+y^2 \le 8-\left(x^2+2y^2\right)+y^2 = \underbrace{8-x^2-y^2}_{f_2(x,y)}$$

the *y*-limits of integration

$x^2+2y^2=4$

$2y^2 = 4-x^2$

$y=\pm\sqrt{\left(4-x^2\right)/2} = g_1(x), g_2(x)$