§3.7 Implicit Differentiation

The implicit differentiation method. Given that $y = y(x)$ is a function of x defined implicitly by the equation $F(x, y) = C$.
Step 1. Take derivative in x both sides of the equation using possibly chain rule.
Step 2. Then, from the resulting equation obtained in Step 1, solve for $y' = dy/dx$ in terms of x and y.
Example 2. Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution. Step 1. Differentiating both sides of the equation with respect to x, we obtain

$$2x + 2y \cdot y' = 0 \tag{1}$$

Step 2. Solve for y' in (1) to have

$$y' = -\frac{x}{y}\ .$$

$\therefore$ The slope at $(3, -4)$ is given by $m = \frac{dy}{dx}|_{(3,-4)} = \frac{3}{4}$. $\square$

Example 3. Find dy/dx if $y^2 = x^2 + \sin xy$.

Solution. Step 1. Differentiating both sides of the equation w. r. t. x, we obtain

$$2yy' = 2x + (y + xy') \cos xy \tag{2}$$

Step 2. Solve for y' in (2) to have

$$y' = \frac{2x + y \cos xy}{2y - x \cos xy}\ .$$

$\square$

Example 4. Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution. Step 1. Differentiating both sides of the equation in x, we have

$$6x^2 - 6y \cdot y' = 0$$

$$x^2 - y \cdot y' = 0 \tag{3}$$

Step 2. Solve for y' in (3) to obtain

$$y' = \frac{x^2}{y} .$$

Step 3. Repeat taking derivative in (3) in x again, we obtain

$$2x - y'^2 - yy'' = 0$$
$$\Rightarrow y'' = \frac{2x - y'^2}{y} = \frac{2xy^2 - x^4}{y^3} .$$

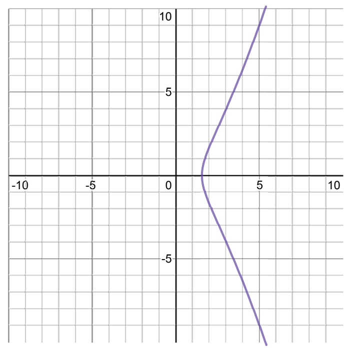


FIGURE 1. Plot of the implicit equation $2x^3 - 3y^2 = 8$ in Ex.4

The following graph for the implicit function $y = y(x)$ is produced from demo $\square$

Example 5. Show that the point $(2, 4)$ lies on the curve $x^3+y^3-9xy = 0$. Then find the tangent and normal to the curve there.

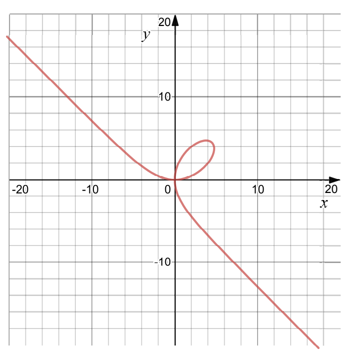


FIGURE 2. Plot of the implicit equation $x^3+y^3-9xy = 0$ in Ex.5

Theorem 3 (The Derivative Rule for Inverses). If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \qquad \text{or} \qquad \left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

Example 2. Let $f(x) = x^3 - 2$, $x > 0$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

[Solution] Step 1. Write $y = f(x) = x^3 - 2$, $x > 0$.
The inverse function $y = f^{-1}(x)$ satisfies $x = y^3 - 2$ by switching x and y in the equation above.
Step 2. In order to evaluate the derivative of $y = f^{-1}$, we apply the formula at the point (a, b) with $b = f(a)$; note here $(a, b) = (2, 6)$.

$$\frac{df^{-1}}{dx}|_{x=b=6} = \frac{1}{\frac{df}{dx}|_{x=f^{-1}(b)}} = \frac{1}{\frac{df}{dx}|_{x=a}}$$
$$= \frac{1}{\frac{df}{dx}|_{x=2}} = \frac{1}{12},$$

where $f'(x) = 3x^2 \Rightarrow f'(2) = 3(2)^2 = 12$.

Derivative of Logarithm. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}, \quad x > 0.$

More generally, $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x}$, $x \neq 0$.
The natural logarithm function $y = \ln x$ is the inverse of $y = e^x$, whose graph is given in Figure 3.

Example 3. Find (a) $\dfrac{d}{dx}\ln(2x)$ (b) $\dfrac{d}{dx}\ln(x^2+3)$.

[answer: (a) $\frac{1}{x}$; (b) $\frac{2x}{x^2+3}$.]

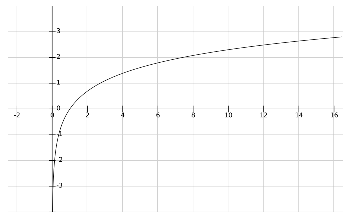


FIGURE 3. Plot of the logarithm function $y = \ln x$

Example 4. A line with slope m passes through the origin and is tangent to the graph $y = \ln x$. What is the value of m?

Example 6*. Find dy/dx if $y = \dfrac{(x^2+1)(x+3)^{1/2}}{x-1}$, $x > 1$.

[Solution] By logarithmic differentiation method

$$\begin{aligned}
\ln y &= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1) \\
\frac{1}{y}\,y' &= \frac{2x}{x^2+1} + \frac{1}{2}\frac{1}{x+3} - \frac{1}{x-1} \\
\Rightarrow y' &= y\left(\frac{2x}{x^2+1} + \frac{1}{2}\frac{1}{x+3} - \frac{1}{x-1}\right) \\
y' &= \frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}\right).
\end{aligned}$$

Example 7*. Differentiate $f(x) = x^x$, $x > 0$.

[Solution] Write $y = x^x$. Then $\ln y = x \ln x$. Implicit differentiation gives

$$\begin{aligned}
\frac{1}{y}y' &= \ln x + x\left(\frac{1}{x}\right) \\
\Rightarrow y' &= (1 + \ln x)x^x.
\end{aligned}$$

§3.9 Inverse Trigonometric Functions

Definition. $\boldsymbol{y = \sin^{-1} x}$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$\boldsymbol{y = \cos^{-1} x}$ is the number in $[0,\pi]$ for which $\cos y = x$.
$\boldsymbol{y = \tan^{-1} x}$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.
$\boldsymbol{y = \cot^{-1} x}$ is the number in $(0,\pi)$ for which $\cot y = x$.
$\boldsymbol{y = \sec^{-1} x}$ is the number in $[0,\pi/2) \cup (\pi/2, \pi]$ for which $\sec y = x$.
$\boldsymbol{y = \csc^{-1} x}$ is the number in $[-\pi/2, 0) \cup (0, \pi/2]$ for which $\csc y = x$.

TABLE 1. Derivative formulae for inverse trigonometric functions

f	f'		f	f'
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$		$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$		$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$		$\csc^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{\lvert x\rvert^2-1}}$

Ex. **Derivative of $\boldsymbol{\sin^{-1} x}$.** $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}, \quad |x| < 1.$

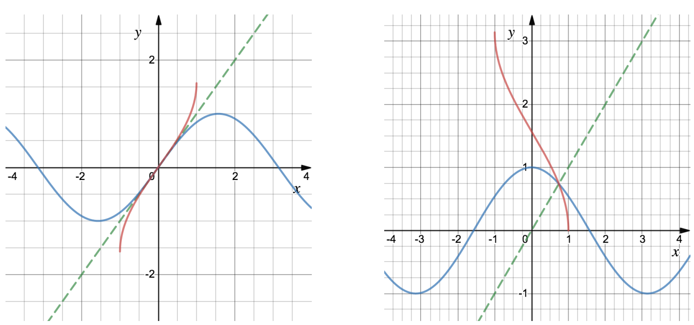
Example 2. Find $\dfrac{d}{dx}\left(\sin^{-1} x^2\right)$.

[answer: $\dfrac{d}{dx}\left(\sin^{-1}(x^2)\right) = \dfrac{2x}{\sqrt{1-x^4}}$.]

Derivative of $\boldsymbol{\tan^{-1} x}$. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$.

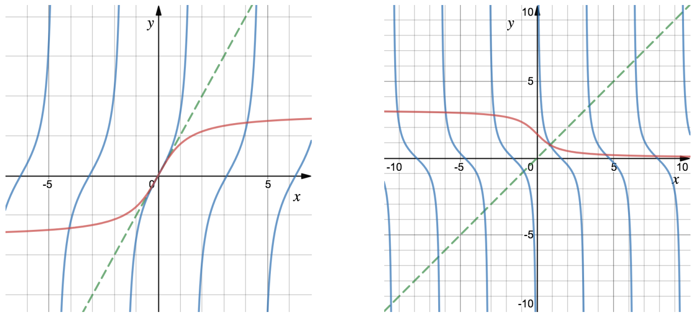
Derivative of $\boldsymbol{\sec^{-1} x}$. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{|x|\sqrt{x^2-1}}, \ |x| > 1.$

Ex. From MML



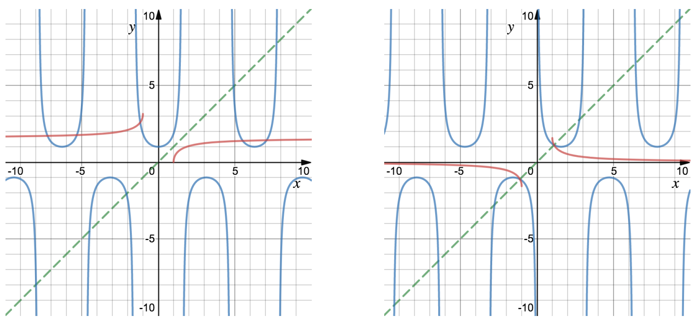
(A) $\arcsin x$ vs $\sin x$ (B) $\arccos x$ vs $\cos x$

FIGURE 4. graphs of $\sin^{-1} x$ and $\cos^{-1} x$



(A) arctan x vs tan x (B) arccot x vs cot x

FIGURE 5. graphs of $\tan^{-1} x$ and $\cot^{-1} x$



(A) arcsec x vs sec x (B) arccsc x vs csc x

FIGURE 6. graphs of $\sec^{-1} x$ and $\csc^{-1} x$

§3.10 Related Rates*

Exercise 6. If $x = y^3 - y$ and $dy/dt = 5$, then what is dx/dt when $y = 2$?

Exercise 10. If $r + s^2 + v^3 = 12$, $dr/dt = 4$, and $ds/dt = -3$, find dv/dt when $r = 3$ and $s = 1$.