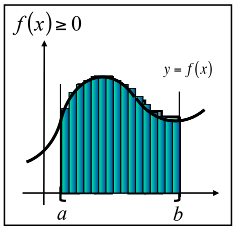
Double Integrals

Recall:

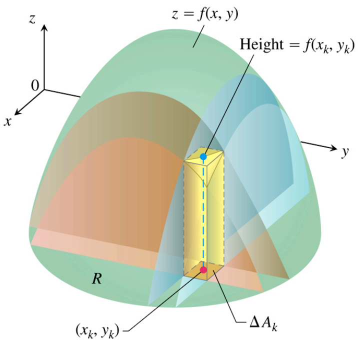
$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x_k$$

If $f(x)\geq 0$ on $[a,b]$ then

$$\int_a^b f(x)dx = \text{area below the graph}$$



For $f(x,y)\geq 0$ over a region R in the plane:

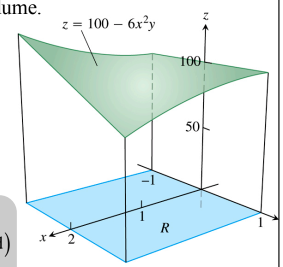


$$\text{Volume} = \lim \sum f(x_k, y_k)\,\Delta A_k = \iint_R f(x,y)\,dA$$

Example

Find $\iint_R f(x,y)dA$ for $f(x,y)=100-6x^2y$ and $R:\ 0\leq x\leq 2,\ -1\leq y\leq 1$

Since $f(x,y)\geq 0$ over R we can use Calc I to find volume.



A. By cross sections ***perpendicular*** to the **x-axis** :

$$V=\int_0^2 A(x)dx$$

where $A(x)$ is { the area of the slice obtained by cutting the solid at location x.

= the area of the region below the curve C_x :

$z=g(y)=f(x,y)$ $\quad$ (x is fixed)

in the (y,z)-plane between $y=-1$ and $y=1$;

i.e.

$$A(x)=\int_{-1}^{1} f(x,y)dy = \left(100y-3x^2y^2\right)\Big|_{y=-1}^{y=1} = \left(100-3x^2\right)-\left(-100-3x^2\right)=200$$

so

$$V=\int_0^2 A(x)dx = \int_0^2 200\,dx = (200x)\Big|_0^2 = \boxed{400}$$

B. By cross sections ***perpendicular*** to the **y-axis** :

$$V=\int_{-1}^{1} A(y)dy$$

where $A(y)$ is { the area of the slice obtained by cutting the solid at location y.

= the area of the region below the curve C_y :

$z=h(x)=f(x,y)$ $\quad$ (y is fixed)

in the (x,z)-plane between $x=0$ and $x=2$;

i.e.

$$A(y)=\int_0^2 f(x,y)dx = \left(100x-2x^3y\right)\Big|_{x=0}^{x=2} = 200-16y$$

so

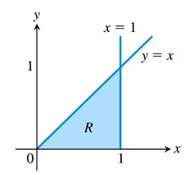
$$V=\int_{-1}^{1} A(y)dy = \int_{-1}^{1}(200-16y)dy = \left(200y-8y^2\right)\Big|_{-1}^{1} = (200-8)-(-200-8)=\boxed{400}$$

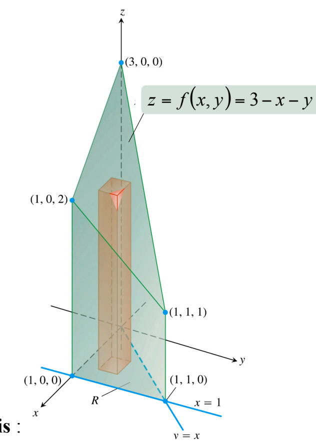
Example

Find the volume of the prism in the picture;

i.e. $\iint_R f(x,y)dA$

for $f(x,y)=3-x-y$ and R the triangle below:





A. By cross sections **perpendicular** to the ***x*-axis** :

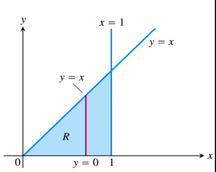
$V=\int_0^1 A(x)dx$, where $A(x)$ is { the area of the slice obtained by cutting the prism at location x.

$A(x)=$ the area of the slice obtained by cutting the prism at location x

= the area of the region below the curve C_x :

$z=g(y)=f(x,y)$ $\quad$ (x is fixed)

in the (y,z)-plane between $y=0$ and $y=x$;



i.e.

$$A(x)=\int_0^x g(y)dy = \int_0^x f(x,y)dy$$

$$=\int_0^x (3-x-y)dy = \left(3y-xy-\tfrac{1}{2}y^2\right)\Big|_{y=0}^{y=x} = 3x-x^2-\tfrac{1}{2}x^2 = 3x-\tfrac{3}{2}x^2$$

so

$$V=\int_0^1 A(x)dx = \left(\tfrac{3}{2}x^2-\tfrac{1}{2}x^3\right)\Big|_0^1 = \boxed{1}$$

B. By cross sections **perpendicular** to the ***y*-axis** :

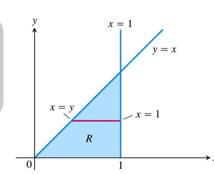
$V=\int_0^1 A(y)dy$, where $A(y)$ is { the area of the slice obtained by cutting the prism at location y.

$A(y)=$ the area of the slice obtained by cutting the prism at location y

= the area of the region below the curve C_y :

$z=h(x)=f(x,y)$ $\quad$ (y is fixed)

in the (x,z)-plane between $x=0$ and $x=y$;



i.e.

$$A(y)=\int_y^1 h(x)dx = \int_y^1 f(x,y)dx$$

$$=\left(3x-\tfrac{1}{2}x^2-yx\right)\Big|_{x=y}^{x=1} = \left(3-\tfrac{1}{2}-y\right)-\left(3y-\tfrac{1}{2}y^2-y^2\right) = \tfrac{5}{2}-4y+\tfrac{3}{2}y^2$$

so

$$V=\int_0^1 A(y)dy = \left(\tfrac{5}{2}y-2y^2+\tfrac{1}{2}y^3\right)\Big|_0^1 = \boxed{1}$$

THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region R: $a \le x \le b, c \le y \le d$, then

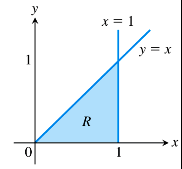
$$\underbrace{\iint_R f(x, y)\, dA}_{\text{double integral}} = \underbrace{\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.}_{\text{iterated integrals}}$$

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then
$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$
2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then
$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Example

Find $\iint_R \frac{\sin x}{x} dA$ for R in the picture:



$$\iint_R \frac{\sin x}{x} dA = \int_0^1 \left(\int_0^x \frac{\sin x}{x} dy \right) dx = \int_0^1 \sin x\, dx = (-\cos x)\big|_0^1 = 1 - \cos(1) \approx 0.46$$

$$\left(\frac{\sin x}{x} y\right)\Big|_{y=0}^{y=x} = \frac{\sin x}{x} x - 0 = \sin x$$

$$\iint_R \frac{\sin x}{x} dA = \int_0^1 \left(\int_y^1 \frac{\sin x}{x} dx \right) dy$$ **cannot do it!**

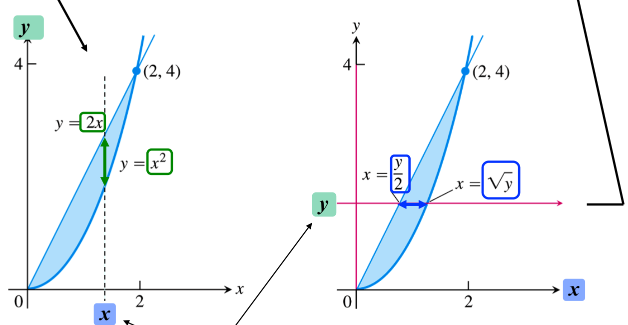
$\int \frac{\sin x}{x} dx$ cannot be expressed in terms of elementary functions.

Example

Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\, dy\, dx \qquad = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\, dx\, dy$$

and write an equivalent integral with the order of integration reversed.



fixed during the first phase of integration