Vector Fields and Line Integrals

Definition
A ***vector field*** over a region D is an assignment of a vector to each point in D.

2D: $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$

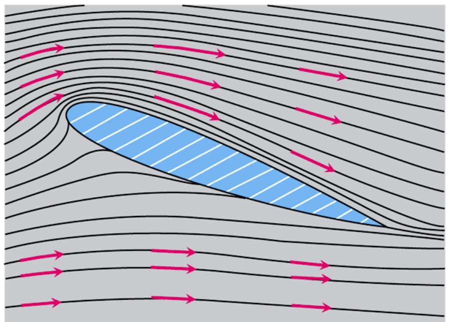


FIGURE 16.6 Velocity vectors of a flow around an airfoil in a wind tunnel.

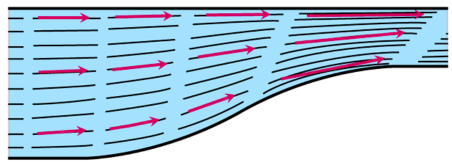


FIGURE 16.7 Streamlines in a contracting channel. The water speeds up as the channel narrows and the velocity vectors increase in length.

3D: $\mathbf{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$

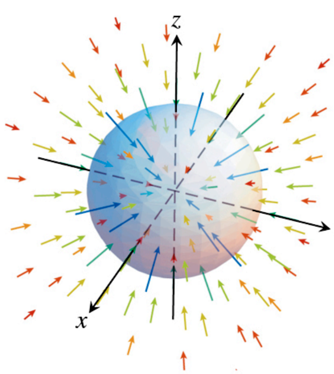


FIGURE 16.8 Vectors in a gravitational field point toward the center of mass that gives the source of the field.

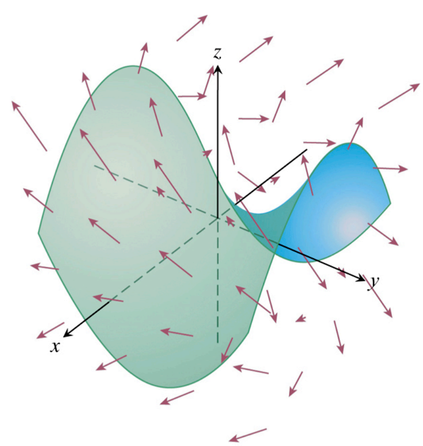
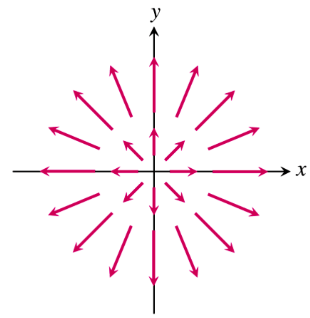


FIGURE 16.9 A surface, like a mesh net or parachute, in a vector field representing water or wind flow velocity vectors. The arrows show the direction and their lengths indicate speed.

Example

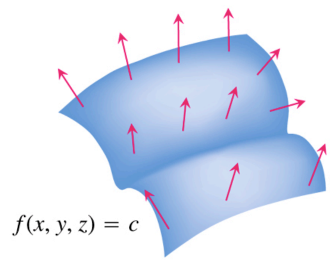
Sketch the vector field given by $\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j}$.



Important Example

The ***gradient field*** of a differentiable function $f(x,y,z)$

$$\mathbf{F}(x,y,z) = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$



Line Integral of a Vector Field

Suppose $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$, is a smooth parametrization of the curve C.
Recall, the unit tangent vector: $\mathbf{T} = d\mathbf{r}/ds = \mathbf{v}/|\mathbf{v}|$.

The ***line integral of a vector field* F *along*** C is the line integral (as defined before) of the ***scalar-valued*** function $\mathbf{F} \bullet \mathbf{T}$ (the scalar component of **F** in the direction **T**):

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds = \int_C \mathbf{F} \bullet \frac{d\mathbf{r}}{ds}\, ds = \int_C \mathbf{F} \bullet d\mathbf{r} = \boxed{\int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt}\, dt}$$

Example

Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ for $\underbrace{\mathbf{F}(x,y,z) = z\mathbf{i} + xy\mathbf{j} - y^2\mathbf{k}}_{\langle z,\ xy, -y^2 \rangle}$, $\underbrace{\mathbf{r}(t) = t^2\mathbf{i} + t\,\mathbf{j} + \sqrt{t}\,\mathbf{k}}_{\langle t^2,\ t,\ \sqrt{t} \rangle}$, $0 \le t \le 1$.

$$\left.\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= \mathbf{F}\left(t^2,\ t,\ \sqrt{t}\right) = \left\langle \sqrt{t},\ t^3, -t^2 \right\rangle \\ \frac{d\mathbf{r}}{dt} &= \left\langle 2t,\ 1,\ 1/2\sqrt{t} \right\rangle \end{aligned}\right\} \quad \mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt} = 2t\sqrt{t} + t^3 - t^2/2\sqrt{t} = \tfrac{3}{2}t\sqrt{t} + t^3$$

So

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^1 \left(\tfrac{3}{2}t^{3/2} + t^3\right) dt = \ldots = \tfrac{17}{20}$$

$$\int_C \mathbf{F}\bullet\mathbf{T}\,ds = \int_C \mathbf{F}\bullet\frac{d\mathbf{r}}{ds}\,ds = \int_C \mathbf{F}\bullet d\mathbf{r} = \boxed{\int_a^b \mathbf{F}(\mathbf{r}(t))\bullet\frac{d\mathbf{r}}{dt}\,dt}$$

Example

Evaluate $\int_C \mathbf{F}\bullet d\mathbf{r}$ for $\mathbf{F}(x,y,z) = z\mathbf{i} + xy\mathbf{j} - y^2\mathbf{k}$ $\langle z,\ xy,\ -y^2\rangle$ and $\mathbf{r}(t) = t^2\mathbf{i} + t\,\mathbf{j} + \sqrt{t}\,\mathbf{k}$, $\langle t^2,\ t,\ \sqrt{t}\rangle$, $0 \le t \le 1$.

$\mathbf{F} = \langle M, N, P\rangle$

$\mathbf{r} = \langle x, y, z\rangle = \langle x(t), y(t), z(t)\rangle$

$d\mathbf{r} = \langle dx, dy, dz\rangle$

scalar functions of x, y, z

$\int_C M\,dx + N\,dy + P\,dz$ **scalar differential form**

To evaluate, express everything in terms of t.

$\mathbf{F} = \langle M, N, P\rangle$

$\mathbf{r} = \langle x, y, z\rangle = \langle x(t), y(t), z(t)\rangle$

$d\mathbf{r} = \langle dx, dy, dz\rangle$

scalar functions of x, y, z

$\int_C M\,dx + N\,dy + P\,dz$ **scalar differential form**

To evaluate, express everything in terms of t.

$dx = x'(t)dt$

$dy = y'(t)dt$

$dz = z'(t)dt$

$\int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\right)dt$ **parametric scalar evaluation**

$(-\sin t)(-\sin t) + (t)(\cos t) + (2\cos t)(1)$

Example

Evaluate $\int_C \underbrace{-y}_{M}dx + \underbrace{z}_{N}dy + \underbrace{2x}_{P}dz$, C is the helix $\mathbf{r}(t) = \underbrace{(\cos t)}_{x(t)}\mathbf{i} + \underbrace{(\sin t)}_{y(t)}\mathbf{j} + \underbrace{t}_{z(t)}\mathbf{k}$, $0 \le t \le 2\pi$.

$dx/dt = -\sin t$

$dy/dt = \cos t$

$dz/dt = 1$

use integration by parts

$$\int_0^{2\pi}\left(\sin^2 t + t\cos t + 2\cos t\right)dt = \ldots = \pi$$

rewrite as $\frac{1}{2}(1-\cos 2t)$

Work in a Force Field

If the vector field $\mathbf{F}$ represents a force throughout a region then the line integral of $\mathbf{F}$ over a smooth curve C parametrized by $\mathbf{r} = \mathbf{r}(t),\ a \le t \le b,$ gives the work done in moving an object from point $A = \mathbf{r}(a)$ to point $B = \mathbf{r}(b)$ along C.

Example

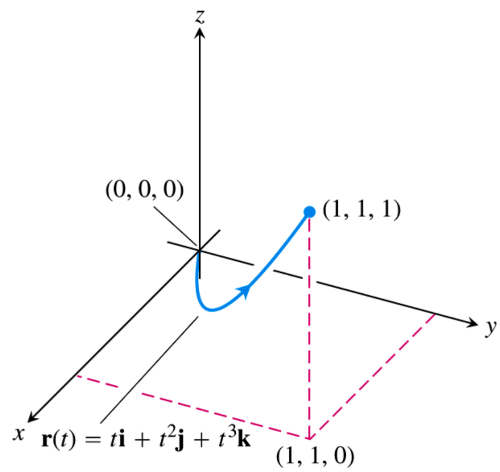
Find the work done by the force $\mathbf{F}(x,y,z) = (y - x^2)\,\mathbf{i} + (z - y^2)\,\mathbf{j} + (x - z^2)\,\mathbf{k}$ along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k},\ 0 \le t \le 1.$

$$W = \int_C \mathbf{F} \bullet \mathbf{T}\, ds = \int_0^1 \boxed{\mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt}}\, dt = \dots = \frac{29}{60}$$

$$\mathbf{F}(\mathbf{r}(t)) = \left\langle 0,\ t^3 - t^4,\ t - t^6 \right\rangle$$

$$\frac{d\mathbf{r}}{dt} = \left\langle 1, 2t, 3t^2 \right\rangle$$

$$\mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt} = \boxed{2t^4 - 2t^5 + 3t^3 - 3t^8}$$



Flow/Circulation **velocity of a fluid flowing through**

If the vector field $\mathbf{F}$ represents ~~a force throughout~~ a region then the line integral of $\mathbf{F}$ over a smooth curve C parametrized by $\mathbf{r} = \mathbf{r}(t),\ a \le t \le b,$ gives the ***flow*** ~~done in moving an object~~ from point $A = \mathbf{r}(a)$ to point $B = \mathbf{r}(b)$ along C.

$$Flow = \int_C \mathbf{F} \bullet \mathbf{T}\, ds$$

If the curve starts and ends at the same point $(A = B)$, the flow is called the ***circulation*** around the curve.

Example

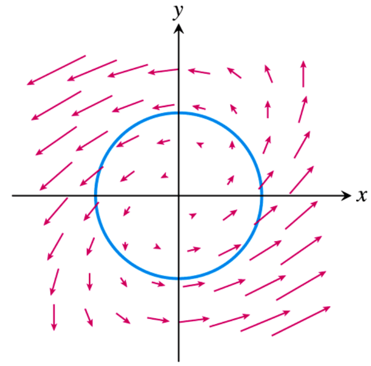
Find the circulation of the field $\mathbf{F}(x,y) = (x - y)\mathbf{i} + (x)\mathbf{j}$ around the circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le 2\pi.$

Example

Find the circulation of the field $\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x)\mathbf{j}$ around the circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le 2\pi$.

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt}\, dt = \left(-\tfrac{1}{2}\sin^2 t + t\right)\Big|_0^{2\pi} = \boxed{2\pi}$$

positive



counterclockwise circulation

$$\mathbf{F}(\mathbf{r}(t)) = \langle \cos t - \sin t,\ \cos t \rangle$$

$$\frac{d\mathbf{r}}{dt} = \langle -\sin t,\ \cos t \rangle$$

$$\mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt} = \underbrace{(\cos t - \sin t)(-\sin t)}_{-\sin t \cos t + \sin^2 t} + \cos^2 t = -\sin t \cos t + 1$$

$$\int \sin t \cos t\, dt = \int \sin t\, d(\sin t) = \tfrac{1}{2}\sin^2 t + C$$