WEAK TYPE ESTIMATES OF RIESZ MEANS ON RIEMANNIAN SYMMETRIC SPACES

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ABSTRACT. Let G be a noncompact semisimple Lie group of rank one, $X = G/K$ its homogeneous space. We investigate the weak type estimates for the Bohner-Riesz means of functions in certain Hardy spaces on X. Solution to this problem relys on an explicit formula for the kernel of the Riesz means. The critical index are the same as in the Euclidean case. The Riesz means on X can also be defined via a weighted Jacobi transform. It is possible we can obtain a weak type estimate for the corresponding maximal operator. We also study the boundedness result for the Hardy spaces on G/K by an atomic decomposition adapted to G.

1. INTRODUCTION

The subject of L^p boundedness for Riesz means is one of the major problems in harmonic analyis. The Bochner-Riesz multiplier $r_\delta = (1 - |x|^2)_+^\delta$. It is known that $\delta(p) = n(1/p - 1/2) - 1/2$ is the critical index for L^p problems (with $1/p - 1/2 \geq 1/(n+1)$). It is showed in [Stein, Taibleson and Weiss] [62] that this is also the critical index for Hardy spaces. Namely, they proved that the maximal function associated to Riesz means maps H^p continuously to weak L^p for $0 < p < 1$. By interpolation we know this implies the L^p boundedness.

We are concerned with obtaining the analogous result on symmetric spaces. Previously, Clerc studied the Riesz means of Hardy spaces on compact Lie groups and he obtained the same critical index for the maximal Bochner-Riesz operator by proving that the maximal operator satisfies the weak type estimates for atoms, which serve as "building blocks" for Hardy spaces on G.

The study of Hardy spaces on a noncompact symmetric space is more difficulty. One of the reason is that the definition of H^p classes on G is relatively not well understood. There are atomic decompositions

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for H^p classes on R^n, compact Lie groups or compact manifolds as well as Heisenberg groups [Coifman-Weiss][Clerc],[Folland-Stein] and [H.P. Liu]. However so far atomic decomposition is known only for K-invariant functions on G as studied in [T.Kawazoe, 1985, 2000]. Also we do not know whether the atomic definition is equivalent to the Poisson function characterization on a symmetric space.

Let $X = G/K$ be the homogeneous space for a connected semisimple Lie group G, where K is a maximal compact subgroup of G. Let $G = KAN$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be Iwasawa decomposition for G and its Lie algebra $\mathfrak{g}$. Let $\mathcal{L}$ be the Laplacian-Beltrami operator on G. In rank one case $\mathcal{L} = \Delta^{-1}\frac{d}{dx}(\Delta\frac{d}{dx})$, where $\Delta(x) = (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1}$ for radial functions.

The sperical functions on G are K-invariant and satisfies

$$\mathcal{L}\phi_\lambda = -(\lambda^2 + \rho^2)\phi_\lambda,$$

with $\rho = \alpha + \beta + 1$, where $2\alpha = m_1 + m_2 - 1$ and $2\beta = m_2 - 1$ and so $\rho = (m_1 + 2m_2)/2$ is equal to half of the sum of the positive roots with multiplicities with respect to the pair $(\mathfrak{g}, \mathfrak{a})$. There are integral formulas for the spectral operator $m(\mathcal{L})$ in [Giulini-Mauceri], for complex or rank one semisimple Lie groups. We shall study the Bochner-Riesz operators near the critical index.

2. Preliminaries

Definition. A function is said to be K bi-invariant if f is invariant under left and right translation by K.

Denote the Fourier (spherical) transform of f by

$$\hat{f} = \int_G f(g)\bar{\phi}_\lambda(g)dg.$$

The inversion formula is given by

$$f(g) = \int_0^\infty \phi_\lambda(g)\hat{f}(\lambda)d\mu(\lambda),$$

where $d\mu(\lambda) = |c(\lambda)|^{-2}d\lambda$ is a measure on R^+. (Gangolli, R. Ann. Math., 93(1971),150-165).

Sperical multiplier. To a function $m \in L^\infty(R^+)$ we associate a map $T_m : C_c(G/K) \to L^2(G/K)$ by

$$T_m f(g) = \int_0^\infty m(\lambda)\phi_\lambda(g)\hat{f}(\lambda)|c(\lambda)|^{-2}d\lambda.$$

($T_m f := m^\vee * f$ can be viewed as a distribution in $S'(G/K)$.)

When $m_z(x) = (1 - \frac{x^2+\rho^2}{R})^z$ we have the Bochner-Riesz multiplier of order z. If $-\mathcal{L} = \int_{\rho^2}^{\infty} \lambda dE_\lambda$ is the spectral resolution of the unique self-adjoint extension of $-\mathcal{L}$ (using the same notation for its extension), then Bochner-Riesz operator can be written as for $\lambda \geq \rho^2$

$$S_R^z = \int_{\rho^2}^{\infty} (1 - \frac{\lambda}{R})_+^z dE_\lambda.$$

We can represent S_R^z by convolution on G: $S_R^z f = f * r_R^z$ with the kernel r_R^z being C^∞ smooth. The spherical transform of this kernel is exactly m_z. In Giulini-Mauceri, it was proved that if $1 \leq p \leq 2$ and if $Rez > \delta(p) := (\frac{1}{p} - \frac{1}{2})(n-1)$, then the maximal operator S_*^z maps $L^p(X)$ continuous into $L^{p+r}(X)$ with r large enough, hence they obtained the almost everywhere convergence for L^p functions.

We are concerned with studying the H^p boundedness of Riesz means for G. There are two ways to address this issue. First we can consider the Riesz mean on the atomic Hardy spaces introduced in T.Kawazoe[2000] for K-invariant functions. The second way is to study the Riesz means for Hardy spaces defined by "Poisson" semigroup $e^{-t\mathcal{L}}$. In this regard, we may refer to the paper on general affine symmetric spaces (which are non-Riemannian) by Hilgert, Olafsson and Orsted [1991].

We mention that in Riemannian case Kawazoe studied in [38] the properties of Poisson kernels related to the Laplacian-Beltrami operator on G/K. The Poisson kernel P is defined as a funciton on $G/K \times K/M$: $P(gK, kM) = \exp(-2\rho(H(g^{-1}k)))$, where G/k is identified with the unit ball in $\mathbb{F}^n$ ($\mathbb{F}^n$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$), and K/M with its boundary, the unit sphere. This suggests that the Hardy classes introduced in the paper by Olafsson et al. are extensions to the non-Riemannian case as they defined the Hardy spaces by Poisson kernels of the above type.

It seems interesing and desirable if we can construct an atomic decomposition for this type of Hardy spaces (which fit naturally with our symmetric spaces). Thereforth we would be able to study in a simpler way the mapping properties of B-R means, multiplier operators for H^p spaces on symmetric spaces.

Examples of semisimple Lie groups of real rank one.
$SL(2,R)$, $SU(n,1)$, $SO(n,1)$, $Sp(n,1)$, the connected Lie group of real type F_4.

Let $X = G/K$. The classification of real rank one noncompact symmetric spaces is known: a. $G = SO_o(n,1)$, $K = SO(n)$, $n \in \mathbb{Z}^+, n \geq 2$
b. $G = SU(n,1)$, $K = S(U(n) \times U(1))$, $n \in \mathbb{Z}^+, n \geq 2$

c. $G = SP(n, 1)$, $K = SP(n) \times SP(1)$, $n \in \mathbb{Z}^+, n \geq 2$
d. $G = F_4(-20)$, $K = \text{spin } 9$.

The first three kinds can be identified as hyperbolic spaces over real, complex or quarternian fields ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$).

Let $X = G/K$ be a Riemannian symmetric space, where G is a connected semisimple Lie group with finite center and K a maximal compact subgroup. In the Lie algebra level, we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with a Cartan involution θ. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Denote by $\mathfrak{a}_C^*$ the complexification of $\mathfrak{a}^*$.

Let $P = MAN$ be a minimal parabolic subgroup of G, where M is the centralizer of A in K. Then the compact $G/P = K/M$ is the maximal boundary in the compactification of G/K.

The spherical function can also be defined via Poisson integral of constant function 1 in $\mathcal{B}(K/M)$ where $\mathcal{B}(K/M)$ is the space of the hyperfunctions on K/M:

$$\phi_\lambda(gK) = \int_K e^{-\langle \lambda+\rho, log(g^{-1}k)\rangle} dk.$$

We have $\lim_{\mathfrak{a}\to\infty} \mathfrak{a}^{-\lambda+\rho}\phi_\lambda(\mathfrak{a}) = c(\lambda)$, Harish-Chandra's c-function.

For elementary spherical functions (K-bi-invariant), there is an explicit formula (G being complex) ([22]):

$$\phi_\lambda(\exp H) = c(i\lambda)\frac{\sum_w (\det w) e^{-w\lambda(H)}}{\prod_{\alpha\in\Sigma_+} \sinh \alpha(H)} \qquad H \in \mathfrak{a} \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*,$$

and

$$c(\lambda) = \pi(\rho)/\pi(i\lambda) \quad \lambda \in \mathfrak{a}^*,$$

where $\pi(z) = \prod_{\alpha\in\Sigma_+} < \alpha, z >$.

In the end of this section we refer to the following paper by S. Ben Saïd, T. Oshima and N.Shimeno, related to our project. "Fatou's theorems and Hardy-type spaces for eigenfunctions of the invariant differential operators on symmetric spaces" in *IMRN*, **16** (2003), 915-931.

3. Riesz means of spherical expansions

For $z \in \mathbb{C}, \Re z \geq 0$, we define the Bochner-Riesz means of order z by

$$S_R^z f(x) = \int_0^\infty (1 - \frac{\lambda^2+\rho^2}{R})_+^z \hat{f}(\lambda)\phi_\lambda(x)|c(\lambda)|^{-2} d\lambda.$$

The Riesz means can be written as $S_R^z f = s_R^z * f$, here s_R^z is a function whose spherical transform is

$$\hat{s}_R^z(\lambda) = (1 - \frac{\lambda^2 + \rho^2}{R})_+^z.$$

Let $\mathcal{F}_c$ be the usual Fourier-cosine transform, $\mathcal{A}$ the Abel transform, then $s_R^z = \mathcal{A}^{-1}\mathcal{F}_c^{-1}(\hat{s}_R^z(\lambda))$.

Let α and 2α be two roots of $(\mathfrak{g}, \mathfrak{a})$ with multiplicities m_α and $m_{2\alpha}$ respectively. Then dimension of G/K is equal to $n = m_\alpha + m_{2\alpha} + 1$. Fix an element H_0 in $\mathfrak{a}$ with $\alpha(H_0) = 1$. Let $G = K\overline{A^+}K$ be Cartan decomposition, where $A^+ = \{\exp(tH_0) : t > 0\}$.

For K-bi-invariant function f, if the spherical transform is given by $\hat{f}(\lambda) = \int_0^\infty f(\exp(tH_0))\phi_{-\lambda}(\exp(tH_0))\triangle(t)dt$ then we have the inversion formula

$$f(t) := f(\exp(tH_0)) = \int_0^\infty \hat{f}(\lambda)\phi_\lambda(t)|c(\lambda)|^{-2}d\lambda,$$

noting that as $\lambda \to \infty$, $|c(\lambda)| \sim \lambda^{-(n-1)/2}$ (when G has rank one).

We need the asymptotic estimates for spherical functions (cf. Stanton-Tomas or Giulini-Mauceri-Meda): Let $\mathcal{J}_\nu(x) = x^{-\nu}J_\nu(x)$, where J_ν is the Bessel function of the first kind and of order ν. We have for $t \leq T_0$ $(T_0 > 1)$

$$\phi_\lambda(\exp(tH_0)) = t^{(n-1)/2}\triangle(t)^{-1/2}(\mathcal{J}_{n/2-1}(\lambda t) + t^2 a(t)\mathcal{J}_{n/2}(\lambda t) + E(\lambda, t)),$$

where a is bounded and smooth, and the error term E verifies

$$|E(\lambda, t)| \leq C \begin{cases} t^4 & \text{if } |\lambda t| \leq 1, \\ t^4(\lambda t)^{-(n+3)/2} & \text{if } |\lambda t| > 1 \end{cases}$$

We also need the estimates for derivatives of $|c(\lambda)|^{-2}$.

$$\frac{d^k}{d\lambda^k}(|c(\lambda)|^{-2}) \leq c_k(1 + |\lambda|)^{n-k-1}.$$

whose estimates are based on the expression

$$c(\lambda) = \Gamma(a/2)\Gamma(b/2)\frac{\Gamma(i\lambda)\Gamma(\frac{a+i\lambda}{2})}{\Gamma(\frac{a}{2} + i\lambda)\Gamma(\frac{\rho+i\lambda}{2})},$$

here $a = m_\alpha$, $b = m_{2\alpha}$ and $\rho = (m_\alpha + 2m_{2\alpha})/2$;

Similarly, we can define the Bochner-Riesz means of order z associated with Jacobi expansion by

$$S_R^z f(x) = \int_0^\infty (1 - \frac{\lambda^2 + \rho^2}{R})_+^z \hat{f}(\lambda)\phi_\lambda^{\alpha,\beta}(x)|c(\lambda)|^{-2}d\lambda.$$

We can write $S_R^z f = s_R^z * f$, where s_R^z is a function whose Jacobi transform is

$$\hat{s}_R^z(\lambda) = (1 - \frac{\lambda^2 + \rho^2}{R})_+^z.$$

We also have the relation between spherical transform and Abel transform via Fourier transform: $\mathcal{F}_c \mathcal{A} s_R^z = \hat{s}_R^z(\lambda)$.

Like in Koornwinder, Jacobi functions and analysis on noncompact semisimple Lie Groups of rank one, in R.Askey et al (eds.), Special functions, D.Reidel Publ. Com., Dordrecht. (1984) and Giulini.S and Mauceri G. 'Almost everywhere convergence of Riesz means on certain noncompact symmetric spaces', Ann. Mat.Pura. Appl.(4)159, (1991). We obtain

Proposition 3.1. *If $\alpha \geq \beta - 1/2$, $\alpha \neq -1/2$, then*

$$|s_R^z(x)| \leq c(z) R^{\alpha+1} (1 + \sqrt{R}|x|)^{-Rez-\alpha-3/2} (\frac{x}{\sinh x})^{\alpha+1/2} (\cosh x)^{\beta+1/2}.$$

Consider the maximal operator

$$S_*^z f(x) = \sup_{R>0} |S_R^z f(x)|, \quad f \in L^p(\Delta(x)dx), 1 \leq p \leq 2.$$

We will use the maximal function Mf:

$$Mf(x) = M_{\alpha,\beta} f(x) = \sup_{r>0} \chi_r * |f|(x).$$

Here $\chi_r = \frac{1}{|B(r)|} \chi_{[-r,r]}$, $\chi_{[-r,r]}$ is the characteristic function of $[-r, r]$.

Note. For certain discrete α, β, Mf has group-theorectical interpretation as the maximal function on symmetric spaces of rank one.

Theorem 3.2. *Let $\alpha \geq \beta \geq -1/2$ and $\alpha \neq -1/2$, the maximal operator M has weak type* $(1, 1)$*,*

$$|\{x : Mf(x) > \lambda\}| \leq C \lambda^{-1} \|f\|_{L^1(\Delta dx)}.$$

Define

$$M_\phi f(x) = \sup_{t>0} \phi(t) \chi_{[-t,t]} * f(x) \tag{1}$$

$$= \sup_{t>0} \phi(t) \int_0^t T_x f(y) \Delta(y) dy. \tag{2}$$

Theorem 3.3. *Let $\alpha \geq \beta - 1/2$, $\alpha \neq -1/2$. i) If $Rez > 0$, then for $f \in L^2(\Delta(x)dx)$,*

$$\|S_*^z f\|_2 \leq c(z) \|f\|_2.$$

ii) If $Rez > \alpha + \frac{1}{2}$, S_*^z is bounded from $L^1(\Delta(x)dx)$ to weak $L^1 + L^q$ for every $q \in [4, \infty]$.

iii) If $Rez > (\frac{2}{p}-1)(\alpha+\frac{1}{2})$ and $1 < p \le 2$, then for every $q \ge \frac{4p}{3p-2}$,

$$\|S_*^z f\|_{p+q} \le c(z)\|f\|_p, \quad f \in L^p(\mathbb{R}^+, \Delta(x)dx).$$

Proof similar to that of Lemma 4.1 in Giulini and Mauceri, 1991. (iii) follows from Stein's complex interpolation.

A direct corollary of the above theorem is the pointwise result.

Theorem 3.4. *Let $1 \le p \le 2$ and $f \in L^p(\mathbb{R}^+, \Delta(x)dx)$. If $Rez > (\frac{2}{p}-1)(\alpha+\frac{1}{2})$, then $\lim_{R\to\infty} S_R^z f(x) = f(x)$ a.e.*

The proof is based on the estimate of the kernel of S_R^z given by

$$S_R^z(x,y) = \int_0^\infty (1 - \frac{\lambda^2+\rho^2}{R})_+^z \phi_\lambda^{\alpha,\beta}(x)\phi_\lambda^{\alpha,\beta}(y)|c(\lambda)|^{-2}d\lambda.$$

and some basic properties in the following lemma concerning measure of geodesic balls in $X = G/K$.

Lemma 3.5. *Define $B(x,r)$ by $B(x,r) = [x-r, x+r]$ if $x - r > 0$; $[0, x+r] := B(x+r)$ if $t - r \neq 0$. Then*

1) If $0 \neg r < \infty, 1 + |B(r)| \sim (\cosh r)^{2\rho}$.

2)

$$|B(r)| \sim r^{2(\alpha+1)} \quad r \le C$$

$$|B(r)| \sim (\cosh r)^{2\rho} \quad r \ge C.$$

3) For $r \le 1$, $x > 2$, $|B(x,r)| \sim r(\cosh x)^{2\rho} \sim re^{2\rho x}$.

4) For $r \le 1$, $x < 2$,

$$|B(x,r)| \sim rx^{2(\alpha+1)} \quad x \ge 2r$$

$$|B(x,r)| \sim |B(r)|, \quad x < 2r.$$

Lemma 3.6. *Let E be a subset of $\mathbb{R}^+$. $\{B(x_k, r_k)\}$ is a covering of E with $r_k \le 1$ for all k. Then there exists a disjoint $\{B(x_i, r_i)\}$ such that*

$$|E| \le C \sum_{i=1}^\infty |B(x_i, r_i)|.$$

Proof of Lemma 3.6 is similar to that of the Vitali covering lemma on space of homogeneous type e.g. [1].

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