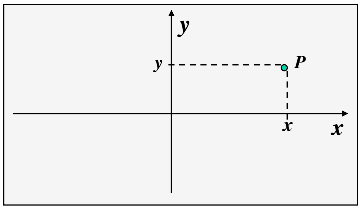
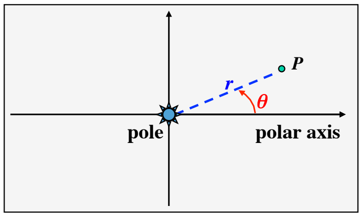
Rectangular Coordinates

$P = (x, y)$



Polar Coordinates

$P = (r, \theta)$



θ = directed angle $\quad r$ = directed distance

Example Plot in polar coordinates:

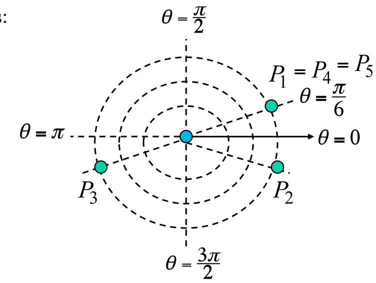
a) $P_1 = \left(3, \frac{\pi}{6}\right)$

b) $P_2 = \left(3, -\frac{\pi}{6}\right)$

c) $P_3 = \left(-3, \frac{\pi}{6}\right)$

d) $P_4 = \left(3, -\frac{11\pi}{6}\right)$

e) $P_5 = \left(-3, \frac{7\pi}{6}\right)$



Every point has infinitely many pole-coordinate representations:

$$(r, \theta) = (r, \theta + 2n\pi) = (-r, \theta + \pi) = (-r, \theta + (2n+1)\pi)$$

e.g. $P_2 = \left(3, \frac{11\pi}{6}\right) = \left(-3, \frac{5\pi}{6}\right) = \left(-3, -\frac{7\pi}{6}\right)$

Coordinate Conversion

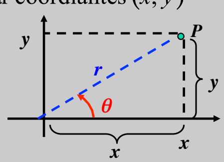
The polar coordinates (r, θ) and the rectangular coordinantes (x, y) are related as follows:

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$



Example Convert to rectangular coordinates:

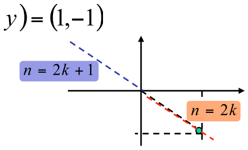
a) $\left(3, \frac{\pi}{6}\right)$ $\quad x = 3\cos\frac{\pi}{6} = \frac{3\sqrt{3}}{2}$ $\quad y = 3\sin\frac{\pi}{6} = \frac{3}{2}$

b) $(5, -\pi)$ $\quad x = 5\cos\pi = -5$ $\quad y = 5\sin\pi = 0$

Example Convert to polar coordinates: $(x, y) = (1, -1)$

$r^2 = 1^2 + (-1)^2 = 2 \qquad r = \pm\sqrt{2}$

$\tan\theta = \frac{-1}{1} = -1 \quad \Rightarrow \quad \theta = -\frac{\pi}{4} + n\pi$



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quadrant IV : $\boxed{-\frac{\pi}{4} + 2k\pi}$

quadrant II : $\boxed{\frac{3\pi}{4} + 2k\pi}$

$\left(-\frac{\pi}{4} + 2k\pi + \pi\right)$

Ans.: $\left(\sqrt{2}, -\frac{\pi}{4} + 2k\pi\right)$ or $\left(-\sqrt{2}, \frac{3\pi}{4} + 2k\pi\right)$

Converting Equations

Example Convert $2x - 3y = 7$ to a polar equation. Solve for r.

$2r\cos\theta - 3r\sin\theta = 7$

$r(2\cos\theta - 3\sin\theta) = 7$

$$\boxed{r = \frac{7}{2\cos\theta - 3\sin\theta}}$$

$x = r\cos\theta$

$y = r\sin\theta$

Example Convert each polar equation to a rectangular equation.

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$

a) $r = 7 \qquad r^2 = 49 \qquad \boxed{x^2 + y^2 = 49}$

b) $\theta = \frac{\pi}{3} \qquad \tan\theta = \tan\frac{\pi}{3} = \sqrt{3}$

$\frac{y}{x} = \sqrt{3} \qquad \boxed{y = \sqrt{3}x}$

c) $\theta = \frac{3\pi}{2} \qquad \tan\theta$ is undefined.

$\frac{y}{x}$ is undefined. $\Leftrightarrow \boxed{x = 0}$

$x = r\cos\theta$

$y = r\sin\theta$

d) $r = 8\sec\theta \qquad r = \frac{8}{\cos\theta} \qquad r\cos\theta = 8 \qquad \boxed{x = 8}$

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e) $r = -2\cos\theta$ (multiply by r)

$r^2 = -2r\cos\theta \qquad \boxed{x^2 + y^2 = -2x}$

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$

$2\sin\theta\cos\theta$

f) $5\sin 2\theta = 1 \qquad 10\sin\theta\cos\theta = 1 \qquad$ (multiply by r^2)

$10\, r\sin\theta\, r\cos\theta = r^2 \qquad \boxed{10xy = x^2 + y^2}$

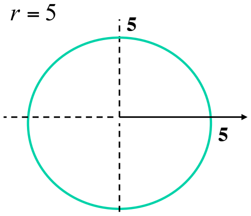
$x = r\cos\theta$

$y = r\sin\theta$

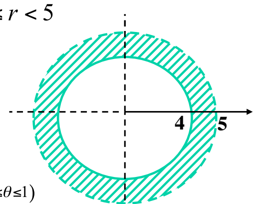
Graphing in Polar Coordinates

Example Graph all points (r,θ) that satisfy the given condition:

a) $r = 5$

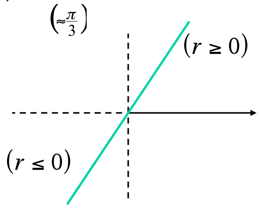


b) $4 \le r < 5$

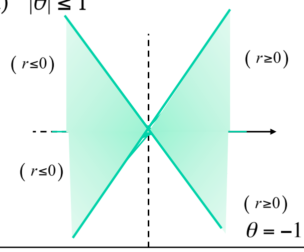


$(-1 \le \theta \le 1)$

c) $\theta = 1$ $\left(\approx \frac{\pi}{3}\right)$



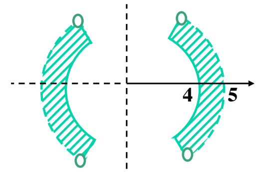
d) $|\theta| \le 1$



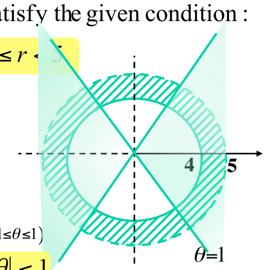
Graphing in Polar Coordinates

Example Graph all points (r,θ) that satisfy the given condition:

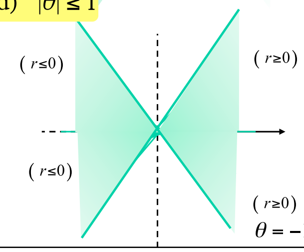
e) $4 \le r < 5$ and $|\theta| \le 1$



b) $4 \le r < 5$



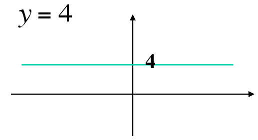
d) $|\theta| \le 1$



f) $r = 4\csc\theta$ $\left(= \dfrac{4}{\sin\theta} \right)$

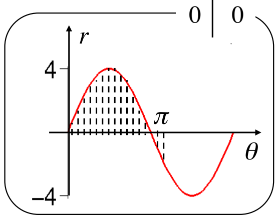
$r\sin\theta = 4$

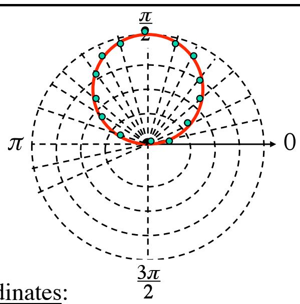
$y = 4$



g) $r = 4\sin\theta$

θ	r
0	0





Or convert to rectangular coordinates:

$r = 4\sin\theta$ $\qquad x^2 + y^2 - 4y = 0$

$r^2 = 4r\sin\theta$ $\qquad x^2 + (y-2)^2 = 4$ ← **Completing the square**

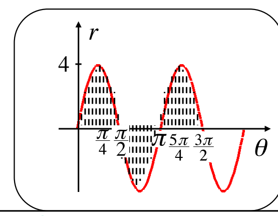
$x^2 + y^2 = 4y$

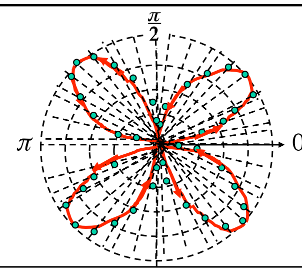
Equation of the circle with radius r

center at (0,0): $x^2 + y^2 = r^2$

center at (h,k): $(x-h)^2 + (y-k)^2 = r^2$

h) $r = 4\sin 2\theta$





Testing for Symmetry

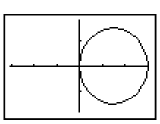
If any of the procedures below results in an equivalent equation, then the corresponding graph exhibits the indicated symmetry.

Replacing	Symmetry w.r.t. the
r by $-r$	pole (the origin)
θ by $-\theta$	polar axis (x-axis)
(r,θ) by $(-r,-\theta)$	line $\theta = \frac{\pi}{2}$ (y-axis)

Note
An equation may fail one of these and the graph may still have the corresponding symmetry!

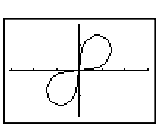
Example Test for symmetry: Equivalent?

a) $r = 3\cos\theta$



			Symmetry?
pole:	$-r = 3\cos\theta$	No.	**Maybe**
polar axis:	$r = 3\cos(-\theta)$	Yes. (cos	**Yes**
line $\theta = \frac{\pi}{2}$:	$-r = 3\cos(-\theta)$	No.	**Maybe**

b) $r^2 = 3\sin 2\theta$



			Symmetry?
pole:	$(-r)^2 = 3\sin 2\theta$	Yes.	**Yes**
polar axis:	$r^2 = 3\sin 2(-\theta)$	No. (si)	**Maybe**
line $\theta = \frac{\pi}{2}$:	$(-r)^2 = 3\sin 2(-\theta)$	No.	**Maybe**

Slope and Tangent Lines

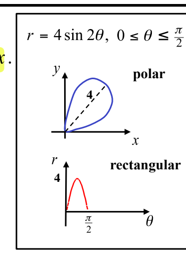
The slope of a line tangent to the polar curve $r = f(\theta)$ is dy/dx.

To find it, parametrize the curve using θ as the parameter:

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

so that

$$m = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$



Example $f(\theta)$

For the polar curve $r = 4\sin 2\theta$, $0 \le \theta \le \frac{\pi}{2}$ find $\qquad f'(\theta) = 8\cos 2\theta$

a) the slopes of the lines tangent to the curve at the indicated points:

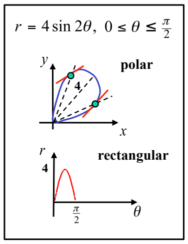
i) $r = 2\sqrt{3}$, ii) the pole $(r = 0)$

b) the points on the curve where the tangent line is: i) horizontal, ii) vertical

$$\frac{dy}{dx}=\frac{\cancel{8}^{\,2}\cos 2\theta\sin\theta+\cancel{4}\sin 2\theta\cos\theta}{\cancel{8}_{\,2}\cos 2\theta\cos\theta-\cancel{4}\sin 2\theta\sin\theta}$$

$$=\frac{\cancel{2}\left(\cos^2\theta-\sin^2\theta\right)\sin\theta+\cancel{2}\sin\theta\cos^2\theta}{\cancel{2}\left(\cos^2\theta-\sin^2\theta\right)\cos\theta-\cancel{2}\sin^2\theta\cos\theta}$$

$$=\frac{\sin\theta}{\cos\theta}\cdot\frac{2\cos^2\theta-\sin^2\theta}{\cos^2\theta-2\sin^2\theta}=\boxed{\tan\theta\cdot\frac{2-\tan^2\theta}{1-2\tan^2\theta}=m(\theta)}$$



a)

i) $r=4\sin 2\theta=2\sqrt{3}\;\Leftrightarrow\;\sin 2\theta=\sqrt{3}/2\;\Leftrightarrow\;2\theta=\frac{\pi}{3},\frac{2\pi}{3}\;\Leftrightarrow\;\theta=\frac{\pi}{6},\frac{\pi}{3}$ **(two points)**

$$m\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{3}\cdot\underset{\frac{6-1}{3-2}}{\boxed{\frac{2-\frac{1}{3}}{1-\frac{2}{3}}}}=\frac{5\sqrt{3}}{3}\qquad m\left(\frac{\pi}{3}\right)=\sqrt{3}\cdot\underset{\frac{-1}{-5}}{\boxed{\frac{2-3}{1-6}}}=\frac{\sqrt{3}}{5}$$

ii) $r=4\sin 2\theta=0\;\Leftrightarrow\;\theta=0,\frac{\pi}{2}$ **(one point visited twice)** **(two tangent lines)**

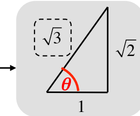
$m(0)=0$ **(horizontal tangent)** $m\left(\frac{\pi}{2}\right)=$ undefined **(vertical tangent)**

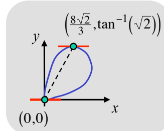
b) the points on the curve where the **tangent line is**: $\boxed{m(\theta)=\tan\theta\,\frac{2-\tan^2\theta}{1-2\tan^2\theta}\quad 0\le\theta\le\frac{\pi}{2}}$

i) **horizontal**

$m=0$ iff $\tan\theta=0$ or $2-\tan^2\theta=0$

$\tan\theta=\sqrt{2}$





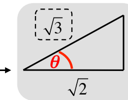
$\boxed{\theta=0}$ or $\boxed{\theta=\tan^{-1}\left(\sqrt{2}\right)}$

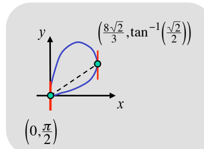
$\boxed{r=0}$ $\boxed{r=8\frac{\sqrt{2}}{\sqrt{3}}\frac{1}{\sqrt{3}}=\frac{8\sqrt{2}}{3}}$ $\leftarrow$ $\boxed{r=4\sin 2\theta}$ $=8\sin\theta\cos\theta$

ii) **vertical**

m is undefined iff $\tan\theta$ is undefined or $1-2\tan^2\theta=0$

$\tan\theta=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\longrightarrow$





$\boxed{\theta=\frac{\pi}{2},}$ or $\boxed{\theta=\tan^{-1}\left(\frac{\sqrt{2}}{2}\right)}$

$\boxed{r=0}$ $\boxed{r=8\frac{1}{\sqrt{3}}\frac{\sqrt{2}}{\sqrt{3}}=\frac{8\sqrt{2}}{3}}$

Tangents at the Pole

Suppose the curve $r=f(\theta)$ passes through the pole when $\theta=\theta_0$; i.e. $f(\theta_0)=0.$

Then
$$m(\theta_0)=\frac{f'(\theta_0)\sin\theta_0+f(\theta_0)\cos\theta_0}{f'(\theta_0)\cos\theta_0-f(\theta_0)\sin\theta_0}=\tan\theta_0\qquad(\text{if } f'(\theta_0)\neq 0)$$

Therefore the polar line $\theta=\theta_0$ is tangent to the curve at the point $(r,\theta)=(0,\theta_0)$.

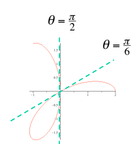
$$\left(\begin{array}{ll}\text{cartesian:} & y=mx\quad\text{where } m=\tan\theta_0,\ \text{if }\tan\theta_0\text{ is defined}\\ \text{or} & x=0\qquad\qquad\qquad\quad,\ \text{otherwise}\end{array}\right)$$

Example

Find all the lines tangent to the polar curve $r=f(\theta)=2\cos 3\theta,\ 0\le\theta\le 2\pi$ at the pole.

Hint: Find the zeros of $f(\theta)$ that are not zeros of $f'(\theta)$.

$f(\theta)=2\cos 3\theta=0$ iff $\theta=\frac{\pi}{6},\frac{\pi}{2}$ (not zeros of $f'(\theta)=-6\sin 3\theta$)



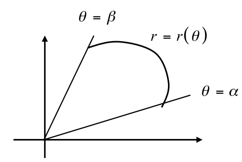
The tangent lines in polar form: $\theta=\frac{\pi}{6},\ \theta=\frac{\pi}{2}$

(cartesian form: $y=\frac{\sqrt{3}}{3}x,\quad x=0$)

Areas and Lengths in Polar Coordinates

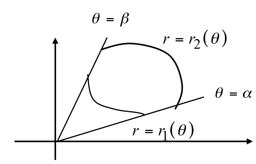
The area of the region bounded by the polar curve $r=r(\theta)\ge 0$ and the rays $\theta=\alpha,\ \theta=\beta$ can be calculated by the formula

$$\boxed{A=\frac{1}{2}\int_{\alpha}^{\beta}r(\theta)^2\,d\theta}$$



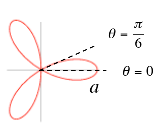
The area of the region bounded by the polar curves $r=r_2(\theta),\ r=r_1(\theta)$, where $r_2(\theta)\ge r_1(\theta)\ge 0$, and the rays $\theta=\alpha,\ \theta=\beta$ can be calculated by the formula

$$\boxed{A=\frac{1}{2}\int_{\alpha}^{\beta}\left[r_2(\theta)^2-r_1(\theta)^2\right]d\theta}$$



Example

Find the area inclosed in one loop of the polar curve $r=a\cos 3\theta\ (a>0)$.



Let A_1 be the area of the top half of the right-hand loop. Then

$$A_1=\frac{1}{2}\int_0^{\pi/6}(a\cos 3\theta)^2\,d\theta=\frac{a^2}{2}\int_0^{\pi/6}\cos^2 3\theta\,d\theta=\frac{a^2}{4}\left(\theta+\tfrac{1}{6}\sin 6\theta\right)\Big|_0^{\pi/6}=\frac{\pi a^2}{24}$$

$$\int\cos^2 3\theta\,d\theta=\frac{1}{2}\int(1+\cos 6\theta)\,d\theta=\tfrac{1}{2}\left(\theta+\tfrac{1}{6}\sin 6\theta\right)$$

And so the area inclosed in one loop is

$$A=2A_1=\boxed{\frac{\pi a^2}{12}}$$