Review Final Exam **Name**
Math 2331 **Id**

Read carefully each problem. Show all your work in order to justify and support your answer. Credits will be given mainly depending on your work, not just an answer. Put a box around the final answer to a question. Use the back of the page if necessary.

(1) Which of the following equations are linear?
a) $xy + 3y = 1$
b) $3x - y + z = 9w$
c) $x\cos 15° + (2 - y)\sin 15° = \sqrt{17}$
d) $5e^x - 11e^y = 0$
e) $e^5 x - e^{11} y = 0$
f) $(x + y)(x - y) = -3$

(2) Write down the coefficient and the augmented matrices for the linear system.
a)

$$\begin{cases} 2x_1 & +x_2 & +x_3 & +2x_4 = 0 \\ x_1 & -x_2 & & +5x_4 = 3 \\ x_1 & -5x_2 & +x_3 & -x_4 = -2 \end{cases}$$

b) Solve the system of equations.

(3) Write the system of linear equations in the form $A\mathbf{x} = \mathbf{b}$ and solve the matrix equation for $\mathbf{x} = [\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}]$ using either Gaussian elimination with back-substitution or Gauss-Jordan elimination.

$$\begin{cases} 3x_1 & & +12x_3 = -6 \\ -9x_1 & & -35x_3 = 2 \\ 18x_1 & +x_2 & +70x_3 = 8 \end{cases}$$

(4) a) Write the coefficient matrix A of the system in (**3**)
b) Find reduced row-echelon form for A using elementary row reductions. Record the elementary matrices $E_1, E_2, \dots, E_k$ corresponding to these elementary row reductions.
c) Find the inverses $E_1^{-1}, E_2^{-1}, \dots, E_k^{-1}$.
(Hint: There are only three kinds of elementary matrices, each of their inverses is easily known; check the Notes or Text if you are not sure)
d) Express A as a product of elementary matrices.
(Hint: Because $A^{-1} = E_k E_{k-1} \dots E_2 E_1$, we have $A = E_1^{-1} E_2^{-1} \dots E_k^{-1}$)

(5) Find the inverse of the matrix (if it exists).

(a) $\begin{pmatrix} 10 & -5 \\ 5 & -3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & -2 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -2 & -1 \\ 3 & -5 & -2 \\ 2 & -5 & -2 \end{pmatrix}$

(6) Solve the inhomogeneous system

$$\begin{cases} x & & -2z = 3 \\ x & -3y & +z = -6 \\ & y & -z = 3 \end{cases}$$

(7) Compute the determinants.

$$(a) \quad \begin{vmatrix} 1 & 1 & 2 \\ 4 & 5 & 6 \\ 2 & 3 & -1 \end{vmatrix} \qquad (b) \quad \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \lambda & 0 \\ 0 & \lambda & 1 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix}$$

(8) [optional*] Find the adjoint $\mathbf{ad(M)}$ of the matrix $M = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 3 & 1 \\ 3 & 1 & -1 \end{pmatrix}$.

Verify that $M\mathbf{ad}(M) = \mathbf{ad}(M)M = \det(M)I_3$.

(9) (i) Which of the following sets of vectors $x = [x_1, x_2, x_3, x_4]^T$ are subspace of $\mathbf{R}^4$?

a) All x such that $x_1 + x_2 = 7x_3$

b) All x such that $x_3 = 0$

c) All x such that $x_1 + x_4 = -12$

d) All x such that each x_i component is positive, that is, the first "I-quadrant" set $= \{x_i \geq 0, i = 1, 2, 3, 4\}$.

(ii) We know that $P_3 = \{f : f \text{ is a polynomial of degree} \leq 3\}$ is a vector space. Which set of functions satisfying the following properties constitutes a subspace of P_3 ?

a) $f(-x) = f(x)$ b) $f(0) + f(1) = 5$ c) $f'(0) = 0$

(10) Which of the following vectors, if any, is in the null space of $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{pmatrix}$?

a) $[-1\ 0\ 1\ 0]^T$ b) $[0\ 2\ 1\ -1]^T$ c) $[0\ 4\ 2\ -2]^T$

(11) Write each vector as a linear combination of the vectors in S if possible. Let $S = \{(1, 2, -2), (2, -1, 1)\}$.

(a) $\mathbf{u} = (-4, -3, 3)$

(b) $\mathbf{w} = (1, -5, -5)$ [From 4.4, #2]

(12) [Testing for liner independence] Determine whether the following set S of vectors is linearly independent or linearly dependent?

(a) $S = \{(-2, 2), (3, 5)\}$ in $\mathbf{R}^2$ (4.4, #29)

(b) $S = \{(-4, -3, 4), (1, -2, 3), (6, 0, 0)\}$ in $\mathbf{R}^3$ (4.4, #37)

(c) $S = \{9, x^2, x^2 + 1\}$ in P_2.

(13) [True or False]

(a) A set of vectors $S = \{v_1, v_2, \ldots, v_k\}$ in a vector space is called linearly dependent when the vector equation $c_1v_2 + c_2v_2 + \cdots + c_kv_k = 0$ has only the trivial solution.
(4.4, #59 (a))

(b) The set $S = \{(1, 0, 0, 0), (0, -2, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ spans $\mathbf{R}^4$.

(c) A set $S = \{v_1, v_2, \ldots, v_k\}$, $k \geq 2$, is linearly independent if and only if at least one of the vectors v_j can be written as a linear combination of the other vectors.
(4.4, #60 (a))

(d) If a subset S spans a vector space V, then every vector in V can be written as a linear combination of the vectors in S.

(14) Determine which of the following statements are equivalent to the fact that a matrix A of size $n \times n$ is invertible?

a) A is nonsingular
b) The row space of A has dimension n
c) The column space of A has dimension n
d) The determinant of A is nonzero
e) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any given $\mathbf{b}$ in $\mathbf{R}^n$
f) The system $A\mathbf{x} = \mathbf{0}$ has nonzero solution
g) The dimension of the null space of A is zero
h) The rows of A are linear independent
i) The columns of A are linear independent
j) The rank of A is n
k) A is row-equivalent to an identity matrix
l) All eigenvalues of A are nonzero
m) A has n linear independent eigenvectors
n) A is similar to an diagonal matrix
o) A can be written as the product of elementary matrices.

(15) * The matrix $A = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$ row reduces to $C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

a) Find the rank and nullity of A.
b) Find a basis of the row space and the column space of A respectively.
c) Find a basis of the null space of A
d) Does the system $A\mathbf{x} = \begin{pmatrix} 109 \\ -217 \\ 66 \end{pmatrix}$ have a solution? (Hint: You can draw a conclusion from the fact that dimension of column space is 3, without having to solve the system. Recall that $rank(A) = dim(Col(A)) = dim(Row(A))$)
e) What is the relation between $rank, dim(null(A))$?(Hint: The theorem states that $rank(A)+dim(null(A)) = n$, the number of columns)

(16) Find all the eigenvalues of the given matrix.

a) $\begin{pmatrix} 1 & -2 & 0 \\ -3 & 1 & 0 \\ -4 & -5 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(17) The matrix $A = \begin{pmatrix} 6 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{pmatrix}$ has eigenvalues 5 and 8.

a) Find the eigenspaces E_5 and E_8 by solving $(\lambda I - A)\mathbf{x} = \mathbf{0}$.
b*) By the theorem in Section 7.3 we know that a symmetric matrix of size n by n is always diagonalizable, equivalently speaking, always has n linear independent eigenvectors. Find an ordered basis consisting of eigenvectors of A.
c*) Specify the matrices P and D in the diagonalization $P^{-1}AP = D$
d*) Find an orthogonal matrix U such that $U^{-1}AU = D$ (Hint: An (real) orthogonal matrix means $U^{-1} = U^T$ or equivalently $U^TU = UU^T = I_n$).

(18) a) Give three distinct examples of elementary matrices and explain how they correspond to row operations for a given matrix of 3 by 3.
b) Factor the matrix into a product of elementary matrices.
$A = \begin{pmatrix} 1 & 6 \\ 6 & 1 \end{pmatrix}$

Solutions

(3) $[62, 12, -16]$

(4) a) $\begin{pmatrix} 3 & 0 & 12 \\ -9 & 0 & -35 \\ 18 & 1 & 70 \end{pmatrix}$

b) The factorization or decomposition of a matrix is not unique in general. Here is an example of the answer. $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$,

$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $E_5 = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

c) In consistent with the answer in (b), we have $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$,

$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_3^{-1} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$,

$E_5^{-1} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(5) (b) not invertible; (c) Form the matrix $[A|I_3] = \left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 3 & -5 & -2 & 0 & 1 & 0 \\ 2 & -5 & -2 & 0 & 0 & 1 \end{array}\right)$

Then use row operation to reduce to $[I_3|B]$. Hence the inverse equals

$B = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & -1 \\ -5 & 1 & 1 \end{pmatrix}$

6) From **5**(b) we know that the coefficient matrix is not invertible, so either the system has no solution or it has infinitely manly solutions. Row reductions for the augmented matrix $\left(\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 1 & -3 & 1 & -6 \\ 0 & 1 & -1 & 3 \end{array}\right)$ give

$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$ Back-substitution shows $x = 3+2t, y = 3+t, z = t$, where t is a parameter of any real values. By the way observe that the augmented matrix is consistent with the coefficient matrix because they have the SAME RANK.

7) (a) -3 (b) $15(\lambda^2-1)$

8*) $\mathbf{ad}(M) = \text{transpose of} \begin{pmatrix} -4 & 3 & -9 \\ 3 & -5 & 4 \\ -5 & 1 & -3 \end{pmatrix}$

A straight forward computation shows $M\mathbf{ad}(M) = -11I_3$.

(9) (i) (a), (b)
(ii) (a), (c).

(10) (b), (c)

(13) (a) False. (b) True. (c) False (d) True.

(14) (a), (b), (c), (d), (e), (g), (h), (i), (j), (k), (ℓ), (o)

(15) a) $rank(A) = 3$ (number of leading 1's in C), nullity of $A = 1$

b) A basis of Row(A) consists of $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

A basis of Col(A) consists of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$

d) Yes.

(16) a) $|\lambda I - A| = (\lambda - 1)(\lambda^2 - 2\lambda - 5)$

(17) (a)&(b) $E_5 = \text{span}\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\}$, $E_8 = \text{span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}$

(c*) $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}$

(d*) The orthogonal matrix U consists of three eigenvectors that are orthogonal in $\mathbf{R}^3$. So we need to orthogonalise the base $u = [-1, 1, 0]^T$, $v = [-1, 0, 1]^T$, $w = [1, 1, 1]$. Since the third vector in E_8 is orthogonal the any vectors in E_5. We only need to orthogonalise the two vectors in E_5 by Grant-Schmidt method. Let

$$u = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \tilde{v} = v - \frac{\langle v, u \rangle}{\langle u, u \rangle} u = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix},$$

Normalise the three vectors $u, \tilde{v}, w$ to obtain

$$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

(18) (a) In the following E_1 (namely, multiplying E_1 on the left on a given matrix) corresponds to the row operation: adding a double of the second row to the third row. E_2: switch the first and the second rows. E_3: multiplying 5 on the second row.

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) Clue: Use a finite number of row operations to reduce $A \to I_2$: $E_k E_{k-1} \cdots E_2 E_1 A = I_2$. Then $A = E_1^{-1} \cdots E_k^{-1}$. (Note: Such E_i may not be unique)