Review Final Exam **Name**
Math 2331 **Id**

Read carefully each problem. Show all your work. Credits will be given mainly depending on your work, not just an answer. Put a box around the final answer to a question. Use the back of the page if necessary.

(1) *Which of the following equations are linear?*
a) $xy + 3y = 1$
b) $3x - y + z = 9w$
c) $x \cos 15° + (2 - y)\sin 15° = \sqrt{17}$
d) $5e^x - 11e^y = 0$
e) $e^5 x - e^{11} y = 0$
f) $(x + y)(x - y) = -3$

(2) *Write down the coefficient and the augmented matrices for the linear system.*
a)
$$\begin{cases} 2x_1 & +x_2 & +x_3 & +2x_4 = 0 \\ x_1 & -x_2 & & +5x_4 = 3 \\ x_1 & -5x_2 & +x_3 & -x_4 = -2 \end{cases}$$
b) Solve the system of equations.

(3) *Write the system of linear equations in the form* $A\mathbf{x} = \mathbf{b}$ *and solve the matrix equation for* $\mathbf{x} = [\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}]$ *using either Gaussian elimination with back-substiution.*
a)
$$\begin{cases} 3x_1 & & +12x_3 = -6 \\ -9x_1 & & -35x_3 = 2 \\ 18x_1 & +x_2 & +70x_3 = 8 \end{cases}$$
b)
$$\begin{cases} 3x_1 & & +12x_3 = -3 \\ -9x_1 & & -35x_3 = 1 \\ 18x_1 & +x_2 & +70x_3 = 4 \end{cases}$$

(4) *a) Write the coefficient matrix* A *of the system in* **3***(a)*
b) Find reduced row-echelon form for A *using elementary row reductions. Record the elementary matrices* $E_1, E_2, \ldots, E_k$ *corresponding to these elementary row reductions.*
c) Find the inverses $E_1^{-1}, E_2^{-1}, \ldots, E_k^{-1}$.
(Hint: There are only three kinds of elementary matrices, each of their inverses is easily known; check the Notes or Text if you are not sure)

d) Express A as a product of elementary matrices.
(Hint: Because $A^{-1} = E_k E_{k-1} \dots E_2 E_1$, we have $A = E_1^{-1} E_2^{-1} \dots E_k^{-1}$)

(5) *Find the inverse of the matrix (if it exists).*

a) $\begin{pmatrix} 10 & -5 \\ 5 & -3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & -2 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & -2 & -1 \\ 3 & -5 & -2 \\ 2 & -5 & -2 \end{pmatrix}$

(6) *Solve the inhomogeneous system*

$$\begin{cases} x & & -2z = 3 \\ x & -3y & +z = -6 \\ & y & -z = 3 \end{cases}$$

(7) *Compute the determinants.*

a)

$$\begin{vmatrix} 1 & 1 & 2 \\ 4 & 5 & 6 \\ 2 & 3 & -1 \end{vmatrix}$$

b)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \lambda & 0 \\ 0 & \lambda & 1 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix}$$

(8) *Find the adjoint* $\mathbf{ad}(\mathbf{M})$ *of the matrix* $M = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 3 & 1 \\ 3 & 1 & -1 \end{pmatrix}$.

Verify that $M\mathbf{ad}(M) = \mathbf{ad}(M)M = \det(M)I_3$.

(9) *(i) Which of the following sets of vectors* $x = [x_1, x_2, x_3, x_4]^T$ *are subspace of* $\mathbf{R}^4$*?*

a) All x such that $x_1 + x_2 = 7x_3$

b) All x such that $x_3 = 0$

c) All x such that $x_1 + x_4 = -12$

d) All x such that each x_i component is positive, that is, the first "I-quadrant" set $= \{x_i \geq 0, i = 1, 2, 3, 4\}$.

(ii) We know that $P_3 = \{f : f \text{ is a polynomial of degree} \leq 3\}$ *is a vector space. Which set of functions satisfying the following properties constitutes a subspace of P_3 ?*

a) $f(-x) = f(x)$ *b)* $f(0) + f(1) = 5$ *c)* $f'(0) = 0$

(10) *Which of the following vectors, if any, is in the null space of* $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{pmatrix}$*?*
a) $[-1\,0\,1\,0]^T$ *b)* $[0\,2\,1\,-1]^T$ *c)* $[0\,4\,2\,-2]^T$

(11) *Determine which of the following statements are equivalent to the fact that a matrix A of size $n \times n$ is invertible?*
a) A is nonsingular
b) The row space of A has dimension n
c) The column space of A has dimension n
d) The determinant of A is nonzero
e) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any given $\mathbf{b}$ in $\mathbf{R}^n$
f) The system $A\mathbf{x} = \mathbf{0}$ has nonzero solution
g) The dimension of the null space of A is zero
h) The rows of A are linear independent
i) The columns of A are linear independent
j) The rank of A is n
k) A is row-equivalent to an identity matrix
l) All eigenvalues of A are nonzero
m) A has n linear independent eigenvectors
n) A is similar to an diagonal matrix
o) A can be written as the product of elementary matrices.

(12) *The matrix* $A = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$ *row reduces to* $C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.
a) Find the rank and nullity of A.
b) Find a basis of the row space and the column space of A respectively.
c) Find a basis of the null space of A
d) Does the system $A\mathbf{x} = \begin{pmatrix} 109 \\ -217 \\ 66 \end{pmatrix}$ *have a solution? (Hint: You can draw a conclusion from the fact that dimension of column space is 3, without having to solve the system. Recall that* $rank(A) = dim(Col(A)) = dim(Row(A))$*)*
e) What is the relation between $rank, dim(null(A))$ *?(Hint: The theorem states that* $rank(A) + dim(null(A)) = n$*, the number of columns)*

(13) *Find all the eigenvalues of the given matrix.*

a) $\begin{pmatrix} 1 & -2 & 0 \\ -3 & 1 & 0 \\ -4 & -5 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(14) *The matrix* $A = \begin{pmatrix} 6 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{pmatrix}$ *has eigenvalues 5 and 8.*

a) Find the eigenspaces E_5 *and* E_8 *by solving* $(\lambda I - A)x = 0$.
b) By the theorem in Section 7.3 we know that a symmetric matrix of size n by n is always diagonalizable, equivalently speaking, always has n linear independent eigenvectors. Find an ordered basis consisting of eigenvectors of A.
c) Specify the matrices P *and* D *in the diagonalization* $P^{-1}AP = D$
d) (optional) Find an orthogonal matrix* U *such that* $U^{-1}AU = D$ *(Hint: An (real) orthogonal matrix means* $U^{-1} = U^T$ *or equivalently* $U^TU = UU^T = I_n$*).*

(15) *a) Give three distinct examples of elementary matrices and explain how they correspond to row operations for a given matrix of 3 by 3.*
b) Factor the matrix into a product of elementary matrices.
$A = \begin{pmatrix} 1 & 6 \\ 6 & 1 \end{pmatrix}$

Solutions

(3) a) $[62, 12, -16]$ *b)* $[31, 6, -8]$

(4) a) $\begin{pmatrix} 3 & 0 & 12 \\ -9 & 0 & -35 \\ 18 & 1 & 70 \end{pmatrix}$

b) The factorization or decomposition of a matrix is not unique in general. Here is an example of the answer. $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$,

$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $E_5 = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

c) In consistent with the answer in (b), we have $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$,

$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_3^{-1} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$,

$E_5^{-1} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(5) (b) not invertible; (c) Form the matrix $[A|I_3] = \begin{pmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 3 & -5 & -2 & | & 0 & 1 & 0 \\ 2 & -5 & -2 & | & 0 & 0 & 1 \end{pmatrix}$

Then use row operation to reduce to $[I_3|B]$. *Hence the inverse equals*

$B = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & -1 \\ -5 & 1 & 1 \end{pmatrix}$

6) From **5***(b) we know that the coefficient matrix is not invertible, so either the system has no solution or it has infinitely manly solutions. Row reductions for the augmented matrix* $\begin{pmatrix} 1 & 0 & -2 & | & 3 \\ 1 & -3 & 1 & | & -6 \\ 0 & 1 & -1 & | & 3 \end{pmatrix}$ *give*

$\begin{pmatrix} 1 & 0 & -2 & | & 3 \\ 0 & 1 & -1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$ *Back-substitution shows* $x = 3+2t, y = 3+t, z = t$, *where* t *is a parameter of any real values. By the way observe that the augmented matrix is consistent with the coefficient matrix because they have the SAME RANK.*

7) (a) -3 *(b)* $15(\lambda^2 - 1)$

8) $\mathbf{ad}(M) =$ *transpose of* $\begin{pmatrix} -4 & 3 & -9 \\ 3 & -5 & 4 \\ -5 & 1 & 3 \end{pmatrix}$

A straight forward computation shows $M\mathbf{ad}(M) = -11I_3$.

(12) a) rank$(A) = 3$ *(number of leading 1's in C), nullity of* $A = 1$

b) A basis of Row(A) consists of $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

A basis of Col(A) consists of $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\begin{pmatrix}2\\1\\1\end{pmatrix}$

c) $\begin{pmatrix}1\\1\\-1\\0\end{pmatrix}$

d) Yes.

(13) a) $|\lambda I - A| = (\lambda - 1)(\lambda^2 - 2\lambda - 5)$

(14) a) $E_5 = span\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\}$, $E_8 = span\{\begin{pmatrix}1\\1\\1\end{pmatrix}\}$

c) $P = \begin{pmatrix}-1 & -1 & 1\\1 & 0 & 1\\0 & 1 & 1\end{pmatrix}$, $D = \begin{pmatrix}5 & 0 & 0\\0 & 5 & 0\\0 & 0 & 8\end{pmatrix}$