

§4.1 Extreme Values of Functions on Closed Intervals (Continued)

**Definition.** An interior point of the domain of a function  $f$  where  $f'$  is zero or undefined is a **critical point** of  $f$ .

Ex. Find all critical points of  $f(x) = x^3 - 3x^2 + 2$ .

*Solution.*  $f'(x) = 3x^2 - 6x$ . To find the critical point(s), we solve  $3x^2 - 6x = 0 \Rightarrow x = 0$  and  $x = 2$ . Therefore,  $x = 0, 2$  are the critical points of the function  $y = f(x)$  on  $\mathbb{R} = (-\infty, \infty)$ .  $\square$

**Exercise 52.** Determine all critical points for  $g(x) = \sqrt{2x - x^2}$ .

**Example 3.** Find the absolute maximum and minimum values of  $f(x) = 10x(2 - \ln x)$  on the interval  $[1, e^2]$ .

**Example 4.** Find the absolute maximum and minimum values of  $f(x) = x^{2/3}$  on the interval  $[-2, 3]$ .

§4.2 The Mean Value Theorem

**Theorem 3 (Rolle's Theorem).** Suppose that  $y = f(x)$  is continuous over the closed interval  $[a, b]$  and differentiable at every point of its interior  $(a, b)$ . If  $f(a) = f(b)$ , then there is at least one number  $c$  in  $(a, b)$  at which  $f'(c) = 0$ .

**Theorem 4 (The Mean Value Theorem).** Suppose  $y = f(x)$  is continuous over a closed interval  $[a, b]$  and differentiable on the interval's interior  $(a, b)$ . Then there is at least one point  $c$  in  $(a, b)$  at which  $\frac{f(b) - f(a)}{b - a} = f'(c)$ .

**Corollary 1.** If  $f'(x) = 0$  at each point  $x$  of an open interval  $(a, b)$ , then  $f(x) = C$  for all  $x$  in  $(a, b)$ , where  $C$  is a constant.

**Corollary 2.** If  $f'(x) = g'(x)$  at each point  $x$  in an open interval  $(a, b)$ , then there exists a constant  $C$  such that  $f(x) = g(x) + C$  for all  $x$  in  $(a, b)$ . That is,  $f - g$  is a constant function on  $(a, b)$ .

§4.3 Monotonic Functions and the First Derivative Test

**Definition.** Let  $f$  be a function defined on an interval  $I$  and let  $x_1$  and  $x_2$  be two distinct points in  $I$ .

1. If  $f(x_2) > f(x_1)$  whenever  $x_1 < x_2$ , then  $f$  is said to be **increasing** on  $I$ .
2. If  $f(x_2) < f(x_1)$  whenever  $x_1 < x_2$ , then  $f$  is said to be **decreasing** on  $I$ .

**Corollary 3.** Suppose that  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . If  $f'(x) > 0$  at each point  $x$  in  $(a, b)$ , then  $f$  is increasing on  $[a, b]$ . If  $f'(x) < 0$  at each point  $x$  in  $(a, b)$ , then  $f$  is decreasing on  $[a, b]$ .

**Example 1.** Find the critical points of  $f(x) = x^3 - 12x - 5$  and identify the open intervals on which  $f$  is increasing and on which  $f$  is decreasing.

**Example 2.** Find the critical points of  $f(x) = x^{1/3}(x - 4)$ . Identify the open intervals on which  $f$  is increasing and decreasing. Find the function's local and absolute extreme values.

**Example 3.** Find the critical points of  $f(x) = (x^2 - 3)e^x$ . Identify the open intervals on which  $f$  is increasing and decreasing. Find the function's local and absolute extreme values.

#### §4.4 Concavity and Curve Sketching

**Definition.** The graph of a differentiable function  $y = f(x)$  is

- (a) **concave up** on an open interval  $I$  if  $f'$  is increasing on  $I$ ;
- (b) **concave down** on an open interval  $I$  if  $f'$  is decreasing on  $I$ .

**The Second Derivative Test for Concavity.** Let  $y = f(x)$  be twice-differentiable on an interval  $I$ . If  $f'' > 0$  on  $I$ , the graph of  $f$  over  $I$  is concave up. If  $f'' < 0$  on  $I$ , the graph of  $f$  over  $I$  is concave down.

**Example 1.** (a) The curve  $y = x^3$  is concave down on  $(-\infty, 0)$  and concave up on  $(0, \infty)$ .  
(b) The curve  $y = x^2$  is concave up on  $(-\infty, \infty)$ .

**Example 2.** Determine the concavity of  $y = 3 + \sin x$  on  $[0, 2\pi]$ .

**Definition.** A point  $(c, f(c))$  where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

**Example 3.** Determine the concavity and find the inflection points of the function  $f(x) = x^3 - 3x^2 + 2$ . See also Ex. in §4.1.

**Example 4.** Determine the concavity and find the inflection points of  $f(x) = x^{5/3}$ .

**Theorem 5 (Second Derivative Test for Local Extrema).** Suppose  $f''$  is continuous on an open interval that contains  $x = c$ .

1. If  $f'(c) = 0$  and  $f''(c) < 0$ , then  $f$  has a local maximum at  $x = c$ .
2. If  $f'(c) = 0$  and  $f''(c) > 0$ , then  $f$  has a local minimum at  $x = c$ .
3. If  $f'(c) = 0$  and  $f''(c) = 0$ , then the test fails. The function  $f$  may have a local maximum, local minimum, or neither.

**Example 8.** Sketch a graph of the function  $f(x) = x^4 - 4x^3 + 10$ .