

**Review Final Exam**  
**Math 2331**

**Name**  
**Id**

*Read carefully each problem. Show all your work in order to justify and support your answer. Credits will be given mainly depending on your work, not just an answer. Put a box around the final answer to a question. Use the back of the page if necessary.*

- (1) Which of the following equations are linear?
- a)  $xy + 3y = 1$
  - b)  $3x - y + z = 9w$
  - c)  $x \cos 15^\circ + (2 - y) \sin 15^\circ = \sqrt{17}$
  - d)  $5e^x - 11e^y = 0$
  - e)  $e^5x - e^{11}y = 0$
  - f)  $(x + y)(x - y) = -3$
- (2) Write down the coefficient and the augmented matrices for the linear system.

a)

$$\begin{cases} 2x_1 & +x_2 & +x_3 & +2x_4 = 0 \\ x_1 & -x_2 & & +5x_4 = 3 \\ x_1 & -5x_2 & +x_3 & -x_4 = -2 \end{cases}$$

b) Solve the system of equations.

- (3) Write the system of linear equations in the form  $A\mathbf{x} = \mathbf{b}$  and solve the matrix equation for  $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$  using either Gaussian elimination with back-substitution or Gauss-Jordan elimination.

$$\begin{cases} 3x_1 & & +12x_3 = -6 \\ -9x_1 & & -35x_3 = 2 \\ 18x_1 & +x_2 & +70x_3 = 8 \end{cases}$$

- (4) a) Write the coefficient matrix  $A$  of the system in (3)  
b) Find reduced row-echelon form for  $A$  using elementary row reductions. Record the elementary matrices  $E_1, E_2, \dots, E_k$  corresponding to these elementary row reductions.  
c) Find the inverses  $E_1^{-1}, E_2^{-1}, \dots, E_k^{-1}$ .  
(Hint: There are only three kinds of elementary matrices, each of their inverses is easily known; check the Notes or Text if you are not sure)  
d) Express  $A$  as a product of elementary matrices.  
(Hint: Because  $A^{-1} = E_k E_{k-1} \dots E_2 E_1$ , we have  $A = E_1^{-1} E_2^{-1} \dots E_k^{-1}$ )
- (5) Find the inverse of the matrix (if it exists).

$$(a) \begin{pmatrix} 10 & -5 \\ 5 & -3 \end{pmatrix}$$

$$(b) \begin{pmatrix} 1 & 0 & -2 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad (c) \begin{pmatrix} 1 & -2 & -1 \\ 3 & -5 & -2 \\ 2 & -5 & -2 \end{pmatrix}$$

(6) Solve the inhomogeneous system

$$\begin{cases} x & -2z = 3 \\ x & -3y & +z = -6 \\ & y & -z = 3 \end{cases}$$

(7) Compute the determinants.

$$(a) \begin{vmatrix} 1 & 1 & 2 \\ 4 & 5 & 6 \\ 2 & 3 & -1 \end{vmatrix} \quad (b) \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \lambda & 0 \\ 0 & \lambda & 1 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix}$$

(8) [optional\*] Find the adjoint  $\mathbf{ad}(\mathbf{M})$  of the matrix  $M = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 3 & 1 \\ 3 & 1 & -1 \end{pmatrix}$ .

Verify that  $M\mathbf{ad}(M) = \mathbf{ad}(M)M = \det(M)I_3$ .

(9) (i) Which of the following sets of vectors  $x = [x_1, x_2, x_3, x_4]^T$  are subspace of  $\mathbf{R}^4$ ?

a) All  $x$  such that  $x_1 + x_2 = 7x_3$

b) All  $x$  such that  $x_3 = 0$

c) All  $x$  such that  $x_1 + x_4 = -12$

d) All  $x$  such that each  $x_i$  component is positive, that is, the first “I-quadrant” set  $= \{x_i \geq 0, i = 1, 2, 3, 4\}$ .

(ii) We know that  $P_3 = \{f : f \text{ is a polynomial of degree } \leq 3\}$  is a vector space. Which set of functions satisfying the following properties constitutes a subspace of  $P_3$ ?

a)  $f(-x) = f(x)$    b)  $f(0) + f(1) = 5$    c)  $f'(0) = 0$

(10) Which of the following vectors, if any, is in the null space of

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{pmatrix}?$$

a)  $[-1 \ 0 \ 1 \ 0]^T$    b)  $[0 \ 2 \ 1 \ -1]^T$    c)  $[0 \ 4 \ 2 \ -2]^T$

(11) Write each vector as a linear combination of the vectors in  $S$  if possible. Let  $S = \{(1, 2, -2), (2, -1, 1)\}$ .

(a)  $\mathbf{u} = (-4, -3, 3)$

(b)  $\mathbf{w} = (1, -5, -5)$

[ From 4.4, #2 ]

- (12) [Testing for linear independence] Determine whether the following set  $S$  of vectors is linearly independent or linearly dependent?
- (a)  $S = \{(-2, 2), (3, 5)\}$  in  $\mathbf{R}^2$  (4.4, #29)
  - (b)  $S = \{(-4, -3, 4), (1, -2, 3), (6, 0, 0)\}$  in  $\mathbf{R}^3$  (4.4, #37)
  - (c)  $S = \{9, x^2, x^2 + 1\}$  in  $P_2$ .
- (13) [True or False]
- (a) A set of vectors  $S = \{v_1, v_2, \dots, v_k\}$  in a vector space is called linearly dependent when the vector equation  $c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$  has only the trivial solution.  
(4.4, #59 (a))
  - (b) The set  $S = \{(1, 0, 0, 0), (0, -2, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$  spans  $\mathbf{R}^4$ .
  - (c) A set  $S = \{v_1, v_2, \dots, v_k\}$ ,  $k \geq 2$ , is linearly independent if and only if at least one of the vectors  $v_j$  can be written as a linear combination of the other vectors.  
(4.4, #60 (a))
  - (d) If a subset  $S$  spans a vector space  $V$ , then every vector in  $V$  can be written as a linear combination of the vectors in  $S$ .
- (14) Determine which of the following statements are equivalent to the fact that a matrix  $A$  of size  $n \times n$  is invertible?
- a)  $A$  is nonsingular
  - b) The row space of  $A$  has dimension  $n$
  - c) The column space of  $A$  has dimension  $n$
  - d) The determinant of  $A$  is nonzero
  - e) The system  $A\mathbf{x} = \mathbf{b}$  has a unique solution for any given  $\mathbf{b}$  in  $\mathbf{R}^n$
  - f) The system  $A\mathbf{x} = \mathbf{0}$  has nonzero solution
  - g) The dimension of the null space of  $A$  is zero
  - h) The rows of  $A$  are linear independent
  - i) The columns of  $A$  are linear independent
  - j) The rank of  $A$  is  $n$
  - k)  $A$  is row-equivalent to an identity matrix
  - l) All eigenvalues of  $A$  are nonzero
  - m)  $A$  has  $n$  linear independent eigenvectors
  - n)  $A$  is similar to a diagonal matrix
  - o)  $A$  can be written as the product of elementary matrices.

(15) \* The matrix  $A = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$  row reduces to  $C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ .

a) Find the rank and nullity of A.

b) Find a basis of the row space and the column space of A respectively.

c) Find a basis of the null space of A

d) Does the system  $A\mathbf{x} = \begin{pmatrix} 109 \\ -217 \\ 66 \end{pmatrix}$  have a solution? (Hint:

You can draw a conclusion from the fact that dimension of column space is 3, without having to solve the system. Recall that  $\text{rank}(A) = \dim(\text{Col}(A)) = \dim(\text{Row}(A))$ )

e) What is the relation between  $\text{rank}, \dim(\text{null}(A))$ ? (Hint: The theorem states that  $\text{rank}(A) + \dim(\text{null}(A)) = n$ , the number of columns)

(16) Find all the eigenvalues of the given matrix.

a)  $\begin{pmatrix} 1 & -2 & 0 \\ -3 & 1 & 0 \\ -4 & -5 & 1 \end{pmatrix}$       b)  $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(17) \* The matrix  $A = \begin{pmatrix} 6 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{pmatrix}$  has eigenvalues 5 and 8.

a) Find the eigenspaces  $E_5$  and  $E_8$  by solving  $(\lambda I - A)x = 0$ .

b) By the theorem in Section 7.3 we know that a symmetric matrix of size  $n$  by  $n$  is always diagonalizable, equivalently speaking, always has  $n$  linear independent eigenvectors. Find an ordered basis consisting of eigenvectors of A.

c) Specify the matrices  $P$  and  $D$  in the diagonalization  $P^{-1}AP = D$

d\*) (optional) Find an orthogonal matrix  $U$  such that  $U^{-1}AU = D$  (Hint: An (real) orthogonal matrix means  $U^{-1} = U^T$  or equivalently  $U^T U = U U^T = I_n$ ).

(18) a) Give three distinct examples of elementary matrices and explain how they correspond to row operations for a given matrix of 3 by 3.

b) Factor the matrix into a product of elementary matrices.

$$A = \begin{pmatrix} 1 & 6 \\ 6 & 1 \end{pmatrix}$$

### Solutions

(3)  $[62, 12, -16]$

$$(4) \text{ a) } \begin{pmatrix} 3 & 0 & 12 \\ -9 & 0 & -35 \\ 18 & 1 & 70 \end{pmatrix}$$

b) The factorization or decomposition of a matrix is not unique in general. Here is an example of the answer.  $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ ,

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c) In consistent with the answer in (b), we have  $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$ ,

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_3^{-1} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$E_5^{-1} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{d) } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(5) \text{ (b) not invertible; (c) Form the matrix } [A|I_3] = \begin{pmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 3 & -5 & -2 & | & 0 & 1 & 0 \\ 2 & -5 & -2 & | & 0 & 0 & 1 \end{pmatrix}$$

Then use row operation to reduce to  $[I_3|B]$ . Hence the inverse equals

$$B = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & -1 \\ -5 & 1 & 1 \end{pmatrix}$$

6) From 5(b) we know that the coefficient matrix is not invertible, so either the system has no solution or it has infinitely many solutions.

Row reductions for the augmented matrix  $\begin{pmatrix} 1 & 0 & -2 & | & 3 \\ 1 & -3 & 1 & | & -6 \\ 0 & 1 & -1 & | & 3 \end{pmatrix}$  give

$$\begin{pmatrix} 1 & 0 & -2 & | & 3 \\ 0 & 1 & -1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ Back-substitution shows } x = 3+2t, y = 3+t, z = t,$$

where  $t$  is a parameter of any real values. By the way observe that the augmented matrix is consistent with the coefficient matrix because they have the SAME RANK.

7) (a)  $-3$  (b)  $15(\lambda^2 - 1)$

8)  $\mathbf{ad}(M) = \text{transpose of } \begin{pmatrix} -4 & 3 & -9 \\ 3 & -5 & 4 \\ -5 & 1 & -3 \end{pmatrix}$

A straight forward computation shows  $M\mathbf{ad}(M) = -11I_3$ .

(15) a)  $\text{rank}(A) = 3$  (number of leading 1's in C), nullity of  $A = 1$

b) A basis of  $\text{Row}(A)$  consists of  $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ .

A basis of  $\text{Col}(A)$  consists of  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

c)  $\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$

d) Yes.

(16) a)  $|\lambda I - A| = (\lambda - 1)(\lambda^2 - 2\lambda - 5)$

(17) a)  $E_5 = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}, E_8 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$

c)  $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}$