Review Test 3 **Name**
Math 5339 **Id**

Read each question carefully. Avoid simple mistakes. (Use the back of the page if necessary). **You must show your work in order to get full credits**.

(1) ([Strauss 2.5, waves vs. diffusions]) The wave equation solution satisfies the Huygens' principle, has finite speed propagation less or equal to c; the singularities transported along characteristics with speed c. the (initial data) gets transported.

The diffusion equation solution has infinite speed propagation; the singularities lost immediately and the data gets lost gradually. This can be observed by the exercises in Sections 2.1 to 2.4.

Ex. [Strauss 2.5, #1] Show that there is no maximum principle for the wave equation.

Ex. [Strauss 2.5, #2] Consider a traveling wave $u(t,x) = g(x - \xi t)$, g being a function of one variable.

(a) If it is a solution of the wave equation $\Box u = 0$, $\Box = \partial_t^2 - c^2\Delta_x$, show that the speed must be $\xi = \pm c$ (except in the trivial case where g a linear function).

(b) If u is a solution of the diffusion equation $u_t = \kappa\Delta u$, find g and show that ξ can be arbitrary real number.

Ex. [Hydrogen atom; Strauss,Section 9.4,9.5] Separate variable to find solutions in $L^2(\mathbb{R}^3)$ of

$$iu_t = -\frac{1}{2}\Delta u - \frac{1}{2}u$$

Writing $u = TX$, $X = X(r)$ radial

$$2iT' = \lambda T$$
$$-\Delta X - \frac{2}{r}X = \lambda X$$

where λ represents the energy of the bound state $u(t,x)$. In 1913, Bohr observed that energy levels of electron in hydrogen atom occur only at values that are related to squares of integers. Via Laguerre equation we find that this has to be the case where

$$\lambda = -1, -\frac{1}{4}, -\frac{1}{9}, -\frac{1}{16}, \ldots$$

Ex. [9.5,#1] Verify the formulas for the first four solutions of the hydrogen atoms. (instructor change 3 to 4)

Ex. Let $X = X(r)$, $r > 0$. For $\lambda > 0$, why would you expect

$$-X_{rr} - \frac{2}{r}X_r - \frac{2}{r}X = \lambda X$$

not to have a solution in L^2? In other words, no positive eigenvalues?

Ex. [Strauss 2.5, #3] (a) Consider $u_t = \kappa\Delta u$, $\kappa = \frac{1}{2}$. Show that the pseudo-conformal transform

$$v(t,x) = \frac{1}{\sqrt{t}} e^{x^2/2t} u(\frac{1}{t}, \frac{x}{t})$$

obeys the "backward" equation $v_t = -\kappa\Delta v$ for $t > 0$.

(b) [Lecture Notes from the Instructor] Show the same property for $u_t = \kappa\Delta u + \lambda u^p$, $\lambda \in \mathbb{R}$, $p \geq 1$.

Ex. [Schrödinger equation] Consider $iu_t = -\frac{1}{2}\Delta u + \lambda|u|^{p-1}u$. Show that the pseudo-conformal transform

$$v(t,x) = \frac{1}{\sqrt{t}} e^{ix^2/2t} u(\frac{1}{t}, \frac{x}{t})$$

obeys the equation $iv_t = -\frac{1}{2}\Delta v + \lambda|v|^{p-1}v$ for all t.

(2) (relation between wave and diffusion equations) Suppose u solves $\Box u = 0$ in $\mathbb{R}$ and u has bounded second derivatives. Then (a)

$$v(t,x) = \frac{c}{\sqrt{4\pi kt}} \int_{\mathbb{R}} e^{-c^2s^2/4kt} u(s,x) ds$$

solves the diffusion equation.

(b) We have $\lim_{t\to 0} v(t,x) = u(0,x)$.

(c)** In virtue of Laplace transform one might be able to derive a formula that maps solutions of heat equation to that of a wave equation. cf. [Strauss, 2.5, #4] and [Lecture Notes]

[Clue: (i) write $v = \int_{\mathbb{R}} H(s,t)u(s,x)ds$; H solves the diffusion equation with constant k/c^2 for $t > 0$.

(ii) Note that $H(s,t)$ is essentially the source function of the diffusion equation with spatial variable s.]

(3) (optional) Let $\mathcal{H} := L^2(\mathbb{R}^n)$ be a *Hilbert* space, that is a vector space with inner product $\langle \cdot, \cdot \rangle$ satisfying

(a) $\langle u, u \rangle \geq 0$

(b) $\langle \alpha u + \beta v, w \rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$

(c) $\langle v, u \rangle = \overline{\langle u, v\rangle}$

These simple and basic properties imply that we can endow a norm $\|u\| := \sqrt{\langle u, u\rangle}$ on $\mathcal{H}$. Also, we have the Cauchy-Schwarz inequality

$$|\langle u, v\rangle| \leq \|u\| \, \|v\| \, .$$

In Quantum mechanics only when an operator A acting in $\mathscr{H}$ is selfadjoint, then it is observable. Only when A and B commute, they are observable simultaneously. The Schr̈dingier representation are:

$$x \mapsto \text{position operator}$$
$$-i\nabla \mapsto \text{momentum operator}$$

Prove the uncertainty principle

$$\|(x-x_0)u\|_2 \, \|(\xi-\xi_0)\hat{u}\|_2 \geq \frac{\pi^2}{16}$$

Remark The UC is the reason for the global in time existence of NLS, cf. [Luis Vega et al].

(4) (angular momentum in quantum mechanics) Ex. [10.7, #1]
Ex. [10.7, #2]
Ex. [10.7, #3]
Ex. [10.7, #5]

(5) (scattering theory vs. soliton or bound sate) If $V(r) = cr^{-1}$ is the Coulomb potential as with the hydrogen atom, then $H_V := -\Delta + V$ has discrete spectrum $\{-1/n^2\}$.
If V has decay at infinity, then there may exist continuous spectrum. Ex. [13.4, #1]
Ex. [13.4, #2]

(6) [equations of elementary particles] Klein-Gordon equation and Yang-Mills equations. Read Section 13.5

(7) [KdV part.I] (Airy's equation) $u_t + u_{xxx} = 0$ Verify that $u(t,x) = e^{i(xy+ty^3)} = e^{i\phi}$ is a solution. Here note that $\phi(t,x,\xi) = \xi(x+t\xi^2)$ so propagation speed depends on the frequency of the wave.

Definition If plane waves of different frequencies travel with different speeds, the PDE is called **dispersive**. (cf. Evans notes)

(8) [Korteweg de Vries] Consider

$$u_t + 6uu_x + u_{xxx} = 0 \qquad \mathbb{R}_+ \times \mathbb{R}$$

symmetry: u a solution $\Rightarrow u(-t,-x)$ also a solution.
Seek a traveling wave solution ansatz $u = v(x-\sigma t)$. Let $\eta = x - \sigma t$ then the symmetry above implies v is even in η.

$$\begin{aligned} &-\sigma v' + 6vv' + v^{'''} = 0 \\ \Rightarrow &-\sigma v' + 3(v^2)' + v^{'''} = 0 \\ \Rightarrow &-\sigma v + 3v^2 + v^{''} = C \quad (**) \end{aligned}$$

To find solution assuming $v, v', v'' \to 0$ as $\eta \to \pm\infty$. these conditions implies $C = 0$.
We write equation (**) of v as a system of ODE

$$\begin{cases} v' = w \\ w' = \sigma v - 3v^2 \end{cases}$$

Two fixed points $(0,0)$, $(0, \sigma/3)$.

Assume $\sigma > 0$, linearizing about $(0,0)$, eigenvalues/eigenfunctions are

$$\begin{aligned} &\sqrt{\sigma} \mid (1, \sqrt{\sigma}) \\ -&\sqrt{\sigma} \mid (1, -\sqrt{\sigma}) \end{aligned}$$

Linearizing about $(0, \sigma/3)$, eigenvalues are $\pm\sqrt{-\sigma}$. There are some details on the homoclinic orbit method with phase plane portrait Evans' notes.

The KdV is completely integrable. Fortunately we can explicitly solve for this homoclinic orbit.

Multiply by v'

$$\begin{aligned} &-\sigma v v' + 3v^2 v' + v' v'' = 0 \\ \Rightarrow &-\frac{1}{2}\sigma(v^2)' + (v^3)' + \frac{1}{2}((v')^2)' = 0 \quad (***) \\ \Rightarrow &-\frac{1}{2}\sigma v^2 + v^3 + \frac{1}{2}(v')^2 = C_2 \\ \Rightarrow &v' = \pm\sqrt{\sigma v^2 - 2v^3} \end{aligned}$$

here again the vanishing condition on v, v' implies $C_2 = 0$.

We obtain from the phase plane with $v(\eta) > 0$

$$\Rightarrow v' = \pm v\sqrt{\sigma\sigma - 2v}$$

Since we seek solution v of even, and v' of odd, solve

$$v'(\eta) = \begin{cases} v\sqrt{\sigma\sigma - 2v} & (-\infty, 0) \\ -v\sqrt{\sigma\sigma - 2v} & (0, \infty) \end{cases}$$

Notice that on each half line, the function has an inverse. $\eta = g(v)$ We obtain on the left branch

$$\begin{aligned} &\frac{d\eta}{dv}(v)\frac{dv}{d\eta}(\eta) = 1 \\ &\frac{d\eta}{dv}(v)\left(v\sqrt{\sigma\sigma - 2v}\right) = 1 \\ &\frac{d\eta}{dv} = \frac{1}{v\sqrt{\sigma - 2v}} \\ \therefore &\eta_\ell(v) = -\frac{2}{\sigma}\tanh^{-1}(\sqrt{1 - 2v/\sigma}) + C_l \end{aligned}$$

$$\text{similarly on the right branch} \quad \eta_r(v) = \frac{2}{\sigma}\tanh^{-1}(\sqrt{1 - 2v/\sigma}) + C_r$$

Note $v_{max} = \sigma/2$

We are ready to find the inverse $v : \mathbb{R} \to (0, v_{max}]$ using simple asymptotics

$$v(\eta) = \frac{\sigma}{2}\operatorname{sech}^2(\sqrt{\sigma}\eta/2)$$

Finally

$$u(t,x) = \frac{\sigma}{2}\,\mathrm{sech}^2(\sqrt{\sigma}/2(x-\sigma t))$$

Remark The higher the soliton the faster it moves because $\sigma = speed = 2v_{max}$.