M 2160 (Elementary Linear Algebra) Instructor: Dr. John S. Zheng
Course web Cengage

§5.1 Length and Dot Product in R^n

Definition. The **length**, or **norm**, of a vector $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ in R^n is $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$. The length of a vector is also called its **magnitude**. If $\|\mathbf{v}\| = 1$, then the vector $\mathbf{v}$ is a **unit vector**.

Example 1. (a) In R^5, the length of $\mathbf{v} = (0, -2, 1, 4, -2)$ is $\|\mathbf{v}\| = 5$.

Theorem 5.1. Let $\mathbf{v}$ be a vector in R^n and let c be a scalar. Then $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$, where $|c|$ is the absolute value of c.

Theorem 5.2. If $\mathbf{v}$ is a nonzero vector in R^n, then the vector $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ has length 1 and has the same direction as $\mathbf{v}$. This vector $\mathbf{u}$ is the **unit vector in the direction of v**.

Example 2. Find the unit vector in the direction of $\mathbf{v} = (3, -1, 2)$, and verify that this vector has length 1.

Definition. The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n is $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.

Example 3. (c) The distance between $\mathbf{u} = (3, -1, 0, -3)$ and $\mathbf{v} = (4, 0, 1, 2)$ is $2\sqrt{7}$.

Definition. The **dot product** of $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ is the scalar quantity $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$.

Example 4. The dot product of $\mathbf{u} = (1, 2, 0, -3)$ and $\mathbf{v} = (3, -2, 4, 2)$ is -7.

Theorem 5.3. If $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors in R^n and c is a scalar, then the properties listed below are true.
1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ 2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ 3. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$ 5. $\mathbf{v} \cdot \mathbf{v} \geq 0$, and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

Example 6. Consider two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n such that $\mathbf{u} \cdot \mathbf{u} = 39$, $\mathbf{u} \cdot \mathbf{v} = -3$, and $\mathbf{v} \cdot \mathbf{v} = 79$. Evaluate $(\mathbf{u} + 2\mathbf{v}) \cdot (3\mathbf{u} + \mathbf{v})$.

Definition. The **angle** θ between two nonzero vectors in R^n can be found using $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$, $0 \leq \theta \leq \pi$.

Example 8. The angle between $\mathbf{u} = (-4, 0, 2, -2)$ and $\mathbf{v} = (2, 0, -1, 1)$ is π.

Definition. Two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are **orthogonal** when $\mathbf{u} \cdot \mathbf{v} = 0$.

Example 9. (b) The vectors $\mathbf{u} = (3, 2, -1, 4)$ and $\mathbf{v} = (1, -1, 1, 0)$ are orthogonal.

Ex. # 41 Find the angle between two vectors.

$$\mathbf{u} = (\cos\frac{\pi}{6}, \sin\frac{\pi}{6})$$
$$\mathbf{v} = (\cos\frac{3\pi}{4}, \sin\frac{3\pi}{4})$$

Ex. # 45 Find the angle between two vectors.

$$\mathbf{u} = (0, 1, 0, 1)$$
$$\mathbf{v} = (3, 3, 3, 3)$$

§5.2 Inner Product Spaces

Definition. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Then the **orthogonal projection** of $\mathbf{u}$ onto $\mathbf{v}$ is $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}$.

Goal:

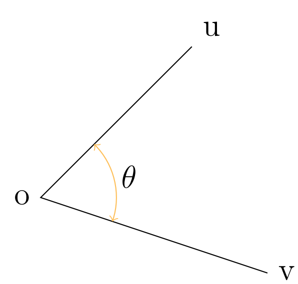
(1) Determine whether a function defines an inner product, and find the inner product of two vectors in $\mathbb{R}^n, M_{mn}, C[a,b]$.
(2) Find an orthogonal projection of a vector onto another vector in an inner product space.

Example 10. Use the Euclidean inner product in R^3 to find the orthogonal projection of $\mathbf{u} = (6, 2, 4)$ onto $\mathbf{v} = (1, 2, 0)$.

§5.3 Orthonormal Bases: Gram-Schmidt Process*

Definition. A set S of vectors in an inner product space V is **orthogonal** when every pair of vectors in S is orthogonal. If, in addition, each vector in the set is a unit vector, then S is **orthonormal**.

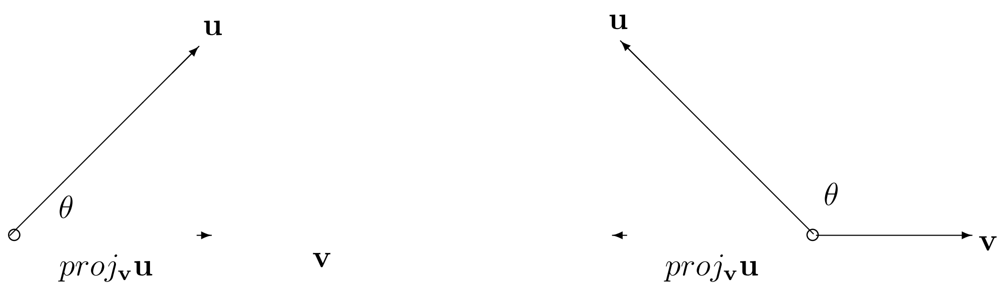
0.1. **Find the angle using inner product.**



(1)
$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|} \Rightarrow \begin{cases} > 0 & \text{if } \theta \text{ is acute} \\ = 0 & \text{if } \theta \text{ is right angle} \\ < 0 & \text{if } \theta \text{ is obtuse} \end{cases}$$

0.2. **Projection formula with inner product.**

$$proj_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}. \tag{2}$$



(A) $proj_{\mathbf{v}}\mathbf{u} = a\mathbf{v}$, $a > 0$

(B) $proj_{\mathbf{v}}\mathbf{u} = a\mathbf{v}$, $a < 0$

0.3. **Gram-Schmidt orthonomalization*.** Ex. # 3. (a) Determine whether the set of vectors in $\mathbb{R}^2$ is orthogonal;
(b) if the set is orthogonal, then determine whether it is also orthonormal, and
(c) determine whether the set is a basis for R^2.

$$\{(\frac{3}{5}, \frac{4}{5}), (-\frac{4}{5}, \frac{3}{5})\} \tag{3}$$

Example 7*. Apply the Gram-Schmidt orthonormalization process to the basis B for $\mathbb{R}^3$.

$$B = \{(1, 1, 0), (1, 2, 0), (0, 1, 2)\} \tag{4}$$

Solution. Applying the Gram-Schmidt orthonormalization process produces

$$\begin{aligned}
w_1 &= v_1 = (1, 1, 0) \\
w_2 &= v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1 = (-\frac{1}{2}, \frac{1}{2}, 0) \\
w_3 &= v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 = (0, 0, 2)
\end{aligned}$$

The set $\tilde{B} = \{w_1, w_2, w_3\}$ is an orthogonal basis for $\mathbb{R}^3$. Normalizing each vector in B' produces the following orthonormal basis

$$\begin{aligned}
u_1 &= (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0) \\
u_2 &= (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0) \\
u_3 &= (0, 0, 1)
\end{aligned}$$

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