§*5.3* Orthonormal Bases: Gram-Schmidt Process* (Continued)

Example 1. Show that the set is an orthonormal basis for R^3.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0), (-\tfrac{\sqrt{2}}{6}, \tfrac{\sqrt{2}}{6}, \tfrac{2\sqrt{2}}{3}), (\tfrac{2}{3}, -\tfrac{2}{3}, \tfrac{1}{3})\}$$

Theorem 5.10. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space V, then S is linearly independent.

Theorem 5.11. If $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, then the coordinate representation of a vector $\mathbf{w}$ relative to B is

$$\mathbf{w} = (\mathbf{w}\cdot\mathbf{v}_1)\mathbf{v}_1 + (\mathbf{w}\cdot\mathbf{v}_2)\mathbf{v}_2 + \cdots + (\mathbf{w}\cdot\mathbf{v}_n)\mathbf{v}_n.$$

Theorem 5.12* (Gram-Schmidt Orthonormalization Process).
1. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be a basis for an inner product space V.
2. Let $B' = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n\}$, where

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1\\
\mathbf{w}_2 &= \mathbf{v}_2 - \frac{\mathbf{v}_2\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1\\
\mathbf{w}_3 &= \mathbf{v}_3 - \frac{\mathbf{v}_3\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3\cdot\mathbf{w}_2}{\mathbf{w}_2\cdot\mathbf{w}_2}\mathbf{w}_2\\
&\ \vdots\\
\mathbf{w}_n &= \mathbf{v}_n - \frac{\mathbf{v}_n\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_n\cdot\mathbf{w}_2}{\mathbf{w}_2\cdot\mathbf{w}_2}\mathbf{w}_2 - \cdots - \frac{\mathbf{v}_n\cdot\mathbf{w}_{n-1}}{\mathbf{w}_{n-1}\cdot\mathbf{w}_{n-1}}\mathbf{w}_{n-1}.
\end{aligned}$$

Then B' is an *orthogonal* basis for V.
2. Let $\mathbf{u}_i = \dfrac{\mathbf{w}_i}{\|\mathbf{w}_i\|}$. Then $B'' = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is an *orthonormal* basis for V. Also, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ for $k = 1, 2, \cdots, n$.

Example 7*. Apply the Gram-Schmidt orthonormalization process to the basis B for R^3.

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1,1,0), (1,2,0), (0,1,2)\}.$$

[Soluiton] From the handout 5.1-5.3, we saw that the Gram-Schmidt yields an orthonormal basis by means of projection

$$\begin{aligned}
\mathbf{u}_1 &= (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)\\
\mathbf{u}_2 &= (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)\\
\mathbf{u}_3 &= (0,0,1)
\end{aligned}$$

such that $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$.

Ex. #15. The set $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ is orthogonal but not orthonormal.

$$\{(\sqrt{3}, \sqrt{3}, \sqrt{3}),\ (-\sqrt{2}, 0, \sqrt{2})\} \tag{1}$$

Indeed,

$$(\sqrt{3}, \sqrt{3}, \sqrt{3}) \cdot (-\sqrt{2}, 0, \sqrt{2}) = -\sqrt{6} + 0 + \sqrt{6} = 0, \tag{2}$$

However,

$$\|\mathbf{v}_1\| = \sqrt{9} = 3 \neq 1$$
$$\|\mathbf{v}_2\| = 2 \neq 1.$$

Normalize *each vector* to obtain an orthonormal set.

$$\mathbf{u_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$$
$$\mathbf{u_2} = \frac{\mathbf{v_2}}{\|\mathbf{v_2}\|} = \frac{1}{2}(-\sqrt{2}, 0, \sqrt{2}) = (-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}).$$

Chapter 6*. Linear Transformations.

Chapter 7. Eigenvalues and Eigenvectors

§7.1 *Eigenvalues and Eigenvectors*

Definition. Let A be an $n \times n$ matrix. The scalar λ is an **eigenvalue** of A when there is a *nonzero* vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is an **eigenvector** of A corresponding to λ.

Example 2. For the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, verify that $\mathbf{x}_1 = (-3, -1, 1)$ and $\mathbf{x}_2 = (1, 0, 0)$ are eigenvectors of A and find their corresponding eigenvalues.

Theorem 7.1. If A is an $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ, together with the zero vector, is a subspace of R^n. This subspace is the **eigenspace** of λ.

Theorem 7.2. Let A be an $n \times n$ matrix.
1. An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$.
2. The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

Example 4. Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$.

Example 5. Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(from the text) Verify that λ_i is an eigenvalue of A and that $\mathbf{x}_i$ is a corresponding eigenvector.
Ex.# 5.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{aligned} \lambda_1 &= 2, \quad x_1 = (1,0,0) \\ \lambda_2 &= -1, \; x_2 = (1,-1,0) \\ \lambda_3 &= 3, \quad x_3 = (5,1,2). \end{aligned}$$

Ex. # 7.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \lambda_1 = 1, \quad x_1 = (1,1,1) \tag{3}$$

Solution. Step 1. Solving $|\lambda - A| = \lambda^3 - 1 = (\lambda - 1)(\lambda^2 + \lambda + 1) = 0$, we obtain $\lambda_1 = 1$ ($\lambda_{2,3} = \frac{-1 \pm \sqrt{3}i}{2}$, complex roots)
Step 2. Solve the homogeneous equation $(\lambda - A)X = 0$ to find $E_1 = \text{span}\{(1,1,1)^T\}$. □

§7.2 Diagonalization

Definition. An $n \times n$ matrix A is **diagonalizable** when A is similar to a diagonal matrix. That is, A is diagonalizable when there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Example 1. The matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ is diagonalizable with $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution. Step 1. Find inverse of P: $P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Step 2. We see that $P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. □

Theorem 7.4. If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.

Theorem 7.5. An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Example 4. Show that the matrix $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$ is diagonalizable. Then find a matrix P such that $P^{-1}AP$ is diagonal.

Example 5.* Show that the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$ is diagonalizable. Then find a matrix P such that $P^{-1}AP$ is diagonal.

Example 6. Show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

Theorem 7.6. If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

Example 7. Determine whether A is diagonalizable.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

[Solution] Because A is a triangular matrix, its eigenvalues are the main diagonal entries $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = -3$. Moreover, since these three are distinct, we conclude from Theorem 7.6 that A is ***diagonalizable***.

Ex.#3. Verify that A is diagonalizable by computing $P^{-1}AP$.

$$A = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$$

12. Show that the matrix is not diagonalizable.

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

Example 10* How about the following: Diagonalizable or not diagonalizable ?

(1) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(2) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$