Finding Absolute Maxima and Minima of f on a (closed bounded) region R

Evaluate f at the
1) ***critical points*** inside R
2) ***boundary points*** *of* R where f has maxima and minima

and choose the highest and the lowest value.

Example

Find all the absolute maxim and minimum of

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2 \qquad f_x = 2 - 2x \qquad f_y = 4 - 2y$$

on the triangular region bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.

1) *Critical points* in R: $(1,2)$; $f(1,2) = 7$

2) *Boundary points*.

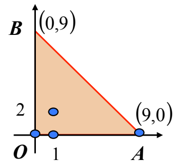
One side at a time:

i) OA $(y = 0)$ $\quad f(x,0) = \underbrace{2 + 2x - x^2}_{g(x)}$

Extreme values of g on $[0,9]$: $g(1) = 3$

$g'(x) = 2 - 2x = 0$ iff $x = 1$ $\quad g(0) = 2$

$g(9) = -61$



ii) OB $(x = 0)$ $\quad f(0,y) = 2 + 4y - y^2 = h(y)$

Extreme values of h on $[0,9]$: $h(2) = 6$

$h'(y) = 4 - 2y = 0$ iff $y = 2$ $\quad h(0) = 2$ (covered in i))

$h(9) = -43 = f(0,9)$

iii) AB $(y = 9 - x)$ $\quad f(x,y) = 2 + 2x + 4(9 - x) - x^2 - (9 - x)^2 = \underbrace{-43 + 16x - 2x^2}_{k(x)}$

Extreme values of k on $[0,9]$

$k'(x) = 16 - 4x = 0$ iff $x = 4$ $\quad k(4) = -11$ $\quad (k(0), k(9)$ covered in i) and ii))

1) *Critical points* in R: $(1,2)$; $f(1,2) = 7$ **Max**

2) *Boundary points*.

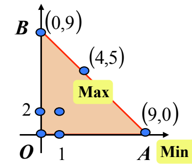
One side at a time:

i) OA $(y = 0)$ $\quad f(x,0) = \underbrace{2 + 2x - x^2}_{g(x)}$

Extreme values of g on $[0,9]$: $g(1) = 3$

$g'(x) = 2 - 2x = 0$ iff $x = 1$ $\quad g(0) = 2$

$g(9) = -61 = f(9,0)$ **Min**



Max: $f(1,2) = 7$
Min: $f(9,0) = -61$

Picture the level curves of f:

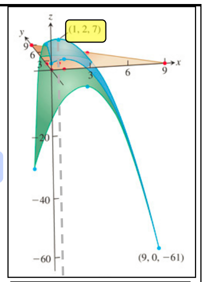
$$L_c = \{(x,y) \mid f(x,y) = c\}$$

e.g. $L_7 = \{(1,2)\}$ (a point),

for $c > 7$, $L_c = \varnothing$ (the empty set),

for $c < 7$, L_c is the circle centered at $(1,2)$ with radius $\sqrt{7-c}$.

$$\begin{aligned} f(x,y) = 2 + 2x + 4y - x^2 - y^2 &= c \\ x^2 - 2x + y^2 - 4y &= 2 - c \\ (x-1)^2 + (y-2)^2 &= 7 - c \end{aligned}$$



Max: $f(1,2) = 7$
Min: $f(9,0) = -61$

Observation

The extreme values occur where the two level curves of f corresponding to the two extreme values $(c = 7, -61)$ touch region R.