Instructions. Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* Put your name on each page of your paper.

(1) [20 Points] Answer the following questions about the differential equation

$$y' = -ty^2. \tag{$*$}$$

(a) Find the general solution of Equation $(*)$. The nonconstant solutions are found by separating the variables and integrating. Separating the variables and writing in differential form gives $y^{-2}\,dy = -t\,dt$. Integrating the left hand side with respect to y and the right hand side with respect to t gives

$$-\frac{1}{y} = -\frac{1}{2}t^2 + c = -\frac{1}{2}(t^2 + b),$$

where $b = -2c$ is an arbitrary constant. Solving for y gives:

$$y = \frac{2}{t^2 + b},$$

so the general solution of $(*)$ is

$$\boxed{y = \frac{2}{t^2 + b}, \text{ for } b \in \mathbb{R}, \text{ and the constant solution } y(t) = 0 \text{ corresponding to } b = \infty.}$$

(b) Find all the constant functions that are solutions of $(*)$. For a separable equation $y' = h(t)g(y)$, the constant solutions are exactly the constant functions $y(t) \equiv c$ where the constant c is determined by $g(c) = 0$ (see Algorithm 1.2.3 (5), Page 20). In this case $g(y) = y^2 = 0$ only when $y = 0$. Thus $y(t) \equiv 0$ is the only constant solution.

(c) Find the solution of Equation $(*)$ with initial condition $y(0) = 4$. Letting $t = 0$, $y = 4$ in the boxed equation, and solving for b gives $b = 1/2$, so the solution of the initial value problem is

$$\boxed{y = \frac{2}{t^2 + 0.5}.}$$

(d) Draw (and clearly label) the solution found in part (c) on the attached direction field. This is the curve passing through $(0, 4)$ above the upper curve drawn on the direction field.

(e) Draw on the direction field (and clearly label) the solution of Equation $(*)$ with initial value $y(0) = -0.5$. This is the curve passing through $(0, -0.5)$ above the lower curve drawn on the direction field.

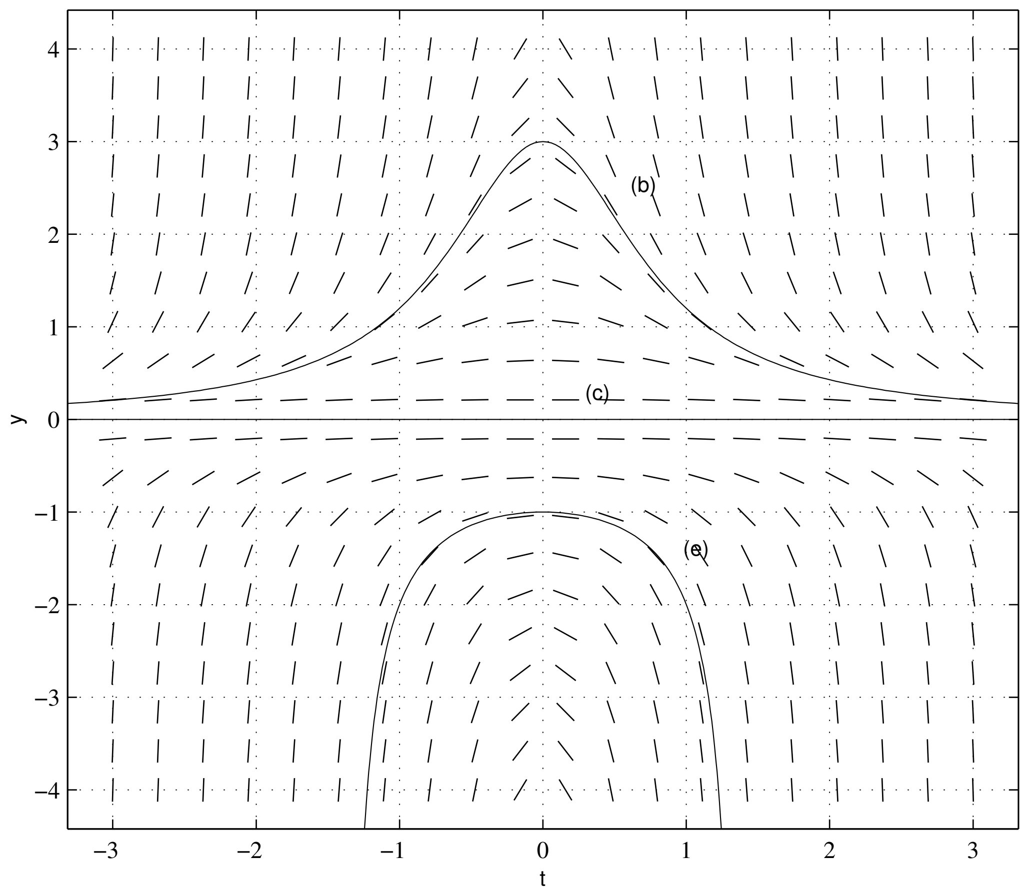
(2) [20 Points] Solve the following initial value problems. Be sure to show all of your work. $y' + 2y = te^{-5t} + e^t, \quad y(0) = -2$ This equation is linear with coefficient function $p(t) = 2$ and forcing function $f(t) = te^{-5t} + e^t$. Then $P(t) = \int 2\,dt = 2t$ and an integrating factor is thus $\mu(t) = e^{2t}$. Multiplying the equation by $\mu(t)$ gives

$$(e^{2t}y)' = e^{2t}\left(te^{-5t} + e^t\right) = te^{-3t} + e^{3t}.$$

Integrate both sides to get

$$e^{2t}y = -\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t} + \frac{1}{3}e^{3t} + C,$$

Direction Field for Prob. 1:
$y' = -ty^2$



and solve for y by multiplying by e^{-2t}:

$$y = -\frac{1}{3}te^{-5t} - \frac{1}{9}e^{-5t} + \frac{1}{3}e^{t} + Ce^{-2t}.$$

Using the initial condition $y = -2$ when $t = 0$ gives that $C = -\frac{20}{9}$. Hence the solution of the initial value problem is

$$\boxed{y = -\frac{1}{3}te^{-5t} - \frac{1}{9}e^{-5t} + \frac{1}{3}e^{t} - \frac{20}{9}e^{-2t}}$$

(3) [20 Points]

(a) Complete the following definition: Suppose $f(t)$ is a continuous function defined for all $t \geq 0$. The **Laplace transform** of f is the function $F(s)$ defined as follows:

$$F(s) = \mathcal{L}\left\{f(t)\right\}(s) = \boxed{\int_0^\infty e^{-st} f(t)\, dt}$$

for all s sufficiently large.

(b) Using your definition compute the Laplace transform of $f(t) = e^{-3t} + 2$. The Laplace transform of $f(t) = e^{-3t} + 2$ is the integral

$$
\begin{aligned}
\mathcal{L}\left\{e^{-3t} + 2\right\}(s) &= \int_0^\infty e^{-st}(e^{-3t} + 2)\,dt \\
&= \int_0^\infty \left(e^{-(s+3)t} + 2e^{-st}\right)\,dt \\
&= -\frac{1}{s+3}e^{-(s+3)t}\Big|_0^\infty + \frac{-2}{s}e^{-st}\Big|_0^\infty \\
&= 0 - \frac{-1}{s+3} + 2\left(0 - \frac{-1}{s}\right) \\
&= \boxed{\frac{1}{s+3} + \frac{2}{s} \quad \text{for } s > 0.}
\end{aligned}
$$

The last evaluation uses the fact (verified in calculus) that $\lim_{t\to\infty} e^{-st} = 0$.

(4) [20 Points] Compute the Laplace transform of each of the following functions.

(a) $f_1(t) = 10t^3e^t$

$$\boxed{F_1(s) = 10 \cdot \frac{3!}{(s-1)^4}}$$

(b) $f_2(t) = e^{5t}f(t)$ where $f(t)$ is the function with Laplace transform $F(s) = \dfrac{s^3}{s^4 - s + 2}$.

$$F_2(s) = F(s-5) = \frac{(s-5)^3}{(s-5)^4 - (s-5) + 2}.$$

(5) [20 Points] Apply Picard's method to compute the first two approximations $y_1(t)$ and $y_2(t)$ to the solution of the initial value problem

$$y' = t^2 + y^2, \quad y(0) = 0.$$

This equation has the form $y' = F(t, y)$ with $F(t, y) = t^2 + y^2$. Thus, the first Picard approximation y_1 is given by

$$y_1(t) = y_0 + \int_0^t F(u, y_0(u))\,du = \int_0^t u^2\,du = \frac{u^3}{3}\Big|_0^t = \frac{t^3}{3},$$

and the second approximation y_2 is given by

$$y_2(t) = \int_0^t F(u, y_1(u))\,du = \int_0^t \left(u^2 + \left(\frac{u^3}{3}\right)^2\right)\,du = \int_0^t \left(u^2 + \frac{u^6}{9}\right)\,du = \frac{t^3}{3} + \frac{t^7}{63}.$$