

Review Final Exam
Math 2331

Name
Id

Read carefully each problem. Show all your work in order to justify and support your answer. Credits will be given mainly depending on your work, not just an answer. Put a box around the final answer to a question. Use the back of the page if necessary.

- (1) Which of the following equations are linear?
- a) $xy + 3y = 1$
 - b) $3x - y + z = 9w$
 - c) $x \cos 15^\circ + (2 - y) \sin 15^\circ = \sqrt{17}$
 - d) $5e^x - 11e^y = 0$
 - e) $e^5x - e^{11}y = 0$
 - f) $(x + y)(x - y) = -3$
- (2) Write down the coefficient and the augmented matrices for the linear system.

a)

$$\begin{cases} 2x_1 & +x_2 & +x_3 & +2x_4 = 0 \\ x_1 & -x_2 & & +5x_4 = 3 \\ x_1 & -5x_2 & +x_3 & -x_4 = -2 \end{cases}$$

b) Solve the system of equations.

- (3) Write the system of linear equations in the form $A\mathbf{x} = \mathbf{b}$ and solve the matrix equation for $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$ using either Gaussian elimination with back-substitution or Gauss-Jordan elimination.

$$\begin{cases} 3x_1 & & +12x_3 = -6 \\ -9x_1 & & -35x_3 = 2 \\ 18x_1 & +x_2 & +70x_3 = 8 \end{cases}$$

- (4) a) Write the coefficient matrix A of the system in (3)
b) Find reduced row-echelon form for A using elementary row reductions. Record the elementary matrices E_1, E_2, \dots, E_k corresponding to these elementary row reductions.
c) Find the inverses $E_1^{-1}, E_2^{-1}, \dots, E_k^{-1}$.
(Hint: There are only three kinds of elementary matrices, each of their inverses is easily known; check the Notes or Text if you are not sure)
d) Express A as a product of elementary matrices.
(Hint: Because $A^{-1} = E_k E_{k-1} \dots E_2 E_1$, we have $A = E_1^{-1} E_2^{-1} \dots E_k^{-1}$)

(5) Find the inverse of the matrix (if it exists).

(a) $\begin{pmatrix} 10 & -5 \\ 5 & -3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & -2 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & -2 & -1 \\ 3 & -5 & -2 \\ 2 & -5 & -2 \end{pmatrix}$

(6) Solve the inhomogeneous system

$$\begin{cases} x & -2z = 3 \\ x & -3y & +z = -6 \\ & y & -z = 3 \end{cases}$$

(7) Compute the determinants.

(a) $\begin{vmatrix} 1 & 1 & 2 \\ 4 & 5 & 6 \\ 2 & 3 & -1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \lambda & 0 \\ 0 & \lambda & 1 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix}$

(8) [optional*] Find the adjoint $\mathbf{ad}(\mathbf{M})$ of the matrix $M = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 3 & 1 \\ 3 & 1 & -1 \end{pmatrix}$.

Verify that $M\mathbf{ad}(M) = \mathbf{ad}(M)M = \det(M)I_3$.

(9) (i) Which of the following sets of vectors $x = [x_1, x_2, x_3, x_4]^T$ are subspace of \mathbf{R}^4 ?

a) All x such that $x_1 + x_2 = 7x_3$

b) All x such that $x_3 = 0$

c) All x such that $x_1 + x_4 = -12$

d) All x such that each x_i component is positive, that is, the first “I-quadrant” set $= \{x_i \geq 0, i = 1, 2, 3, 4\}$.

(ii) We know that $P_3 = \{f : f \text{ is a polynomial of degree } \leq 3\}$ is a vector space. Which set of functions satisfying the following properties constitutes a subspace of P_3 ?

a) $f(-x) = f(x)$ b) $f(0) + f(1) = 5$ c) $f'(0) = 0$

(10) Which of the following vectors, if any, is in the null space of

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{pmatrix}?$$

a) $[-1 \ 0 \ 1 \ 0]^T$ b) $[0 \ 2 \ 1 \ -1]^T$ c) $[0 \ 4 \ 2 \ -2]^T$

(11) Write each vector as a linear combination of the vectors in S if possible. Let $S = \{(1, 2, -2), (2, -1, 1)\}$.

(a) $\mathbf{u} = (-4, -3, 3)$

(b) $\mathbf{w} = (1, -5, -5)$ [From 4.4, #2]

- (12) [Testing for linear independence] Determine whether the following set S of vectors is linearly independent or linearly dependent?
- (a) $S = \{(-2, 2), (3, 5)\}$ in \mathbf{R}^2 (4.4, #29)
 - (b) $S = \{(-4, -3, 4), (1, -2, 3), (6, 0, 0)\}$ in \mathbf{R}^3 (4.4, #37)
 - (c) $S = \{9, x^2, x^2 + 1\}$ in P_2 .
- (13) [True or False]
- (a) A set of vectors $S = \{v_1, v_2, \dots, v_k\}$ in a vector space is called linearly dependent when the vector equation $c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$ has only the trivial solution.
(4.4, #59 (a))
 - (b) The set $S = \{(1, 0, 0, 0), (0, -2, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ spans \mathbf{R}^4 .
 - (c) A set $S = \{v_1, v_2, \dots, v_k\}$, $k \geq 2$, is linearly independent if and only if at least one of the vectors v_j can be written as a linear combination of the other vectors.
(4.4, #60 (a))
 - (d) If a subset S spans a vector space V , then every vector in V can be written as a linear combination of the vectors in S .
- (14) Determine which of the following statements are equivalent to the fact that a matrix A of size $n \times n$ is invertible?
- a) A is nonsingular
 - b) The row space of A has dimension n
 - c) The column space of A has dimension n
 - d) The determinant of A is nonzero
 - e) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any given \mathbf{b} in \mathbf{R}^n
 - f) The system $A\mathbf{x} = \mathbf{0}$ has nonzero solution
 - g) The dimension of the null space of A is zero
 - h) The rows of A are linear independent
 - i) The columns of A are linear independent
 - j) The rank of A is n
 - k) A is row-equivalent to an identity matrix
 - l) All eigenvalues of A are nonzero
 - m) A has n linear independent eigenvectors
 - n) A is similar to a diagonal matrix
 - o) A can be written as the product of elementary matrices.

(15) * The matrix $A = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$ row reduces to $C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

a) Find the rank and nullity of A.

b) Find a basis of the row space and the column space of A respectively.

c) Find a basis of the null space of A

d) Does the system $A\mathbf{x} = \begin{pmatrix} 109 \\ -217 \\ 66 \end{pmatrix}$ have a solution? (Hint:

You can draw a conclusion from the fact that dimension of column space is 3, without having to solve the system. Recall that $\text{rank}(A) = \dim(\text{Col}(A)) = \dim(\text{Row}(A))$)

e) What is the relation between $\text{rank}, \dim(\text{null}(A))$? (Hint: The theorem states that $\text{rank}(A) + \dim(\text{null}(A)) = n$, the number of columns)

(16) Find all the eigenvalues of the given matrix.

a) $\begin{pmatrix} 1 & -2 & 0 \\ -3 & 1 & 0 \\ -4 & -5 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(17) The matrix $A = \begin{pmatrix} 6 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{pmatrix}$ has eigenvalues 5 and 8.

a) Find the eigenspaces E_5 and E_8 by solving $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

b*) By the theorem in Section 7.3 we know that a symmetric matrix of size n by n is always diagonalizable, equivalently speaking, always has n linear independent eigenvectors. Find an ordered basis consisting of eigenvectors of A.

c*) Specify the matrices P and D in the diagonalization $P^{-1}AP = D$

d*) Find an orthogonal matrix U such that $U^{-1}AU = D$ (Hint: An (real) orthogonal matrix means $U^{-1} = U^T$ or equivalently $U^T U = U U^T = I_n$).

(18) a) Give three distinct examples of elementary matrices and explain how they correspond to row operations for a given matrix of 3 by 3.

b) Factor the matrix into a product of elementary matrices.

$$A = \begin{pmatrix} 1 & 6 \\ 6 & 1 \end{pmatrix}$$

Solutions

(3) $[62, 12, -16]$

$$(4) \text{ a) } \begin{pmatrix} 3 & 0 & 12 \\ -9 & 0 & -35 \\ 18 & 1 & 70 \end{pmatrix}$$

b) The factorization or decomposition of a matrix is not unique in general. Here is an example of the answer. $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$,

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c) In consistent with the answer in (b), we have $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$,

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_3^{-1} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$E_5^{-1} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{d) } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(5) \text{ (b) not invertible; (c) Form the matrix } [A|I_3] = \begin{pmatrix} 1 & -2 & -1 & | & 1 & 0 & 0 \\ 3 & -5 & -2 & | & 0 & 1 & 0 \\ 2 & -5 & -2 & | & 0 & 0 & 1 \end{pmatrix}$$

Then use row operation to reduce to $[I_3|B]$. Hence the inverse equals

$$B = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & -1 \\ -5 & 1 & 1 \end{pmatrix}$$

6) From 5(b) we know that the coefficient matrix is not invertible, so either the system has no solution or it has infinitely many solutions.

Row reductions for the augmented matrix $\begin{pmatrix} 1 & 0 & -2 & | & 3 \\ 1 & -3 & 1 & | & -6 \\ 0 & 1 & -1 & | & 3 \end{pmatrix}$ give

$$\begin{pmatrix} 1 & 0 & -2 & | & 3 \\ 0 & 1 & -1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ Back-substitution shows } x = 3+2t, y = 3+t, z = t,$$

where t is a parameter of any real values. By the way observe that the augmented matrix is consistent with the coefficient matrix because they have the SAME RANK.

7) (a) -3 (b) $15(\lambda^2 - 1)$

8*) $\mathbf{ad}(M) = \text{transpose of } \begin{pmatrix} -4 & 3 & -9 \\ 3 & -5 & 4 \\ -5 & 1 & -3 \end{pmatrix}$

A straight forward computation shows $M\mathbf{ad}(M) = -11I_3$.

(9) (i) (a), (b)

(ii) (a), (c).

(13) (a) False. (b) True. (c) False (d) True.

(14) (a), (b), (c), (d), (e), (g), (h), (i), (j), (k), (ℓ), (m), (o)

(15) a) $\text{rank}(A) = 3$ (number of leading 1's in C), nullity of $A = 1$

b) A basis of $\text{Row}(A)$ consists of $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

A basis of $\text{Col}(A)$ consists of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$

d) Yes.

(16) a) $|\lambda I - A| = (\lambda - 1)(\lambda^2 - 2\lambda - 5)$

(17) (a)&(b) $E_5 = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}, E_8 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$

(c*) $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}$

(d*) The orthogonal matrix U consists of three eigenvectors that are orthogonal in \mathbf{R}^3 . So we need to orthogonalise the base $u = [-1, 1, 0]^T$, $v = [-1, 0, 1]^T$, $w = [1, 1, 1]$. Since the third vector in E_8 is orthogonal the any vectors in E_5 . We only need to orthogonalise the two vectors

in E_5 by Gram-Schmidt method. Let

$$u = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \tilde{v} = v - \frac{\langle v, u \rangle}{\langle u, u \rangle} u = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix},$$

Normalise the three vectors u, \tilde{v}, w to obtain

$$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$