Line Integral of a Vector Field

Suppose $\mathbf{r} = \mathbf{r}(t),\ a \le t \le b$, is a smooth parametrization of the curve C.
Recall, the unit tangent vector: $\mathbf{T} = d\mathbf{r}/ds = \mathbf{v}/|\mathbf{v}|$.

The ***line integral of a vector field*** $\mathbf{F}$ over C is the line integral (as defined before) of the ***scalar-valued*** function $\mathbf{F} \bullet \mathbf{T}$

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds = \int_C \mathbf{F} \bullet \frac{d\mathbf{r}}{ds}\, ds = \int_C \mathbf{F} \bullet d\mathbf{r} = \boxed{\int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt}\, dt}$$

Example

Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ for $\mathbf{F}(x,y,z) = z\mathbf{i} + xy\mathbf{j} - y^2\mathbf{k}$ $\langle z,\ xy,\ -y^2 \rangle$ and $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \sqrt{t}\mathbf{k},\ 0 \le t \le 1.$ $\langle t^2,\ t,\ \sqrt{t} \rangle$

$\mathbf{F} = \langle M, N, P \rangle$
$\mathbf{r} = \langle x, y, z \rangle = \langle x(t), y(t), z(t) \rangle$
$d\mathbf{r} = \langle dx, dy, dz \rangle$

scalar functions of x, y, z

$\int_C M\,dx + N\,dy + P\,dz$ **scalar differential form**

To evaluate, express everything in terms of t.

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$\mathbf{r} = \langle x, y, z \rangle = \langle x(t), y(t), z(t) \rangle$
$d\mathbf{r} = \langle dx, dy, dz \rangle$

scalar functions of x, y, z

$\int_C M\,dx + N\,dy + P\,dz$ **scalar differential form**

To evaluate, express everything in terms of t.

$dx = x'(t)dt$
$dy = y'(t)dt$
$dz = z'(t)dt$

$$\int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt} \right) dt$$ **parametric scalar evaluation**

$$(-\sin t)(-\sin t) + (t)(\cos t) + (2\cos t)(1)$$

Example

Evaluate $\int_C \underset{M}{-y}\,dx + \underset{N}{z}\,dy + \underset{P}{2x}\,dz$, C is the helix $\mathbf{r}(t) = \underset{x(t)}{(\cos t)}\mathbf{i} + \underset{y(t)}{(\sin t)}\mathbf{j} + \underset{z(t)}{t}\mathbf{k},\ 0 \le t \le 2\pi.$

$dx/dt = -\sin t$
$dy/dt = \cos t$
$dz/dt = 1$

use integration by parts

$$\int_0^{2\pi} \left(\sin^2 t + t\cos t + 2\cos t \right) dt = \ldots = \pi$$

rewrite as $\frac{1}{2}(1 - \cos 2t)$

Work in a Force Field

If the vector field $\mathbf{F}$ represents a force throughout a region then the line integral of $\mathbf{F}$ over a smooth curve C parametrized by $\mathbf{r} = \mathbf{r}(t),\ a \le t \le b$, gives the work done in moving an object from point $A = \mathbf{r}(a)$ to point $B = \mathbf{r}(b)$ along C.

Example

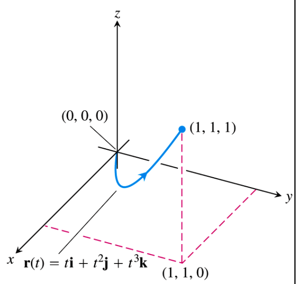
Find the work done by the force $\mathbf{F}(x,y,z) = (y - x^2)\,\mathbf{i} + (z - y^2)\,\mathbf{j} + (x - z^2)\,\mathbf{k}$ along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k},\ 0 \le t \le 1$.

$$W = \int_C \mathbf{F} \bullet \mathbf{T}\, ds = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt}\, dt = \ldots = \tfrac{29}{60}$$

$$\mathbf{F}(\mathbf{r}(t)) = \langle 0,\ t^3 - t^4,\ t - t^6 \rangle$$

$$\frac{d\mathbf{r}}{dt} = \langle 1, 2t, 3t^2 \rangle$$

$$\mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt} = 2t^4 - 2t^5 + 3t^3 - 3t^8$$



Flow/Circulation

velocity of a fluid flowing through

If the vector field $\mathbf{F}$ represents ~~a force throughout~~ a region then the line integral of $\mathbf{F}$ over a smooth curve C parametrized by $\mathbf{r} = \mathbf{r}(t),\ a \le t \le b$, gives the ***flow*** ~~done in moving an object~~ from point $A = \mathbf{r}(a)$ to point $B = \mathbf{r}(b)$ along C.

$$Flow = \int_C \mathbf{F} \bullet \mathbf{T}\, ds$$

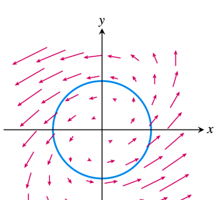
If the curve starts and ends at the same point $(A = B)$, the flow is called the ***circulation*** around the curve.

Example

Find the circulation of the field $\mathbf{F}(x,y) = (x - y)\mathbf{i} + (x)\mathbf{j}$ around the circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le 2\pi$.

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt}\, dt = \left(-\tfrac{1}{2}\sin^2 t + t \right)_0^{2\pi} = 2\pi$$

positive



counterclockwise circulation

$$\mathbf{F}(\mathbf{r}(t)) = \langle \cos t - \sin t,\ \cos t \rangle$$

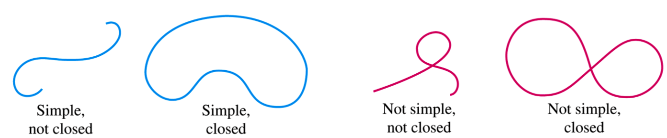
$$\frac{d\mathbf{r}}{dt} = \langle -\sin t,\ \cos t \rangle$$

$$\mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt} = (\cos t - \sin t)(-\sin t) + \cos^2 t = -\sin t \cos t + 1$$
$$-\sin t\cos t + \sin^2 t$$

$$\int \sin t \cos t\, dt = \int \sin t\, d(\sin t) = \tfrac{1}{2}\sin^2 t + C$$

Flux Across a Simple Closed Plane Curve

A plane curve is
- ***simple*** if it does not cross itself
- ***closed***, or a ***loop***, if it starts and ends at the same point.



Flux Across a Plane Curve

If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ then the ***flux*** of $\mathbf{F}$ across C is

$$\int_C \mathbf{F} \bullet \mathbf{n}\, ds$$

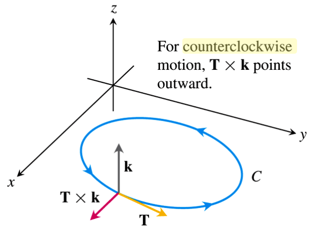
where $\mathbf{n}$ is the outward-pointing unit normal vector on C.

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}$$

$$\boxed{\mathbf{n} = \mathbf{T}\times\mathbf{k}} = \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\right)\times\mathbf{k} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}$$

$$\mathbf{F}\bullet\mathbf{n} = \langle M(x,y), N(x,y)\rangle \cdot \left\langle \frac{dy}{ds}, -\frac{dx}{ds}\right\rangle$$

$$= M(x,y)\frac{dy}{ds} - N(x,y)\frac{dx}{ds}$$



counterclockwise parametrization

$$\int_C \mathbf{F}\bullet\mathbf{n}\,ds = \int_C \left(M(x,y)\frac{dy}{ds} - N(x,y)\frac{dx}{ds}\right)ds = \oint_C M(x,y)dy - N(x,y)dx$$

Flux Across a Plane Curve

If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ then the ***flux*** of $\mathbf{F}$ across C is

$$\int_C \mathbf{F}\bullet\mathbf{n}\,ds = \oint_C M dy - N dx$$

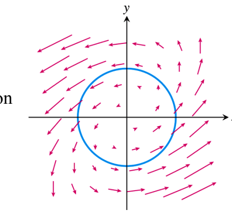
Evaluate with any smooth parametrization that traces C ***counterclockwise*** exactly once.

where $\boxed{\mathbf{n}}$ is the outward-pointing unit normal vector on C.

Example

Find the flux of $\mathbf{F}(x,y) = \underbrace{(x-y)}_{M}\mathbf{i} + \underbrace{(x)}_{N}\mathbf{j}$ across the circle $x^2 + y^2 = 1$.

$$\text{Flux} = \oint_C (x-y)dy - x\,dx = \int_a^b \left((x-y)\frac{dy}{dt} - x\frac{dx}{dt}\right)dt$$



where a, b, $x(t)$ and $y(t)$ come from any smooth parametrization that traces the circle counterclockwise exactly once; e.g. with $x = \cos t, y = \sin t,\ 0 \le t \le 2\pi$ we get

$$= \int_0^{2\pi} \big((\cos t - \sin t)(\cos t) - (\cos t)(-\sin t)\big)dt$$

$$= \int_0^{2\pi} \cos^2 t\,dt = \tfrac{1}{2}\int_0^{2\pi}(1+\cos 2t)\,dt = \tfrac{1}{2}\left(t + \tfrac{1}{2}\sin 2t\right)\Big|_0^{2\pi} = \pi$$

A positive answer indicates that the net flow across the curve is outward. (A net inward flow would have given a negative flux.)

$$\text{Flux} = \int_C \mathbf{F}\bullet\mathbf{n}\,ds = \oint_C M dy - N dx$$

where $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$

Evaluate with any smooth parametrization that traces C ***counterclockwise*** exactly once.