

§2.3 The Inverse of a Matrix

Definition. An $n \times n$ matrix A is **invertible** (or **nonsingular**) when there exists an $n \times n$ matrix B such that $AB = BA = I_n$, where I_n is the identity matrix of order n . The matrix B is the (multiplicative) **inverse** of A . A matrix that does not have an inverse is **noninvertible** (or **singular**).

Theorem 2.7. If A is an invertible matrix, then its inverse is unique. The inverse of A is denoted by A^{-1} .

Example 3. Find the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$.

Example 4. Show that the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$ has no inverse.

Theorem 2.8. If A is an invertible matrix, k is a positive integer, and c is a nonzero scalar, then A^{-1} , A^k , cA , and A^T are invertible and the statements below are true.

1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = A^{-1}A^{-1} \cdots A^{-1} = (A^{-1})^k$
3. $(cA)^{-1} = \frac{1}{c}A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$

Theorem 2.9. If A and B are invertible matrices of order n , then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem 2.10. If C is an invertible matrix, then the properties below are true.

1. If $AC = BC$, then $A = B$.
2. If $CA = CB$, then $A = B$.

Theorem 2.11. If A is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Example 8. Use an inverse matrix to solve the system

$$2x + 3y + z = -1$$

$$3x + 3y + z = 1$$

$$2x + 4y + z = -2$$

§2.4 Elementary Matrices

Definition. An $n \times n$ matrix is an **elementary matrix** when it can be obtained from the identity matrix I_n by a single elementary row operation.

Example 1. Which of the matrices below are elementary? For those that are, describe the corresponding elementary row operation.

$$\begin{array}{lll} \text{(a)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{(b)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \text{(c)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \text{(d)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \text{(e)} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} & \text{(f)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{array}$$

Theorem 2.12. Let E be the elementary matrix obtained by performing an elementary row operation on I_m . If that same elementary row operation is performed on an $m \times n$ matrix A , then the resulting matrix is the product EA .

Example 3. Find a sequence of elementary matrices that can be used to write the matrix

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix} \text{ in row-echelon form.}$$

Definition. Let A and B be $m \times n$ matrices. Matrix B is **row-equivalent** to A when there exists a finite number of elementary matrices E_1, E_2, \dots, E_k such that $B = E_k E_{k-1} \cdots E_2 E_1 A$.

Theorem 2.13. If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

Theorem 2.14. A square matrix A is invertible if and only if it can be written as the product of elementary matrices.

Example 4. Find a sequence of elementary matrices whose product is the nonsingular matrix $A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}$.

Definition. If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U , then $A = LU$ is an ***LU*-factorization** of A .

Example 6*. Find an LU -factorization of the matrix $A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}$.

Example 7. Solve the linear system.

$$x_1 - 3x_2 = -5$$

$$x_2 + 3x_3 = -1$$

$$2x_1 - 10x_2 + 2x_3 = -20$$