

§5.3 The Definite Integral (Continued)

**Exercise 20.** Evaluate  $\int_{-1}^1 (1 - |x|) dx$ .

*Solution.* Method I. The definite integral is equal to the area of under the graph of the “hat-function”  $y = 1 - |x|$ . So, we only need to find the area of the triangle  $= \frac{1}{2}bh = \frac{1}{2}(2)(1) = 1$ .

Method II. Note that  $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$  We split the integral into two parts:

$$\begin{aligned} \int_{-1}^1 (1 - |x|) dx &= \int_{-1}^0 (1 - |x|) dx + \int_0^1 (1 - |x|) dx \\ &= \int_{-1}^0 (1 - (-x)) dx + \int_0^1 (1 - x) dx \\ &= \left(x + \frac{x^2}{2}\right)\Big|_{-1}^0 + \left(x - \frac{x^2}{2}\right)\Big|_0^1 = -\left(-\frac{1}{2}\right) + \frac{1}{2} = 1. \end{aligned}$$

□

**Definition.** If  $f$  is integrable on  $[a, b]$ , then its **average value on  $[a, b]$** , which is also called its **mean**, is  $\text{av}(f) = \frac{1}{b-a} \int_a^b f(x) dx$ .

**Example 5.** Find the average value of  $f(x) = \sqrt{4 - x^2}$  on  $[-2, 2]$ .

*Solution.* The integral of  $f(x)$  over  $[-2, 2]$  is equal to the area of a semi-circle of radius 2. Hence we have  $\int_{-2}^2 \sqrt{4 - x^2} dx = \frac{1}{2}\pi R^2 = \frac{1}{2}\pi(2)^2 = 2\pi$ . By the definition,

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{2 - (-2)} (2\pi) = \frac{\pi}{2}.$$

□

Video [The Integral \(28 min\)](#) Definition of the integral. Signed area. Interval additivity property.

§5.4 The Fundamental Theorem of Calculus

**Theorem 3 (The Mean Value Theorem for Definite Integrals).** If  $f$  is continuous on  $[a, b]$ , then at some point  $c$  in  $[a, b]$ ,  $f(c) = \frac{1}{b-a} \int_a^b f(x) dx$ .

**Theorem 4 (The Fundamental Theorem of Calculus, Part 1).** If  $f$  is continuous on  $[a, b]$ , then  $F(x) = \int_a^x f(t) dt$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  and its derivative is  $f(x)$ :  $F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$ .

**Example 2.** Find  $dy/dx$  if

$$(a) y = \int_a^x (t^3 + 1) dt \quad (b) y = \int_x^5 3t \sin t dt \quad (c) y = \int_1^{x^2} \cos t dt \quad (d) y = \int_{1+3x^2}^4 \frac{1}{2+e^t} dt.$$

*Solution.* (a) The F.T.C. part I says that the derivative of  $F(x) = \int_a^x f(t) dt$  w.r.t.  $x$  is equal to the integrand function evaluated at  $x$ , namely,  $F'(x) = f(x)$ . Thus

$$\frac{dy}{dx} = x^3 + 1.$$

$$(b) y = -\int_5^x 3t \sin t dt \Rightarrow \frac{dy}{dx} = -3x \sin x.$$

(c) Write the function  $y$  as a composite function  $y = \int_1^u \cos t dt$ , with  $u = x^2$ . By chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (\cos u)(2x) = 2x \cos x^2.$$

$$(d) \text{ Chain rule yields } y'(x) = \frac{dy}{dx} = -6x \frac{1}{2+e^{1+3x^2}} = -\frac{6x}{2+e^{3x^2+1}}. \quad \square$$

**Exercise 80.** Find  $f(4)$  if  $\int_0^x f(t) dt = x \cos(\pi x)$ .

*Solution.* F.T.C  $\Rightarrow$

$$\frac{dy}{dx} \left( \int_0^x f(t) dt \right) = f(x).$$

On the other hand  $\frac{dy}{dx}(x \cos(\pi x)) = \cos(\pi x) - \pi x \sin(\pi x)$ . Hence

$$\begin{aligned} f(x) &= \cos(\pi x) - \pi x \sin(\pi x) \\ \Rightarrow f(4) &= \cos(4\pi) - \pi 4 \sin(4\pi) = 1. \end{aligned}$$

$\square$

**Theorem 4 Continued (The Fundamental Theorem of Calculus, Part 2).** If  $f$  is continuous over  $[a, b]$  and  $F$  is any antiderivative of  $f$  on  $[a, b]$ , then  $\int_a^b f(x) dx = F(b) - F(a)$ .

**Example 3.** Calculate (a)  $\int_{-\pi/4}^0 \sec x \tan x dx$  (c)  $\int_1^4 \left( \frac{3}{2} \sqrt{x} - \frac{4}{x^2} \right) dx$  (d)  $\int_0^1 \frac{dx}{x^2 + 1}$ .

*Solution.* (a)

$$\begin{aligned} \int_{-\pi/4}^0 \sec x \tan x dx &= \sec(x) \Big|_{-\pi/4}^0 \\ &= \sec(0) - \sec(-\pi/4) = \frac{1}{\cos 0} - \frac{1}{\cos(-\pi/4)} = 1 - \sqrt{2}. \end{aligned}$$

(c)

$$\begin{aligned}\int_0^1 \frac{dx}{x^2 + 1} &= \tan^{-1}(x) \Big|_0^1 \\ &= \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}.\end{aligned}$$

□

**Theorem 5 (The Net Change Theorem).** The net change in a differentiable function  $F(x)$  over an interval  $[a, b]$  is the integral of its rate of change:  $F(b) - F(a) = \int_a^b F'(x)dx$ .

**Example 8.** Find the area of the region between the  $x$ -axis and the graph of  $f(x) = x^3 - x^2 - 2x$ ,  $-1 \leq x \leq 2$ .

*Solution.* Factorize  $f(x) = x^3 - x^2 - 2x = x(x+1)(x-2)$ . We see that  $y = f(x)$  is positive over  $[-1, 0]$  and negative over  $[0, 2]$ . So the area of the region between the  $x$ -axis and the graph of  $f(x)$  is given by

$$\begin{aligned}A &= \int_{-1}^2 |f(x)|dx = \int_{-1}^0 f(x)dx - \int_0^2 f(x)dx \\ &= \left( \frac{x^4}{4} - \frac{x^3}{3} - x^2 \right) \Big|_{-1}^0 - \left( \frac{x^4}{4} - \frac{x^3}{3} - x^2 \right) \Big|_0^2 \\ &= \frac{5}{12} - \left( -\frac{8}{3} \right) = \frac{37}{12}.\end{aligned}$$

□

Video [The Fundamental Theorem of Calculus \(26 min\)](#) Average value theorem. The function  $F(x) = \int_a^x f(s) ds$ . The fundamental theorem of calculus.

### §5.5 Indefinite Integrals and the Substitution Method

**Example 1.** Find the integral  $\int (x^3 + x)^5 (3x^2 + 1) dx$ .

*Solution.* Let  $u = x^3 + x$ , then  $du = u' dx = (3x^2 + 1) dx$ . We have

$$\int (x^3 + x)^5 (3x^2 + 1) dx = \int u^5 du = \frac{u^6}{6} + C = \frac{(x^3 + x)^6}{6} + C.$$

□

**Theorem 6 (The Substitution Rule).** If  $u = g(x)$  is a differentiable function whose range is an interval  $I$ , and  $f$  is continuous on  $I$ , then  $\int f(g(x)) \cdot g'(x) dx = \int f(u) du$ .

**Example 4.** Find  $\int \cos(7\theta + 3) d\theta$ .

[Answer]  $\frac{\sin(7\theta+3)}{7} + C$

**Example 5.** Find  $\int x^2 e^{x^3} dx$ .

[Answer]  $\frac{e^{x^3}}{3} + C$

**Example 6.** Evaluate  $\int x\sqrt{2x+1} dx$ .

[Answer]  $\frac{(2x+1)^{3/2}}{3} + C$

*Solution.* Let  $u = 2x + 1$ , then  $x = \frac{u-1}{2}$ ,  $dx = \frac{1}{2}du$ .

$$\begin{aligned}\int x\sqrt{2x+1}dx &= \int \left(\frac{u-1}{2}\right)u^{1/2}\frac{du}{2} \\ &= \frac{1}{4} \int (u-1)u^{1/2}du = \frac{1}{4} \left( \int u^{3/2}du - \int u^{1/2}du \right) \\ &= \frac{1}{4} \left[ \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C \right] = \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C.\end{aligned}$$

□

### Integrals of the Tangent, Cotangent, Secant, and Cosecant Functions.

$$\int \tan x dx = \ln |\sec x| + C$$

$$\int \sec x dx = \ln |\sec x + \tan x| + C$$

$$\int \cot x dx = \ln |\sin x| + C$$

$$\int \csc x dx = -\ln |\csc x + \cot x| + C$$

Video [Change of Variables \(Substitution\) \(21 minutes\)](#)

Differentials. Using basic “ $u$ -substitutions” to find indefinite integrals and compute definite integrals.

### §5.6 Definite Integral Substitutions and the Area Between Curves

**Theorem 7 (Substitution in Definite Integrals).** If  $g'$  is continuous on the interval  $[a, b]$  and  $f$  is continuous on the range of  $g(x) = u$ , then  $\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$ .

**Example 1.** Evaluate  $\int_{-1}^1 3x^2\sqrt{x^3+1}dx$ .

*Solution.* Let  $u = g(x) = x^3 + 1$ , then  $du = u'dx = 3x^2dx$ . We have

$$\begin{aligned}\int_{-1}^1 3x^2\sqrt{x^3+1}dx &= \int_0^2 \sqrt{u}du \\ &= \frac{2}{3}u^{3/2}\Big|_0^2 = \frac{2}{3}(2^{3/2}) = \frac{4\sqrt{2}}{3}.\end{aligned}$$

□

**Example 2.** Evaluate (a)  $\int_{\pi/4}^{\pi/2} \cot \theta \csc^2 \theta d\theta$ .

*Solution.* Let  $u = \cot \theta$ , then  $du = u'd\theta = -\csc^2 \theta d\theta$ .

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \cot \theta \csc^2 \theta d\theta &= -\int_1^0 u du \\ &= -\frac{u^2}{2}\Big|_1^0 = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}.\end{aligned}$$

□

**Definition.** If  $f$  and  $g$  are continuous with  $f(x) \geq g(x)$  throughout  $[a, b]$ , then the **area of the region between the curves  $y = f(x)$  and  $y = g(x)$  from  $a$  to  $b$**  is the integral of  $(f - g)$  from  $a$  to  $b$ :  $A = \int_a^b [f(x) - g(x)]dx$ .

**Example 4.** Find the area of the region bounded above by the curve  $y = 2e^{-x} + x$ , below by the curve  $y = e^x/2$ , on the left by  $x = 0$ , and on the right by  $x = 1$ .

*Solution.*

$$\begin{aligned}A &= \int_0^1 (2e^{-x} + x) - \frac{e^x}{2} dx = -2e^{-x} + \frac{x^2}{2} - \frac{e^x}{2}\Big|_0^1 \\ &= 3 - \frac{e}{2} - \frac{2}{e}.\end{aligned}$$

□

**Example 5.** Find the area of the region enclosed by the parabola  $y = 2 - x^2$  and the line  $y = -x$ .

**Examples 6 & 7.** Find the area of the region in the first quadrant that is bounded above by  $y = \sqrt{x}$  and below by the  $x$ -axis and the line  $y = x - 2$ .

Video [Areas Between Curves \(19 minutes\)](#)