Definition

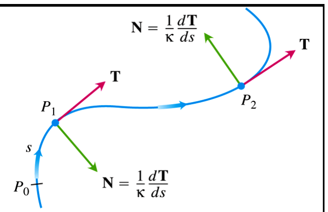
The ***principal unit normal*** vector is defined by

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$$

(This **is** a unit vector since by definition $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$)

An alternative formula:

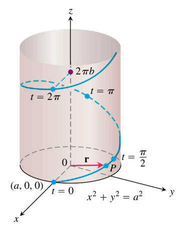
$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \text{where } \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$



Example

Find $\mathbf{T}, \mathbf{N}$ and κ for the helix

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + (bt)\mathbf{k}, \quad a, b \geq 0, \;\; a^2+b^2 \neq 0$$



Solution

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-a\sin t)\boldsymbol{i} + (a\cos t)\boldsymbol{j} + b\boldsymbol{k}$$

$$|\mathbf{v}| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2+b^2}$$

so $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \left(-\dfrac{a}{\sqrt{a^2+b^2}}\sin t\right)\mathbf{i} + \left(\dfrac{a}{\sqrt{a^2+b^2}}\cos t\right)\mathbf{j} + \dfrac{b}{\sqrt{a^2+b^2}}\mathbf{k}$

$$\frac{d\mathbf{T}}{dt} = \left(-\frac{a}{\sqrt{a^2+b^2}}\cos t\right)\mathbf{i} + \left(-\frac{a}{\sqrt{a^2+b^2}}\sin t\right)\mathbf{j} + 0\mathbf{k}$$

$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{a^2}{a^2+b^2}\cos^2 t + \frac{a^2}{a^2+b^2}\sin^2 t} = \sqrt{\frac{a^2}{a^2+b^2}} = \frac{a}{\sqrt{a^2+b^2}}$$

so $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = (-\cos t)\mathbf{i} + (-\sin t)\mathbf{j}$

and $\kappa = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right| = \dfrac{1}{\sqrt{a^2+b^2}}\dfrac{a}{\sqrt{a^2+b^2}} = \dfrac{a}{a^2+b^2}$

$$\boxed{\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\right|\frac{1}{|\mathbf{v}|}} \qquad \left(\frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{|\mathbf{v}|}\right) \qquad \left(s(t) = \int_{t_0}^{t}|\mathbf{v}|\,d\tau\right)$$

(chain rule)

Tangential and Normal Components of Acceleration

It is often useful to write the acceleration vector of a moving object as

$$\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}$$

the (common) direction of $\dfrac{d\mathbf{T}}{ds}$ and $\dfrac{d\mathbf{T}}{dt}$

We will derive formulas for a_{T} and a_{N} - the ***tangential*** and ***normal*** components of **a**.

Recall that **T is the direction of v** i.e $\mathbf{v} = v_{\mathrm{T}}\mathbf{T}$

where $v_{\mathrm{T}} = |\mathbf{v}| \quad (v_{\mathrm{N}} = 0)$

$\kappa|\mathbf{v}|^2\mathbf{N}$

So

$$\mathbf{a} = \frac{d}{dt}\mathbf{v} = \frac{d}{dt}(|\mathbf{v}|\mathbf{T}) = \left(\frac{d}{dt}|\mathbf{v}|\right)\mathbf{T} + |\mathbf{v}|\frac{d\mathbf{T}}{dt}$$

$$= \left|\frac{d\mathbf{T}}{dt}\right|\mathbf{N}$$

$$= \kappa|\mathbf{v}|$$

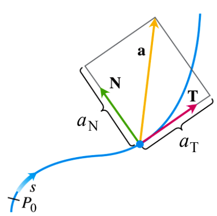
So

$$\boxed{a_{\mathrm{T}} = \frac{d}{dt}|\mathbf{v}| \quad \text{and} \quad a_{\mathrm{N}} = \kappa|\mathbf{v}|^2}$$

or

$$\boxed{a_{\mathrm{T}} = \frac{d^2 s}{dt^2} \quad \text{and} \quad a_{\mathrm{N}} = \kappa\left(\frac{ds}{dt}\right)^2}$$

$$\left(\frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{|\mathbf{v}|}\right) \qquad \left(s(t) = \int_{t_0}^{t}|\mathbf{v}|\,d\tau\right)$$



1

Note

1) If $\kappa = 0$ (no turning) then $a_{\mathrm{N}} = 0$, and so the direction of $\mathbf{a}$ is $\mathbf{T}$.

2) Otherwise, a_{N} is proportional to the square of the speed (κ = coeff. of proportionality).

If the object's speed is constant, then $a_{\mathrm{T}} = 0$, and so the direction of $\mathbf{a}$ is $\mathbf{N}$.