CS280 - Computer Vision - Assignment 1

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January 27, 2016

1 Perspective Projection

1. Let's consider plane P and a line l on P. Line l is defined by the following equation

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

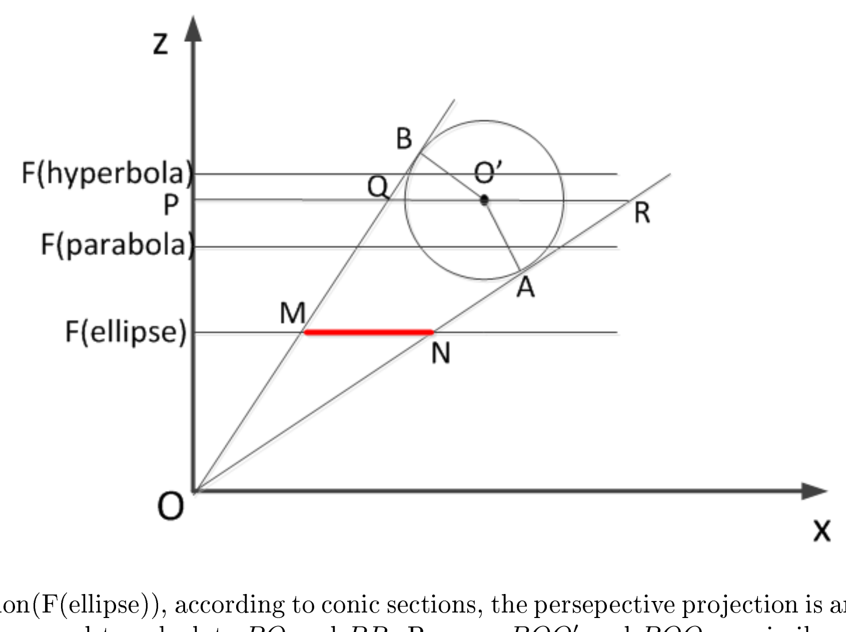
The vanishing point of line l is $\left(f\frac{a}{c}, f\frac{b}{c}, f\right)$. Let $n_P$be the normal vector of plane P and we also know that the direction vector of line l is $d_l = (a, b, c)$. We have

$$d_l \cdot n_P = 0$$

Moreover, the vanishing lien of P is the intersection of the projection plane $z = f$ and the plane P'that is parallel to P and contains the origin. Since P' is parallel to P, we have $n_{P'} = n_P$. Moreover, since P'contains the origin we know that the vanishing line $(x, y, f) \cdot n_{P'} = (x, y, f) \cdot n_P = 0$. Since we have

$$\left(f\frac{a}{c}, f\frac{b}{c}, f\right) \cdot n_P = \frac{f}{c}(a, b, c) \cdot n_P = 0$$

Therefore, $\left(f\frac{a}{c}, f\frac{b}{c}, f\right)$ is on the vanishing line of P.
2.



Under typical condition(F(ellipse)), according to conic sections, the persepective projection is an ellipse. To calculate the major axis MN we need to calculate PQ and PR. Because BQO' and PQO are similar trangulars, we have

$$\frac{QB}{r} = \frac{PQ}{Z}$$

we have

$$QB = \frac{r \cdot PQ}{Z}$$

$$O'Q^2 = r^2 + QB^2 = r^2\left(1 + \frac{PQ^2}{Z^2}\right)$$

Then we have

$$(X - PQ)^2 = r^2\left(1 + \frac{PQ^2}{Z^2}\right)$$

Then,

$$PQ = \frac{Z^2X^2 \pm rZ\sqrt{Z^2 + X^2 - r^2}}{Z^2 - r^2}$$

To calculate PR, we can use

$$\frac{AR}{r} = \frac{PR}{Z}$$

then we have

$$AR = r\frac{PR}{Z}$$

similarly we get

$$(PR - X)^2 = r^2\left(1 + \frac{PR^2}{Z^2}\right)$$

Solve the above equation, we will get

$$PQ = \frac{Z^2X^2 \pm rZ\sqrt{Z^2 + X^2 - r^2}}{Z^2 - r^2}$$

Therefore, we have

$$QR = \frac{2rZ\sqrt{Z^2 + X^2 - r^2}}{Z^2 - r^2}$$

By similarity we get

$$a = MN = QR\frac{f}{Z} = \frac{2rf\sqrt{Z^2 + X^2 - r^2}}{Z^2 - r^2}$$

By analogy, since $Y = 0$,we will have

$$b = \frac{2rf\sqrt{Z^2 - r^2}}{Z^2 - r^2}$$

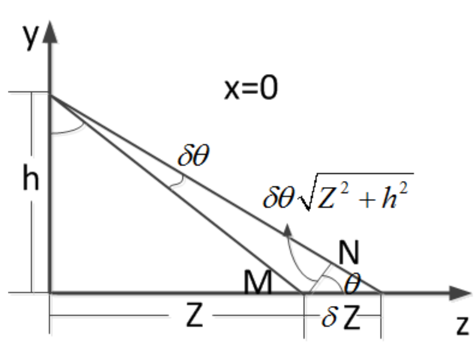
By the definition of eccentricity, we have

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{Z^2 - r^2}{Z^2 + X^2 - r^2}} = \frac{X}{\sqrt{Z^2 + X^2 - r^2}}$$

If the image plane intersects with only one of the segment OA and OB, and $O'$is on the other side of O, it will form a parabola. (F(parabola))

If the image plane intersects with only one of the segment OA and OB, and $O'$is on the same side of O, it will form a hyperbola. (F(hyperbola))

3.



Let h be the height of the observer's eyes, for sufficiently small $\delta\theta$ we get

$$M\bar{N} = \delta\theta\sqrt{Z^2 + h^2}$$

$$\delta\mathrm{Z}{=}\frac{M\bar{N}}{\cos\theta} = \frac{\sqrt{Z^2 + h^2}}{h}\delta\theta\sqrt{Z^2 + h^2} = \frac{Z^2 + h^2}{h}\delta\theta$$

where $\delta\theta = 1'$.

2 Rotations

1. Let the unit vector $\hat{s}$ be

$$\hat{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

Then, the corss product can be expressed as

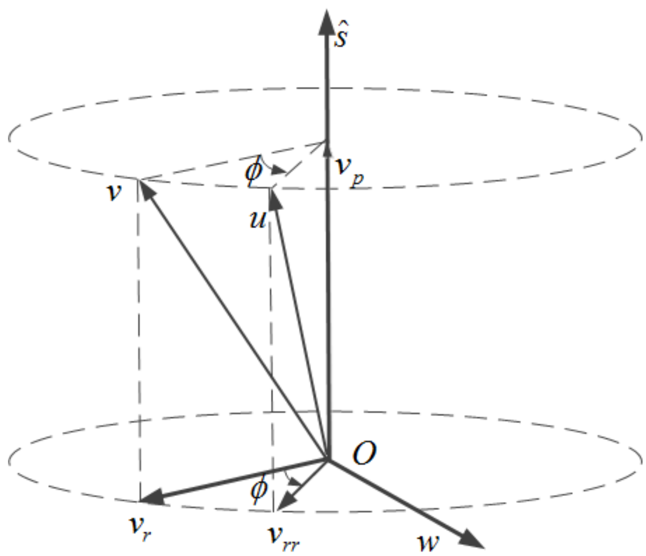
$$\hat{s} \times b = \begin{bmatrix} -s_3 b_2 + s_2 b_3 \\ s_3 b_1 - s_1 b_3 \\ -s_2 b_1 + s_1 b_2 \end{bmatrix} = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Thus we have

$$S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

is a skew symmetric matrix.

2. Let's consider the rotation $u = Rv$ as the following figure.



Let $w = \hat{s} \times v = Sv$ where S is defined as the previous question. From the above figure we have

$$v_p = (\hat{s} \cdot v)\,\hat{s}$$

$$v_r = v - v_p$$

$$w = Sv$$

$$v_{rr} = v_r \cos\phi + w \sin\phi$$

$$u = v_{rr} + v_p$$

Thus we have

$$u = (v - (\hat{s} \cdot v)\,\hat{s}) \cos\phi + Sv \sin\phi + (\hat{s} \cdot v)\,\hat{s} = v \cos\phi + \hat{s}\hat{s}^T v\,(1 - \cos\phi) + Sv \sin\phi$$

Because $u = Rv$, we have

$$R = I \cos\phi + \hat{s}\hat{s}^T\,(1 - \cos\phi) + S \sin\phi$$

Here

$$\hat{s}\hat{s}^T = \begin{bmatrix} s_1^2 & s_1 s_2 & s_1 s_3 \\ s_1 s_2 & s_2^2 & s_2 s_3 \\ s_1 s_3 & s_2 s_3 & s_3^2 \end{bmatrix}$$

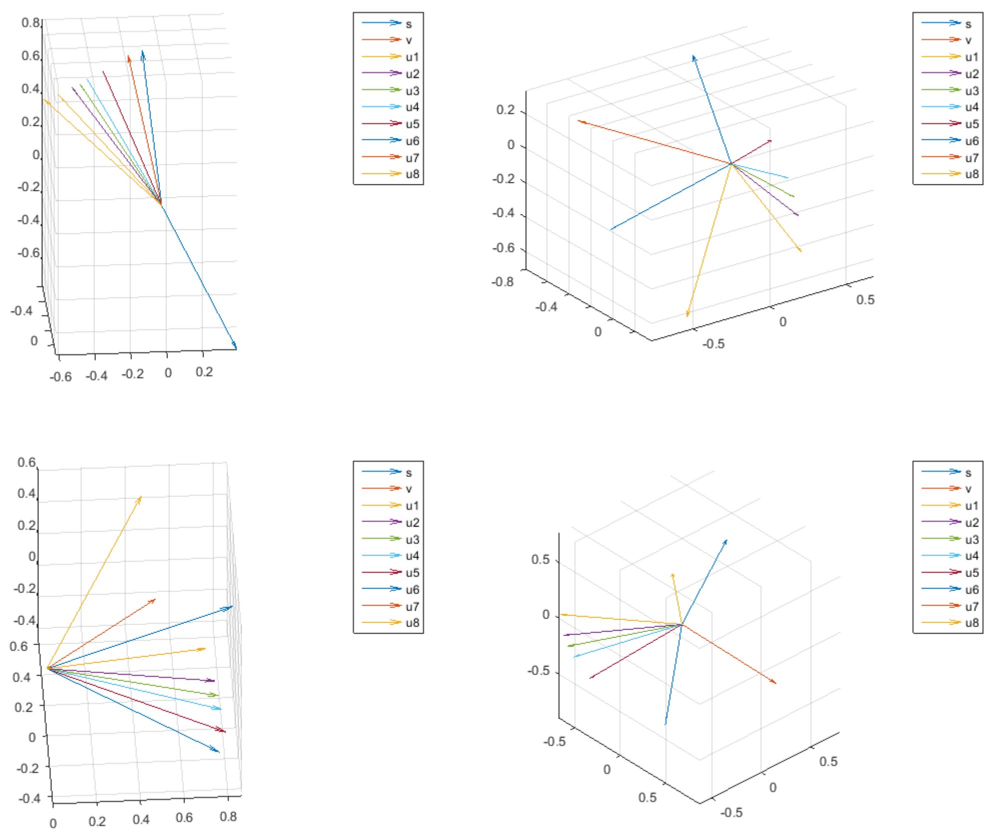
$$S^2 = \begin{bmatrix} -s_3^2 - s_2^2 & s_1 s_2 & s_1 s_3 \\ s_1 s_2 & -s_1^2 - s_3^2 & s_2 s_3 \\ s_1 s_3 & s_2 s_3 & -s_2^2 - s_1^2 \end{bmatrix}$$

and $s_1^2 + s_2^2 + s_3^2 = 1$, we have $\hat{s}\hat{s}^T = S^2 + I$.
Thus we have

$$R = I\cos\phi + S^2 + I - S^2\cos\phi - I\cos\phi + S\sin\phi$$

$$R = I + (\sin\phi)\, S + (1 - \cos\phi)\, S^2$$

3.The figures are show as following.



4. Since we have

$$R = I + (\sin\phi)\, S + (1 - \cos\phi)\, S^2$$

Given λ_S and v_S are eigenvalue and corresponding eigenvectors of matrix S. We know that the eigenvalue of R is $\lambda_R = 1 + \lambda_S \sin\phi + \lambda_S^2 (1 - \cos\phi)$. To see the eigen vectors, we have

$$Rv_R = v_R + (\sin\phi)\, Sv_R + (1 - \cos\phi)\, S^2 v_R = \lambda_R v_S$$

because R is a matrix polynomial of matrix S. To calculate the eigenvalues and eigen vectors of S.

$$\det(\lambda_S I - S) = 0$$

$$det\left(\left[\begin{array}{ccc}\lambda_S & s_3 & -s_2\\ -s_3 & \lambda_S & s_1\\ s_2 & -s_1 & \lambda_S\end{array}\right]\right)=\lambda_S^3+\left(s_1^2+s_2^2+s_3^2\right)\lambda_S=\lambda_S^3+\lambda_S=0$$

We have

$$\lambda_S=0,i,-i$$

Thus,

$$\lambda_{R,1}=1$$

$$\lambda_{R,2}=1+i\sin\phi-1+\cos\phi=\cos\phi+i\sin\phi$$

$$\lambda_{R,3}=1-i\sin\phi-1+\cos\phi=\cos\phi-i\sin\phi$$

To ensure $\hat{u}\times\hat{v}=\hat{s}$ for unit vectors $\hat{u}$,$\hat{v}$ and $\hat{s}$, we have $\hat{u}\cdot\hat{v}=\hat{v}\cdot\hat{s}=\hat{u}\cdot\hat{s}=0$, i.e. $\hat{u}$,$\hat{v}$ and $\hat{s}$, are orthogonal. Since

$$S\hat{s}=0$$

we know that $\hat{s}$ is the eigen vector for $\lambda_{R,1}=0$.
According to wikipedia page of cross product, we have if $\hat{u}\times\hat{v}=\hat{s}$,

$$S=\hat{v}\hat{u}^T-\hat{u}\hat{v}^T$$

Then the eigen vectors for $\lambda_S=\pm i$ can be found by

$$Sv_S=\hat{v}\hat{u}^Tv_S-\hat{u}\hat{v}^Tv_S=\pm iv_S$$

Let's take $\lambda_S=i$ as an example,

$$\hat{v}\hat{u}^Tv_S-\hat{u}\hat{v}^Tv_S=iv_S$$

we have

$$\left(\hat{u}^T\hat{v}\right)\left(\hat{u}^Tv_S\right)-\left(\hat{u}^T\hat{u}\right)\left(\hat{v}^Tv_S\right)=i\hat{u}^Tv_S\Rightarrow\hat{v}^Tv_s=-i\hat{u}^Tv_S$$

We have

$$v_S\left(\hat{v}+i\hat{u}\right)=0$$

$v_S=\hat{v}-i\hat{u}$ is a solution.
For $\lambda_S=-i$, by analogy, we get

$$v_S=\hat{v}+i\hat{u}$$

5. We have

$$trace\left(R\right)=\lambda_{R,1}+\lambda_{R,2}+\lambda_{R,3}=2\cos\phi+1$$

Then, we have

$$\cos\phi=\frac{1}{2}\left(trace\left(R\right)-1\right)$$

6.The code is attached in the report as below.

```
function [s,phi] = rot_to_ax_phi(R)
%base on part5 we can use cos(phi)=0.5(trace(R)-1)
phi=acos(0.5*trace(R)-0.5);
[v,d]=eig(R);
for i=1:3
    if (d(i,i)>=0.99999)&&(d(i,i)<=1.0001)
        s=v(:,i)/norm(v(:,i));
    end
end
end
```

3 Make yourself famous.

3.1 Affine Transform.

We have

$$v_i = \begin{bmatrix} \frac{V_{ix}}{V_{iz}} \\ \frac{V_{iy}}{V_{iz}} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{V_{iz}} & 0 & 0 \\ 0 & \frac{1}{V_{iz}} & 0 \\ 0 & 0 & \frac{1}{V_{iz}} \end{bmatrix} \begin{bmatrix} V_{ix} \\ V_{iy} \\ V_{iz} \end{bmatrix} = \begin{bmatrix} \frac{1}{V_{iz}} & 0 & 0 \\ 0 & \frac{1}{V_{iz}} & 0 \\ 0 & 0 & \frac{1}{V_{iz}} \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_{ix} \\ u_{iy} \\ 1 \end{bmatrix}$$

$$= \frac{1}{V_{iz}} \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{ix} \\ u_{iy} \\ 1 \end{bmatrix} = \hat{H} u_i = T(u_i)$$

After we get $\hat{H}$ we can easily recover matrix H by dividing all components of matrix $\hat{H}$ by its lower right cornor components.

For affine transform, the constraint for H is $h_{31} = h_{32} = 0$.

We can first solve $\hat{H}^*$first

$$\hat{H}^* = \arg\max \left\{ \sum_{i=1}^{4} \left\| \hat{H} u_i - v_i \right\|^2 : \ \hat{h}_{31} = \hat{h}_{32} = 0 \right\}$$

Plug in the matrix formulation and the constraints we get

$$\hat{H}^* = \arg\max \sum_{i=1}^{4} \left[\left(\hat{h}_{11} u_{ix} + \hat{h}_{12} u_{iy} + \hat{h}_{13} - v_{ix} \right)^2 + \left(\hat{h}_{21} u_{ix} + \hat{h}_{22} u_{iy} + \hat{h}_{23} - v_{iy} \right)^2 + \left(\hat{h}_{33} - 1 \right)^2 \right]$$

which is convex when $u_i \in R^3_+$.Take derivative of the 7 components of $\hat{H}$ we get

$$2\sum_{i=1}^{4} u_{ix} \left(\hat{h}_{11} u_{ix} + \hat{h}_{12} u_{iy} + \hat{h}_{13} - v_{ix} \right) = 0, \ 2\sum_{i=1}^{4} u_{iy} \left(\hat{h}_{11} u_{ix} + \hat{h}_{12} u_{iy} + \hat{h}_{13} - v_{ix} \right) = 0,$$

$$2\sum_{i=1}^{4} \left(\hat{h}_{11} u_{ix} + \hat{h}_{12} u_{iy} + \hat{h}_{13} - v_{ix} \right) = 0, \ 2\sum_{i=1}^{4} u_{ix} \left(\hat{h}_{21} u_{ix} + \hat{h}_{22} u_{iy} + \hat{h}_{23} - v_{iy} \right) = 0,$$

$$2\sum_{i=1}^{4} u_{iy} \left(\hat{h}_{21} u_{ix} + \hat{h}_{22} u_{iy} + \hat{h}_{23} - v_{iy} \right) = 0, \ 2\sum_{i=1}^{4} \left(\hat{h}_{21} u_{ix} + \hat{h}_{22} u_{iy} + \hat{h}_{23} - v_{iy} \right) = 0$$

$$4 \left(\hat{h}_{33} - 1 \right) = 0$$

Obvious we have $\hat{h}^*_{33} = 1$. Then

$$H^* = \hat{H}^*$$

For the other variables, the matrix form of the equations is

$$\begin{bmatrix} \sum_{i=1}^4 u_{ix}^2 & \sum_{i=1}^4 u_{ix} u_{iy} & \sum_{i=1}^4 u_{ix} & 0 & 0 & 0 \\ \sum_{i=1}^4 u_{iy} u_{ix} & \sum_{i=1}^4 u_{iy}^2 & \sum_{i=1}^4 u_{iy} & 0 & 0 & 0 \\ \sum_{i=1}^4 u_{ix} & \sum_{i=1}^4 u_{iy} & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sum_{i=1}^4 u_{ix}^2 & \sum_{i=1}^4 u_{ix} u_{iy} & \sum_{i=1}^4 u_{ix} \\ 0 & 0 & 0 & \sum_{i=1}^4 u_{iy} u_{ix} & \sum_{i=1}^4 u_{iy}^2 & \sum_{i=1}^4 u_{iy} \\ 0 & 0 & 0 & \sum_{i=1}^4 u_{ix} & \sum_{i=1}^4 u_{iy} & 4 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^4 u_{ix} v_{ix} \\ \sum_{i=1}^4 u_{iy} v_{ix} \\ \sum_{i=1}^4 v_{ix} \\ \sum_{i=1}^4 u_{ix} v_{iy} \\ \sum_{i=1}^4 u_{iy} v_{iy} \\ \sum_{i=1}^4 v_{iy} \end{bmatrix}$$

Solve the above linear equation we can get our H^*.

In affine transform, the focal point is assume to be infinite far away from the image plane. Under affine transformation, parallel lines are still in parallel. This can be observed from the results. But in reality the focal point of the images is not infinitely far away. Thus we can not perfectly transform the points from one image to the other. Mathematically, from the analysis we can see that V_{iz} is fixed to be 1 which is not the general case.

3.2 Homography

For homography, there is no constraint on matrix H .We have

$$v_{ix} = \frac{h_{11}u_{ix} + h_{12}u_{iy} + h_{13}}{h_{31}u_{ix} + h_{32}u_{iy} + 1}$$

$$v_{iy} = \frac{h_{21}u_{ix} + h_{22}u_{iy} + h_{23}}{h_{31}u_{ix} + h_{32}u_{iy} + 1}$$

The above 2 equations is in linear form of the variables in H,

$$\begin{bmatrix} u_{ix} & u_{iy} & 1 & 0 & 0 & 0 & -u_{ix}v_{ix} & -u_{iy}v_{ix} \\ 0 & 0 & 0 & u_{ix} & u_{iy} & 1 & -u_{ix}v_{iy} & -u_{iy}v_{iy} \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} v_{ix} \\ v_{iy} \end{bmatrix}$$

To solve the 8variables we need four points. Let

$$a_i = \begin{bmatrix} u_{ix} & u_{iy} & 1 & 0 & 0 & 0 & -u_{ix}v_{ix} & -u_{iy}v_{ix} \\ 0 & 0 & 0 & u_{ix} & u_{iy} & 1 & -u_{ix}v_{iy} & -u_{iy}v_{iy} \end{bmatrix}, \; b_i = \begin{bmatrix} v_{ix} \\ v_{iy} \end{bmatrix}$$

we have

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

3.3 Images

My source image is a traditional Chinese seal carved by me. The two characters on the seal represent a state of free and inner peace. The edges of the seal are approximately parallel.

The target image is a photo of my home town.

The results of affine transform are:

The result for homography transform are listed as below:

3.4 Code

The matlab functions are listed as below:

3.4.1 H=affine_solve(u,v)

```
function H = affine_solve(u,v)
%u,v are 2*4 matrices
H=zeros(3,3);
H(3,3)=1;
U=zeros(3,3);
U(1,1)=norm(u(1,:))^2;
U(2,2)=norm(u(2,:))^2;
U(3,3)=4;
U(1,2)=u(1,:)*(u(2,:)');
U(2,1)=U(1,2);
U(1,3)=sum(u(1,:));
U(3,1)=U(1,3);
U(2,3)=sum(u(2,:));
U(3,2)=U(2,3);
A=[U,zeros(3,3);zeros(3,3),U];
b=zeros(6,1);
b(1)=u(1,:)*(v(1,:)');
b(2)=u(2,:)*(v(1,:)');
b(3)=sum(v(1,:));
b(4)=u(1,:)*(v(2,:)');
b(5)=u(2,:)*(v(2,:)');
b(6)=sum(v(2,:));
x=A\b;
H(1,1)=x(1);
H(1,2)=x(2);
H(1,3)=x(3);
H(2,1)=x(4);
H(2,2)=x(5);
```

```
H(2,3)=x(6);
end
```

3.4.2 H=homography_solve(u,v)

```
function H = homography_solve(u,v)
H=zeros(3,3);
b=zeros(8,1);
b(1)=v(1,1);b(2)=v(2,1);
b(3)=v(1,2);b(4)=v(2,2);
b(5)=v(1,3);b(6)=v(2,3);
b(7)=v(1,4);b(8)=v(2,4);
A=zeros(8,8);
for i=1:4
    A((i-1)*2+1,1:3)=[u(:,i)' 1];
    A(2*i,4:6)=[u(:,i)' 1];
    A((i-1)*2+1,7:8)=-1*v(1,i)*u(:,i)';
    A(i*2,7:8)=-1*v(2,i)*u(:,i)';
end
x=A\b;
H(1,1)=x(1);
H(1,2)=x(2);
H(1,3)=x(3);
H(2,1)=x(4);
H(2,2)=x(5);
H(2,3)=x(6);
H(3,1)=x(7);
H(3,2)=x(8);
H(3,3)=1;
```

3.4.3 v=homography_transform(u,H)

```
function v = homography_transform(u,H)
n=size(u,2);
ul=[u;ones(1,n)];
V=H*ul;
v=zeros(2,n);
v(1,:)=V(1,:)./V(3,:);
v(2,:)=V(2,:)./V(3,:);
end
```

Other code used to generate the new images is in the submitted folder.