study group
Date: September 17, 2015

Homework 1 Solutions

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Problem 1.

The main idea is to create a recurrence relation that describes the algorithm and figure out asymptoptic behavior.
We will first describe the recurrence relation for the algorithm. Then we will prove that the runtime of this algorithm is loglinear.
Recurrence relation:

$T(n) = T(\frac{n}{4}) + T(\frac{3n}{4}) + \theta(n)$

where $T(\frac{n}{4})$ is the time cost of finding the median of medians

where $T(\frac{3n}{4})$ is the time cost of the recurrence in which we reduce the problem size to $\frac{3n}{4}$ (we can eliminate a quarter of all values every iteration of the algorithm).

and $\theta(n)$ is the time to find the median of every group of four.

Next we will calculate the upper bound of this algorithm:

We know $T(x) = x^{\rho}(1 + \int_1^x \frac{g(u)}{u^{\rho+1}} du)\,, where\, \rho = \sum_{i=0}^{k} a_i b_i^{\rho} = 1$

from problem 2 (see below for proof of this).

We can calculate ρ

$\rho = \sum_{i=1}^{n} a_i b_i^{\rho} = (\frac{1}{4})^1 + (\frac{3}{4})^1 = 1$

Thus

$T(x) = x(1 + \int_1^x \frac{g(u)}{u^{1+1}} du) = x(1 + \int_1^x \frac{1}{u} du)$

$= xlnx$

The runtime for this algorithm is loglinear.

Self-evaluation: 3 points. We believe that the above is a complete and correct proof.

Problem 2.

$T(x) = x^\rho(1 + \int_1^x \frac{g(u)}{u^{\rho+1}} du)\,, where\, \rho = \sum_{i=0}^k a_i b_i^\rho = 1$
The proof of the above statement is as follows:

First we show that $g(x) = x^\rho \int_{b_i x}^x \frac{g(u)}{u^{\rho+1}} du$

We will show this through induction:

Assume $g(x) \geq c_1 g(x)$ $cg(x) \leq g(\lambda x) \leq c^{'} g(x)$ or in other words, g is scale free.

$x^\rho \int_{b_i x}^x \frac{g(u)}{u^{\rho+1}} du \geq x^\rho (x - b_i x) \frac{cg(x)}{max(x^{\rho+1}, (b_i x)^{\rho+1})}$

$= \frac{(1-b_i)cg(x)}{max(1, b_i^{\rho+1})} \geq c_1 g(x)$

This will hold as long as $c_1 \leq \frac{(1-b_i)c}{max(1, b_i^{\rho+1})}$

Now assuming $g(x) \leq c_2 g(x)$, and knowing $cg(x) \leq g(\lambda x) \leq c^{'} g(x)$

$x^\rho \int_{b_i x}^x \frac{g(u)}{u^{\rho+1}} du \leq x^\rho (x - b_i x) \frac{c^{'} g(x)}{min(x^{\rho+1}, (b_i x)^{\rho+1})} = \frac{(1-b_i)c^{'} g(x)}{min(1, b_i^{\rho+1})}$

$\leq c_2 g(x)$

This will hold as long as $c_2 \geq \frac{(1-b_i)c^{'}}{min(1, b_i^{\rho+1})}$

Now we can prove that $T(x) = x^\rho(1 + \int_1^x \frac{g(u)}{u^{\rho+1}} du)\,, where\, \rho = \sum_{i=0}^k a_i b_i^\rho = 1$

First assume $T(x) \geq c_3 T(x)$

$T(x) = \sum_{i=1}^k a_i T(b_i x) + g(x)$

$\geq \sum_{i=1}^k a_i c_3 (b_i x)^\rho (1 + \int_{b_i x}^x \frac{g(u)}{u^{\rho+1}} du) + g(x)$

$= \sum_{i=1}^k a_i c_3 (b_i x)^\rho (1 + \int_1^x \frac{g(u)}{u^{\rho+1}} du - \int_{b_i x}^x \frac{g(u)}{u^{\rho+1}} du) + g(x)$

$= \sum_{i=1}^k a_i c_3 (b_i x)^\rho (1 + \int_1^x \frac{g(u)}{u^{\rho+1}} du - \sum_{i=1}^k a_i c_3 (b_i x)^\rho \int_{b_i x}^x \frac{g(u)}{u^{\rho+1}} du) + g(x)$

$= c_3 x^\rho (1 + \int_1^x \frac{g(u)}{u^{\rho+1}} du) - (\sum_{i=1}^k a_i c_3 b_i^\rho x^\rho \int_{b_i x}^x \frac{g(u)}{u^{\rho+1}} du) + g(x)$

$\geq c_3 x^\rho (1 + \int_1^x \frac{g(u)}{u^{\rho+1}} du) - c_3 \sum_{i=1}^k a_i b_i^\rho c_2 g(x) + g(x)$

$= c_3 x^\rho (1 + \int_1^x \frac{g(u)}{u^{\rho+1}} du) - (c_2 c_3 - 1) * g(x)$

$\geq c_3 x^\rho (1 + \int_1^x \frac{g(u)}{u^{\rho+1}} du)$

$= c_3 * T(x)$

This will hold as long as g(x) is nonnegative.

Next, we will assume $T(x) \leq c_4 T(x)$ and use the same logic as we just did above.

$T(x) = \sum_{i=1}^{k} a_i T(b_i x) + g(x)$

$\leq \sum_{i=1}^{k} a_i c_4 (b_i x)^\rho (1 + \int_{b_i x}^{x} \frac{g(u)}{u^{\rho+1}} du) + g(x)$

$= \sum_{i=1}^{k} a_i c_4 (b_i x)^\rho (1 + \int_{1}^{x} \frac{g(u)}{u^{\rho+1}} du - \int_{b_i x}^{x} \frac{g(u)}{u^{\rho+1}} du) + g(x)$

$= \sum_{i=1}^{k} a_i c_4 (b_i x)^\rho (1 + \int_{1}^{x} \frac{g(u)}{u^{\rho+1}} du - \sum_{i=1}^{k} a_i c_4 (b_i x)^\rho \int_{b_i x}^{x} \frac{g(u)}{u^{\rho+1}} du) + g(x)$

$= c_4 x^\rho (1 + \int_{1}^{x} \frac{g(u)}{u^{\rho+1}} du) - (\sum_{i=1}^{k} a_i c_4 b_i^\rho x^\rho \int_{b_i x}^{x} \frac{g(u)}{u^{\rho+1}} du) + g(x)$

$\leq c_4 x^\rho (1 + \int_{1}^{x} \frac{g(u)}{u^{\rho+1}} du) - c_4 \sum_{i=1}^{k} a_i b_i^\rho c_1 g(x) + g(x)$

$= c_4 x^\rho (1 + \int_{1}^{x} \frac{g(u)}{u^{\rho+1}} du) - (c_1 c_4 - 1) * g(x)$

$\leq c_4 x^\rho (1 + \int_{1}^{x} \frac{g(u)}{u^{\rho+1}} du)$

$= c_4 * T(x)$

Using the conclusions from these two steps, we have proved that
$T(x) = x^\rho (1 + \int_{1}^{x} \frac{g(u)}{u^{\rho+1}} du)\, , where\, \rho = \sum_{i=0}^{k} a_i b_i^\rho = 1$

Self-evaluation: 3 point. We believe that the above is a complete and correct proof.

Problem 3.

Because we need to achieve $O(logn)$ runtime we have to use a modified binary search that narrows down our search space in one array at a time. The pseudocode for the algorithm is described below:

$find(x, y, i)$
  $if\, x.length == 0$
    $return y[i]$
  $if\, y.length == 0$
    $return x[i]$
  $if\, i == 1$
    $return\, min(x[i], y[i])$
  $a = min(\frac{1}{2}i, x.length)$
  $b = min(\frac{1}{2}i, y.length)$
  $if\, x[a] \leq y[b]$
    $find(x[a+1 : end], y, i - a)$
  $else$
    $find(x, y[b+1 : end], i - b)$

Self-evaluation: 3 point. We believe the algorithm we have is correct in the context of the problem.
