MPC Formulation for Miniature Autonomous Race Cars

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Agenda

MPCC - Model Predictive Contouring Control

Original MPCC Formulation

Reference Trajectory as Splines & Why?

MPCC formulation with Splines

Results & Observations

References

MPCC - Model Predictive Contouring Control

MPCC is a control scheme based on minimisation of a cost function which reflects the trade-off between the competing objectives of accuracy and traversal time.

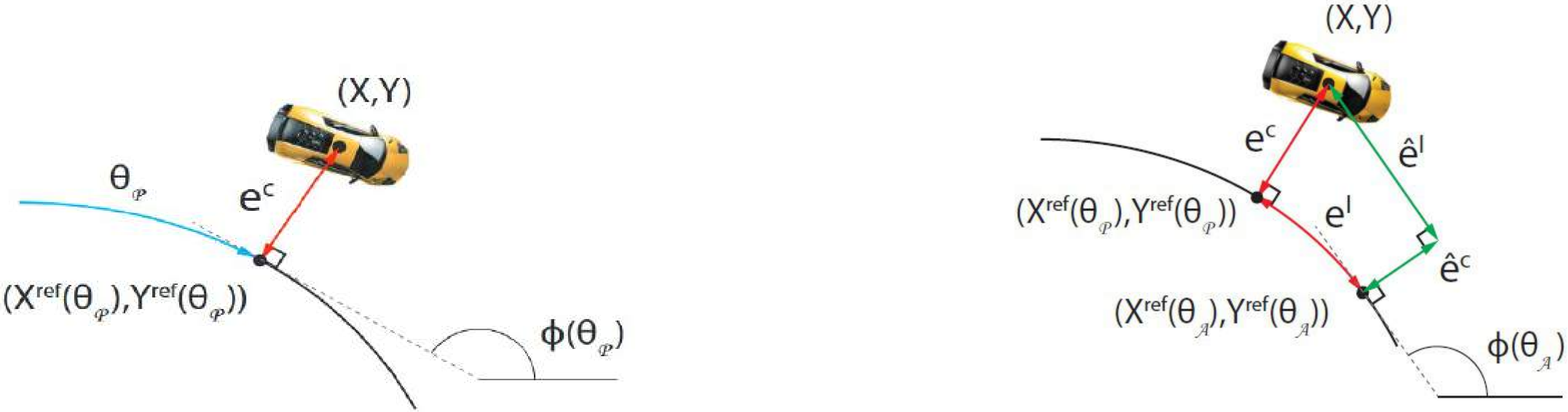


Figure 1.Contouring error e^c (left) and lag error e^l (right) with linear approximations $\hat{e}^c$ and $\hat{e}^l$. Retrieved from [1]

$$e^l(X,Y,\theta_A) \triangleq |\theta_A - \theta_P|$$

$$e^c(X,Y,\theta_P) \triangleq sin(\phi(\theta_P))(X - X^{ref}(\theta_P)) - cos(\phi(\theta_P))(Y - Y^{ref}(\theta_P))$$

$$\hat{e}^c(X,Y,\theta_A) \triangleq sin(\phi(\theta_A))(X - X^{ref}(\theta_A)) - cos(\phi(\theta_A))(Y - Y^{ref}(\theta_A))$$

$$\hat{e}^l(X,Y,\theta_A) \triangleq -cos(\phi(\theta_A))(X - X^{ref}(\theta_A)) - sin(\phi(\theta_A))(Y - Y^{ref}(\theta_A))$$

Original MPCC Formulation

$$\min_{x,u} \sum_{k=1}^{N} \|\hat{e}_k^c(X_k, Y_k)\|_{Q_1}^2 + \quad \|\hat{e}_k^l(X_k, Y_k)\|_{Q_2}^2 - q\theta_k + \|\Delta T_k\|_{R1}^2 + \|\Delta\delta_k\|_{R2}^2 + \|\Delta\theta_k\|_{R3}^2$$

subject to

$$x(0) = x_0$$

$$x_{k+1} = f(x_k, u_k),\ \forall k = 0, \ldots, N-1$$

$$\theta_{k+1} = \theta_k + \Delta\theta_k\ ,\ \forall k = 0, \ldots, N-1$$

$$x_{min} \leq x_k \leq x_{max},\ \forall k = 1, \ldots, N$$

$$u_{min} \leq u_k \leq u_{max},\ \forall k = 0, \ldots, N$$

$$(X_k - X_k^{ref})^2 + (Y_k - Y_k^{ref})^2 \leq r^2,\ \forall k = 1, \ldots, N$$

$$\hat{e}^c(X,Y) \triangleq sin(\phi^{ref})(X - X^{ref} - \nabla X^{ref}(\theta - \hat{\theta})) - cos(\phi^{ref})(Y - Y^{ref} - \nabla Y^{ref}(\theta - \hat{\theta}))$$

$$\hat{e}^l(X,Y) \triangleq -cos(\phi^{ref})(X - X^{ref} - \nabla X^{ref}(\theta - \hat{\theta})) - sin(\phi^{ref})(Y - Y^{ref} - \nabla Y^{ref}(\theta - \hat{\theta}))$$

State Vector x: $[X, Y, \psi, V_X, V_Y, \omega, T, \delta, \theta]^T$ $\quad \{Q1, Q2, q, R1, R2, R3\} \in \mathbb{R}$

Control Vector u:$[\Delta T, \Delta\delta, \Delta\theta]^T$

Reference Trajectory as Splines Why?

- **Use Splines to parameterize the path.**
  - Change the spline parameters to change the shape of the path.
  - Easier to sample discrete coordinates to represent a path.
- **Splines are basically piecewise polynomials.**
  - Break the curve into multiple segments, each segment is a polynomial interpolated to form a continuous curve.
- **Cubic splines** are the most common lowest order piecewise polynomials.
  - The continuity at the control points of piecewise cubic splines is also defined.

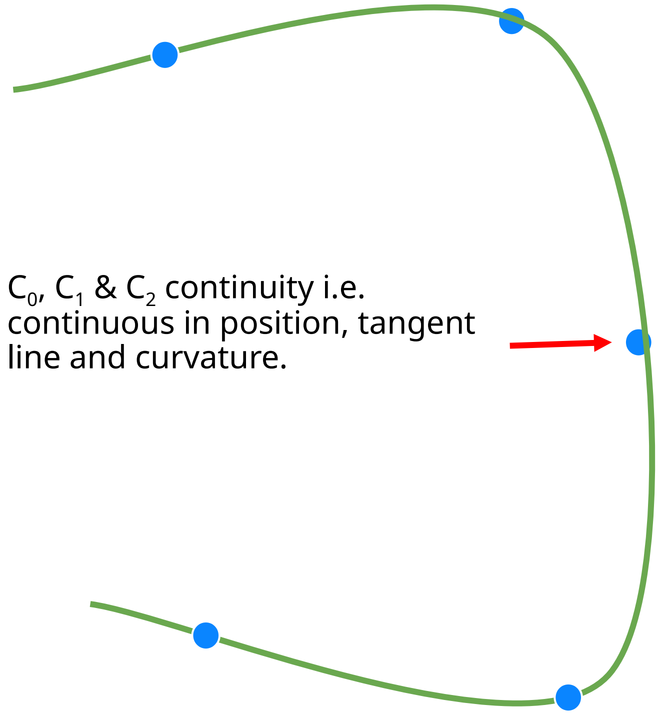


Figure 2. Spline representing a curve in green, blue dots are controls points where polynomials are interpolated

Reference path representation with Cubic Splines

Center track is parameterized by θ where,
L: Length of the track

$$\theta \in [0, L]$$

$$X^{ref}(\theta_A) = a_x + b_x\theta_A + c_x\theta_A^2 + d_x\theta_A^3$$
$$Y^{ref}(\theta_A) = a_y + b_y\theta_A + c_y\theta_A^2 + d_y\theta_A^3$$

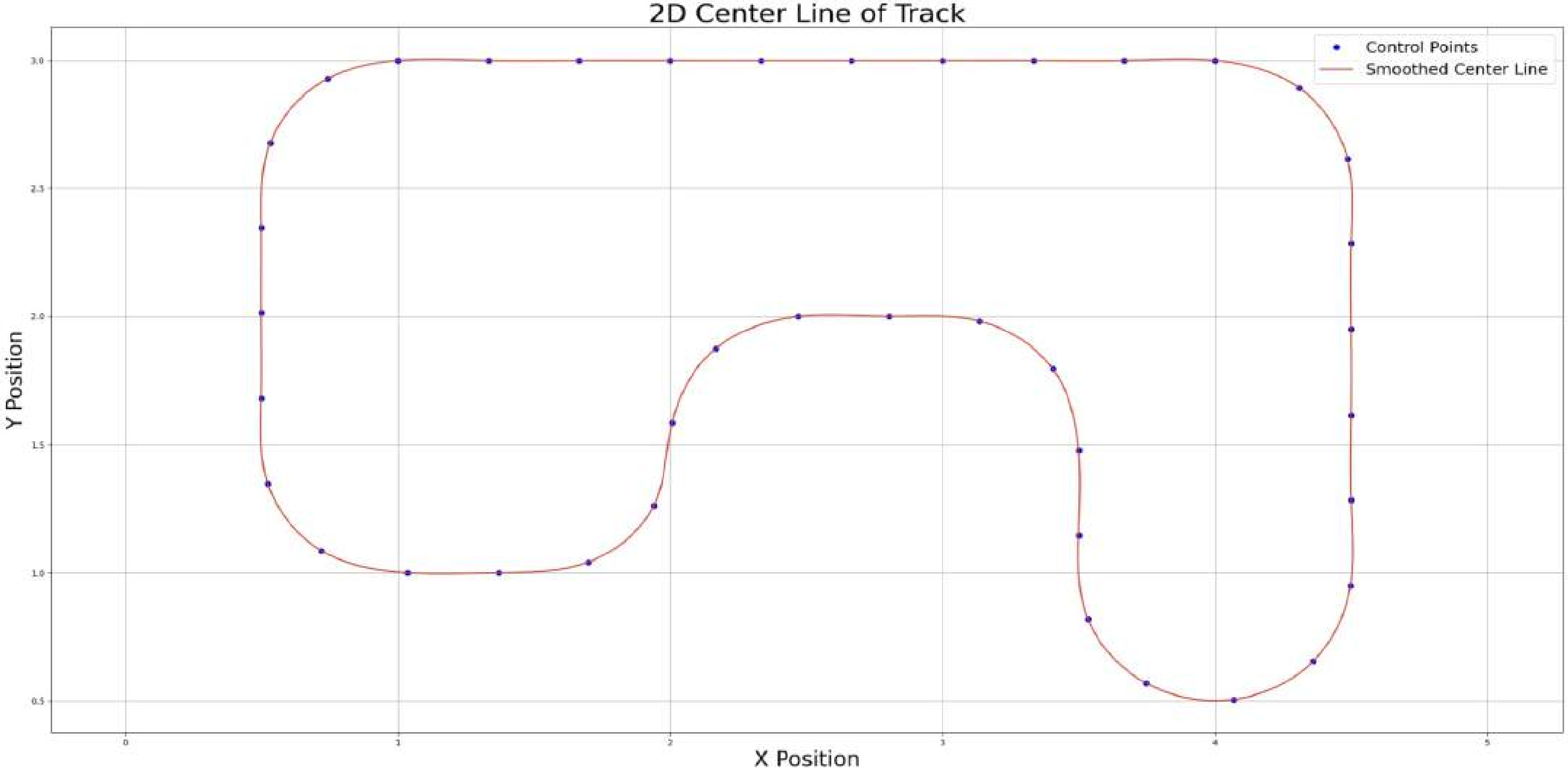
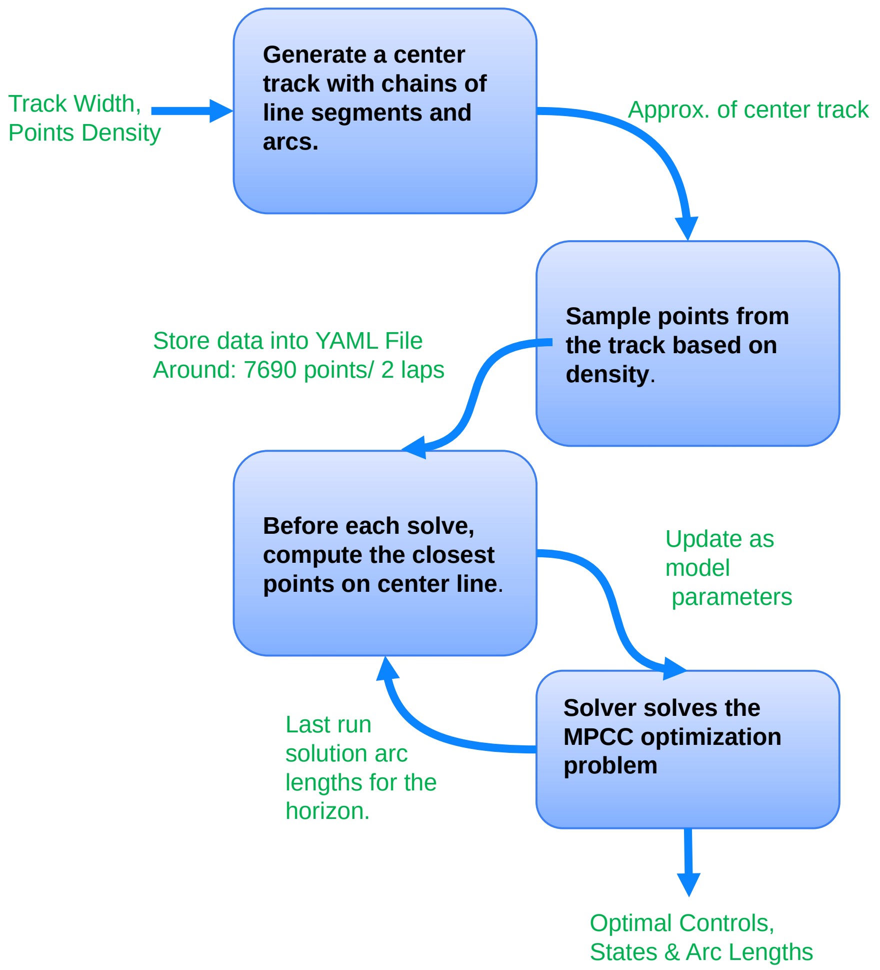
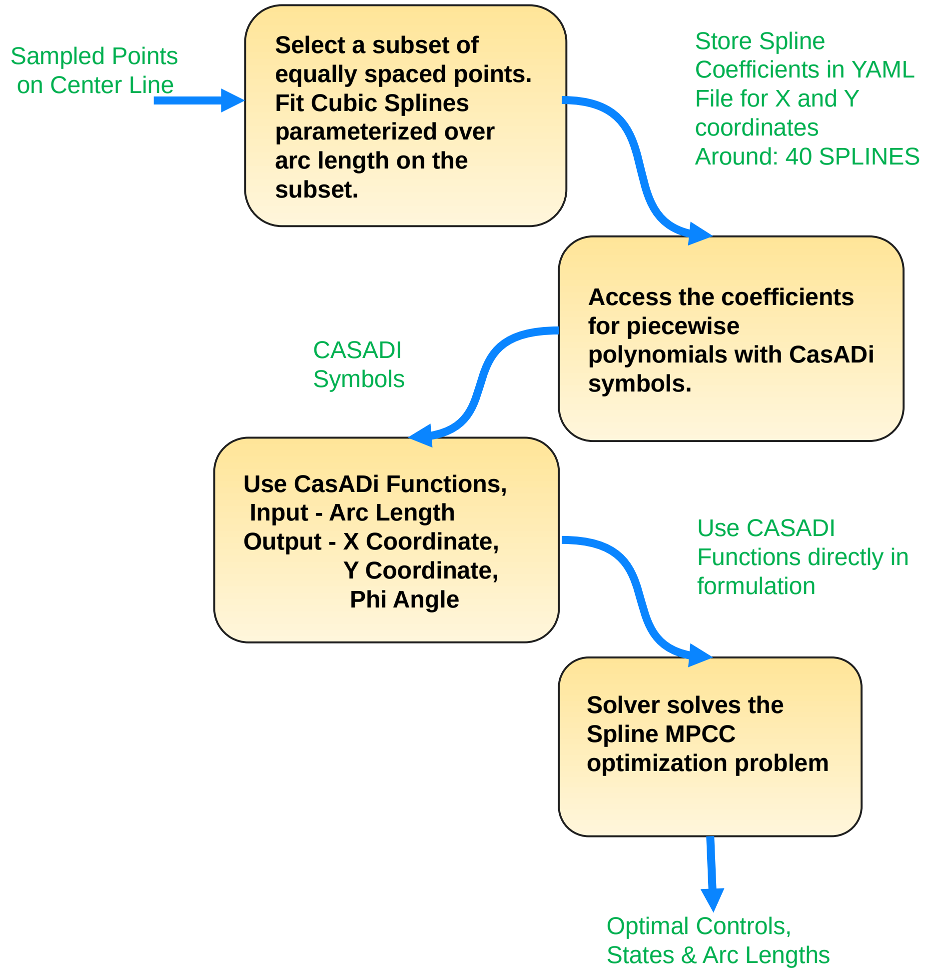


Figure 3. Result of the cubic spline interpolator fitting across the sampled control points

ORIGINAL MPCC CONTROLLER



SPLINE BASED MPCC CONTROLLER



MPCC formulation with Splines

$$\min_{x,u} \sum_{k=1}^{N-1} \|\hat{e}^l_k(X_k, Y_k, \theta_{A,k})\|^2_{Q2} \quad - q\theta_{A,k} + \|\Delta T_k\|^2_{R1} + \|\Delta\delta_k\|^2_{R2} - q_N\theta_{A,N}$$

subject to

$$x(0) = x_0$$
$$x_{k+1} = f(x_k, u_k),\ \forall k = 0, \ldots, N-1$$
$$\theta_{A,k+1} = \theta_{A,k} + \Delta\theta_k,\ \forall k = 0, \ldots, N-1$$
$$x_{min} \le x_k \le x_{max},\ \forall k = 1, \ldots, N$$
$$u_{min} \le u_k \le u_{max},\ \forall k = 0, \ldots, N$$
$$(X_k - X_k^{ref}(\theta_{A,k}))^2 + (Y_k - Y_k^{ref}(\theta_{A,k}))^2 \le r^2,\ \forall k = 1, \ldots, N$$

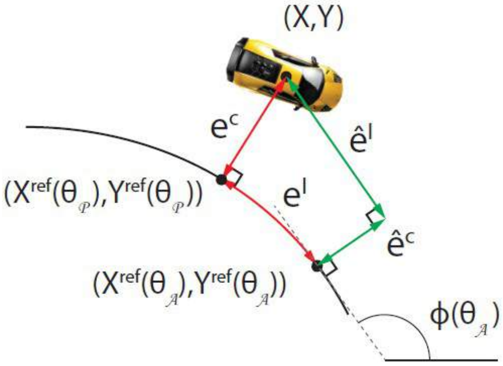


Figure 4. Retrieved from [1]

$$\theta_A \approx \theta_P$$
$$\hat{e}_l(X_k, Y_k, \theta_{A,k}) \approx 0$$
$$\hat{e}_c(X_k, Y_k, \theta_{A,k}) \approx \text{ track constraints}$$

where,

$$\hat{e}^c(X, Y, \theta_A) \triangleq sin(\phi^{ref}(\theta_A))(X - X^{ref}(\theta_A)) - cos(\phi^{ref}(\theta_A))(Y - Y^{ref}(\theta_A))$$
$$\hat{e}^l(X, Y, \theta_A) \triangleq -cos(\phi^{ref}(\theta_A))(X - X^{ref}(\theta_A)) - sin(\phi^{ref}(\theta_A))(Y - Y^{ref}(\theta_A))$$
$$X^{ref}(\theta_A) = a_x + b_x\theta_A + c_x\theta_A^2 + d_x\theta_A^3$$
$$Y^{ref}(\theta_A) = a_y + b_y\theta_A + c_y\theta_A^2 + d_y\theta_A^3$$
$$\phi^{ref}(\theta_A) = \frac{\partial Y^{ref}(\theta_A)}{\partial X^{ref}(\theta_A)}$$
$$0 \le \theta_A \le L$$

- **Original MPCC formulation:**

$$\min_{x,u} \sum_{k=1}^{N} \|\hat{e}_k^c(X_k, Y_k)\|_{Q_1}^2 + \quad \|\hat{e}_k^l(X_k, Y_k)\|_{Q_2}^2 - q\theta_k + \|\Delta T_k\|_{R1}^2 + \|\Delta\delta_k\|_{R2}^2 + \|\Delta\theta_k\|_{R3}^2$$

- **MPCC formulation with spline:**

$$\begin{aligned}
&\min_{x,u} \sum_{k=1}^{N-1} \|\hat{e}_k^l(X_k, Y_k, \theta_{A,k})\|_{Q_2}^2 && - q\theta_{A,k} + \|\Delta T_k\|_{R1}^2 + \|\Delta\delta_k\|_{R2}^2 - q_N\theta_{A,N} \\
&\text{subject to} && x(0) = x_0 \\
& && x_{k+1} = f(x_k, u_k),\ \forall k = 0, \ldots, N-1 \\
& && \theta_{A,k+1} = \theta_{A,k} + \Delta\theta_k,\ \forall k = 0, \ldots, N-1 \\
& && x_{min} \le x_k \le x_{max},\ \forall k = 1, \ldots, N \\
& && u_{min} \le u_k \le u_{max},\ \forall k = 0, \ldots, N \\
& && (X_k - X_k^{ref}(\theta_{A,k}))^2 + (Y_k - Y_k^{ref}(\theta_{A,k}))^2 \le r^2,\ \forall k = 1, \ldots, N
\end{aligned}$$

- **Configuration for the ocp solver**

Integrator : RK4

NLP solver: SQP_RTI

Hessian Approximation: Gauss- Newton

RESULTS & OBSERVATIONS

Comparison of Lag Errors for Formulations

```
Spline MPCC Lag Error: 0.00228387
Spline MPCC Lag Error: 3.52428e-07
Spline MPCC Lag Error: 7.66978e-07
Spline MPCC Lag Error: 8.04986e-07
Spline MPCC Lag Error: 9.07669e-08
Spline MPCC Lag Error: 1.24219e-07
Spline MPCC Lag Error: 3.79848e-08
Spline MPCC Lag Error: 2.51437e-08
Spline MPCC Lag Error: 1.67111e-06
Spline MPCC Lag Error: 2.62932e-06
Spline MPCC Lag Error: 3.65649e-05
Spline MPCC Lag Error: 7.53367e-05
Spline MPCC Lag Error: 0.000149174
Spline MPCC Lag Error: 0.000605955
Spline MPCC Lag Error: 0.00149517
Spline MPCC Lag Error: 0.00300158
Spline MPCC Lag Error: 0.00484851
Spline MPCC Lag Error: 0.0077087
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MPCC Lag Error: 0.02299
MPCC Lag Error: 0.024019
MPCC Lag Error: 0.0226927
MPCC Lag Error: 0.0156915
MPCC Lag Error: 0.00502496
MPCC Lag Error: 0.0050866
MPCC Lag Error: 0.00435448
MPCC Lag Error: 0.00207409
MPCC Lag Error: 0.00580607
MPCC Lag Error: 0.00794931
MPCC Lag Error: 0.00841238
MPCC Lag Error: 0.0084915
MPCC Lag Error: 0.00620773
MPCC Lag Error: 0.00652064
MPCC Lag Error: 0.0084661
MPCC Lag Error: 0.00895306
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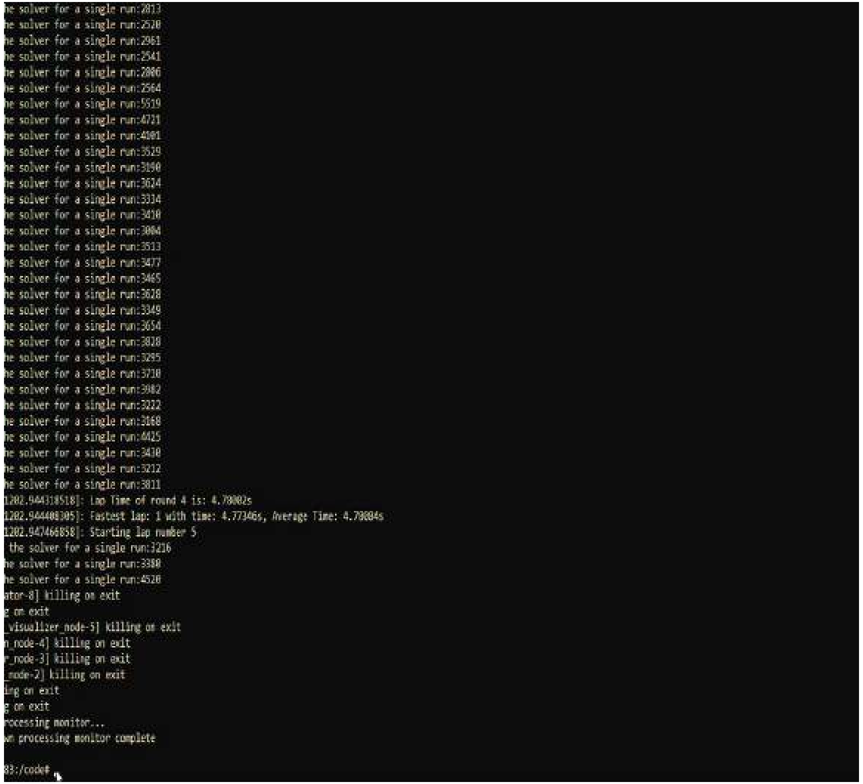
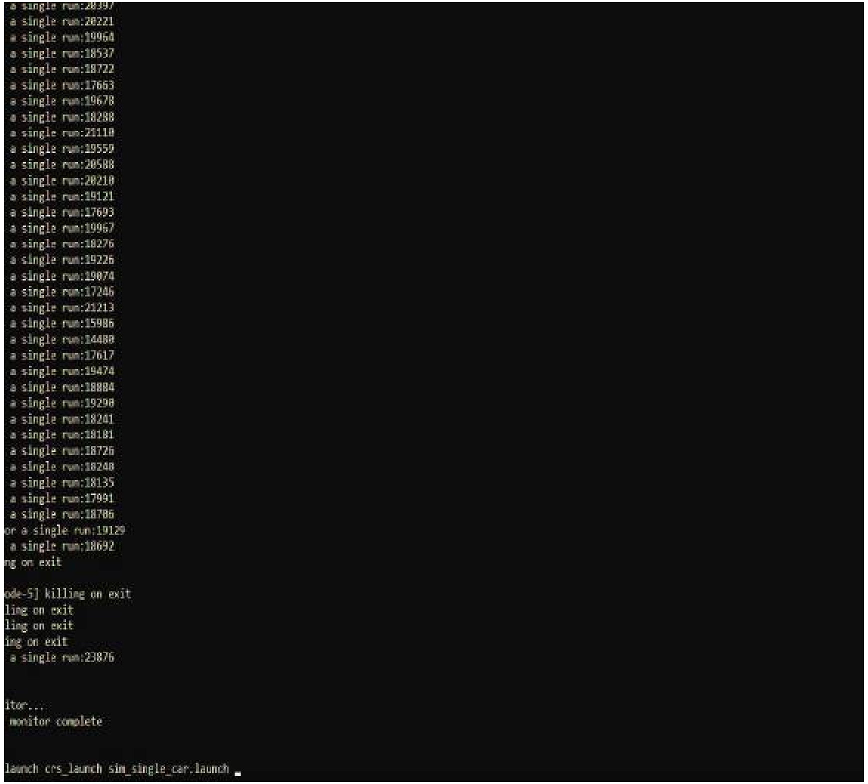
Figure 5. Lag error for Spline MPCC and original MPCC

- The lag error values for Spline MPCC is comparatively very small.
- Could be further reduced by increasing the number of piecewise cubic splines for center track representation.

Simulation





Original MPCC

MPCC with Spline

Performance of the MPCC Formulations

Solver Frequency (Hz) (1 / Sampling Time (Ts))	Horizon Length (N)	Time taken by solver for single solve (mSec)		Observed results with Spline MPCC Formulation
		Original MPCC formulation	Spline MPCC formulation	
25	30 40 45	3.181 3.498 3.626	QP fails	QP fails to generate a solution..
30	30 40 45	3.501 3.726 3.987	17.756 19.798 22.224	Optimal Solution is achieved. At continuous curved section the velocity tends to be slower.
	[illegible]	[illegible]	[illegible]	At straight section there is oscillatory

Lap Times of MPCC Formulations

Frequency of solver: 30 Hz
Horizon Length: 40

MPCC Formulation	Lap time (seconds)	Fastest Lap Time (seconds)
Original MPCC	4.77 - 4.85	4.77 - 4.80
Spline MPCC	4.62 - 5.01	4.62 - 4.71

Observations

- The controller has more freedom to move across the whole of the track width.
- The optimal **trajectory is fairly smoother** around steep curves.
- Having removed the contouring cost and cost on virtual velocity, there are **lesser weights to be tuned**.
- The lookahead is observed to be larger than original formulation.
- **Warm starting is not really required** for the Spline MPCC formulation.
- The lap times of both the formulations is comparable, but the Spline MPCC is producing **better fastest lap times**.

Possible Improvements

- Initial Guess could have an impact on the performance of the QP Solver.
- Could try using some globalization technique such as line search algorithm to vary the step size.
- Increasing the number of piecewise splines, would give better approximation for the center track.
- Update in model and track parameters would result in better results.

References

- *Liniger, A., Domahidi, A., & Morari, M. (2015). Optimization-based autonomous racing of 1: 43 scale RC cars. Optimal Control Applications and Methods, 36(5), 628-647.[1]*
- *Carron, A., Bodmer, S., Vogel, L., Zurbrügg, R., Helm, D., Rickenbach, R., ... & Zeilinger, M. N. (2022). Chronos and CRS: Design of a miniature car-like robot and a software framework for single and multi-agent robotics and control. arXiv preprint arXiv:2209.12048.[2]*
- *Lyons, Lorenzo & Ferranti, Laura. (2023). Curvature-Aware Model Predictive Contouring Control.[3]]*
- *Kelly, Matthew O', and Hongrui Zheng. "Raceline Optimization." Learn, f1tenth.org/learn.html. Accessed 2 July 2023. [4]*

Thank you

Questions?

Performance of the MPCC Formulations

Frequency (Hz)	Horizon Length	Time taken by solver for single solve (mSec)			Observed results with Spline MPCC Formulation
		Original MPCC formulation	Spline MPCC formulation	Spline MPCC with lag error as constraint	
<=25	30 40 45	3.481 3.498	QP fails 18.836 20.352	2.646 4.232 4.645	QP fails to generate a solution.
30	30 40 45	2.501 3.426 3.587	14.756 20.798 22.224	2.697 3.623 3.864	Optimal Solution is achieved. At continuous curved section the velocity tends to be slower.
40	30 40 45	2.362 3.891 4.032	15.495 19.752 24.054	2.524 3.910 4.473	At straight section there is oscillatory behaviour. With increased N the velocity is faster.

Lap Times of the MPCC Formulations

Frequency of solver: 30 Hz
Horizon Length: 40

MPCC Formulation	Lap time (seconds)	Fastest Lap Time (seconds)
Original MPCC	4.77 - 4.85	4.77 - 4.80
Spline MPCC	4.62 - 5.01	4.62 - 4.71
Spline MPCC with lag error as constraint	4.56 – 4.65	4.53 – 4.61