Class 5 | Random Vector and Matrices

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Contents

Notation

Given n random variables $X_1, X_2, \dots, X_n$,

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

Here, X is a called a n-dimensional **random vector**.

- $F_X(x) = F_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n)$
- In a similar way we can define PDF.
- Expectation $E[X] = \begin{bmatrix} E[X_1] & E[X_2] & \cdots & E[X_n] \end{bmatrix}^\top$

Similarly we can extend to a ***random matrix***, $M_{m\times n}$ and generalize expectation for M (expectation of each entry in the matrix).

Correlation and Covariance

Take $R_X = E[XX^\top]$ (outer product).

$$R_X = E\left[\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}\right]$$

where $[R_X]_{ij} = E[X_i X_j]$.

Covariance Matrix

$C_X = E[(X - E[X])(X - E[X])^\top]$.

$$\therefore [C_X]_{ij} = Cov(X_i, X_j)$$

Remark. $C_X = R_X - E[X]E[X^\top]$

Remark. *X is n-dimensional random vector and $Y = AX + b,\ A \in R^{m\times n}, b \in R^m$.*

$$\therefore C_Y = AC_XA^\top$$

Properties of Covariance Matrix

- C_X is symmetric
- C_X is positive definite

Example 1. *X, Y are independent $U(0,1)$ random vectors. $U = [X \quad X+Y]^\top$ and $V = [X \quad Y \quad X+Y]^\top$.*

Determine whether C_U and C_V are positive definite.

Solution:

$Var(X) = Var(Y) = \frac{1}{12}$. And $Cov(X, X+Y) = \frac{1}{12}$ and $Var(X+Y) = \frac{1}{6}$.

Then, compute C_U and C_V.

Cross-Correlation

Define $R_{XY} = E[XY^\top]$, then cross correlation can be defined as:

$$C_{XY} = E[(X - E[X])(Y - E[Y])^\top]$$

Method of Trasform for Random Vectors

- $X :=$ n-dimensional random vector with PDF $f_X(x)$
- $G : R^n \to R^n$ be a continuous and invertible function with continuous partial derivative and let $\mathcal{H} = G^{-1}$.
- $Y = G(X) \implies X = \mathcal{H}(Y)$.

  PDF of Y is thus, given as

$$f_Y(y) = f_X(\mathcal{H}(y)) \, |\det J|$$

  where J is the Jacobian. $J_{ij} = \frac{\partial \mathcal{H}_i}{\partial y_j}$.

Example 2. *A:= fixed invertible n by n matrix and b:= fixed, n-dimensional vector. Define* $Y = AX + b$. *Find PDF of* Y.

Solution:

$X = A^{-1}(Y - b) = \mathcal{H}(Y)$.

$J = \frac{\partial \mathcal{H}}{\partial Y} = A^{-1}$

$\therefore |J| = \det(A^{-1}) = \frac{1}{\det(A)}$.

$f_Y(y) = f_X(A^{-1}(y - b)) \frac{1}{\det(A)}$

Standard Normal Vector

Let $X_i \sim N(0, 1)$, assume X_i's are iid.

Denote standard Normal Vector $Z = [Z_1 \quad Z_2 \quad \cdots \quad Z_n]^\top$ where Z_i's are iid and standard normal. Then, $f_Z(z) = \prod_i f_{Z_i}(z_i)$.

Recall, $X = \sigma Z + \mu$ then $X \sim N(\mu, \sigma^2)$.

Normal Random vector X with mean vector m and covariance matrix C.

Let $Z \sim N(0, I)$ and $X = AZ + m$ where $AA^\top = A^\top A = C$.

Claim: $X \sim N(m, C)$.

Proof. $E[X] = E[AZ + m] = AE[Z] + m = 0 + m = m$

Since, C is symmetric and positive definite. We can do eigenvalue decomposition. Since, C is symmetric, it is guaranteed to have n-independent (orthogonal) eigenvalues.

$C = QDQ^\top$ where columns of Q are eigenvectors of C and D is a diagonal matrix (with the eigenvalues).

$CQ = QD \implies Cq_i = \lambda_i q_i$.

Now, define A as $QD^{1/2}Q^\top$

$$C = AA^\top = (QD^{1/2}Q^\top)(QD^{1/2}Q^\top) = QDQ^\top$$

Note, $D^{1/2}$ always exists because all eigenvalues are ≥ 0.

$$\begin{aligned} f_X(x) &= \frac{1}{\det A} f_Z(A^{-1}(X-m)) \\ &= \frac{1}{(2\pi)^{n/2}(\det A)} \exp\left\{-\frac{1}{2}(A^{-1}(X-m))^\top (A^{-1}(X-m))\right\} \\ &= \frac{1}{(2\pi)^{n/2}|\det A|} \exp\left\{-\frac{1}{2}(X-m)^\top C^{-1}(X-m)\right\} \end{aligned}$$

□

Probability Bounds

Union Bound, Bode's inequality

For any $A_1, A_2, \cdots, A_n$

$$P(\cup_i A_i) \leq \sum_i P(A_i)$$

Proof. By PIE,

$$P(\cup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) + ... + (-1)^{n-1} P(\cap_{i=1}^n A_i)$$

Thus, $P(\cup_i A_i) \leq \sum_i P(A_i)$ and $P(\cup_i A_i) \geq \sum_i P(A_i) - \sum P(A_i \cap A_j)$ and so on.

□

Markov Inequality

$X :=$ positive continuous random variable

$$\begin{aligned}
E[X] &= \int_0^\infty x f_X(x)dx \\
&\geq \int_a^\infty x f_X(x)dx \\
&\geq \int_a^\infty a f_X(x)dx \\
&= a\int_a^\infty f_X(x)dx \\
&= aP(X \geq a)
\end{aligned}$$

$$\implies P(X \geq a) \leq \frac{E[X]}{a}, \quad a > 0$$

Chebyshev's Inequality

Define $Y = (X - E[X])^2$, X is any random variable. Y is therefore a non-negative random variable.

Applying Markov Inequality on Y,

$$\begin{aligned}
P(Y \geq b^2) &\leq \frac{E[Y]}{b^2} = \frac{Var(X)}{b^2} \\
\implies P((X - E[X])^2 \geq b^2) &\leq \frac{Var(X)}{b^2} \\
\implies P(|X - E[X]| \geq b) &\leq \frac{Var(X)}{b^2}
\end{aligned}$$

Cauchy Schwarz Inequality

For any two random variables X and Y

$$|E[XY]| \leq \sqrt{E[X^2]E[Y^2]}$$

where equality holds iff $X = \alpha Y,\ \alpha \in R$

Jensen's Inequality

Take a convex function $f(x)$ then

$$E[f(X)] \geq f(E[X])$$

Law of Large Numbers

For iid random variables $X_1, X_2, \ldots, X_n$, the sample mean denoted by $\overline{X} = \sum X/n$.

Let $E[X_i] = \mu < \infty$ (finite mean).

Then, for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(|\overline{X} - \mu| \geq \epsilon) = 0$$

Proof. Assume $Var(X) = \sigma^2$.

By Chebyshev's inequality,

$$\begin{aligned} P(|\overline{X} - \mu| \geq \epsilon) &\leq \frac{Var(\overline{X})}{\epsilon^2} \\ &= \frac{Var(X)}{n\epsilon^2} \to 0 \text{ as } n \to \infty \end{aligned}$$

□

Central Limit Theorem

- $X_i :=$ iid
- $E[X_i] = \mu < \infty$
- $Var(X_i) = \sigma^2 < \infty$
- Take sample mean $\overline{X}$.

Take Normalized random vector

$$Z_n = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

Z_n converges in distribution to the standard normal random variable i.e.,

$$\lim_{n\to\infty} P(Z_n \leq x) = \Phi(x) \quad \forall x \in R$$