Class 3

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Contents

Continuous Random Variable

X is continuous if CDF is continuous $\forall x \in R$.

PDF defined as:

$$f_X(x) = \lim_{\Delta \to 0^+} \frac{P(x < X \le x + \Delta)}{\Delta} \tag{1}$$

Also,

$$P(x < X \le x + \Delta) = F_X(x + \Delta) - F_X(x) \tag{2}$$

From, (1) and (2) we have

$$f_X(x) = \frac{dF_X(x)}{dx}$$

- Some Properties of PDF and CDF
- Expectation and Variance for continuous R.V.

Transformation of Random Variables

$X \sim U(0,1)$ and $Y = e^X$

$\therefore R_X = [0,1]$ and $R_Y = [1,e]$.

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(e^X \le y) = P(X \le ln(y)) \end{aligned}$$

$$\therefore F_Y(y) = \begin{cases} 0 & y < 1 \\ ln(y) & 1 \le y \le e \\ 1 & y > e \end{cases}$$

Method of Transform

- X : continuous random variable
- $g : R \to R$
  - ⋆ Assume g to be strictly monotonic differentiable function
- $Y = g(X)$

$$\therefore f_Y(y) = \begin{cases} \frac{f_X(x_1)}{|g'(x_1)|} = f_X(x_1)\left|\frac{dx_1}{dy}\right| & \text{where } g(x_1) = y \\ 0 & \text{if } g(x) = y \text{ does not have solution} \end{cases}$$

Example 1. *Take X such that $f_X(x) = 4x^3$ when $0 < x \le 1$ and 0 otherwise. Take $Y = \frac{1}{X}$ which is a strictly decreasing (hence, monotonic) function.*

$$x_1 = \frac{1}{y} \implies \frac{dx_1}{dy} = -\frac{1}{y^2}$$

$$\therefore f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} = \frac{4x_1^3}{y^2} = 4\left(\frac{1}{y}\right)^3 \frac{1}{y^2} = \frac{4}{y^5} \quad \textit{when } y \ge 1$$

A General Method of Transform

- X : continuous random variable
- $g : R \to R$ and $Y = g(X)$

- Partition range R_x into finite intervals such that $g(x)$ is strictly monotonic and differentiable in each partition

$$\therefore f_Y(y) = \begin{cases} \sum_{i=1}^{n} \frac{f_X(x_1)}{|g'(x_1)|} = \sum_{i=1}^{n} f_X(x_1) \left|\frac{dx_1}{dy}\right| & \text{where } g(x_1) = y \\ 0 & \text{if } g(x) = y \text{ does not have solution} \end{cases}$$

Example 2. *Take $X \sim N(0,1)$. So, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$*

Now, $Y = X^2$. This is strictly decreasing in $(-\infty, 0)$ and strictly increasing in $(0, \infty)$. Therefore, we can partition the range and apply the general method of transformation.

$x^2 = y \implies x = \pm\sqrt{y}$. Let $x_1 = \sqrt{y}$ and $x_2 = -x_1$.

$$\therefore \frac{dx_1}{dy} = \frac{1}{2\sqrt{y}}, \ \frac{dx_2}{dy} = -\frac{1}{2\sqrt{y}}$$

Thus, we obtain:

$$\begin{aligned} f_Y(y) &= \frac{f_X(\sqrt{y})}{|2\sqrt{y}|} + \frac{f_X(-\sqrt{y})}{|-2\sqrt{y}|} \\ &= \frac{1}{2\sqrt{2\pi y}}[e^{-\frac{y}{2}} + e^{-\frac{y}{2}}] \\ &= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} \quad R_y = (0, \infty) \end{aligned}$$

Important Probability Distributions

- Uniform Distribution $U(a, b)$
- Normal (Gaussian) Distribution
  - ⋆ Standard Normal R.V. $Z \sim N(0,1)$
    - CDF of Z denoted by Φ and it does not have a closed form solution.
  - ⋆ General Normal R.V. $X \sim N(\mu, \sigma^2)$
    - Transform to standard normal using $X = \sigma Z + \mu, \ \sigma > 0$
    - PDF given as

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

    - CDF given as $\Phi\left(\frac{x-\mu}{\sigma}\right)$.

Joint R.V.

Discrete

- Joint PMF $P_{XY}(x, y)$

  - ⋆ Marginal PMF (PDF)
  - ⋆ Joint and Marginal CDF
  - ⋆ Independent when $P_{XY}(x, y) = P_X(x)P_Y(y)$. Also holds for joint CDF.
- ∘ Conditional PMF (or PDF) $P_{X|Y}(x|y)$
  - ⋆ Conditional Expectation
  - ⋆ Law of Total probability and total expectation

$$E[X] = \sum_i E[X|B_i]P(B_i)$$

  where B_i are partitions of sample space.
- ∘ Law of Unconscious Statistician
- ∘ Law of Iterated Expectation
- ∘ Conditional Variance

$$Var(X|Y) = E[X^2|Y] - E[X|Y]^2$$

- ∘ Law of Total Variance

$$Var(X) = E[Var(X|Y)] + Var(E[X|Y])$$

Continuous

- ∘ Joint and Marginal PMF
- ∘ Joint and Marginal CDF
  - ⋆ $F_{XY}(\infty, \infty) = 1$
  - ⋆ $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0$
  - ⋆ $P(x_1 \leq X \leq x_2,\ y_1 \leq Y \leq y_2) = F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1)$
- ∘ Conditional PDF (and CDF) and independence
  - ⋆ $F_{X|Y}(x|y) = P(X \leq x \mid Y = y)$

Example 3. *X is continuous R.V. and event $A : a < X < b$.*

Therefore,

$$F_{X|A}(x) \begin{cases} 1 & x > b \\ \frac{F_X(x)-F_X(a)}{F_X(b)-F_X(a)} & a \leq x < b \\ 0 & x < a \end{cases}$$

and

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(A)} & a \leq x \leq b \\ 0 & otherwise \end{cases}$$

Function of two continuous R.V.

X, Y are jointly cont. R.Vs.

Let $(z, w) = g(x, y) = (g_1(x, y), g_2(x, y))$ where $g : R^2 \to R^2$ is a cont. 1-1 invertible function with cont. partial derivatives. Let $h = g^{-1}$.

$(x, y) = h(z, w) = (h_1(z, w), h_2(z, w))$

Then z, w are jointly continuous and their joint PDF $f_{ZW}(z, w)$ is given by

$$f_{ZW}(z, w) = f_{XY}(h_1(z, w), h_2(z, w))\ |J|$$

where J is Jacobian,

$$J = \begin{pmatrix} \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \\ \\ \frac{\partial h_2}{\partial z} & \frac{\partial h_2}{\partial w} \end{pmatrix}$$

.

Example 4. *X, Y are independent standard normal R.V.*

$Z = 2X - Y$ and $W = -X + Y$.

$\implies$ *$X = Z + W$ and $Y = 2W + Z$*

$$f_{XY}(x, y) = \frac{1}{2\pi} \exp\left\{-\frac{x^2 + y^2}{2}\right\}$$

$$\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We invert this matrix and get $h(z, w)$ (we already found that when we represented X and Y in terms of Z and W).

Computing $|J|$

$$|J| = \det \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = 1$$

$$\therefore f_{ZW}(z, w) = \frac{1}{2\pi} \exp\left\{-\frac{(z + w)^2 + (z + 2w)^2}{2}\right\}$$

Covariance and Correlation

Covariance

$Cov(X, Y) = E\left[(X - E[X])(Y - E[Y])\right] = E[XY] - E[X]E[Y]$

- $Var(X) = Cov(X, X)$
- Commutative function
- $Cov(aX, Y) = a\ Cov(X, Y)$

- $Cov(X+c, Y) = Cov(X, Y)$
- $Cov(X+Y, Z) = Cov(X, Z) + Cov(Y, Z)$
- $Var(aX+bY) = a^2\ Var(X) + b^2\ Var(Y) + 2ab\ Cov(X, Y)$

Correlation

Correlation $\rho_{XY} = \rho(X, Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\ Var(Y)}}$

- $-1 \leq \rho_{XY} \leq 1$.
- If $\rho = 1$ then $Y = aX + b,\ a > 0$
- If $\rho = -1$ then $Y = aX + b,\ a < 0$
- $\rho(aX+b, cY+d) = \rho(X, Y)\ a, c > 0$
- $\rho = 0$ then X, Y are uncorrelated
  - ⋆ For uncorrelated, $Var(X+Y) = Var(X) + Var(Y)$.
- $\rho > 0$ then X, Y are positively correlated
- $\rho < 0$ then X, Y are negatively correlated

Bivariate Normal

X, Y are said to have bivariate normal distribution with parameters $\mu_x, \sigma_x^2, \mu_y, \sigma_y^2$.

Their joint PDF is

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right.$$