Class 9

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Contents

Recap

$$
\begin{aligned}
D_f(p\|q) &= \int q(x) \sup_t \left\{ t\frac{p(x)}{q(x)} - f^*(t) \right\} dx \\
&= \sup_t \int q(x) \left\{ T_1(x)\frac{p(x)}{q(x)} - f^*(T_1(x)) \right\} \\
&\geq \int q(x) T_1(x) \frac{p(x)}{q(x)} - f^*(T_1(x))
\end{aligned}
$$

$$
\therefore D_f(p\|q) = \sup_{T:X\to\mathbb{R}} \int_X \left(T(x)p(x) - f^*(T(x))q(x)\right) dx
$$

Using parameterized family of function T_ϕ to approximate the optimal function T, can minimize an f-divergence between p and p_θ (**GAN problem**)

$$
\begin{aligned}
\theta_s &= arg\ \min_\theta \sup_\phi \left[E_{x\sim p} T_\phi(x) - E_{x\sim p_\theta} f^*(T_\phi(x)) \right] \\
&= arg\ \min_\theta \sup_\phi \left[E_{x\sim p} T_\phi(x) - E f^*(T_\phi(g_\theta(z))) \right]
\end{aligned}
$$

where $z \to g_\theta(z) = x$

In 2014 paper, *I. Goodfellow et al, Generative Adversarial Networks*

<u>Parameterizing:</u> $T_\phi(x) = \log(d_\phi(x))$

$$
\theta_f = arg\ \min_\theta \sup_\phi \left[E_{x\sim p} \log(d_\phi(x)) + E_{z\sim q} \log(1 - d_\phi(g_\theta(z))) \right]
$$

Algorithm 1: GAN

for *training iterations* **do**
  for k *steps* **do**
    Sample minibatch of m noise sample $\{z^1, z^2, \dots, z^m\}$ from priors $p_g(z)$
    Sample minibatch of m examples $\{x^{(1)}, x^{(2)}, \dots, x^{(m)}\}$ from data generating distribution $p_{data}(x)$
    Update the discriminator by ascending its stochastic gradient

$$u^i \leftarrow \nabla_\phi \left(1/m \sum_{i=1}^{m} \log d_\phi(x^i) + \log(1 - d_\phi(g_\theta(z^i))) \right) \quad (\phi^{i+1} \leftarrow \phi^i + \eta u^i)$$

  end
  Sample minibatch of m noise samples $\{z^1, \dots, z^m\}$
  Update the generator by descending

$$v_i = \nabla_\theta 1/m \log(1 - d_\phi(g_\theta(z^i))) \quad (\theta^{i+1} \leftarrow \theta^i - \eta v^i)$$

end

Algorithm for GAN

Proposition 1. *For G fixed, optimal discriminator D is*

$$D^*_G(x) = \frac{p_{data}(x)}{p_{data}(x) + p_g(x)}$$

Proof. Recall for Two-player game

$$\min_G \max_D V(D, G) = E_{x \sim p_{data}(x)} \log D(x) + E_{z \sim p_Z(z)} [\log(1 - D(G(z)))]$$

Therefore, for a given G, discriminator D wants to maximize $V(G, D)$

$$V(G, D) = \int_x p_{data}(x) \log(D(x))dx + p_g(x) \log(1 - D(x))dx$$

For any $(a, b) \in \mathbb{R}^2$ the function $f(y) : y \to a \log y + b \log(1 - y)$ achieves max in $[0, 1]$ at $\frac{a}{a+b}$.

$f'(y) = \frac{a}{y} + \frac{b(-1)}{1-y} = 0 \implies y = \frac{a}{a+b}$ $\square$