Sol. Ans. (a). Use Art. 4.8.

Ex. 3. Solution of $(x-a)p^2+(x-y)p-y=0$ is

(a) $y=cx-(ac^2)/(c+1)$ (b) $y=cy+(ac^2)/(x+1)$

(c) $y=cx-(ac^2)/(y+1)$ (d) $y=cy-(ac^2)/(c+1)$ **[Garhwal 2010]**

Sol. Ans. (a) Re-writing given equation, $y(p+1)=xp(p+1)-ap^2$ or $y=xp-(ap^2)/(p+1)$ which is in Clariaut's form. Hence the required solution is $y=cx-(ac^2)/(c+1)$

Ex. 4. Write the general form of the differential equation of the first order and n^{th} degree in p. **(Madras 2012)**

Hint. Refer Art. 4.1. The required form is given by

$P_0p^n+P_1p^{n-1}+\ldots+P_{n-1}p+P_n=0$, $P_0, P_1, \ldots, P_n$ being functions of x and y.

5

Linear Differential Equations With Constant Coefficients

PART 1: USUAL METHODS OF SOLVING LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

5.1 Some useful results

Let D stand for d/dx ; D^2 for d^2/dx^2; and so on. The symbols D, D^2, etc., are called operators. The index of D indicates the number of times the operation of differentiation must be carried out. For example, D^3x^4 shows that we must differentiate x^4 three times. Thus, $D^3x^4 = 24x$. The following results are valid for such operators.

1. $D^m + D^n = D^n + D^m$
2. $D^mD^n = D^nD^m = D^{m+n}$
3. $D(u+v) = Du + Dv$, where u and v are functions of x.
4. $(D-\alpha)(D-\beta) = (D-\beta)(D-\alpha)$, where α and β are constants.

Negative index of *D*. D^{-1} is equivalennt to an integration. For example, $D^{-1}x=\int x\,dx = x^2/2$. But it is important to note that the main object of D^{-1} is to find an integral but not the complete integral. Consequently the arbitrary constant which arises in integration must be omitted. The index of D^{-1}, say $(D^{-1})^5$ is denoted by D^{-5} The negative index of D indicates the number of times the operation of integration is to be carried out. For example,

$$D^{-2}x=\int\left[\int x\,dx\right]dx=\int(x^2/2)\,dx = x^3/6$$

It is usual to write $1/D^m$ for D^{-m}. It is to be remembered that $DD^{-1}=1$ and the symbol D with negative indices also satisfy the above four results. Furthermore we write $(d^2y/dx^2)+a_1(dy/dx)+a_2y = (D^2+a_1D+a_2)y = f(D)y$, where $f(D)$ is the operator now. If $f_1(D)$ and $f_2(D)$ be two operators, then $f_1(D)f_2(D)$ is also an operator such that $f_1(D)f_2(D) = f_2(D)f_1(D)$.

If u be a function of x and k be a constant then $f(D)(ku) = kf(D)u$.

From the above discussion we note that the symbol D obviously satisfies the fundamental laws of algebra and hence it can be regarded as an algebraic quantity in several respects.

5.2 Linear differential equations with constant coefficients

A linear differential equation with constant coefficients is that in which the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together, and the coefficients are all constants.

The general from of the equation is $$\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_ny=X, \quad \ldots (1)$$

where X is a function of x only and $a_1, a_2 \ldots a_n$ are constants.

Using the symbols $D, D^2, \ldots, D^n$ of Art. 5.1, (1) becomes

$$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \ldots + a_n y = X \quad \text{or} \quad (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n) y = X \qquad \ldots (2)$$

or $$f(D) y = X \qquad \ldots (3)$$

where $$f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n \qquad \ldots (4)$$

and $f(D)$ now acts as operator and operates on y to yield X. The forms (2) and (3) are called the symbolic forms of the given equation (1).

Consider the differential equation $$f(D)\, y = 0, \qquad \ldots (5)$$

obtained on replacing the right hand member of (3) by zero. We will, now, show that if $y_1, y_2, \ldots, y_n$ are n linearly independent solutions of (5) then, $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ is also a solution of (5), $c_1, c_2, \ldots, c_n$ being arbitrary constants.

Since $y_1, y_2, \ldots, y_n$ are solutions of (5), $f(D)\, y_1 = 0, \quad f(D)\, y_2 = 0, \quad \ldots f(D) y_n = 0 \qquad \ldots (6)$

If $c_1, c_2, \ldots, c_n$ are an arbitrary constants, we get

$$f(D)\,(c_1 y_1 + c_2 y_2 + \ldots c_n y_n) = f(D)\,(c_1 y_1) + f(D)\,(c_2 y_2) + \ldots + f(D)\,(c_n y_n)$$
$$= c_1 f(D)\, y_1 + c_2 f(D)\, y_2 + \ldots + c_n f(D)\, y_n = c_1 .0 + c_2 .0 + \ldots + c_n .0 = 0, \text{ using (6)}$$

This proves the statement made above.

Since the general solution of a differential equaiton of the n^{th} order contains n arbitrary constants, we conclude that $$c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = u, \text{ say}$$

is the general solution of (5).

Thus, $$f(D)\, u = 0. \qquad \ldots (7)$$

Again, let v be any particular solution of (3) and hence $$f(D)\, v = X. \qquad \ldots (8)$$

Now, we have $$f(D)\,(u + v) = f(D)\, u + f(D)\, v = 0 + X, \text{ using (7) and (8)}$$

This shows that $(u + v)$, *i.e.*, $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n + v$ is the general solution of (3), *i.e.*, (1) containing n arbitrary constants $c_1, c_2, \ldots, c_n$. The part $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$, is known as the *Complementary Function* (C.F.) and v, not involving any arbitrary constant, is called the *Particular Integral (P.I.) or particular solution (P.S.)*

Thus, the general solution of (1) is y = C.F. + P.I., where C.F. involves n arbitrary constants and P.I. does not involve any arbitrary constant.

Remarks. It should be remembered that P.I. appears due to X in (1). Hence if a linear differential equation with constant coefficients is given with $X = 0$, then its general solution will not involve P.I. and so for such differential equations the general solution will be given by y = C.F.

5.3 To find complementary function (C.F.) of the given equation [Mumbai 2010]

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n) y = X \quad \text{or} \quad f(D) y = X. \qquad \ldots (1)$$

By definition, C.F. of (1) is the general solution of

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n) y = 0 \quad \text{or} \quad f(D) y = 0. \qquad \ldots (2)$$

Let (2) be equivalent to $$(D - m_1)(D - m_2) \ldots (D - m_n) y = 0 \qquad \ldots (3)$$

Then solution of any one of the equations

$$(D - m_1) y = 0, \quad (D - m_2) y = 0, \quad \ldots, \quad (D - m_n) y = 0 \qquad \ldots (4)$$

is also a solution of (3) because if $\phi_r(x)$, $1 \le r \le n$ be a solution of $(D - m_r) y = 0 \qquad \ldots (5)$

then, $$(D - m_1)(D - m_2) \ldots (D - m_n) \phi_r(x)$$
$$= (D - m_1) \ldots (D - m_{r-1})(D - m_r)(D - m_{r+1}) \ldots (D - m_n) \phi_r(x)$$

$$= (D - m_1) \ldots (D - m_{r-1})(D - m_{r+1}) \ldots (D - m_n)(D - m_r) \phi_r(x)$$
$$= 0, \text{ since } \phi_r(x) \text{ is a solution of (5)} \Rightarrow (D - m_r) \phi_r(x) = 0$$

We now proceed to find the general solution of (5), *i.e.*,

$$(dy/dx) - m_r y = 0 \quad \text{or} \quad (1/y)\, dy = m_r\, dx.$$

Integrating, $$\log y - \log c_r = m_r x \quad \text{or} \quad y = c_r e^{m_r x}, \qquad \ldots (6)$$

where c_r is the constant of integration.

Since the general solution of (5) is given by (6), we can assume that a solution of the equation (2) is of the form $y = e^{mx}$. Then since $y = e^{mx}$, $Dy = m e^{mx}$, $D^2 y = m^2 e^{mx}, \ldots, D^n y = m^n e^{mx}$, so (2) becomes.

$$(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n) e^{mx} = 0$$

Cancelling e^{mx} as $e^{mx} \neq 0$ for any m, we obtain

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n = 0 \qquad \ldots (7)$$

Equation (7) is called the *auxiliary equation* (A. E.). Replacing m by D in (7), we have

$$D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n = 0 \qquad \ldots (8)$$

Clearly, equaiton (7) gives the same values of m as equation (8) gives of D. In practice, we may also take equation (8) as the auxiliary equation which is obtained by equating to zero the symbolic coefficient of y in (1) or (2). Here D is considered as an algebraic quantity.

(7) or (8) will give, in genral, n roots say, $m_1, m_2, \ldots, m_n$.

Case I. When all the roots of the A.E. (7) or (8) are real and different.

Let $m_1, m_2, m_3, \ldots, m_n$ be the n real and different roots of (8). Then $y = e^{m_1 x}, y = e^{m_2 x}, \ldots, y = e^{m_n x}$ are n independent solutions of (2). So the solution of (2) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}, \; c_1, c_2, \ldots, c_n \text{ being arbitrary constants} \qquad \ldots (9)$$

Case II. When the auxilary equaiton has equal roots

Let the roots m_1 and m_2 of the A.E. (7) or (8) be equal. Then the general solution (9) becomes

$$y = (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

or $$y = A e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x} \quad \text{where} \quad A = c_1 + c_2. \qquad \ldots (10)$$

This solution contains $(n - 1)$ arbitrary constants. $A, c_3, c_4, \ldots, c_n$ and not n. But we know that C.F. of (1), *i.e.*, solution of (1) must contain as many arbitrary constants as is the order of the given differential equation. Hence (10) is not the general solution of (2). To obtain the general solution of (2), consider the differential equation

$$(D - m_1)^2\, y = 0 \quad \text{or} \quad (D - m_1)(D - m_1)\, y = 0, \qquad \ldots (11)$$

in which the two roots are equal.

Let $$(D - m_1)\, y = v. \qquad \ldots (12)$$

Then, $(11) \Rightarrow (D - m_1) v = 0$ or $dv/dx = m_1 v$ or $(1/v)\, dv = m_1 dx$.

Integrating, $$\log v - \log c_1 = m_1 x \quad \text{or} \quad v = c_1 e^{m_1 x} \qquad \ldots (13)$$

Using (13), (12), becomes $(D - m_1)\, y = c_1 e^{m_1 x}$ or $Dy - m_1 y = c_1 e^{m_1 x}$

or $$(dy/dx) - m_1 y = c_1 e^{m_1 x}, \text{ which is linear equation} \qquad \ldots (14)$$

Its I.F. $= e^{\int (-m_1) dx} = e^{-m_1 x}$ and so its solution is

$$y e^{-m_1 x} = \int \{(c_1 e^{m_1 x}).e^{-m_1 x})\} dx + c_2 = c_1 \int dx + c_2 = c_1 x + c_2$$

or $$y = (c_1 x + c_2) e^{m_1 x} \quad \text{or} \quad y = (c_2 + c_1 x)\, e^{m_1 x}$$

Hence the general solution of (2) in this case is of the form

$$y = (c_2 + c_1 x)\, e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$$

Similarly, if three roots of the A.E. are equal, say $m_1 = m_2 = m_3$, then the general solution of (2) is of the form

$$y = (c_3 + c_2 x + c_1 x^2)\, e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x} \qquad \dots (15)$$

Case III. When the A.E. has complex roots

Let the two roots of the A.E. be complex, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$. Then the corresponding part of the C.F.

$= c_1' e^{(\alpha + i\beta)x} + c_2' e^{(\alpha - i\beta)x}$, c_1', c_2' being arbitrary constants

$= e^{\alpha x} (c_1' e^{i\beta x} + c_2' e^{-i\beta x}) = e^{\alpha x} [c_1' (\cos \beta x + i \sin \beta x) + c_2' (\cos \beta x - i \sin \beta x)]$

[As by Euler's theorem, $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$]

$= e^{\alpha x} [(c_1' + c_2') \cos \beta x + i (c_1' - c_2') \sin \beta x] = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$,

[Taking $c_1 = c_1' + c_1'$ and $c_2 = i(c_1' - c_2')$]

where c_1 and c_2 are arbitrary constants.

Similarly, if the complex roots are repeated, say $\alpha + i\beta$ and $\alpha - i\beta$ occur twice, then the corresponding part of C.F. is of the form $e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]$ and so on.

Remark. After suitably adjusting the constants, $e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$ may also be written as $c_1 e^{\alpha x} \cos (\beta x + c_2)$ or $c_1 e^{\alpha x} \sin (\beta x + c_2)$.

Case IV. When the A.E. has surd roots.

Let the two roots of the A.E. be surds, say $m_1 = \alpha + \sqrt{\beta}, m_2 = \alpha - \sqrt{\beta}$. Then the corresponding part of the C.F. of (1)

$= c_1' e^{(\alpha + \sqrt{\beta})x} + c_2' e^{(\alpha - \sqrt{\beta})x} = e^{\alpha x} [c_1' e^{\sqrt{\beta} x} + c_2' e^{-\sqrt{\beta} x}]$

$= e^{\alpha x} [c_1' (\cosh x\sqrt{\beta} + \sinh x\sqrt{\beta}) + c_2' (\cosh x\sqrt{\beta} - \sinh x\sqrt{\beta})]$

[$\because e^{\theta} = \cosh\theta + \sinh\theta$ and $e^{-\theta} = \cosh\theta - \sinh\theta$]

$= e^{\alpha x} [(c_1' + c_2') \cosh x\sqrt{\beta} + (c_1' - c_2') \sinh x\sqrt{\beta})] = e^{\alpha x} [c_1 \cosh x\sqrt{\beta} + c_2 \sinh x\sqrt{\beta})]$,

where c_1 and c_2 are arbitrary constants given by $c_1 = c_1' + c_2'$ and $c_2 = c_1' - c_2'$.

Similarly, if the surd roots are repeated, say $\alpha + \sqrt{\beta}$ and $\alpha - \sqrt{\beta}$ occur twice, then the corresponding part of C.F. is of the form

$$e^{\alpha x} [(c_1 + c_2 x) \cosh x\sqrt{\beta} + (c_3 + c_4 x) \sin x\sqrt{\beta}] \text{ and so on.}$$

Remark. After suitably adjusting the constants, $e^{\alpha x} (c_1 \cosh x\sqrt{\beta} + c_2 \sinh x\sqrt{\beta})$ may be written as $c_1 e^{\alpha x} \cosh (x\sqrt{\beta} + c_2)$ or $c_1 e^{\alpha x} \sinh (x\sqrt{\beta} + c_2)$.

5.4 Working rule for finding C.F. of the given equation

$$(d^n y / dx^n) + a_1 (d^{n-1} y/dx^{n-1}) + a_2 (d^{n-2} y/dx^{n-2}) + \dots + a_n y = X \qquad \dots (1)$$

Step I. Re-write the equation (1) in the symbolic form

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n)\, y = X. \qquad \dots (2)$$

Step II. The auxiliary equation is

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0. \qquad \dots (3)$$

or

$$D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n = 0. \qquad \dots (3)'$$

Step III. From the roots of A.E. (3) or (3)$'$, write down the corresponding part of the C.F. as given in the following table

S. No.	*Corresponding part of C.F.*	*Nature of roots of auxiliary equation (A.E.)*
1.	(i) One real root m_1	$c_1 e^{m_1 x}$
	(ii) Two real and different roots m_1, m_2	$c_1 e^{m_1 x} + c_2 e^{m_2 x}$
	(iii) Three real and different roots m_1, m_2, m_3	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$
2.	(i) Two real and equal roots m_1, m_1	$(c_1 + c_2 x) e^{m_1 x}$
	(ii) Three real and equal roots m_1, m_1, m_1	$(c_1 + c_2 x + c_3 x^2) e^{m_1 x}$
3.	(i) One pair of complex roots $\alpha \pm i\beta$	$e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$ or $c_1 e^{\alpha x} \cos (\beta x + c_2)$ or $c_1 e^{\alpha x} \sin (\beta x + c_2)$
	(ii) Two pairs of complex and equal roots $\alpha \pm i\beta$, $\alpha \pm i\beta$	$e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]$
4.	(i) One pair of surd roots $\alpha \pm \sqrt{\beta}$	$e^{\alpha x} (c_1 \cosh x\sqrt{\beta} + c_2 \sinh x\sqrt{\beta})$ or $c_1 e^{\alpha x} \cosh (x\sqrt{\beta} + c_2)$ or $c_1 e^{\alpha x} \sinh (x\sqrt{\beta} + c_2)$
	(ii) Two pairs of surd and equal roots $\alpha \pm \sqrt{\beta}$, $\alpha \pm \sqrt{\beta}$	$e^{\alpha x} [(c_1 + c_2 x) \cosh x\sqrt{\beta} + (c_3 + c_4 x) \sinh x\sqrt{\beta}]$

5.5 Solved examples based on Art 5.4

Ex. 1. *Solve* $(d^3y/dx^3) + 6\,(d^2y/dx^2) + 11\,(dy/dx) + 6y = 0$.

Sol. The given equation can be re-written as

$$(D^3 + 6D^2 + 11D + 6)\, y = 0, \text{ where } D \equiv d/dx \qquad \dots (1)$$

The auxiliary equation of (1) is $D^3 + 6D^2 + 11D + 6 = 0$

or $(D + 1)(D + 2)(D + 3) = 0$ so that $D = -1, -2, -3$.

$\therefore$ The required general solution is $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$, c_1, c_2, c_3 being arbitrary constants.

Ex. 2. *Solve* $(D^3 + 3D^2 + 3D + 1)\, y = 0$ **[Delhi Maths. (G) 1994]**

Sol. The auxiliary equation is $D^3 + 3D^2 + 3D + 1 = 0$ or $(D + 1)^3 = 0 \Rightarrow -1, -1, -1$.

$\therefore$ The required solution is $y = (c_1 + c_2 x + c_3 x^2)\, e^{-x}$, c_1, c_2, c_3 being arbitrary constants.

Ex. 3 *Solve* $(d^4y/dx^4) - (d^3y/dx^3) - 9(d^2y/dx^2) - 11(dy/dx) - 4y = 0$. **[Delhi Maths. (G) 1997]**

Sol. Let $D \equiv d/dx$. Then the given equation can be written as

$(D^4 - D^3 - 9D^2 - 11D - 4)\, y = 0$ or $(D + 1)^3 (D - 4) = 0$ so that $D = 4, -1, -1, -1$.

$\therefore$ The required solution is $y = c_1 e^{4x} + (c_2 + c_3 x + c_4 x^2)\, e^{-x}$, c_1, c_2, c_3, c_4 being arbitrary constants.

Ex. 4. *Solve* (a) $(D^4 - 5D^2 + 4)\, y = 0$

(b) $(D^4 + 2D^3 - 3D^2 - 4D + 4)\, y = 0$

(c) $(D^3 - 3D^2 + 2D)\, y = 0$

Sol. (a) Here auxiliary equation is $D^4 - 5D^2 + 4 = 0$

or $(D^2 - 4)(D^2 - 1) = 0$ or $D^2 = 4$ or 1 so that $D = 2, -2, 1, -1$.

$\therefore$ The required general solution is $y = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{x} + c_4 e^{-x}$, c_1, c_2, c_3, c_4 being arbitrary constants

(b) Here auxiliary equation is $D^4 + 2D^3 - 3D^2 - 4D + 4 = 0$ or $(D - 1)(D^3 + 3D^2 - 4) = 0$

or $(D - 1)\{(D - 1)(D^2 + 4D + 4)\}$ or $(D - 1)^2 (D + 2)^2 = 0$ so that $D = 1, 1, -2, -2$.

$\therefore$ The required solution is $y = (c_1 + c_2 x)\, e^{x} + (c_3 + c_4 x)^{-2x}$, c_1, c_2, c_3 being arbitrary constants.

(c) Here the auxiliary equation is $D^3 - 3D^2 + 2D = 0$ or $D(D^2 - 3D + 2) = 0$

or $D(D-1)(D-2) = 0$ so that $D = 0, 1, 2$

Hence the required solution is $y = c_1 e^{0x} + c_2 e^x + c_3 e^{2x}$

or $y = c_1 + c_2 e^x + c_3 e^{2x}$, c_1, c_2, c_3 being arbitrary constants

Ex. 5. *Solve* $(D^3 - 8)y = 0$.

Sol. (a) Here auxiliary equation is $D^3 - 8 = 0$ or $(D-2)(D^2 + 2D + 4) = 0$ so that

$D = 2$ or $D = \{-2 \pm (4-16)^{1/2}\}/2$ or $D = 2, -1 \pm i\sqrt{3}$

$\therefore$ The required solution is $y = c_1 e^{2x} + e^{-x}\{c_2 \cos(x\sqrt{3}) + c_3 \sin(x\sqrt{3})\}$,

c_1, c_2, c_3 being arbitrary constants

Ex. 6. *Solve* (i) $d^4y/dx^4 + m^4 y = 0$

(ii) $d^4y/dx^4 + y = 0$ **[(I.A.S.Prel 2001; Agra 2006)]**

Sol. (i) Let $D \equiv d/dx$. Then, the given equation can be rewritten as $(D^4 + m^4)y = 0$

Its auxiliary equation is $D^4 + m^4 = 0$ or $(D^2 + m^2)^2 - (\sqrt{2}Dm)^2 = 0$

or $(D^2 + m^2 + \sqrt{2}Dm)(D^2 + m^2 - \sqrt{2}Dm) = 0 \Rightarrow D^2 + m^2 + \sqrt{2}\,Dm = 0$ or $D^2 + m^2 - \sqrt{2}\,Dm = 0$

$\therefore$ $D = \{-\sqrt{2}m \pm (2m^2 - 4m^2)^{1/2}\}/2 = -(m/\sqrt{2}) \pm i(m/\sqrt{2})$,

and $D = \{\sqrt{2}m \pm (2m^2 - 4m^2)^{1/2}\}/2 = m/\sqrt{2} \pm i(m/\sqrt{2})$

Hence the required general solution is $y = e^{-(mx/\sqrt{2})}\{c_1 \cos(mx/\sqrt{2}) + c_2 \sin(mx/\sqrt{2})\}$

$+ e^{mx/\sqrt{2}}\{c_3 \cos(mx/\sqrt{2}) + c_4 \sin(mx/\sqrt{2})\}$, c_1, c_2, c_3, c_4 being arbitrary constants

(ii) This is a particular case of part (i). Here $m = 1$. Solution is

$y = e^{-(x/\sqrt{2})}\{c_1 \cos(x/\sqrt{2}) + c_2 \sin(x/\sqrt{2})\} + e^{x/\sqrt{2}}\{c_3 \cos(x/\sqrt{2}) + c_4 \sin(x/\sqrt{2})\}$.

Ex. 7. *Solve* (i) $(D^4 - m^4)y = 0$

(ii) $(D^4 - 81)y = 0$

Sol. (i) Here auxiliary equation is $D^4 - m^4 = 0$ or $(D^2 - m^2)(D^2 + m^2) = 0$

Hence $D = m, -m, \pm im$. Now the part of C.F. corresponding to roots $m, -m$ is $c_1 e^{mx} + c_2 e^{-mx}$ and the part of the C.F. corresponding to roots $0 \pm mi$ is given by (noting that $\alpha = 0$ and $\beta = m$ in S.No. 3 (i) of table of Art. 5.4) $e^{0x}(c_3 \cos mx + c_4 \sin mx)$, *i.e.*, $c_3 \cos mx + c_4 \sin mx$.

Solution is $y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx$, c_1, c_2, c_3, c_4 being arbitrary constants.

(ii) Take $m = 3$ in part, (i). **Ans.** $y = c_1 e^{3x} + c_2 e^{-3x} + c_3 \cos 3x + c_4 \sin 3x$

Ex. 8. (a) *Solve* $(D^2 + 1)^2 y = 0$, *where* $D \equiv d/dx$. **[I.A.S. Prel. 1993]**

Sol. Here auxiliary equation is $(D^2 + 1)^2 = 0$ so that $D^2 + 1 = 0$ (twice)

Hence $D = 0 \pm i$ (twice). Therefore, required solution is

$y = e^{0x}\{(A_1 + A_2 x)\cos x + (A_3 + A_4 x)\sin x\}$

or $y = (A_1 + A_2 x)\cos x + (A_3 + A_4 x)\sin x$, A_1, A_2, A_3, A_4 being arbitrary constants

Ex. 8. (b) *Find the primtive of* $(D^2 - 2D + 5)^2 y = 0$. **[I.A.S. Prel. 1995]**

Sol. Here auxiliary equation is $(D^2 - 2D + 5)^2 = 0$ so that

$D^2 - 2D + 5 = 0$ (twice) and hence $D = (2 \pm \sqrt{-16})/2 = 1 \pm 2i$ (twice)

$\therefore$ Required solution is $y = e^x\{(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x\}$,

c_1, c_2, c_3 and c_4 being arbitrary constants.

Ex. 9. *Solve* $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$ **[I.A.S. Prel. 1996]**

Sol. Here A.E. is $D(D^3 - 6D^2 + 12D - 8) = 0$ or $D\{D^2(D-2) - 4D(D-2) + 4(D-2)\} = 0$

or $D(D-2)(D^2 - 4D + 4) = 0$ or $D(D-2)^3 = 0$ so that $D = 0, 2, 2, 2$

$\therefore$ Required solution is $y = c_1 e^{0x} + (c_2 + c_3 x + c_4 x^2)e^{2x}$ or $y = c_1 + (c_2 + c_3 x + c_4 x^2)e^{2x}$,

c_1, c_2, c_3 and c_4 being arbitrary constants

Ex. 10. *Solve* $(D^6 - 1)y = 0$.

Sol. Here the auxiliary equation is $D^6 - 1 = 0$ or $(D^2)^3 - (1)^3 = 0$

or $(D^2 - 1)(D^4 + D^2 + 1) = 0$ or $(D^2 - 1)\{(D^4 + 2D^2 + 1) - D^2\} = 0$

or $(D^2 - 1)\{(D^2 + 1)^2 - D^2\} = 0$

or $(D-1)(D+1)(D^2 + 1 + D)(D^2 + 1 - D) = 0$

$\therefore$ $D = 1, -1, \dfrac{-1 \pm (1-4)^{1/2}}{2}, \dfrac{1 \pm (1-4)^{1/2}}{2} = 1, -1, -\dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}, \dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}$

Hence the solution is $y = c_1 e^x + c_2 e^{-x} + e^{-x/2}[c_3 \cos(x\sqrt{3}/2) + c_4 \sin(x\sqrt{3}/2)]$

$+ e^{x/2}[c_5 \cos(x\sqrt{3}/2) + c_6 \sin x\sqrt{3}/2)]$, $c_1, c_2, c_3, c_4, c_5, c_6$ being arbitrary constants

Ex. 11. *Solve* (a) $(D^4 + 8D^2 + 16)y = 0$

(b) $(D^2 + D + 1)^2 = 0$. **[I.A.S. (Prel.) 1994]**

Sol. (a) Here the auxiliary equation is $D^4 + 8D^2 + 16 = 0$ or $(D^2 + 4)^2 = 0$ or $D^2 + 4 = 0$ (twice) so that $D = 0 \pm 2i$ (twice). Here complex roots of A.E. are repeated twice. Hence, the required general solution is

$y = e^{0x}[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x]$

or $y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$, c_1, c_2, c_3 and c_4 being arbitrary constants

(b) Ans. $y = e^{-x/2}[(c_1 + c_2 x)\cos(x\sqrt{3}/2) + (c_3 + c_4 x)\sin(x\sqrt{3}/2)]$.

Ex. 12. *Solve (a)* $(D^2 + D + 1)^2 (D - 2)y = 0$.

(b) $(D^2 + 1)^2 (D^2 + D + 1)y = 0$

(c) $(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0$

Sol. (a) Here the auxiliary equation is $(D^2 + D + 1)^2 (D - 2)y = 0$

$\Rightarrow D^2 + D + 1 = 0$ (twice) or $D - 2 = 0$

$\Rightarrow D = [-1 \pm \sqrt{(1-4)}]/2$ (twice) or $D = 2$

$\Rightarrow D = (-1/2) \pm i(\sqrt{3}/2)$ (twice) or $D = 2$.

So required general solution is $y = c_1 e^{2x} + e^{-x/2}[(c_2 + c_3 x)\cos(x\sqrt{3}/2)$

$+ (c_4 + c_5 x)\sin(x\sqrt{3}/2)]$, c_1, c_2, c_3, c_4 and c_5 being arbitrary constants.

(b) Here the auxiliary equation is $(D^2 + 1)^2 (D^2 + D + 1) = 0$

$\therefore$ $D = 0 \pm i$ (twice), $\{-1 \pm (1-4)^{1/2}\}/2$ *i.e.*, $0 \pm i$ (twice), $-(1/2) \pm i(\sqrt{3}/2)$.

Hence the required general solution is $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$

$+ e^{-x/2}[c_5 \cos(x\sqrt{3}/2) + c_6 \sin(x\sqrt{3}/2)]$, $c_1, c_2, c_3, c_4, c_5, c_6$ being arbitrary constants.

(c) Here the auxiliary solution is $(D^2 + 1)^3 (D^2 + D + 1)^2 = 0 \Rightarrow D^2 + 1 = 0$ (thrice), $D^2 + D + 1 = 0$ (twice). Hence $D = 0 \pm i$ (thrice), $-(1/2) \pm i(\sqrt{3}/2)$ (twice).

Solution is $y = (c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x + e^{-x/2}[(c_7 + c_8 x)\cos(\sqrt{3}x/2)$

$+ (c_9 + c_{10} x)\sin(\sqrt{3}x/2)]$, where $c_1, c_2, \ldots, c_{10}$ are arbitrary constants.

Ex. 13. (a) *Solve* $(d^4y/dx^4) - 4(d^3y/dx^3) + 8(d^2y/dx^2) - 8(dy/dx) + 4y = 0$ **[Rohilkhand 1995]**

Sol. Let $D \equiv d/dx$. Then the given equation, becomes $(D^4 - 4D^3 + 8D^2 - 8D + 4)\, y = 0$

Here the auxiliary equation is $D^4 - 4D^3 + 8D^2 - 8D + 4 = 0$

or $(D^2 + 4/D^2) - 4\,(D + 2/D) + 8 = 0$, on dividing both sides by D^2

Let $D + 2/D = z$ so that $D^2 + 4/D^2 = z^2 - 4$. ... (1)

Using (2), (1) becomes $(z^2 - 4) - 4z + 8 = 0$ or $z^2 - 4z + 4 = 0$ ... (2)

or $(z - 2)^2 = 0$ or $[D + (2/D) - 2]^2 = 0$, using (2)

or $(D^2 - 2D + 2)^2 = 0$ $\Rightarrow$ $D = [2 \pm (4 - 8)]/2$ (twice), *i.e.*, $D = 1 \pm i$ (twice)

Hence the required general solution is $y = e^x\,[(c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x]$,

c_1, c_2, c_3 and c_4 being arbitrary constants.

Ex. 13. (b) *Solve* $(D^4 + 2D^3 + 3D^2 + 2D + 1)\, y = 0$.

Sol. A.E. is $D^4 + 2D^3 + 3D^2 + 2D + 1 = 0$ or $(D^4 + 2D^3 + D^2) + (2D^2 + 2D + 1) = 0$

or $(D^2 + D)^2 + 2\,(D^2 + D) + 1 = 0$ or $(D^2 + D + 1)^2 = 0$ or $D^2 + D + 1 = 0$ (twice)

$\therefore$ $D = \{-1 \pm (1 - 4)^{1/2}\}/2 = -(1/2) \pm i(\sqrt{3}/2)$ (twice). The required general solution is

$$y = e^{-x/2}\,[(c_1 + c_2x)\cos(\sqrt{3}x/2) + (c_3 + c_4x)\sin(\sqrt{3}x/2)]$$

c_1, c_2, c_3 and c_4 being arbitrary constants.

Ex. 14 (a). *Solve* $(d^2y/dx^2) + 4y = 0$, *given that* $y = 2$ *and* $dy/dx = 0$ *when* $x = 0$.

Sol. Let $D \equiv d/dx$. Then the equation is $(D^2 + 4)\, y = 0$.

Its auxiliary equation is $D^2 + 4 = 1$ so that $D = 0 \pm 2i$. ... (1)

Hence the general solution of (1) is $y = c_1 \cos 2x + c_2 \sin 2x$, ... (2)

where c_1 and c_2 are arbitrary constants. These constants will be determined by using the given conditioins of the problem, namely,

$y = 2$ when $x = 0$ ... (3)

and $dy/dx = 0$ when $x = 0$ ... (4)

Now, from (2), $dy/dx = -2c_1 \sin 2x + 2c_2 \cos 2x$. ... (5)

Using condition (3), (2) gives $2 = c_1$ so that $c_1 = 2$

Using condition (4), (5) gives $0 = 2c_2$ so that $c_2 = 0$.

Putting $c_1 = 2$, $c_2 = 0$ in (2), the required solution is $y = 2 \cos 2x$.

Ex. 14 (b). *Solve* $(d^2y/dx^2) + y = 0$ *given* $y = 2$ *for* $x = 0$ *and* $y = -2$ *for* $x = \pi/2$.

Sol. Proceed as in part (*a*). **Ans.** $y = 2(\cos x - \sin x)$.

Ex. 15 (a). *Solve* $l(d^2\theta/dt^2) + g\theta = 0$ *given that* $\theta = \theta_0$ *and* $d\theta/dt = 0$ *when* $t = 0$.

Sol. Let $D \equiv d/dt$. Then the given equation can be written as $[D^2 + (g/l)]\,\theta = 0$. ... (1)

Its auxiliary equation is $D^2 + (g/l) = 0$ so that $D = 0 \pm i(g/l)^{1/2}$

The general solution of (1) is $\theta = c_1 \cos\{t\sqrt{(g/l)}\} + c_2 \sin\{t\sqrt{(g/l)}\}$, ... (2)

where c_1 and c_2 are arbitrary constants.

From (2), $d\theta/dt = -c_1\sqrt{(g/l)}\sin\{t\sqrt{g/l}\} + c_2\sqrt{(g/l)}\cos\{t\sqrt{g/l}\}$ ... (3)

Given that $\theta = \theta_0$ when $t = 0$ ... (4)

and $d\theta/dt = 0$ when $t = 0$. ... (5)

Using the condition (4), $(2) \Rightarrow \theta_0 = c_1$ so that $c_1 = \theta_0$.

Using the condition (5), $(3) \Rightarrow 0 = c_2\sqrt{(g/l)}$ so that $c_2 = 0$.

Putting $c_1 = \theta_0$, $c_2 = 0$ in (2), the required solution is $\theta = \theta_0 \cos\{t\sqrt{(g/l)}\}$.

Ex. 15 (b). *Find the solution of* $(d^2i/dt^2) + (R/L)\,(di/dt) + (1/LC)\, i = 0$, *where* $R^2C = 4L$ *and* R, C, L *are constants.*

Sol. Let $D \equiv d/dt$. Then the given equation can be written as $[(D^2 + (R/L)\,D + (1/LC)]\, i = 0$.

Here the auxiliary equation is $D^2 + (R/L)\,D + (1/LC) = 0$

so that $D = [-(R/L) \pm \{(R^2/L^2) - (4/LC)\}^{1/2}]/2 = -(R/2L)$, as $R^2C = 4L$

Thus, $D = -(R/2L)$ (twice). Hence the required general solution is

$y = (c_1 + c_2t)\, e^{-t\,(R/2L)}$, c_1, c_2 being arbitrary constants.

Exercise 5(A)

Solve the following differential equations :

1. (a) $(D^3 + 6D^2 + 11D + 6)\, y = 0$ **(Meerut 2010)**
   (b) $d^2y/dx^2 + 2(dy/dx) + 5y = 0$ **(Guwahati 2007)** **Ans.** $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$
   (c) $d^3y/dx^3 - 6(d^2y/dx^2) + 9\,(dy/dx) = 0$ **(Pune 2010)** **Ans.** $y = e^{-x}\,(c_1 \cos 4x + c_2 \sin 4x)$
2. $(D^3 + 6D^2 + 12D + 8)\, y = 0$. **Ans.** $y = c_1 + (c_2 + xc_3)\, e^{3x}$
3. $(d^2y/dx^2) + 2p\,(dy/dx) + (p^2 + q^2)\, y = 0$. **Ans.** $y = (c_1 + c_2x + c_3x^2)\, e^{-2x}$
4. $(D^4 - 2D^3 + 5D^2 - 8D + 4)\, y = 0$. **Ans.** $y = e^{-px}\,(c_1 \cos qx + c_2 \sin qx)$
5. $(D^4 + D^2 + 1)\, y = 0$. **Ans.** $y = (c_1 + c_2x)\, e^x + c_3 \cos 2x + c_4 \sin 2x$
   Ans. $y = e^{x/2}\,[c_1 \cos(x\sqrt{3}/2) + c_2 \sin(x\sqrt{3}/2)] + e^{-x/2}[c_3 \cos(x\sqrt{3}/2) + c_4 \sin(x\sqrt{3}/2)]$
6. $(D^4 + 4D^3 - 5D^2 - 36D - 36)\, y = 0$.
7. $(D^4 - 7D^3 + 18D^2 - 20D + 8)\, y = 0$. **Ans.** $y = c_1e^{-3x} + c_2e^{3x} + (c_3 + c_4x)\, e^{-2x}$
8. $(D^2 \pm w^2)\, y = 0$, $w \neq 0$. **Ans.** $y = c_1e^x + (c_2 + c_3x + c_4x^2)\, e^{2x}$
   Ans. $y = c_1 \cos wx + c_2 \sin wx + c_3\, e^{wx} + c_4 e^{-wx}$
9. $\{D^3 + D^2\,(2\sqrt{3} - 1) + D\,(3 - 2\sqrt{3}) - 3\}\, y = 0$ **Ans.** $y = c_1\, e^x + (c_2 + c_3x)\, e^{-x\sqrt{3}}$
10. (a) $(D^5 - 13D^3 + 26D^2 + 82D + 104)\, y = 0$
    Ans. (a) $y = c_1\, e^{-4x} + e^{-x}\,(c_2 \cos x + c_3 \sin x) + e^{3x}\,(c_4 \cos 2x + c_5 \sin 2x)$
    (b) $(D^6 + 9D^4 + 24D^2 + 16)\, y = 0$ (b) $y = c_1 \cos x + c_2 \sin x + (c_3 + c_4x) \cos 2x + (c_5 + c_6x) \sin 2x$
11. $d^2x/dt^2 - 3\,(dx/dt) + 2x = 0$ given when $t = 0, x = 0$ and $dx/dt = 0$ **Ans.** $x = 0$

5.6 The symbolic function $1/f(D)$.

Definition. The expression $\dfrac{1}{f(D)}X$ is defined to be that function of x which when operated upon by $f(D)$ gives X.

For example, $\dfrac{1}{D^2 + 3D}(2 + 6x) = x^2$ $[\because (D^2 + 3D)\, x^2 = 2 + 6x]$

The operator $1/f(D)$, according to this definition, is the inverse of the operator $f(D)$.

Thus, $\dfrac{1}{D}X = \int X\, dx$. **[Remember]**

5.7 Determination of the particular integral (P.I.) of

$$f(D)\, y = X. \qquad \text{... (1)}$$

In view of the definition 5.6, it follows that, Particular integral of (1) $= \dfrac{1}{f(D)}X$. ... (2)

5.8 General method of getting particular integral

Theorem If X is a function of x, then $\dfrac{1}{D - \alpha}X = e^{\alpha x}\int Xe^{-\alpha x}\, dx$.

Proof. Let
$$y = \frac{1}{D-\alpha} X$$

On operating by $(D-\alpha)$, we get
$$(D-\alpha)\, y = X$$

or
$$\left(\frac{d}{dx} - \alpha\right) y = X \quad \text{or} \quad \frac{dy}{dx} - \alpha y = X$$

which is a linear differential equation whose I.F. $= e^{-\int \alpha\, dx} = e^{-\alpha x}$ and hence its solution is given by

$ye^{-\alpha x} = \int X e^{-\alpha x}\, dx$, after omitting constant of integration, since P.I. is required

$\therefore$
$$y = e^{\alpha x} \int X e^{-\alpha x}\, dx$$

Thus,
$$\frac{1}{D-\alpha} X = e^{\alpha x} \int X e^{-\alpha x}\, dx \qquad \text{... (1)}$$

Similarly
$$\frac{1}{D+\alpha} X = e^{-\alpha x} \int X e^{\alpha x}\, dx. \qquad \text{... (2)}$$

Remark 1. Since we require only a particular integral, we shall never add a constant of integration after integration is performed in connection with any method of finding P.I. Hence P.I. will never contain any arbitrary constant.

Remark 2. The above method can be used to evaluate P.I. in any problem. Since shorter methods depending upon the special form of function X are available (to be discussed later on), the above general method, however, must be used for problems in which X is of the form sec ax, cosec ax, $\sec^2 ax$, $\operatorname{cosec}^2 ax$, tan ax, cot ax or any other form not covered by shorter methods (employed for special forms).

5.9 Corollary. *If n is a positive integer, then*
$$\frac{1}{(D-\alpha)^n} e^{\alpha x} = \frac{x^n}{n!} e^{\alpha x}$$

Proof L.H.S.
$$= \frac{1}{(D-\alpha)^n} e^{\alpha x} = \frac{1}{(D-\alpha)^{n-1}} \frac{1}{D-\alpha} e^{\alpha x} = \frac{1}{(D-\alpha)^{n-1}} e^{\alpha x} \int e^{\alpha x} e^{-\alpha x}\, dx$$

[Using the theorem of Art. 5.8 with $X = e^{\alpha x}$]
$$= \frac{1}{(D-\alpha)^{n-1}} e^{\alpha x} x = \frac{1}{(D-\alpha)^{n-2}} \frac{1}{D-\alpha} x e^{\alpha x} = \frac{1}{(D-\alpha)^{n-2}} e^{\alpha x} \int x e^{\alpha x} e^{-\alpha x}\, dx$$

[Using the theorem of Art. 5.8 with $X = xe^{\alpha x}$]
$$= \frac{1}{(D-\alpha)^{n-2}} e^{\alpha x} \int x\, dx = \frac{1}{(D-\alpha)^{n-2}} e^{\alpha x} \cdot \frac{x^2}{2!} \qquad \text{... (i)}$$
$$= \frac{1}{(D-\alpha)^{n-3}} \frac{1}{D-\alpha} e^{\alpha x} \frac{x^2}{2!} = \frac{1}{(D-\alpha)^{n-3}} e^{\alpha x} \int \{e^{\alpha x} (x^2/2!)\, e^{-\alpha x}\}\, dx$$

[Using the theorem of Art. 5.8 with $X = e^{\alpha x} (x^2/2!)$]
$$= \frac{1}{(D-\alpha)^{n-3}} \frac{e^{\alpha x}}{2!} \int x^2\, dx = \frac{1}{(D-\alpha)^{n-3}} \frac{x^3}{3!} e^{\alpha x} \qquad \text{... (ii)}$$

Continuing as before and noting (*i*) and (*ii*), we finally obtain
$$\frac{1}{(D-\alpha)^n} e^{\alpha x} = \frac{1}{(D-\alpha)^{n-n}} \frac{x^n}{n!} e^{\alpha x} = \frac{x^n}{n!} e^{\alpha x} \qquad \text{... (iii)}$$

5.10 Working rule of finding the particular integral (P.I.) i.e. $\dfrac{1}{f(D)} X$.

There are two following ways to obtain P.I.

Method I. The oeprator $1/f(D)$ may be factored into linear factors;

Then,
$$\text{P.I.} = \frac{1}{D-\alpha_1} \frac{1}{D-\alpha_2} \frac{1}{D-\alpha_3} \cdots \frac{1}{D-\alpha_n} X$$

On operating with the first symbolic factor, beginning at the right, there is obtained (keeping the result (1) of theorem 5.8)
$$\text{P.I.} = \frac{1}{D-\alpha_1} \frac{1}{D-\alpha_2} \cdots \frac{1}{D-\alpha_{n-1}} e^{\alpha_n x} \int X e^{-\alpha_n x}\, dx;$$

then, on operating with the second and remaining factors in succession, taking them from right to left, required P.I. can be obtained.

Method II. The operator $1/f(D)$ may be decomposed into its partial fractions, then
$$\text{P.I.} = \left[\frac{A_1}{D-\alpha_1} + \frac{A}{D-\alpha_2} + \dots + \frac{A_n}{D-\alpha_n}\right] X = A_1 e^{\alpha_1 x} \int X e^{-\alpha_1 x}\, dx + \dots + A_n e^{\alpha_n x} \int X e^{-\alpha_n x}\, dx.$$

Of these two methods, the latter is generally used in practice.

5.11 Solved examples based on Art. 5.10

Ex. 1. *Solve $(D^2 + a^2)\, y = \cot ax$.* **[Delhi Maths. (G) 2005]**

Sol. Here the auxiliary equation is $D^2 + a^2 = 0$ so that $D = 0 \pm ia$.

$\therefore$ C.F. $= e^{0x} (c_1 \cos ax + c_2 \sin ax) = c_1 \cos ax + c_2 \sin ax$, c_1, c_2 being arbitrary constants

Now,
$$\text{P.I.} = \frac{1}{D^2 + a^2} \cot ax = \frac{1}{(D+ai)(D-ia)} \cot ax$$

[$\because D^2 + a^2 = D^2 - (ia)^2 = (D + ai)(D - ai)$]
$$= \frac{1}{2ia}\left[\frac{1}{D-ia} + \frac{1}{D+ia}\right] \cot ax, \text{ on resolving into partial fractions}$$

Now,
$$\frac{1}{D-ia} \cot ax = e^{iax} \int e^{-iax} \cot ax\, dx = e^{iax} \int (\cos ax - i \sin ax) \frac{\cos ax}{\sin ax}\, dx$$

[$\because$ by Euler's theorem, $e^{-iax} = \cos ax - i \sin ax$]
$$= e^{iax} \int \left(\frac{\cos^2 ax}{\sin ax} - i \cos ax\right) dx = e^{iax} \int \left(\frac{1 - \sin^2 ax}{\sin ax} - i \cos ax\right) dx$$
$$= e^{iax} \int (\operatorname{cosec} ax - \sin ax - i \cos ax)\, dx = e^{iax} [(1/a) \log \tan (ax/2) + (1/a) \cos ax - (i/a) \sin ax]$$
$$= e^{iax} [(1/a) \log \tan (ax/2) + (1/a)(\cos ax - i \sin ax)]$$
$$= e^{iax} [(1/a) \log \tan (ax/2) + (1/a)\, e^{-iax}], \text{ by Euler's theorem}$$

$\therefore$
$$\frac{1}{D-ia} \cot ax = \frac{1}{a}\left[e^{iax} \log \tan \frac{ax}{2} + 1\right]. \qquad \text{... (2)}$$

Replacing i by $-i$ in (2), $\dfrac{1}{D-ia}\cot ax=\dfrac{1}{a}\left[e^{-iax}\log\tan\dfrac{ax}{2}+1\right]$ ...(3)

Using (2) and (3), (1) reduces to

$$\text{P.I.}=\frac{1}{2ia}\left[\frac{1}{a}\{e^{iax}\log\tan\frac{ax}{2}+1\}-\frac{1}{a}\{e^{-iax}\log\tan\frac{ax}{2}+1\}\right]$$

$$=\frac{1}{a^2}\cdot\frac{e^{iax}-e^{-iax}}{2i}\log\tan\frac{ax}{2}=\frac{1}{a^2}\sin ax\log\tan\frac{ax}{2}.$$

Hence the required general solution is $y=\text{C.F.}+\text{P.I.}$, i.e.,
$y=c_1\cos ax+c_2\sin ax+(1/a^2)\sin ax\log\tan(ax/2)$, where c_1 and c_2 are arbitrary constants.

Ex. 2. *Solve* $(D^2+a^2)y=\tan ax.$ **(Meerut 1996)**

Sol. Here the auxiliary equation $D^2+a^2=0$ gives $D=\pm ia$

$\therefore$ $\text{C.F.}=c_1\cos ax+c_2\sin ax$, c_1, c_2 is being arbitrary constants.

Now, $$\text{P.I.}=\frac{1}{D^2+a^2}\tan ax=\frac{1}{(D+ai)(D-ai)}\tan ax=\frac{1}{2ia}\left[\frac{1}{D-ia}-\frac{1}{D+ia}\right]\tan ax. \quad ...(1)$$

Now, $\dfrac{1}{D-ia}\tan ax=e^{iax}\int e^{-iax}\tan ax\,dx=e^{iax}\int(\cos ax-i\sin ax)\dfrac{\sin ax}{\cos ax}dx$, by Euler's theorem

$$=e^{iax}\int\left[\sin ax-i\frac{1-\cos^2 ax}{\cos ax}\right]dx=e^{iax}\int[\sin ax-i(\sec ax-\cos ax)]dx$$

$$=e^{iax}\left[-\frac{\cos ax}{a}-\frac{i}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)+i.\frac{\sin ax}{a}\right]=-\frac{e^{iax}}{a}\left[(\cos ax-i\sin ax)+i\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right]$$

$$=-\frac{e^{iax}}{a}\left[e^{-iax}+i\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right],\text{ by Euler's theorem}$$

Thus, $$\frac{1}{D-ia}\tan ax=-\frac{1}{a}\left[1+ie^{iax}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right] \quad ...(2)$$

Replacing i by $-i$ in (2), $$\frac{1}{D+ia}\tan ax=-\frac{1}{a}\left[1-ie^{-iax}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right] \quad ...(3)$$

$\therefore$ By (1), (2) and (3), we have

$$\text{P.I.}=\frac{1}{2ia}\left\{-\frac{i}{a}(e^{iax}+e^{-iax})\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right\}=-(1/a)^2\cos ax\log\tan(\pi/4+ax/2).$$

$\therefore$ The required solution is $y=\text{C.F.}+\text{P.I.}$, that is,

$$y=c_1\cos ax+c_2\sin ax-(1/a^2)\cos ax\log\tan(\pi/4+ax/2).$$

Ex. 3. *Solve* $(D^2+a^2)y=\sec ax.$ **[Rohilkhand 1995, Purvanchal 2007, Agra 2006, Kanpur 1997, Delhi, Maths (G) 2006, Lucknow 1995]**

Sol. Here the auxiliary equation is $D^2+a^2=0.$ so that $D=\pm ia.$

and hence $\text{C.F.}=c_1\cos ax+c_2\sin ax$, c_1, c_2 being arbitrary constants.

$$\text{P.I.}=\frac{1}{D^2+a^2}\sec ax=\frac{1}{2ia}\left[\frac{1}{D-ia}-\frac{1}{D+ia}\right]\sec ax. \quad ...(1)$$

Now, $$\frac{1}{D-ia}\sec ax=e^{iax}\int e^{-iax}\sec ax\,dx=e^{iax}\int\frac{\cos ax-i\sin ax}{\cos ax}dx$$

$$=e^{iax}\int(1-i\tan ax)\,dx=e^{iax}\{x+(i/a)\log\cos ax\} \quad ...(2)$$

Replacing i by $-i$ in (2), $$\frac{1}{D+ia}\sec ax=e^{-iax}[x-(i/a)\log\cos ax] \quad ...(3)$$

$\therefore$ From (1), (2) and (3), we get

$$\text{P.I.}=(1/2ia)\,[e^{iax}\{x+(i/a)\log\cos ax\}-e^{-iax}\{x-(i/a)\log\cos ax\}]$$

$$=x\frac{(e^{iax}-e^{-iax})}{2ia}+\frac{1}{a^2}(\log\cos ax)\cdot\frac{(e^{iax}+e^{-iax})}{2}$$

$$=(x/a)\sin ax+(1/a^2)\cos ax\log\cos ax$$

$\therefore$ Hence the required solution is $y=\text{C.F.}+\text{P.I.}$, that is,
$y=c_1\cos ax+c_2\sin ax+(x/a)\sin ax+(1/a^2)\cos ax\log\cos ax.$

Ex. 4. *Solve (a)* $(d^2y/dx^2)+y=\sec^2 x.$
(b) $(d^2y/dx^2)+y=\text{cosec}^2 x.$ **[Delhi Maths. (Prog) 2009]**

Sol. *(a)* Let $D\equiv d/dx$. Then the given equation is $(D^2+1)\,y=\sec^2 x.$
Its auxiliary equation is $D^2+1=0$ so that $D=\pm i$ and $\text{C.F.}=c_1\cos x+c_2\sin x.$

and $$\text{P.I.}=\frac{1}{D^2+1}\sec^2 x=\frac{1}{(D-i)(D+i)}\sec^2 x=\frac{1}{2i}\left[\frac{1}{D-i}-\frac{1}{D+i}\right]\sec^2 x$$

Now, $$\frac{1}{D-i}\sec^2 x=e^{ix}\int e^{-ix}\sec^2 x\,dx=e^{ix}\int\frac{\cos x-i\sin x}{\cos^2 x}dx$$

$$=e^{ix}\int(\sec x-i\sec x\tan x)\,dx=e^{ix}[\log(\sec x+\tan x)-i\sec x] \quad ...(2)$$

Replacing i by $-i$ in (2), $$\frac{1}{D+i}\sec^2 x=e^{-ix}[\log(\sec x+\tan x)+i\sec x] \quad ...(3)$$

From (1) (2), and (3), we have

$$\text{P.I.}=(1/2i)\,[e^{ix}\log(\sec x+\tan x)-ie^{ix}\sec x-e^{-ix}\log(\sec x+\tan x)-e^{-ix}\sec x]$$

$$=\frac{e^{ix}-e^{-ix}}{2i}\log(\sec x+\tan x)-\frac{e^{ix}+e^{-ix}}{2}\sec x$$

$$=\sin x\log(\sec x+\tan x)-\cos x\sec x=\sin x\log(\sec x+\tan x)-1.$$

Hence the required solution. $y=c_1\cos x+c_2\sin x+\sin x\log(\sec x+\tan x)-1.$

(b) Do yourself **Ans.** $y=c_1\cos x+c_2\sin x-\cos x\log(\text{cosec } x-\cot x)-1.$

Exercise 5 (B)

Solve the following differential equations :

1. $(D^2+4)\,y=\tan 2x$ **Ans.** $y=c_1\cos 2x+c_2\sin x-(1/4)\times\cos 2x\log\tan(\pi/4+x)$
2. $(D^2+1)\,y=\text{cosec } x.$ **Ans.** $y=c_1\cos x+c_2\sin x-x\cos x+\sin x\log\sin x$
3. **(b)** $(D^2+9)\,y=\sec 3x.$ **[Meerut 2007; Rajasthan 2010]**
3. **(b)** $(D^2+9)\,y=\sec 3x.$ **[Meerut 2007]**
Ans. $y=c_1\cos 3x+c_2\sin 3x+(x/3)\times\sin 3x+(1/9)\times\cos 3x\log\cos 3x$
4. $(D^2+a^2)\,y=\text{cosec } ax.$ **Ans.** $y=c_1\cos ax+c_2\sin ax+(1/a^2)\sin ax\log\sin ax-(x/a)\cos ax$

5. $d^2x/dy^2 + 4x = \tan 2y$. **Ans.** $x = c_1 \cos 2y + c_2 \sin 2y - (1/4) \times \cos 2y \tan (y + \pi/4)$

Hint: It is same as Ex. 1 by interchanging x and y.

6. $(D^2 - 3D + 2)\, y = \sin e^{-x}$ **Ans.** $y = c_1 e^x + c_2 e^{2x} - e^{2x} \sin e^{-x}$

7. $(D^2 - 9D + 18) y = e^{e^{-3x}}$ **Ans.** $y = c_1 e^{3x} + c_2 e^{6x} + (1/9) \times e^{6x} e^{e^{-3x}}$

5.12 Short methods for finding the particular integral of given equation $f(D)\, y = X$, when X is of certain special form.

The general method of finding P.I. given in Art. 5.8 leads to cumbersome calculations in most of the problems. However, the P.I. can be obtained by methods that are shorter than general methods provided X is one of the following special forms :

Form I. When $X = e^{ax}$, where a is any constant.

Form II. When $X = \sin ax$ or $\cos ax$

Form III. When $X = x^n$ or *a polynomial* $a_0 x^n + a_1 x^{n-1} + \ldots\ldots\ldots + a_{n-1} x + a_n$ where n is any positive integer

Form IV. When $X = x^n V$, where V is a function of x and n is positive integer.

5.13 Short method of finding P.I. when $X = e^{ax}$, where 'a' is constant.

Formula IA. $\quad \text{P.I.} = \dfrac{1}{f(D)} e^{ax} = \dfrac{1}{f(a)} e^{ax}$, when $f(a) \neq 0$.

Formula IIA. $\quad \text{P.I.} = \dfrac{1}{(D-a)^n} e^{ax} = \dfrac{x^n}{n!} e^{ax}$, $n = 1, 2, 3, \ldots$

Proof of formula IA. Let $\quad f(D) = D^n + c_1 D^{n-1} + c_2 D^{n-2} + \ldots + c_{n-1} D + c_n$.

But $\; D\, e^{ax} = a\, e^{ax},\; D^2 e^{ax} = a^2 e^{ax},\; \ldots,\; D^{n-1} e^{ax} = a^{n-1} e^{ax},\; D^n e^{ax} = a^n e^{ax}$.

$$\therefore \quad f(D)\, e^{ax} = (D^n + c_1 D^{n-1} + \ldots c_{n-1} D + c_n)\, e^{ax}$$

$$= (a^n + c_1 a^{n-1} + \ldots + c_{n-1} a + c_n)\, e^{ax} = f(a)\, e^{ax}$$

Thus, $\quad f(D)\, e^{ax} = f(a)\, e^{ax}$.

Operating upon both sides by $1/f(D)$, we get

$$\frac{1}{f(D)} f(D) \cdot e^{ax} = \frac{1}{f(D)} f(a) e^{ax} \quad \text{or} \quad e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$$

or
$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \text{ provide } f(a) \neq 0.$$

Proof of formula IIA. Refer corollary in Art. 5.8.

5.14 Working rule for finding P.I. of $f(D)\, y = X$ when $X = e^{ax}$

Step I. If $f(a) \neq 0$, then use formula IA of Art. 5.13. Note that we put a for D in $f(D)$ provided $f(a) \neq 0$.

Step II. If $f(a) = 0$, then the following two cases arise.

Case (i) If $f(D) = (D - a)^n$, where $n = 1, 2, 3, \ldots$ Then we shall use formula IIA of Art. 5.13.

Case (ii) If $f(D) = (D - a)^r \phi(D)$, where $\phi(a) \neq 0$ and $r = 1, 2, 3, \ldots$ Then we use formulas IA and IIA of Art. 5.13 in succession as below :

$$\text{P.I.} = \frac{1}{(D-a)^r \phi(D)} e^{ax} = \frac{1}{(D-a)^r} \frac{1}{\phi(a)} e^{x}, \text{ using formula IA of Art. 5.13}$$

$$= \frac{1}{\phi(a)} \frac{1}{(D-a)^r} e^{ax} = \frac{1}{\phi(a)} \frac{x^r}{r!} e^{ax}, \text{ using formula IIA of Art. 5.13} \qquad \ldots (1)$$

Alternative form of above result (1). Since $f(D) = (D-a)^r \phi(D)$, $f^{(r)}(D) = r!\, \phi(D)$ + terms containing $(D-a)$ and its higher powers. So $f^{(r)}(a) = r!\, \phi(a)$. Hence, (1) takes new form

$$P.I. = (x^r e^{ax}) / f^{(r)}(a) = x^r \left[f^{(r)}(a)\right]^{-1} e^{ax} \qquad \ldots (2)$$

Note. As a particular case, if $a = 0$ so that $X = e^{0 \cdot x} = 1$, then formulae I and II of Art. 5.13 take the following forms :

Formula IB. $\dfrac{1}{f(D)} 1 = \dfrac{1}{f(D)} e^{0 \cdot x} = \dfrac{1}{f(0)} e^{0 \cdot x} = \dfrac{1}{f(0)}$, if $f(0) \neq 0$

Formula IIB. $\dfrac{1}{D^n} 1 = \dfrac{1}{(D-0)^n} e^{0 \cdot x} = \dfrac{x^n}{n!} e^{0 \cdot x} = \dfrac{x^n}{n!}$, $n = 1, 2, 3 \ldots$

5.15 Solved examples based on working rule 5.14

Ex. 1. *Solve the following differential equations :*

(a) $(D^2 - 3D + 2)\, y = e^{3x}$. **[I.A.S. (Preliminary) 1993, Meerut 1994]**

(b) $(4D^2 + 12D + 9)\, y = 144\, e^{-3x}$. **[Rohilkhand 1992, 93]**

(c) $[D^2 + 2pD + (p^2 + q^2)]\, y = e^{ax}$.

(d) $D^2 (D+1)^2\, (D^2 + D + 1)^2 y = e^x$

Sol. (a) Here the auxiliary equation is $\quad D^2 - 3D + 2 = 0 \quad$ so that $\quad D = 1, 2$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{2x}$, c_1, c_2 being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} e^{3x} = \frac{1}{3^2 - (3 \times 3) + 2} e^{3x} = \frac{1}{2} e^{3x}$$

$\therefore$ The required general solution is $\quad y = c_1 e^x + c_2 e^{2x} + (1/2)\, e^{3x}$.

(b) Here the A.E. is $\quad (2D + 3)^2 = 0 \quad$ so that $\quad D = -3/2, -3/2$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{-3x/2}$, c_1, c_2 being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{4D^2 + 12D + 9} 144 e^{-3x} = 144 \frac{1}{(2D+3)^2} e^{-3x} = \frac{144}{(-6+3)^2} e^{-3x} = 16 e^{-3x}$$

Hence the required solution is $\quad y = (c_1 + c_2 x)\, e^{-3x/2} + 16\, e^{-3x}$.

(c) Here the auxiliary equation is $\quad D^2 + 2pD + (p^2 + q^2) = 0$

Solving
$$D = \frac{-2p \pm \sqrt{4p^2 - 4(p^2 + q^2)}}{2} = -p \pm iq \text{ (complex roots)}$$

$\therefore$ C.F. $= e^{-px}\, (c_1 \cos qx + c_2 \sin qx)$, c_1, c_2 being arbitrary constants

and
$$\text{P.I.} = \frac{1}{D^2 + 2pD + (p^2 + q^2)} e^{ax} = \frac{1}{a^2 + 2pa + p^2 + q^2} e^{ax} = \frac{e^{ax}}{(p+a)^2 + q^2}$$

$\therefore$ Required solution is $\quad y = e^{-p}\, (c_1 \cos qx + c_2 \sin qx) + e^{ax}/\{(p+a)^2 + q^2\}$

(d) Here the auxiliary equation is $\quad D^2 (D+1)^2\, (D^2 + D + 1)^2 = 0$

Solving, we get $\; D = 0, \; 0, \; -1, \; -1, \; -(1/2) \pm i\,(\sqrt{3}/2), \; -(1/2) \pm i\,(\sqrt{3}/2)$.

$\therefore$ C.F. $= (c_1 + c_2 x) e^{2x} + (c_3 + c_4 x) e^{-x} + e^{-x/2} [(c_5 + c_6 x) \cos(\sqrt{3}x/2)]$

$+ [(c_7 + c_8 x) \sin(\sqrt{3}x/2)]$, $c_1, c_2, \ldots, c_7, c_8$ being arbitrary constants

and P.I. $= \dfrac{1}{D^2 (D+1)^2 (D^2 + D + 1)^2} e^x = \dfrac{1}{1^2 (1+1)^2 (1^2 + 1 + 1)^2} e^x = \dfrac{e^x}{36}$

Hence the required solution is $y = c_1 + c_2 x + (c_3 + c_4 x) e^{-x}$

$+ e^{-x/2} [(c_5 + c_6 x) \cos(\sqrt{3}x/2)] + [(c_7 + c_8 x) \sin(\sqrt{3}x/2)] + (1/36) e^x$.

Ex. 2. *Solve the following differential equations :* **(Madras 2012)**

(a) $(9D^2 - 12D + 4) y = e^{2x/3}$.

(b) $(D^2 + 4D + 4) y = e^{2x} - e^{-2x}$ or $(D^2 + 4D + 4) y = 2 \sinh 2x$.

Sol. (a) Here the auxiliary equation is $9D^2 - 12D + 4 = 0$.

or $(3D - 2)^2 = 0$ so that $D = 2/3, 2/3$

$\therefore$ C.F. $= (c_1 + c_2 x) e^{2x/3}$, c_1, c_2 being arbitrary constants.

and P.I. $= \dfrac{1}{(9D^2 - 12D + 4)} e^{2x/3} = \dfrac{1}{(3D-2)^2} e^{2x/3} = \dfrac{1}{9} \dfrac{1}{\{D - (2/3)\}^2} e^{2x/3}$

$= \dfrac{1}{9} \cdot \dfrac{x^2}{2!} e^{2x/3}$, as $\dfrac{1}{(D-a)^n} e^{ax} = \dfrac{x^n}{n!} e^{ax}$

$\therefore$ Solution is $y = (c_1 + c_2 x) e^{2x/3} + (1/9) \times x^2 e^{2x/3}$. c_1, c_2, being arbitrary constants.

(b) Given equation is $(D^2 + 4D + 4) y = 2 \sinh 2x$.

or $(D+2)^2 y = e^{2x} - e^{-2x}$, as $\sinh 2x = (e^{2x} - e^{-2x})/2$.

Here the auxiliary equation is $(D+2)^2 = 0$ so that $D = -2, -2$

$\therefore$ C.F. $= (c_1 + c_2 x) e^{-2x}$, c_1, c_2 being arbitrary constants.

and P.I. $= \dfrac{1}{(D+2)^2} (e^{2x} - e^{-2x}) = \dfrac{1}{(D+2)^2} e^{2x} - \dfrac{1}{(D+2)^2} e^{-2x}$

$= \dfrac{1}{(2+2)^2} e^{2x} - \dfrac{x^2}{2!} e^{-2x} = \dfrac{1}{16} e^{2x} - \dfrac{x^2}{2} e^{-2x}$.

Hence the required solution is $y = (c_1 + c_2 x) e^{-2x} + (1/16) \times e^{2x} - (x^2/2) \times e^{-2x}$

Ex. 3. *Solve the following differential equations :*

(a) $(D+2)(D-1)^3 y = e^x$.

(b) $(D-1)^2 (D^2+1)^2 y = e^x$

Sol. (a) Here auxiliary equation is $(D+2)(D-1)^3 = 0$ so that $D = -2, 1, 1, 1$.

$\therefore$ C.F. $= c_1 e^{-2x} + (c_2 + c_3 x + c_4 x^2) e^x$, c_1, c_2, c_3 being arbitrary constants

P.I. $= \dfrac{1}{(D-1)^3} \dfrac{1}{D+2} e^x = \dfrac{1}{(D-1)^3} \cdot \dfrac{1}{1+2} e^x$

$= \dfrac{1}{3} \dfrac{1}{(D-1)^3} e^x = \dfrac{1}{3} \cdot \dfrac{x^3}{3!} e^x$, as $\dfrac{1}{(D-a)^n} e^{ax} = \dfrac{x^n}{n!} e^{ax}$

$\therefore$ The required solution is $y = c_1 e^{-2x} + (c_2 + c_3 x + c_4 x^2) e^x + (1/18) x^3 e^x$.

(b) **Ans.** $y = (c_1 + c_2 x) e^x + (c_3 + c_4 x) \cos x + (c_5 + c_6 x) \sin x + (x^2/8) e^x$.

Ex. 4. *Solve :* (a) $(D^2 + D - 2) y = e^x$ **[I.A.S. (Prel.) 1997]**

(b) $(D-1)(D^2 - 2D + 2) y = e^x$ **[I.A.S. 2002]**

(c) $(D^3 - D) y = e^x + e^{-x}$ or $(D^3 - D) y = 2 \cosh x$ **[I.A.S. Prel. 1993]**

Sol. (a) Here the auxiliary equation is $D^2 + D - 2 = 0$ so that $D = -2, 1$

Hence C.F. $= c_1 e^{-2x} + c_2 e^x$, c_1, c_2 being arbitrary constants.

P.I. $= \dfrac{1}{D^2 + D - 2} e^x = \dfrac{1}{(D-1)(D+2)} e^x = \dfrac{1}{D-1} \cdot \dfrac{1}{1+2} e^x = \dfrac{1}{3} \dfrac{1}{D-1} e^x = \dfrac{1}{3} \times \dfrac{x}{1!} e^x = \dfrac{x}{3} e^x$

So required solution is $y = c_1 e^{-2x} + c_2 e^x + (x/3) \times e^x$

(b) Here the auxiliary equation is $(D-1)(D^2 - 2D + 2) = 0$

Hence $D = 1, \{2 \pm (4-8)^{1/2}\}/2$ i.e. $D = 1, 1 \pm i$

Hence C.F. $= c_1 e^x + e^x (c_2 \cos x + c_3 \sin x)$, c_1, c_2, c_3 being arbitrary constants

and P.I. $= \dfrac{1}{D-1} \cdot \dfrac{1}{D^2 - 2D + 2} e^x = \dfrac{1}{D-1} \dfrac{1}{1-2+2} e^x = \dfrac{1}{D-1} e^x = \dfrac{x}{1!} e^x$

So the required solution is $y = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x) + x e^x$

i.e. $y = e^x (c_1 + c_2 \cos x + c_3 \sin x + x)$.

(c) Hence the auxiliary equation is $D^3 - D = 0$ so that $D = 0, 1, -1$

Hence C.F. $= c_1 e^{0 \cdot x} + c_2 e^x + c_2 e^{-x} = c_1 + c_2 e^x + c_3 e^{-x}$

P.I. $= \dfrac{1}{D^3 - D} (e^x + e^{-x}) = \dfrac{1}{D(D-1)(D+1)} (e^x + e^{-x})$

$= \dfrac{1}{D-1} \dfrac{1}{D(D+1)} e^x + \dfrac{1}{D+1} \dfrac{1}{D(D-1)} e^{-x} = \dfrac{1}{D-1} \dfrac{1}{1 \times 2} e^x + \dfrac{1}{D+1} \dfrac{1}{(-1) \times (-2)} e^{-x}$

$= \dfrac{1}{2} \dfrac{1}{D-1} e^x + \dfrac{1}{2} \dfrac{1}{D+1} e^{-x} = \dfrac{1}{2} \dfrac{x}{1!} e^x + \dfrac{1}{2} \dfrac{x}{1!} e^{-x} = \dfrac{x(e^x + e^{-x})}{2}$

$\therefore$ Required solution is $y = c_1 + c_2 e^x + c_3 e^{-x} + (x/2) \times (e^x + e^{-x})$

Ex. 5. *Solve (a)* $(D^4 - 2D^3 + 5D^2 - 8D + 4) y = e^x$ **[I.A.S. 1991]**

(b) $(D^4 + D^3 + D^2 - D - 2) y = e^x$ **[Bhopal 1993; Lucknow 1994]**

Sol. (a) A.E. is $D^4 - 2D^3 + 5D^2 - 8D + 4 = 0$ or $(D-1)(D^3 - D^2 + 4D - 4) = 0$

or $(D-1)[D^2(D-1) + 4(D-1)] = 0$ or $(D-1)^2 (D^2+4) = 0$ so that $D = 1, 1, \pm 2i$.

$\therefore$ C.F. $= (c_1 + c_2 x) e^x + c_3 \cos 2x + c_4 \sin 2x$, c_1, c_2, c_3, c_4 being arbitrary constants.

and P.I. $= \dfrac{1}{(D-1)^2 (D^2+4)} e^x = \dfrac{1}{(D-1)^2} \dfrac{1}{5} e^x = \dfrac{1}{5} \dfrac{x^2}{2!} e^x$

The solution is $y = (c_1 + c_2 x) e^x + c_3 \cos 2x + c_4 \sin 2x + (x^2 e^x)/10$.

(b) The given equation is $(D^4 + D^3 + D^2 + D - 2) y = e^x$.

or $\{D^3(D-1) + 2D^2(D-1) + 3D(D-1) + 2(D-1)\} y = e^x$

or $(D-1)(D^3 + 2D^2 + 3D + 2) y = e^x$

or $(D-1)\{(D^2(D+1) + D(D+1) + 2(D+1)\} y = e^x$

or $(D-1)(D+1)(D^2 + D + 2) y = e^x$

Here auxiliary equation is $(D-1)(D+2)(D^2 + D + 2) = 0$

$\therefore D = 1, -1, \{-1 \pm (1-8)^{1/2}\}/2$ or $D = 1, -1, -(1/2) \pm i(\sqrt{7}/2)$

and C.F. $= c_1e^{x} + c_2e^{-x} + e^{-x/2}\{c_3\cos(\sqrt{7}x/2) + c_4\sin(\sqrt{7}x/2)\}$,

c_1, c_2, c_3 and c_4 being arbitrary constants.

Also, $$\text{P.I.} = \frac{1}{(D-1)(D+1)(D^2+D+2)}e^{x} = \frac{1}{(D-1)}\frac{1}{(1+1)(1+1+2)}e^{x}$$

$$= \frac{1}{8}\frac{1}{D-1}e^{x} = \frac{1}{8}\frac{x}{1!}e^{x} = \frac{x}{8}e^{x}, \quad \text{as} \quad \frac{1}{(D-a)^n}e^{ax} = \frac{x^n e^{ax}}{n!}$$

Required solution is $$y = c_1e^{x} + c_2e^{-x} + e^{-x/2}\{c_3\cos(\sqrt{7}x/2) + c_4\sin(\sqrt{7}x/2)\} + (x/8)e^{x}.$$

Ex. 6. *Solve : (a)* $(D^2 - 3D + 2)\,y = e^{x} + e^{2x}$. **[Delhi Maths (G) 1996]**

(b) $(D^2 - 3D + 2)\,y = \cosh x$. **[I.A.S. Prel. 2005]**

(c) $(D^3 - 5D^2 + 7D - 3)\,y = e^{2x}\cosh x$

(d) $(d^3y/dx^3) - y = (e^{x} + 1)^2$. **[Delhi Maths (H) 1993, 1996]**

Sol. (*a*) Here the auxiliary equation is $D^2 - 3D + 2 = 0$ so that $D = 1, 2$.

$\therefore$ C.F. $= c_1e^{x} + c_2e^{2x}$, c_1, c_2 being arbitrary constants.

and $$\text{P.I.} = \frac{1}{D^2-3D+2}(e^{x} + e^{2x}) = \frac{1}{(D-1)(D-2)}e^{x} + \frac{1}{(D-2)(D-1)}e^{2x}$$

$$= \frac{1}{D-1}\frac{1}{1-2}e^{x} + \frac{1}{D-2}\frac{1}{2-1}e^{2x} = -\frac{x}{1!}e^{x} + \frac{x}{1!}e^{2x}$$

Hence the general solution is $y = c_1e^{x} + c_2e^{2x} - xe^{x} + xe^{2x}$.

(*b*) Here auxiliary equation is $D^2 - 3D + 2 = 0$ so that $D = 1, 2$

$\therefore$ C.F. $= c_1e^{x} + c_2e^{2x}$, c_1, c_2 being arbitrary constants

and $$\text{P.I} = \frac{1}{D^2-3D+2}\cosh x = \frac{1}{(D-1)(D-2)}\frac{(e^{x}+e^{-x})}{2}$$

$$= \frac{1}{2}\frac{1}{(D-1)(D-2)}e^{x} + \frac{1}{2}\frac{1}{(D-1)(D-2)}e^{-x}$$

$$= \frac{1}{2}\frac{1}{D-1}\frac{1}{1-2}e^{x} + \frac{1}{2}\frac{1}{(-2)\times(-3)}e^{-x} = -\frac{1}{2}\frac{1}{D-1}e^{x} + \frac{1}{2}\times\frac{1}{6}e^{-x}$$

$$= -\frac{1}{2}\times\frac{x}{1!}e^{x} + \frac{1}{12}e^{-x} = -\frac{x}{2}e^{x} + \frac{1}{12}e^{-x}$$

$\therefore$ the required solution is $y = c_1e^{x} + c_2e^{2x} - (x/2)\times e^{x} + (1/12)\times e^{-x}$

(*c*) Re-writing, the given equation becomes

$\{D^2(D-1) - 4D(D-1) + 3(D-1)\}y = e^{2x}\cosh x$ or $(D-1)^2(D-3)\,y = e^{2x}\cosh x$.

Here auxiliary equation is $(D-1)^2(D-3) = 0$ so that $D = 1, 1, 3$.

$\therefore$ C.F. $= (c_1 + c_2x)\,e^{x} + c_3e^{3x}$, c_1, c_2, c_3 being arbitrary constants

$$\text{P.I.} = \frac{1}{(D-1)^2(D-3)}e^{2x}\cosh x = \frac{1}{(D-1)^2(D-3)}\frac{e^{2x}(e^{x}+e^{-x})}{2}$$

$$= \frac{1}{2}\frac{1}{D-3}\frac{1}{(D-1)^2}e^{3x} + \frac{1}{2}\frac{1}{(D-1)^2}\frac{1}{D-3}e^{x}$$

$$= \frac{1}{8}\frac{1}{D-3}e^{3x} - \frac{1}{4}\frac{1}{(D-1)^2}e^{x} = \frac{1}{8}\frac{x}{1!}e^{3x} - \frac{1}{4}\frac{x}{2!}e^{x}$$

So the required solution is $y = (c_1 + c_2x)\,e^{x} + c_3e^{3x} + (x/8)\,e^{3x} - (x^2/8)\,e^{x}$.

(*d*) Let $D \equiv d/dx$. Then the given equation reduces to

$(D^3 - 1)\,y = (e^{x} + 1)^2$ or $(D-1)(D^2 + D + 1)\,y = e^{2x} + 2e^{x} + 1$.

Here auxiliary equation is $(D-1)(D^2+D+1) = 0$ so that $D = 1, -(1/2) \pm i\,(\sqrt{3}/2)$.

$\therefore$ C.F. $= c_1e^{x} + e^{-(x/2)}\left[c_2\cos(\sqrt{3}x/2) + c_3\sin(\sqrt{3}x/2)\right]$

$$\text{P.I.} = \frac{1}{D^3-1}(e^{2x} + 2e^{x} + 1) = \frac{1}{D^3-1}e^{2x} + 2\frac{1}{D^3-1}e^{x} + \frac{1}{D^3-1}e^{0\cdot x}$$

$$= \frac{1}{2^3-1}e^{2x} + 2\frac{1}{(D-1)(D^2+D+1)}e^{x} + \frac{1}{(0)^3-1}e^{0\cdot x}$$

$$= \frac{1}{7}e^{2x} + 2\frac{1}{D-1}\frac{1}{1^2+1+1}e^{x} - 1 = \frac{1}{7}e^{2x} + \frac{2}{3}\frac{x}{1!}e^{x} - 1$$

So solution is $y = c_1e^{x} + e^{-(x/2)}\,[c_2\cos(\sqrt{3}x/2) + c_3\sin(\sqrt{3}x/2)] + (1/7)\times e^{2x} + (2x/3)\times e^{x} - 1$,

c_1, c_2, c_3 being arbitrary constants.

Ex. 7. *If* $(d^2x/dt^2) + (g/b)\,(x - a) = 0$, *(a, b and g being constants) and* $x = a'$ *and* $dx/dt = 0$ *when* $t = 0$, *show that* $x = a + (a' - a)\cos t\sqrt{(g/b)}$. **[I.A.S. 1994, Kurushetra 1994]**

Sol. With $D \equiv d/dt$, given equation is $\{D^2 + (g/b)\}\,x = ga/b$. ... (1)

Here auxiliary equation $D^2 + g/b = 0$ gives $D = 0 \pm i\sqrt{(g/b)}$.

$\therefore$ C.F. $= c_1\cos t\sqrt{(g/b)} + c_2\sin t\sqrt{(g/b)}$, c_1, c_2, being arbitrary constants

$$\text{P.I.} = \frac{1}{D^2+(g/b)}\frac{ga}{b} = \frac{ga}{b}\frac{1}{D^2+(g/b)}e^{0\cdot t} = \frac{ga}{b}\frac{1}{0+(g/b)}e^{0\cdot t} = a.$$

So general solution is $x = c_1\cos t\sqrt{(g/b)} + c_2\sin t\sqrt{(g/b)} + a$. ...(2)

From (2), $dx/dt = -\,c_1\sqrt{(g/b)}\sin t\sqrt{(g/b)} + c_2\sqrt{(g/b)}\cos t\sqrt{(g/b)}$ ...(3)

Given $x = a'$ when $t = 0$. So $(2) \Rightarrow a' = c_1 + a$ or $c_1 = a' - a$. ...(4)

Given $dx/dt = 0$ when $t = 0$. So $(3) \Rightarrow 0 = c_2\sqrt{g/b}$ or $c_2 = 0$. ...(5)

Substituting the values of c_1 and c_2 in (2), the required solution is

$$x = (a' - a)\cos t\sqrt{g/b} + a, \text{ as required.}$$

Ex. 8. *Solve* $(D^2 - 6D + 8)\,y = (e^{2x} + 1)^2$. **[Delhi Maths (G) 2006]**

Sol. Here auxiliary equation is $D^2 - 6D + 8 = 0$ giving $D = 2, 4$.

Hence C.F. $= c_1e^{2x} + c_2e^{4x}$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2-6D+8}(e^{2x}+1)^2 = \frac{1}{(D-2)(D-4)}(e^{4x} + 2e^{2x} + 1)$$

$$= \frac{1}{(D-4)(D-2)}e^{4x} + 2\frac{1}{(D-2)(D-4)}e^{2x} + \frac{1}{(D-2)(D-4)}e^{0\cdot x}$$

$$= \frac{1}{D-4}\frac{1}{4-2}e^{4x} + 2\frac{1}{D-2}\frac{1}{2-4}e^{2x} + \frac{1}{(0-2)(0-4)}e^{0\cdot x}$$

$$= \frac{1}{2}\frac{x}{1!}e^{4x} - \frac{x}{1!}e^{2x} + \frac{1}{8} = \frac{1}{8}(4xe^{4x} - 8xe^{2x} + 1).$$

$\therefore$ Required solution is $\qquad y = c_1e^{2x} + c_2e^{4x} + (1/8) \times (4\,xe^{4x} - 8xe^{2x} + 1).$

Ex. 9. *Find the solution of the equation $(D^2 - 1) = 1$, which vanishes when $x = 0$ and tends to a finite limit as $x \to -\infty$. D stands for d/dx.*

Sol. Here auxiliary equation. is $\qquad D^2 - 1 = 0 \qquad$ so that $\qquad D = 1, -1.$

$\therefore$ C.F. $= c_1e^x + c_2e^{-x}$, c_1, c_2 being arbitrary constants.

and $\qquad$ P.I. $= \dfrac{1}{D^2-1}1 = \dfrac{1}{D^2-1}e^{0.x} = \dfrac{1}{0^2-1}e^{0.x} = -1.$

So the general solution is $\qquad y = c_1\,e^x + c_2\,e^{-x} - 1.$ $\qquad$ ... (1)

Given $y = 0$ when $x = 0$. So $\qquad (1) \Rightarrow 0 = c_1 + c_2 - 1 \qquad$ or $\qquad c_1 + c_2 = 1.$ $\qquad$ ... (2)

Multiplying both sies of (1) by e^x, $\qquad ye^x = c_1e^{2x} + c_2 - e^x.$ $\qquad$ ... (3)

We know that $\qquad e^x = 0 \qquad$ as $\qquad x \to -\infty$ $\qquad$ ... (4)

Taking limit of both sides of (3) as $x \to -\infty$ and using (4) and the given fact that y in finite, we get $\qquad$ (finite) $\times\, 0 = c_1 \times 0 + c_2 - 0 \qquad$ so that $\qquad c_2 = 0.$ $\qquad$ ... (5)

Solving (2) and (5), $c_1 = 1$, $c_2 = 0$. Hence, from (1), $y = e^x - 1$, which is the required solution.

Exercise 5(C)

Solve the following differential equations.

1. $(D-3)^2y = 2e^{4x}$ **[Guwahati 2007]** $\qquad$ **Ans.** $y = (c_1 + c_2\,x)\,e^{3x} + 2e^{4x}$
2. $d^2y/dx^2 - 3\,(dy/dx) + 2y = e^{3x}$. **[Merrut 1994]** $\qquad$ **Ans.** $y = c_1e^x + c_2e^{2x} + (1/12)\,e^{3x}$
3. $(D^2 + D + 1)\,y = e^{-x}$ $\qquad$ **Ans.** $y = e^{-x/2}\,[c_1\cos(x\sqrt{3}/2) + c_2\sin(x\sqrt{3}/2)] + e^{-x}$
4. $(D^2 + 5D + 6)\,y = e^{2x}$ $\qquad$ **Ans.** $y = c_1e^{-2x} + c_2e^{-3x} + (1/20)e^{2x}$
5. $(D^2 - 1)\,y = \cosh x$. **[Utkal 2003; I.A.S. 2008]** $\qquad$ **Ans.** $y = c_1e^x + c_2e^{-x} + (x/2) \times \sinh x$
6. $(D^3 + 3D^2 + 3D + 1)\,y = e^{-x}$. **[Pune 2010]** $\qquad$ **Ans.** $y = (c_1 + c_2x + c_3x^2)\,e^{-x} + (x^3/6)\,e^{-x}$
7. $(D^3 - D^2 - 4D + 4)\,y = e^{3x}$. $\qquad$ **Ans.** $y = c_1e^x + c_2e^{2x} + c_3e^{-2x} + (1/10)\,e^{3x}$
8. $(D^3 + 1)\,y = (e^x + 1)^2$. **Ans.** $y = c_1\,e^{-x} + e^{x/2}\,[c_2\cos(x\sqrt{3}/2) + c_3\sin(x\sqrt{3}/2)] + 1 + e^x + (1/9)e^{2x}$
9. $(D^2 - 2k\,D + k^2)\,y = e^{kx}$. $\qquad$ **Ans.** $y = (c_1 + c_2x)\,e^{kx} + (x^2/2)\,e^{kx}$
10. $(D^2 - 3D + 2)\,y = e^x$, given $y = 3$ and $dy/dx = 3$ when $x = 0$. $\qquad$ **Ans.** $y = 2e^x + e^{2x} - xe^x$
11. $(D^2 - a^2)\,y = \cosh ax$. $\qquad$ **Ans.** $y = c_1e^{ax} + c_2e^{-ax} + (x/2a)\sinh ax$
12. $(D^2 + 4D + 4)\,y = 2\sinh 2x$. **[Garhwal 2010]** $\qquad$ **Ans.** $y = (c_1 + c_2x)\,e^{-2x} + (1/16) \times e^{2x} - (x^2/2) \times e^{-2x}$

5.16 Short method of finding P.I. when $X = \sin ax$ or $X = \cos ax$.

Case I. When $f(D)$ can be expressed as $\phi\,(D^2)$ and $\phi\,(-a^2) \neq 0$, we shall use the following formulas

$$\frac{1}{\phi(D^2)}\sin ax = \frac{1}{\phi(-a^2)}\sin ax \qquad \text{and} \qquad \frac{1}{\phi(D^2)}\cos ax = \frac{1}{\phi(-a^2)}\cos ax$$

Thus, the rule is to replace D^2 by $-a^2$.

Proof of the above formula. By successive differentiation, we have

$$D\sin ax = a\cos ax$$

$$D^2\sin ax = -a^2\sin ax \Rightarrow (D^2)^1\sin ax = (-a^2)^1\sin ax \qquad \text{... (A}_1\text{)}$$

$$D^3\sin ax = -a^3\cos ax$$

$$D^4\sin ax = a^4\sin ax \Rightarrow (D^2)^2\sin ax = (-a^2)^2\sin ax \qquad \text{... (A}_2\text{)}$$

....

$$D^{2n}\sin ax = -a^{2n}\sin ax \Rightarrow (D^2)^n\sin ax = (-a^2)^n\sin ax. \qquad \text{... (A}_n\text{)}$$

Let $\qquad \phi\,(D^2) = (D^2)^n + a_1(D^2)^{n-1} + \dots + a_{n-1}\,(D^2)^1 + a_n.$ $\qquad$ ... (1)

Then, from (A_1), (A_2), ..., (A_n) and (1), it follows that $\phi\,(D)^2\sin ax = \phi\,(-a^2)\sin ax$. ... (2)

Operating upon both sides of (2) by $1/\phi\,(D^2)$, we have

$$\frac{1}{\phi(D^2)}\phi(D^2)\sin ax = \frac{1}{\phi(D^2)}\phi(-a^2)\sin ax \qquad \text{or} \qquad \sin ax = \phi(-a^2)\,\frac{1}{\phi(D^2)}\sin ax$$

Dividing both sides by $\phi\,(-a^2)$ which is not zero, we get

$$\frac{1}{\phi(D^2)}\sin ax = \frac{1}{\phi(-a^2)}\sin ax, \text{ provided } \phi(-a^2) \neq 0.$$

Similarly, we have $\qquad \dfrac{1}{\phi(D^2)}\cos ax = \dfrac{1}{\phi(-a^2)}\cos ax$, provided $\phi(-a^2) \neq 0$.

An important sub case. If $f(D)$ contains odd powers also, it can be put in the form $f(D) = f_1(D^2) + D\,f_2(D^2)$, where $f_1\,(-a^2) \neq 0$ and $f_2\,(-a^2) \neq 0$. Then

$$\text{P.I.} = \frac{1}{f_1(D^2) + D\,f_2(D^2)}\sin ax = \frac{1}{f_1(-a^2) + D\,f_2(-a^2)}\sin ax$$

[Use case I so that replace D^2 by $-a^2$]

$$= \frac{1}{p + qD}\sin ax\text{, where } p = f_1(-a^2) \text{ and } q = f_2(-a^2)$$

$$= (p - qD)\cdot\frac{1}{(p-qD)(p+qD)}\sin ax = (p-qD)\frac{1}{p^2 - q^2D^2}\sin ax$$

$$= (p - qD)\frac{1}{p^2 - q^2(-a^2)}\sin ax = \frac{1}{p^2 + q^2a^2}(p\sin ax - qD\sin ax)$$

$$= \frac{1}{p^2 + q^2a^2}\,(p\sin ax - qa\cos ax)\text{, as } D\sin ax = \frac{d}{dx}\sin ax.$$

Similarly, $\qquad$ P.I. $= \dfrac{1}{f_1(D^2) + D\,f_2(D^2)}\cos ax \qquad$ can be evaluated.

Case II. When $f(D)$ can be expressed as $\phi\,(D^2)$ where $\phi\,(-a^2) = 0$.

Then we shall use the following formula (for proof refer Art. 5.20).

$$\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V\text{, where } V \text{ is a function of } x \qquad \text{... (}i\text{)}$$

The above result says that e^{ax} which is on the right of $1/f(D)$ may be taken out to the left provided D is replaced by $D + a$.

We shall now evaluate $\dfrac{1}{D^2 + a^2}\sin ax$ and $\dfrac{1}{D^2 + a^2}\cos ax$. **[Pune 2010']**

Note that here $\qquad f(D^2) = D^2 + a^2 \qquad$ and $\qquad f(-a^2) = -a^2 + a^2 = 0.$

But $\qquad \dfrac{1}{D^2 + a^2}\sin ax$ = Imaginary part of $\dfrac{1}{D^2 + a^2}(\cos ax + i\sin ax)$

Thus, $\qquad \dfrac{1}{D^2 + a^2}\sin ax$ = Imaginary part of $\dfrac{1}{D^2 + a^2}e^{aix}$ $\qquad$ ... (1)

Now, $\dfrac{1}{D^2+a^2}e^{aix} = \dfrac{1}{D^2+a^2}e^{aix}\cdot 1$, taking $V = 1$

$= e^{aix}\dfrac{1}{(D+ai)^2+a^2}1$, by formula (*i*) of case II.

$= e^{aix}\dfrac{1}{D(D+2ai)^2}e^{0.x} = \dfrac{1}{2ia}e^{aix}\dfrac{1}{D}1$ by formula IA of Art 5.13.

$= \dfrac{1}{2ia}e^{aix}x = \dfrac{x}{2ia}(\cos ax + i\sin ax) = \dfrac{x}{2a}\sin ax + \dfrac{1}{i}\dfrac{x}{2a}\cos ax = \dfrac{x}{2a}\sin ax - i\dfrac{x}{2a}\cos ax$ ...(2)

Using (2), (1) reduces to

$$\frac{1}{D^2+a^2}\sin ax = \text{imaginary part of}\left[\frac{x}{2a}\sin ax - i\frac{x}{2a}\cos ax\right] = -\frac{x}{2a}\cos ax.$$

Similarly, $$\frac{1}{D^2+a^2}\cos ax = \text{Real part of }\frac{1}{D^2+a^2}e^{aix} \quad ...(3)$$

$\therefore$ (2) and (3) $\Rightarrow$ $\dfrac{1}{D^2+a^2}\cos ax$ = Real part of $\left[\dfrac{x}{2a}\sin ax - i\dfrac{x}{2a}\cos ax\right] = \dfrac{x}{2a}\sin ax$

Remark. Note carefully and remember the following formulas :

$$\frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax = \frac{x}{2}\int \sin ax\,dx. \quad ...(4)$$

$$\frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax = \frac{x}{2}\int \cos ax\,dx. \quad ...(5)$$

5.17 Solved examples based on Art. 5.16

Ex. 1. *Solve the following differential equations*

(a) $(D^2+1)\,y = \cos 2x$

(b) $(D^2+9)\,y = \cos 4x$.

Sol. (*a*) Here the auxiliary eqution is $D^2+1=0$ so that $D = \pm i$,

$\therefore$ C.F. $= c_1\cos x + c_2\sin x$, c_1, c_2 being arbitrary constants.

Now, P.I. $= \dfrac{1}{D^2+1}\cos 2x = \dfrac{1}{-2^2+1}\cos 2x = -\dfrac{1}{3}\cos 2x$.

The required general solution is $y = c_1\cos x + c_2\sin x - (1/3)\cos 2x$.

(*b*) Try yourself. **Ans.** $y = c_1\cos 3x + c_2\sin 3x - (1/7)\cos 4x$

Ex. 2. (*a*) *Solve* $(D^2-3D+2)\,y = \sin 3x$. **[Delhi Maths (G) 1996]**

(*b*) $(D^2-4D+4)\,y = \sin 2x$. **[G.N.D.U. (Amritsar) 2010]**

Sol. (*a*) Here auxiliary equation $D^2-3D+2=0$ gives $D = 1, 2$.

C.F. $= c_1e^x + c_2e^{2x}$, c_1, c_2 being arbitrary constants.

and P.I. $= \dfrac{1}{D^2-3D+2}\sin 3x = \dfrac{1}{-3^2-3D+2}\sin 3x$

$= -\dfrac{1}{3D+7}\sin 3x = -(3D-7)\dfrac{1}{(3D-7)(3D+7)}\sin 3x$

$= -(3D-7)\dfrac{1}{9D^2-49}\sin 3x = -(3D-7)\dfrac{1}{9(-3^2)-49}\sin 3x$

$= (1/130)\times(3D-7)\sin 3x = (1/130)\times(9\cos 3x - 7\sin 3x)$.

$\therefore$ Solution is $y = c_1e^x + c_2e^{2x} + (1/130)\times(9\cos 3x - 7\sin 3x)$.

(*b*) **Ans.** $y = (c_1 + c_2x)\,e^{2x} + (3\sin 2x + 8\cos 2x)/25$

Ex. 3. *Solve* $(D^2+a^2)\,y = \sin ax$. **[Lucknow 1997, Delhi 1997, Meerut 1995]**

(*b*) $(D^2+a^2)\,y = \cos ax$

Sol. (*a*) Here the auxiliary equation is $D^2+a^2=0$ so that $D = \pm ia$.

$\therefore$ C.F. $= c_1\cos ax + c_2\sin ax$, c_1, c_2 being arbitrary constants.

and P.I. $= \dfrac{1}{D^2+a^2}\sin ax$ = Imaginary part of $\dfrac{1}{D^2+a^2}(\cos ax + i\sin ax)$

or P.I. = Imaginary part of $\dfrac{1}{D^2+a^2}e^{iax}$, by Euler's theorem. ... (1)

Now, $\dfrac{1}{D^2+a^2}e^{iax} = \dfrac{1}{D^2+a^2}e^{iax}\cdot 1 = e^{iax}\dfrac{1}{(D+ia)^2+a^2}\cdot 1$

$\left[\because \text{From Art. 5.20, } \dfrac{1}{f(D)}e^{ax}V = e^{ax}\dfrac{1}{f(D+a)}V\right]$

$= e^{iax}\dfrac{1}{D^2+2iaD}e^{iax} = \dfrac{1}{D}\dfrac{1}{D+2ia}e^{0.x} = e^{iax}\dfrac{1}{D}\dfrac{1}{0+2ia}e^{0.x}$

$= \dfrac{1}{2ia}e^{iax}\dfrac{1}{D}1 = \dfrac{1}{2ia}e^{iax}x = \dfrac{x}{2ia}(\cos ax + i\sin ax)$

$= (x/2a)\sin ax - i\,(x/2a)\cos ax$, as $(1/i) = -i$.

$\therefore$ From (1), P.I. = Imaginary part of $\left(\dfrac{x}{2a}\sin ax - \dfrac{ix}{2a}\cos ax\right) = -\dfrac{x}{2a}\cos ax$

Hence the required general solution of the given equation is

y = C.F. + P.I. or $y = c_1\cos ax + c_2\sin ax - (x/2a)\cos ax$.

Note: You can also use remark of Art. 4.15 to write $\dfrac{1}{D^2+a^2}\sin ax = -\dfrac{x}{2a}\cos ax$

(*b*) Proceed as in part (*a*). C.F. is same as in part (*a*).

P.I. $= \dfrac{1}{D^2+a^2}\cos ax$ = Real part of $\dfrac{1}{D^2+a^2}(\cos ax + i\sin ax)$

or P.I. = Real part of $\dfrac{1}{D^2+a^2}e^{iax}$ ...(1)′

Now , we have $\dfrac{1}{D^2+a^2}e^{iax} = \left(\dfrac{x}{2a}\right)\sin ax - \left(i\dfrac{x}{2a}\right)\cos ax$, do as in part (*a*).

Hence from (1)′, P.I. = real part of $\left(\dfrac{x}{2a}\sin ax - \dfrac{ix}{2a}\cos ax\right) = \dfrac{x}{2a}\sin ax$

$\therefore$ The required solution is $y = c_1\cos ax + c_2\sin ax + (x/2a)\sin ax$

Note: You can also use the result $\dfrac{1}{D^2+a^2}\cos ax = \dfrac{x}{2a}\sin ax$

Ex. 4. *Solve the following differential equations:* **[I.A.S. Prel. 2006, Rajsthan 2010, Purvanchal 1996]**

(a) $(D^3 + a^2 D)\, y = \sin ax$

(b) $(D^3 + 9D)\, y = \sin 3x$

(c) $(d^3x/dy^3) + b^2(dx/dy) = \sin by$

Sol. (a) Here auxiliary equation is $D^3 + a^2 D = 0$ so that $D = 0, 0 \pm ia$

$\therefore$ C.F. $= c_1 e^{0x} + e^{0x}(c_2 \cos ax + c_3 \sin ax) = c_1 + c_2 \cos ax + c_3 \sin ax$,

where c_1, c_2 and c_3 arbitrary constants.

$$\text{P.I.} = \frac{1}{D^3 + a^2 D}\sin ax = \frac{1}{D^2+a^2}\frac{1}{D}\sin ax = \frac{1}{D^2+a^2}\left(-\frac{1}{a}\cos ax\right)$$

$$= -\frac{1}{a}\left[\text{Real part of } \frac{1}{D^2+a^2}(\cos ax + i\sin ax)\right]$$

$$= -\frac{1}{a}\left[\text{Real part of } \frac{1}{D^2+a^2}e^{iax}\right], \text{ by Euler's theorem}$$

$$= -\frac{1}{a}\left[\text{Real part of }\left(\frac{x}{2a}\sin ax - \frac{ix}{2a}\cos ax\right)\right]$$

$$\left[\text{As in Ex. 3. (a), prove that } \frac{1}{D^2+a^2}e^{iax} = \frac{x}{2a}\sin ax - \frac{ix}{2a}\cos ax\right]$$

$= -(1/a)\,(x/2a)\sin ax = -(x/2a^2)\sin ax$.

Hence the general solution is $y = c_1 + c_2 \cos ax + c_3 \sin ax - (x/2a^2)\sin ax$.

(b) Do as in part (a) **Ans.** $y = c_1 + c_2 \cos 3x + c_3 \sin 3x - (x/18)\sin 3x$.

(c) Proceed as in part (a). Here y is independent and x is dependent variable

Ans. $x = c_1 + c_2 \cos by + c_3 \sin by - (y/2b^2)\sin by$

Ex. 5. *Solve* $(D-1)^2(D^2+1)^2 y = \sin x$. **[Gorakhpur 1994]**

Sol. Here the auxiliary equation is $(D-1)^2(D^2+1)^2 = 0$ so that $D = 1, 1, 0 \pm i, 0 \pm i$.

$\therefore$ C.F. $= (c_1 + c_2 x)e^x + (c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x$,

where c_1, c_2, c_3, c_4, c_5 and c_6 arbitrary constants.

$$\text{P.I.} = \frac{1}{(D-1)^2(D^2+1)^2}\sin x = \frac{1}{(D^2+1)^2}(D+1)^2\frac{1}{(D+1)^2(D-1)^2}\sin x$$

$$= \frac{1}{(D^2+1)^2}(D^2+2D+1)\frac{1}{(D^2-1)^2}\sin x = \frac{1}{(D^2+1)^2}\frac{(D^2+2D+1)}{(-1^2-1)^2}\sin x$$

$$= \frac{1}{4}\frac{1}{(D^2+1)^2}(D^2\sin x + 2D\sin x + \sin x)$$

$$= \frac{1}{4}\frac{1}{(D^2+1)^2}(-\sin x + 2\cos x + \sin x) = \frac{1}{2}\frac{1}{(D^2+1)^2}\cos x$$

$$= \frac{1}{2}\text{Real part of }\frac{1}{(D^2+1)^2}e^{ix}\cdot 1 = \frac{1}{2}\text{Real part of } e^{ix}\frac{1}{[(D+i)^2+1]}\cdot 1$$

$$\left[\because \text{From Art. 5.20, } \frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V\right]$$

$$= \frac{1}{2}\text{Real part of } e^{ix}\frac{1}{D^2+2iD}\cdot 1 = \frac{1}{2}\text{Real part of } e^{ix}\frac{1}{D(D+2i)}e^{0x}$$

$$= \frac{1}{2}\text{Real part of } e^{ix}\frac{1}{D}\frac{1}{0+2i}e^{0x} = \frac{1}{2}\text{Real part of }\frac{e^{ix}}{2i}\frac{1}{D}\cdot 1$$

$= -(i/4) \times$ Real part of $x(\cos x + i\sin x) = (x/4) \times \sin x$

Solution is $y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x + (x/4)\sin x$.

Ex. 6. *Solve* $(d^4y/dx^4) - m^4 y = \sin mx$. **[I.A.S. 1991]**

Sol. Given $(D^4 - m^4)\, y = \sin mx$, where $D \equiv d/dx$ ...(1)

whose auxiliary equation is $D^4 - m^4 = 0$ giving $D = \pm m, 0 \pm im$.

C.F. $= c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx$, c_1, c_2, c_3, c_4 being arbitrary constants

$$\text{P.I.} = \frac{1}{(D^2+m^2)}\frac{1}{(D^2-m^2)}\sin mx = \frac{1}{D^2+m^2}\frac{1}{-m^2-m^2}\sin mx$$

$$= -\frac{1}{2m^2}\frac{1}{D^2+m^2}\sin mx = -\frac{1}{2m^2}\times\left(-\frac{x}{2m}\cos mx\right) = \frac{x}{4m^3}\cos mx.$$

So solution is $y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx + (x/4m^3)\cos mx$.

Ex. 7. (a) *Find the solution of* $(d^2y/dx^2) + 4y = 8\cos 2x$, *given that* $y = 0$ *and* $dy/dx = 0$, *when* $x = 0$. **[I.A.S. 1995]**

(b) *Solve* $(D^2 + 4)\, y = \sin 2x$, *given that when* $x = 0$, $y = 0$ *and* $dy/dx = 2$. **[I.A.S. 1994]**

Sol. (a) Re-writing the given equation, we get $(D^2 + 4)\, y = 8\cos 2x$ ...(1)

Also given that when $x = 0$, $y = 0$ ...(2)

and when $x = 0$, $dy/dx = 0$. ...(3)

The auxiliary equation of (1) is $D^2 + 4 = 0$ so that $D = \pm 2i$

C.F. $= e^{0x}(c_1 \cos 2x + c_2 \sin 2x) = c_1 \cos 2x + c_2 \sin 2x$, c_1, c_2 being arbitrary constants

Also, $$\text{P.I.} = 8\frac{1}{D^2+2^2}\cos 2x = 8\cdot\frac{x}{(2\times 2)}\sin 2x = 2x\sin 2x.$$

Solution of (1) is $y = c_1 \cos 2x + c_2 \sin 2x + 2x\sin 2x$. ...(4)

Putting $x = 0$ and $y = 0$ (due to (2)), (4) yields $c_1 = 0$. Then (4) gives

$y = c_2 \sin 2x + 2x\sin 2x = (c_2 + 2x)\sin 2x$. ...(5)

From (5) $dy/dx = 2\sin 2x + 2(c_2 + 2x)\cos 2x$ ...(6)

Putting $x = 0$ and $dy/dx = 0$, (6) yields $0 = 2c_2 \Rightarrow c_2 = 0$.

Hence from (5), the required solution is $y = 2x\sin 2x$.

(b) Proceed as in part (a). Solution is $y = (1/8)\,(9\sin 2x - 2x\cos 2x)$

Ex. 8. *Solve* $(d^2y/dx^2) - 8\,(dy/dx) + 9y = 40\sin 5x$.

Sol. Let $D \equiv d/dx$. Then given equation becomes $(D^2 - 8D + 9)\, y = 40\sin 5x$. ...(1)

Here auxiliary equation $$D^2 - 8D + 9 = 0 \Rightarrow D = \frac{8 \pm \sqrt{(64-36)}}{2} = \frac{8 \pm 2\sqrt{7}}{2} = 4 \pm \sqrt{7}.$$

$\therefore$ C.F. $= e^{4x}(c_1 \cosh x\sqrt{7} + c_2 \sinh x\sqrt{7})$, c_1, c_2 being arbitrary constants

and $$\text{P.I.} = \frac{1}{D^2 - 8D + 9}40\sin 5x = 40\frac{1}{-5^2 - 8D + 9}\sin 5x = 40\frac{1}{-8(D+2)}\sin 5x$$

$$= -5(D-2)\frac{1}{(D-2)(D+2)}\sin 5x = -5(D-2)\frac{1}{D^2-4}\sin 5x = -5(D-2)\frac{1}{-5^2-4}\sin 5x$$

$$= (5/29) \times (D \sin 5x - 2 \sin 5x) = (5/29) \times (5 \cos 5x - 2 \sin 5x).$$

Hence the required general solution is $\quad y = \text{C.F.} + \text{P.I.}$

or $\quad y = e^{4x}(c_1 \cosh x\sqrt{7} + c_2 \sin x\sqrt{7}) + (5/29) \times (5 \cos 5x - 2 \sin 5x).$

Ex. 9. *Solve $(d^2y/dx^2) + 2(dy/dx) + 10y + 37 \sin 3x = 0$, and find the value of y when $x = \pi/2$ if it is given that $y = 3$ and $dy/dx = 0$ when $x = 0$.* **[I.A.S. 1997, Lucknow 1996]**

Sol. Re-writing, the given equation is $\quad (D^2 + 2D + 10)\, y = -37 \sin 3x$., $D \equiv d/dx$

Its auxiliary equation $D^2 + 2D + 10 = 0 \Rightarrow D = \dfrac{-2 \pm \sqrt{(4-40)}}{2} = \dfrac{-2 \pm 6i}{2} = -1 \pm 3i.$

$\therefore$ C.F. $= e^{x}(c_1 \cos 3x + c_2 \sin 3x)$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2+2D+10}(-37\sin 3x) = -37\frac{1}{-3^2+2D+10}\sin 3x$$

$$= -3\frac{1}{2D+1}\sin 3x = -37(2D-1)\frac{1}{(2D-1)(2D+1)}\sin 3x$$

$$= -37(2D-1)\frac{1}{4D^2-1}\sin 3x = -37(2D-1)\frac{1}{4(-3^2)-1}\sin 3x$$

$$= (2D-1)\sin 3x = 6 \cos 3x - \sin 3x.$$

Hence the general solution of the given equation is

$$y = e^{x}(c_1 \cos 3x + c_2 \sin 3x) + 6 \cos 3x - \sin 3x. \qquad \text{... (1)}$$

Differentiating both sides of (1) w.r.t. 'x', we have

$$dy/dx = e^{-x}(-3\, c_1 \sin 3x + 3\, c_2 \cos 3x) - e^{x}(c_1 \cos 3x + c_2 \sin 3x) - 18 \sin 3x - 3 \cos 3x. \qquad \text{... (2)}$$

It is given that $y = 3$, $dy/dx = 0$ when $x = 0$. So (1) and (2) give

$3 = c_1 + 6 \quad$ and $\quad 0 = 3\, c_2 - c_1 - 3 \quad$ so that $\quad c_1 = -3, \quad c_2 = 0.$

$\therefore$ From (1), $\quad y = -3\, e^{x} \cos 3x + 6 \cos 3x - \sin 3x. \qquad$ ... (3)

Putting $x = \pi/2$ in (3), the corresponding value of y is given by

$y = -3\, e^{\pi/2} \cos (3\pi/2) + 6 \cos (3\pi/2) - \sin (3\pi/2) = 1$, as $\cos (3\pi/2) = 0$.

Ex. 10. *Find the integral of the equation $(d^2x/dt^2) + 2n \cos \alpha\, (dx/dt) + n^2 x = a \cos nt$, which is such that when $t = 0$, $x = 0$ and $dx/dt = 0$.*

Sol. Let $D \equiv d/dt$. Then the given equation can be written as

$$[D^2 + (2n \cos \alpha)\, D + n^2]\, x = a \cos nt.$$

Its auxiliary equation is $\quad D^2 + (2n \cos \alpha)\, D + n^2 = 0.$

$$\therefore\ D = \frac{-2n\cos\alpha \pm \sqrt{4n^2\cos^2\alpha - 4n^2}}{2} = -n\cos\alpha \pm \sqrt{(-n^2\sin^2\alpha)} = -n \cos \alpha \pm i\, n \sin \alpha.$$

$\therefore \quad$ C.F. $= e^{-nt \cos \alpha}[c_1 \cos (nt \sin \alpha) + c_2 \sin (nt \sin \alpha)]$, c_1, c_2 being arbitrary constants

and $\quad \text{P.I.} = \dfrac{1}{D^2+(2n\cos\alpha)D+n^2}\, a\cos nt = a\dfrac{1}{-n^2+(2n\cos\alpha)D+n^2}\cos nt$

$$= \frac{a}{2n\cos\alpha}\frac{1}{D}\cos nt = \frac{a}{2n\cos\alpha}\int \cos nt\, dt = \frac{a \sin nt}{2n^2 \cos\alpha}.$$

[Note that here $(1/D)$ stands for integration w.r.t. 't']

Hence the general solution of the given equation is $\quad x = \text{C.F.} + \text{P.I.}$

or $\quad x = e^{-nt \cos \alpha}[c_1 \cos (nt \sin \alpha) + c_2 \sin (nt \sin \alpha)] + (a \sin nt)/(2n^2 \cos \alpha) \qquad$ ... (1)

Differentiating both sides of (1) w.r.t. 't', we have

$$dx/dt = e^{-nt \cos \alpha}\, n \sin \alpha\, [-c_1 \sin (nt \sin \alpha) + c_2 \cos (nt \sin \alpha)]$$

$$- n \cos \alpha\, e^{-nt \cos \alpha}[c_1 \cos (nt \sin \alpha) + c_2 \sin (nt \sin \alpha)] + (a \cos nt)/(2n \cos \alpha) \qquad \text{... (2)}$$

Since, it is given that $x = 0$, $dx/dt = 0$ when $t = 0$, so from (1) and (2)

$0 = c_1 \quad$ and $\quad 0 = c_2 n \sin \alpha - n\, c_1 \cos \alpha + a/(2n \cos \alpha)$

$\Rightarrow \quad c_1 = 0 \quad$ and $\quad c_2 = -a/(2n^2 \sin \alpha \cos \alpha) = -a/(n^2 \sin 2\alpha)$.

Putting the above values of c_1 and c_2 in (1), the required solution is

$$x = [-a/(n^2 \sin 2\alpha)]\, e^{-nt \cos \alpha} \sin (nt \sin \alpha) + (a \sin nt)/(2n^2 \cos \alpha).$$

Ex. 11. *Solve $(D^2 + 4)\, y = \sin^2 x$.* **[I.A.S. Prel. 1998; Merrut 1996]**

Sol. Here auxiliary equation is $\quad D^2 + 4 = 0 \quad$ so that $\quad D = \pm 2i$.

Hence $\quad$ C.F. $= \cos 2x + c_2 \sin 2x$, c_1, c_2 being arbitrary constants

Also $\quad \text{P.I.} = \dfrac{1}{D^2+4}\sin^2 x = \dfrac{1}{2}\dfrac{1}{D^2+4}(1-\cos 2x) = \dfrac{1}{2}\left[\dfrac{1}{D^2+4}e^{0.x} - \dfrac{1}{D^2+4}\cos 2x\right]$

$$= \frac{1}{2}\left[\frac{1}{0+4}e^{0.x} - \left(\frac{x}{2\times 2}\right)\sin 2x\right] = \frac{1}{8} - \frac{1}{8}x\sin 2x.$$

$\therefore$ Solution is $\quad y = c_1 \cos 2x + c_2 \sin 2x + (1/8) - (x/8) \sin 2x.$

Exercise 5(D)

Solve the following differential equations :

1. $y'' + y = \sin x$ **[Delhi B.Sc. (Hons) II 2011]** **Ans.** $y = c_1 \cos x + c_2 \sin x - (x/2) \times \cos x$
2. $(D^3 + D^2 - D - 1)\, y = \cos 2x$. **[Kanpur 2006]** **Ans.** $y = c_1 e^{x} + (c_2 + c_3 x)\, e^{-x} - (1/25)(2 \sin 2x + \cos 2x)$
3. $(D^2 - 5D + 6)\, y = \sin 3x$. **[Gorakhpur 1995]** **Ans.** $y = c_1 e^{2x} + c_2 e^{3x} + (1/78)(5 \cos 3x - \sin 3x)$
4. $(D^2 + D + 1)\, y = \sin 2x$. **Ans.** $y = e^{-x/2}\left\{c_1 \cos(x\sqrt{3}/2) + c_2 \sin(x\sqrt{3}/2)\right\} - (1/13)(2\cos 2x + 3\sin 2x)$
5. $(D^2 - 4)\, y = \cos^2 x$. **Ans.** $y = c_1 e^{2x} + c_2 e^{-2x} - (1/16)(2 + \cos 2x)$
6. $(D^2 - a^2)\, y = \cos mx$. **Ans.** $y = c_1 e^{ax} + c_2 e^{-ax} - [1/(m^2 + a^2)] \cos mx$
7. $(D^3 + 1)\, y = \cos 2x$. **Ans.** $y = c_1 e^{-x} + e^{x/2}\{c_2 \cos(x\sqrt{3}/2) + c_3 \sin(x\sqrt{3}/2)\} + (\cos 2x - 8 \sin 2x)/65$
8. $(D^2 + 9)\, y = \cos 2x + \sin 2x$. **Ans.** $y = c_1 \cos 3x + c_2 \sin 3x + (1/5)(\cos 2x + \sin 2x)$
9. (*a*) $(D^2 + 4)\, y = \sin 2x$. **[Pune 2011]** **Ans.** $y = c_1 \cos 2x + c_2 \sin 2x - (x/4) \times \cos 2x$

   (*b*) $(D^2 + 4)\, y = 4 + \sin^2 x$. **[Guwahati 2007]** **Ans.** $y = c_1 \cos 2x + c_2 \sin 2x + 9/8 - (x/8) \times \sin 2x$
10. $d^2x/dt^2 + 4x = a \sin t \cos t$. **Ans.** $x = c_1 \cos 2t + c_2 \sin 2t - (at/8) \times \cos 2t$
11. $(D^2 + 1)\, y = \sin x \sin 2x$ **Ans.** $y = c_1 \cos x + c_2 \sin x + (x/4) \times \sin x + (1/16) \times \cos 3x$
12. $(D^2 + 1)\, y = \cos x \sin 3x$ **Ans.** $y = c_1 \cos x + c_2 \sin x - (1/30) \times \sin 4x - (1/6) \times \sin 2x$
13. $(D^4 - 1)\, y = \sin 2x$. **[Delhi Maths.(G) 2000]** **Ans.** $y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + (1/15) \times \sin 2x$
14. $(D^3 + 1)\, y = \cos 2x$ **Ans.** $y = c_1 e^{-x} + e^{x/2}\{c_2 \cos(x\sqrt{3}/2) + c_3 \sin(x\sqrt{3}/2)\} + (1/65) \times (\cos 2x - 8 \sin 2x)$ **[Meerut 2008, 10]**
15. $d^2y/dx^2 - (dy/dx) - 2y = \sin 2x$. **Ans.** $y = c_1 e^{-x} + c_2 e^{2x} + (\cos 2x - 3\sin 2x)/20$

5.18 Short method of finding P.I. when $X = x^m$, m being a positive integer.

Working rule for evaluating $\{1/f(D)\}\,x^m$.

Step I. Bring out the lowest degree term from $f(D)$ so that the remaining factor in the denominator is of the form $[1+\phi(D)]^n$ or $[1-\phi(D)]^n$, n being a positive integer.

Step II. We take $[1+\phi(D)]^n$ or $[1-\phi(D)]^n$ in the numerator so that it takes the form $[1+\phi(D)]^{-n}$ or $[1-\phi(D)]^{-n}$.

Step III. We expand $[1 \pm \phi(D)]^{-n}$ by the binomial theorem, namely

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

In particular the following binomial expansions should be remembered.

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$; $\quad (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$; $\quad (1-x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

In any case, the expansion is to be carried upto D^m, since $D^{m+1}x^m = 0$, $D^{m+2}x^m = 0$, and all the higher differential coefficients of x^m vanish.

Remark. If we are given a polynomial $x^m + a_1x^{m-1} + \dots + a_{m-1}x + a_m$ of degree m in place of x^m, then also we proceed to evaluate $[1/f(D)]\,(x^m + a_1x^{m-1} + \dots + a_m)$ in the same manner as we did for $\{1/f(D)\}\,x^m$. Also, if X is a constant, the above method can be used.

5.19 Solved examples based on Art. 5.18

Ex. 1. *Solve $(D^4 - D^2)\,y = 2$.* **[Agra 2005]**

Sol. Here auxiliary equation is $D^4 - D^2 = 0$ or $D^2(D^2-1) = 0 \Rightarrow D = 0, 0, 1, -1$

$\therefore$ C.F. $= (c_1 + c_2x)\,e^{0 \cdot x} + c_3e^x + c_4e^{-x} = c_1 + c_2x + c_3e^x + c_4e^{-x}$,

where c_1, c_2, c_3 and c_4 are arbitrary constants.

$$\text{P.I.} = \frac{1}{D^4 - D^2}\,2 = -\frac{2}{D^2}\,\frac{1}{(1-D^2)}\,1 = -\frac{2}{D^2}(1-D^2)^{-1}\,1$$

$$= -\frac{2}{D^2}(1 + D^2 + D^4 + \dots)\,1 = -\frac{2}{D^2}\,1 = -\frac{2}{D}\,x = (-2)\times\frac{x^2}{2} = -x^2$$

$\therefore$ Solution is $\qquad y = c_1 + c_2x + c_3e^x + c_4e^{-x} - x^2$

Ex. 2. *Find the particular integral of $(D^2 + D)\,y = x^2 + 2x + 4$.* **[I.A.S. Prel. 1994]**

Sol. The required particular integral

$$= \frac{1}{D^2 + D}(x^2 + 2x + 4) = \frac{1}{D(1+D)}(x^2 + 2x + 4) = \frac{1}{D}(1+D)^{-1}(x^2 + 2x + 4)$$

$= (1/D)\,(1 - D + D^2 - D^3 + \dots)\,(x^2 + 2x + 4)$

$= (1/D)\,[(x^2 + 2x + 4) - D\,(x^2 + 2x + 4) + D^2\,(x^2 + 2x + 4)]$

$= (1/D)\,[x^2 + 2x + 4 - (2x + 2) + 2] = (1/D)\,(x^2 + 4) = x^3/3 + 4x$.

Ex. 3. *Solve $(D^4 - a^4)\,y = x^4$.* **[Delhi Maths Hons. 1994]**

Sol. Here the auxiliary equation is $\quad D^4 - a^4 = 0 \quad$ or $\quad (D^2 - a^2)(D^2 + a^2) = 0$

so that $\quad D^2 - a^2 = 0 \quad$ or $\quad D^2 + a^2 = 0 \quad$ and hence $\quad D = a, -a, 0 \pm ia$.

$\therefore$ C.F. $= c_1e^{ax} + c_2e^{-ax} + c_3\cos ax + c_4\sin ax$, c_1, c_2 being arbitrary constants

and $$\text{P.I.} = \frac{1}{D^4 - a^4}\,x^4 = \frac{1}{-a^4(1 - D^4/a^4)}\,x^4 = -\frac{1}{a^4}\left(1 - \frac{D^4}{a^4}\right)^{-1}x^4$$

$= -(1/a^4)\,\{1 + (D^4/a^4) + \dots\}\,x^4 = -(1/a^4)\,[x^4 + (1/a^4) \times D^4x^4]$

$= -(1/a^4)\,[x^4 + (1/a^4) \times 24]$, as $D^4x^4 = 4 \times 3 \times 2 \times 1 = 24$

$\therefore$ Solution is $\qquad y = c_1e^{ax} + c_2e^{-ax} + c_3\cos ax + c_4\sin ax - (1/a^4)\,(x^4 + 24/a^4)$.

Ex. 4. *Solve $(D^3 + 8)\,y = x^4 + 2x + 1$.* **[Delhi Maths Hons. 1998]**

Sol. Here the auxiliary equation is $\quad D^3 + 2^3 = 0 \quad$ or $\quad (D+2)(D^2 - 2D + 4) = 0$

so that $\quad D = -2,\ \{2 \pm (4 - 16)^{1/2}\}/2 = -2,\ 1 \pm i\sqrt{3}$.

$\therefore$ C.F. $= c_1e^{-2x} + e^x\,(c_2\cos x\sqrt{3} + c_3\sin x\sqrt{3})$, c_1, c_2, c_3 being arbitrary constans

$$\therefore \quad \text{P.I.} = \frac{1}{D^3 + 8}(x^4 + 2x + 1) = \frac{1}{8(1 + D^3/8)}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 + \frac{D^3}{8}\right)^{-1}(x^4 + 2x + 1) = \frac{1}{8}\left(1 - \frac{D^3}{8} + \dots\right)(x^4 + 2x + 1)$$

$= (1/8) \times [\,(x^4 + 2x + 1) - (1/8) \times D^3(x^4 + 2x + 1)]$

$= (1/8) \times [(x^4 + 2x + 1) - (1/8)\,(24x)] = (1/8) \times (x^4 + 2x + 1 - 3x) = (x^4 - x + 1)/8$.

$\therefore$ Required solution is $\qquad y = c_1e^{-2x} + e^x\,(c_2\cos x\sqrt{3} + c_3\sin x\sqrt{3}) + (x^4 - x + 1)/8$.

Ex. 5. *Solve (a) $(D^2 + 2D + 2)\,y = x^2$.*

(b) $(D^2 - 4D + 4)\,y = x^2$.

Sol. (a) Here the auxiliary equation is $\quad D^2 + 2D + 2 = 0 \quad$ so that $\quad D = -1 \pm i$.

$\therefore$ C.F. $= e^{-x}\,(c_1\cos x + c_2\sin x)$, c_1, c_2 being arbitrary constants

and $$\text{P.I.} = \frac{1}{D^2 + 2D + 2}\,x^2 = \frac{1}{2[1 + (D + D^2/2)]}\,x^2 = \frac{1}{2}\{1 + (D + D^2/2)\}^{-1}$$

$= (1/2)\,\{1 - (D + D^2/2) + (D + D^2/2)^2 - \dots\}\,x^2 = (1/2)\,\{1 - (D + D^2/2) + D^2 + \dots\}\,x^2$

$= (1/2)\,\{1 - D + D^2/2 + \dots\}\,x^2 = (1/2)\,(x^2 - 2x + 1)$

$\therefore$ The required solution is $\qquad y = e^{-x}\,(c_1\cos x + c_2\sin x) + (x^2 - 2x + 1)/2$.

(b) Try yourself. $\qquad$ **Ans.** $y = (c_1 + c_2x)\,e^{2x} + (2x^2 + 4x + 3)/4$.

Ex. 6. *Solve (a) $(D^3 + 3D^2 + 2D)\,y = x^2$.* **[Delhi Maths(G) 2006]**

(b) $(D^3 + 3D^2 + 2D)\,y = x$.

Sol. (a) Here the auxiliary equation is $\quad D^3 + 3D^2 + 2D = 0 \quad$ or $\quad D(D^2 + 3D + 2) = 0$

or $\quad D(D+1)(D+2) = 0 \quad$ so that $\quad D = 0, -1, -2$.

$\therefore$ C.F. $= c_1 + c_2e^{-x} + c_3e^{-2x}$, where c_1, c_2, c_3 are arbitrary constants.

and $$\text{P.I.} = \frac{1}{D^3 + 3D^2 + 2D}\,x^2 = \frac{1}{2D[1 + (3/2)D + (1/2)D^2]}\,x^2$$

$$= \frac{1}{2D}\left[1 + \left(\frac{3D}{2} + \frac{D^2}{2}\right)\right]^{-1}x^2 = \frac{1}{2D}\left[1 - \left(\frac{3D}{2} + \frac{D^2}{2}\right) + \left(\frac{3D}{2} + \frac{D^2}{2}\right)^2 - \dots\right]x^2$$

$$= \frac{1}{2D}\left[1 - \left(\frac{3D}{2} + \frac{D^2}{2}\right) + \left(\frac{9D^2}{4} + \dots\right) + \dots\right]x^2 = \frac{1}{2D}\left(1 - \frac{3D}{2} + \frac{7D^2}{4} + \dots\right)x^2$$

$$= \frac{1}{2D}\left[x^2 - \frac{3}{2}(2x) + \frac{7}{4}(2)\right] = \frac{1}{2}\int\left(x^2 - 3x + \frac{7}{2}\right)dx = \frac{x^3}{6} - \frac{3x^2}{4} + \frac{7x}{4}.$$

$\therefore$ The required solution $\qquad y = c_1 + c_2e^{-x} + c_2e^{-2x} + (x^3/6) - (3x^2/4) + (7x/4)$.

(b) Try yourself. $\qquad$ **Ans.** $y = c_1 + c_2e^{-x} + c^2e^{-2x} + (x^2/4) - (3x/4)$.

Ex. 7. *Solve* $(D^4 - 2D^3 + D^2)\, y = x$.

Sol. Here the auxiliary equation is $D^4 - 2D^3 + D^2 = 0$ giving $D = 0, 0, 1, 1$.

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{0 \cdot x} + (c_3 + c_4 x)\, e^x = c_1 + c_2 x + (c_3 + c_4 x)\, e^x$,

where c_1, c_2, c_3 and c_4 are arbitrary constants.

$$\text{P.I.} = \frac{1}{D^4 - 2D^3 + D^2}x = \frac{1}{D^2(D^2 - 2D + 1)}x = \frac{1}{D^2(1-D)^2}x$$

$$= \frac{1}{D^2}(1-D)^{-2}x = \frac{1}{D^2}(1 + 2D + \ldots)x = \frac{1}{D^2}(x+2)$$

$$= \frac{1}{D}\int (x+2)\,dx = \frac{1}{D}\left(\frac{x^2}{2} + 2x\right) = \int\left(\frac{x^2}{2} + 2x\right)dx = \frac{x^3}{6} + x^2.$$

$\therefore$ The required solution is $y = c_1 + c_2 x + (c_3 + c_4 x)\, e^x + (x^3/6) + x^2$.

Ex. 8. *Solve* $(D^4 + D^2 + 16)\, y = 16x^2 + 256$.

Sol. Here the auxiliary equation is $D^4 + D^2 + 16 = 0$ or $(D^2 + 4)^2 - (D\sqrt{7})^2 = 0$

or $(D^2 + D\sqrt{7} + 4)(D^2 - D\sqrt{7} + 4) = 0 \Rightarrow D^2 + D\sqrt{7} + 4 = 0$ or $D^2 - D\sqrt{7} + 4 = 0$

$$\therefore\ D = \frac{-\sqrt{7} \pm \sqrt{7-16}}{2}, \frac{\sqrt{7} \pm \sqrt{7-16}}{2} = -\frac{\sqrt{7}}{2} \pm \frac{3i}{2}, \frac{\sqrt{7}}{2} \pm \frac{3i}{2}$$

$$\text{C.F.} = e^{-x\sqrt{7}/2}(c_1 \cos 3x/2 + c_2 \sin 3x/2) + e^{x\sqrt{7}/2}(c_3 \cos 3x/2 + c_4 \sin 3x/2),$$

c_1, c_2, c_3 and c_4 being arbitrary constants.

$$\text{and P.I.} = \frac{1}{D^4 + D^2 + 16}(16x^2 + 256) = \frac{1}{16\{1 + (D^2 + D^4)/16\}}(16x^2 + 256)$$

$$= \frac{1}{16}\left[1 + \left(\frac{D^2}{16} + \frac{D^4}{16}\right)\right]^{-1}(16x^2 + 256) = \frac{1}{16}\left[1 - \left(\frac{D^2}{16} + \frac{D^4}{16}\right) + \ldots\right](16x^2 + 256)$$

$$= (1/16) \times [(16x^2 + 256) - (1/16) \times (32)] = x^2 + (127/8)$$

Hence the required solution is
$$y = e^{-x\sqrt{7}/2}(c_1 \cos 3x/2 + c_2 \sin 3x/2) + e^{x\sqrt{7}/2}(c_3 \cos 3x/2 + c_4 \sin 3x/2) + x^2 + (127/8)$$

Ex. 9. *Solve the equation* $(d^2y/dx^2) = a + bx + cx^2$, *given that* $dy/dx = 0$ *when* $x = 0$ *and* $y = d$, *when* $x = 0$.

Sol. Let $D \equiv d/dx$. Then, we have $D^2 y = a + bx + cx^2$. The A.E. is $D^2 = 0$ so that $D = 0, 0$. Hence C.F. $= (c_1 + c_2 x)\, e^{0 \cdot x} = c_1 + c_2 x$, c_1 and c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2}(a + bx + cx^2) = \frac{1}{D}\int (a + bx + cx^2)\,dx = \frac{1}{D}\left(ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3\right)$$

$$= \int\left(ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3\right)dx = \frac{1}{2}ax^2 + \frac{1}{6}bx^3 + \frac{1}{12}cx^4.$$

$\therefore$ The general solution is $y = c_1 + c_2 x + (1/2)\, ax^2 + (1/6)\, bx^3 + (1/12)\, cx^4$. ... (1)

From (1), $(dy/dx) = c_2 + ax + 2\, bx^2 + (1/3)\, cx^3$. ... (2)

Putting $x = 0$ and $y = d$ in (1), we get $c_1 = d$. Next, putting $x = 0$ and $dy/dx = 0$ in (2), we get $c_2 = 0$. Putting values of c_1 and c_2 in (1), the desired solution is

$$y = d + (1/2) \times ax^2 + (1/6) \times bx^3 + (1/12) \times cx^4.$$

Ex. 10. *Find solution of* $(D^3 - D^2 - D - 2)\, y = x$. **[Agra 2005]**

Sol. Auxiliary equation of the given equation is given by

$D^3 - D^2 - D - 2 = 0$, or $(D - 2)(D^2 + D + 1) = 0$

$\therefore$ $D = 2, (-1 \pm \sqrt{-3})/2$ i.e., $D = 2, (-1/2) \pm i\,(\sqrt{3}/2)$.

$\therefore$ C.F. $= c_1 e^{2x} + e^{-x/2}\{c_3 \cos(x\sqrt{3}/2) + c_4 \sin(x\sqrt{3}/2)\}$, c_1, c_2 being arbitrary constants

$$\text{P.I.} = \frac{1}{D^3 - D^2 - D - 2}x = \frac{1}{-2\{1 + (D + D^2 - D^3)/2\}}x$$

$$= -\frac{1}{2}\{1 + \frac{1}{2}(D + D^2 - D^3)\}^{-1}x = -\frac{1}{2}\{1 - \frac{1}{2}(D + D^2 - D^3) + \ldots\}x$$

$$= -\frac{1}{2}\left(x - \frac{1}{2}Dx\right) = -\frac{1}{2}\left(x - \frac{1}{2} \times \frac{x^2}{2}\right) = -\frac{1}{8}(4x - x^2)$$

$\therefore$ The required solution is $y = c_1 e^{2x} + e^{-x/2}\{c_3 \cos(x\sqrt{3}/2) + c_4 \sin(x\sqrt{3}/2)\} - (1/8) \times (4x - x^2)$.

Ex. 11. *Solve* $(D^3 - D^2 - 6D)\, y = x^2 + 1$. **[Agra 1995, Garhwal 1996; Bangalore 1995, Lucknow 1992, Allahabad 1994]**

Sol. The auxiliary equation is $D^3 - D^2 - 6D = 0$ giving

$D(D^2 - D - 6) = 0$ or $D(D - 3)(D + 2) = 0$ so that $D = 0, 3, -2$

$\therefore$ C.F. $= c_1 e^{0 \cdot x} + c_2 e^{3x} + c_3 e^{-2x} = c_1 + c_2 e^{3x} + c_3 e^{-2x}$.

$$\text{P.I.} = \frac{1}{D^3 - D^2 - 6D}(x^2 + 1) = \frac{1}{-6D(1 + D/6 - D^2/6)}(x^2 + 1) = -\frac{1}{6D}\left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right)\right]^{-1}(x^2 + 1)$$

$$= -\frac{1}{6D}\left[1 - \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 \ldots\right](x^2 + 1) = -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{7}{36}D^2 + \ldots\right](x^2 + 1)$$

$$= -(1/6) \times (1/D)\,\{x^2 + 1 - (1/6) \times D(x^2 + 1) + (7/36) \times D^2(x^2 + 1)\}$$

$$= -\frac{1}{6D}\left[x^2 + 1 - \frac{1}{6}(2x + 0) + \frac{7}{36}(2 + 0) + \ldots\right] = -\frac{1}{6D}\left[x^2 - \frac{1}{3}x + \frac{25}{18}\right] = -\frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right].$$

Hence the required solution is $y =$ C.F. + P.I., that is,

$$y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - (1/18) \times (x^3 - x^2/2 + 25\,x/6)$$

Exercise 5(E)

Solve the following differential equations:

1. $(D^2 + D - 6)\, y = x$. **Ans.** $y = c_1 e^{2x} + c_2 e^{-3x} - (1/36)(6x + 1)$
2. $(D^2 - 2D + 1)\, y = x - 1$. **Ans.** $y = (c_1 + c_2 x)\, e^x + x + 1$
3. $(D^2 - 4D + 4)\, y = x^2$. **Ans.** $y = (c_1 + c_2 x)\, e^{2x} + (1/4)(x^2 + 2x + 3/2)$
4. $(D^2 - 4)\, y = x^2$. **Ans.** $y = c_1 e^{2x} + c_2 e^{-2x} - (1/8)(2x^2 + 1)$
5. $(D^3 - 8)\, y = x^3$. **Ans.** $y = c_1 e^{2x} + e^{-x}(c_2 \cos x\sqrt{3} + c_3 \sin x\sqrt{3}) - (1/32)(3 + 4x^3)$
6. $(D^2 + 2D + 1)\, y = 2x + x^2$. **Ans.** $y = (c_1 + c_2 x)\, e^{-x} + x^2 - 2x + 2$
7. $(x^3 + 8)\, y = x^4 + 2x + 1$. **Ans.** $y = c_1 e^{-2x} + e^x(c_2 \cos x\sqrt{3} + c_3 \sin x\sqrt{3}) + (1/8)(x^4 - x + 1)$
8. $(D^4 + 8D^2 + 16)\, y = 16x + 256$. **Ans.** $y = (c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x + x + 16$
9. $(D^4 - 2D^3 + 5D^2 - 8D + 4)\, y = x^2$. **Ans.** $y = (c_1 + c_2 x)\, e^x + c_3 \cos 2x + c_4 \sin 2x + (1/4)(x_2 + 4x + 11/2)$
10. $(D^2 - 1)\, y = 1$ **Ans.** $y = c_1 e^x + c_2 e^{-x} - 1$

Ans. $y = (c_1 + c_2 x) e^{-x} + x^2 - 2x + 2$

11. $(D^2 + 2D + 1) y = 2x + x^2$.
12. $(D^2 + 2D - 20) y = (x + 1)^2$

Ans. $y = e^{-x} \{c_1 \cosh (x\sqrt{21}) + c_2 \sinh (x\sqrt{21})\} - (1/20) \times \{x^2 + (11/5) x + 33/25\}$

13. $(D^3 - 3D^2 - 6D + 8) y = x$. **Ans.** $y = c_1 e^{x} + c_2 e^{4x} + c_3 e^{-2x} + (4x + 3)/32$

14. $(D^4 + D^3 + D^2) y = x^2 (a + bx)$. **Ans.** $y = e^{-x/2} \{c_1 \cos (x\sqrt{3}/2) + c_2 \sin (x\sqrt{3}/2)\} + c_3 + c_4 x$

[G.N.D.U. Amritsar 2010] $+ (1/20) \times bx^5 + (1/12) \times (a - 3b) x^4 - (1/3) \times ax^3 - 3bx^2$

5.20 Short method of finding P.I. when $X = e^{ax} V$, where V is any function of x.

Theorem. $\dfrac{1}{f(D)} e^{ax} V = e^{ax} \dfrac{1}{f(D+a)} V$, V **being a function of** x.

Proof. By successive differentiation, we have

$$D (e^{ax} V) = e^{ax} DV + a e^{ax} V = e^{ax} (D + a) V,$$

$$D^2 (e^{ax} V) = e^{ax} D^2 V + a e^{ax} DV + a e^{ax} DV + a^2 e^{ax} V$$

$$= e^{ax} (D^2 + 2aD + a^2) V = e^{ax} (D + a)^2 V.$$

Similarly, $\qquad D^3 (e^{ax} V) = e^{ax} (D + a)^3 V,$

....

$$D^n (e^{ax} V) = e^{ax} (D + a)^n V.$$

$\therefore$
$$f(D)\, e^{ax} V = e^{ax} f(D + a)\, V. \qquad \dots (1)$$

The above result (1) is true for any function of x. Taking $\{1/f(D + a)\}\, V$ in place of V in (1), we have

$$f(D)\left\{e^{ax} \frac{1}{f(D+a)} V\right\} = e^{ax} f(D+a) \left\{\frac{1}{f(D+a)} V\right\}$$

or
$$e^{ax} V = f(D) \left\{e^{ax} \frac{1}{f(D+a)} V\right\}. \qquad \dots (2)$$

Operating by $1/f(D)$ on both sides of (2), we have

$$\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V. \qquad \dots (3)$$

Working rule. Read carefully the above formula, (3). Accordingly, e^{ax} which is on the right of $1/f(D)$ may be taken to the left provided D is replaced by $D + a$. After applying the above formula, $\{1/f(D + a)\}\, V$ is evaluated by short methods of Art 5.16 or 5.18 as the case may be.

5.21 Solved examples based on Art. 5.20

Ex. 1. *Solve* $(D^2 - 2D + 1) y = x^2 e^{3x}$.

[Purvanchal 2007, Delhi 1993, 2005, 08; Agra 2003]

Sol. The auxiliary equation of the given equation $\qquad D^2 - 2D + 1 = 0 \Rightarrow D = 1, 1$.

$\therefore$ C.F. $= (c_1 + c_2 x) e^{x}$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} x^2 e^{3x} = \frac{1}{(D-1)^2} x^2 e^{3x} = e^{3x} \frac{1}{(D+3-1)^2} x^2$$

$$= e^{3x} \frac{1}{(D+2)^2} x^2 = e^{3x} \frac{1}{4(1+D/2)^2} x^2 = \frac{1}{4} e^{3x} \left(1 + \frac{D}{2}\right)^{-2} x^2$$

$$= \frac{1}{4} e^{3x} \left[1 - \frac{D}{2} + \frac{(-2)(-3)}{2!} \frac{D^2}{4} + \dots\right] x^2 = \frac{1}{4} e^{3x} \left(1 - \frac{D}{2} + \frac{3}{4} D^2 + \dots\right) x^2$$

$$= (1/4) \times e^{3x} \{x^2 - (1/2) \times (2x) + (3/4) \times 2\} = (1/8) \times e^{3x} (2x^2 - 4x + 3).$$

$\therefore$ Required solution is $\qquad y = (c_1 + c_2 x) e^{x} + (1/8) \times e^{3x} (2x^2 - 4x + 3)$.

Ex. 2. *Solve (a)* $(D^2 - 2D + 1) y = x^2 e^{x}$

(b) $(D^2 - 6D + 9) y = x^2 e^{3x}$ **[G.N.D.U. Amritsar 2010]**

Sol. *(a)* Here the auxiliary equation $\qquad D^2 - 2D + 1 = 0 \Rightarrow D = 1, 1$.

$\therefore$ C.F. $= (c_1 + c_2 x) e^{x}$, c_1, c_2 being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} e^{x} x^2 = \frac{1}{(D-1)^2} e^{x} x^2 = e^{x} \frac{1}{(D+1-1)^2} x^2$$

$$= e^{x} \frac{1}{D^2} x^2 = e^{x} \frac{1}{D} \int x^2\, dx = e^{x} \frac{1}{D} \frac{x^3}{3} = e^{x} \int \frac{x^3}{3} dx = \frac{e^{x}}{3} \cdot \frac{x^4}{4}.$$

Hence the required solution is $\qquad y = (c_1 + c_2 x) e^{x} + (1/12) \times x^4 e^{x}$.

(b) Try yourself. **Ans.** $y = (c_1 + c_2 x) e^{3x} + (x^4/12) \times e^{3x}$

Ex. 3. *Find the particular solution of* $(D - 1)^2 y = e^{x} \sec^2 x \tan x$. **[Kuvempa 2005]**

Sol.
$$\text{P.I.} = \frac{1}{(D-1)^2} e^{x} \sec^2 x \tan x = e^{x} \frac{1}{(D+1-1)^2} \tan x \sec^2 x$$

$$= e^{x} \frac{1}{D} \int \tan x \sec^2 x\, dx = e^{x} \frac{1}{D} \left(\frac{\tan^2 x}{2}\right) = \frac{e^{x}}{2} \int \tan^2 x\, dx$$

$$= \frac{e^{x}}{2} \int (\sec^2 x - 1)\, dx = \frac{e^{x}}{2} (\tan x - x)$$

Ex. 4. *Solve* $(D - a)^2 y = e^{ax} f''(x)$ **[Agra 2006]**

Sol. Here auxiliary equation of the given equation is $\quad (D - a)^2 = 0 \quad$ so that $\quad D = a, a$.

$\therefore$ C.F. $= (c_1 + c_2 x) e^{ax}$, c_1 and c_2 being arbitrary constants

$$\text{P.I.} = \frac{1}{(D-a)^2} e^{ax} f''(x) = e^{ax} \frac{1}{(D+a-a)^2} f''(x) = e^{ax} \frac{1}{D} \frac{1}{D} f''(x) = e^{ax} \frac{1}{D} f'(x) = e^{ax} \int f'(x)\, dx$$

$\therefore$ The required solution is $\qquad y = (c_1 + c_2 x) e^{ax} + e^{ax} \int f'(x)\, dx$

Ex. 5. *Solve* $(D^3 - 3D - 2) y = 540 x^3 e^{-x}$ **[Lucknow 1994]**

Sol. Here the auxiliary equation of the given equation is $\qquad D^3 - 3D - 2 = 0$

or $\qquad (D - 2)(D^2 + 2D + 1) = 0 \qquad$ so that $\qquad D = 2, -1, -1$.

$\therefore$ C.F. $= c_1 e^{2x} + (c_2 + c_3 x) e^{-x}$, c_1, c_2, c_3 being arbitrary constants and

$$\text{P.I.} = \frac{1}{D^3 - 3D - 2} 540\, x^3 e^{-x} = 540\, e^{-x} \frac{1}{(D-1)^3 - 3(D-1) - 2} x^3 = 540\, e^{-x} \frac{1}{D^3 - 3D^2} x^3$$

$$= 540 e^{-x} \frac{1}{-3D^2 (1 - D/3)} x^3 = -180 e^{-x} \frac{1}{D^2} \left(1 - \frac{D}{3}\right)^{-1} x^3 = -180 e^{-x} \frac{1}{D^2} \left(1 + \frac{D}{3} + \frac{D^2}{9} + \frac{D^3}{27} + \dots\right) x^3$$

$$= -180 e^{-x} \frac{1}{D^2} \left(x^3 + x^2 + \frac{2}{3} x + \frac{2}{9}\right) = -180 e^{-x} \frac{1}{D} \left(\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{3} + \frac{2x}{9}\right)$$

$$= -180 e^{-x} \left(\frac{x^5}{20} + \frac{x^4}{12} + \frac{x^3}{9} + \frac{x^2}{9}\right) = -e^{-x} (9x^5 + 15x^4 + 20x^3 + 20x^2).$$

$\therefore$ Solution is $\qquad y = c_1 e^{2x} + (c_2 + c_3 x) e^{-x} - e^{-x} (9x^5 + 15x^4 + 20x^3 + 20x^2)$.

Ex. 6. *Solve* $(D^2+3D+2)y=e^{2x}\sin x$. **[Delhi Maths (Prog.) 2007; Delhi Maths(H) 1996]**

Sol. Here the auxiliary equation $D^2+3D+2=0 \Rightarrow D=-2,-1$.

$\therefore$ C.F. $=c_1e^{-2x}+c_2e^{-x}$, where c_1 and c_2 are arbitrary constants

$$\text{P.I.}=\frac{1}{D^2+3D+2}e^{2x}\sin x=e^{2x}\frac{1}{(D+2)^2+3(D+2)+2}\sin x=e^{2x}\frac{1}{D^2+7D+12}\sin x$$

$$=e^{2x}\frac{1}{-1^2+7D+12}\sin x=e^{2x}\frac{1}{11+7D}\sin x=e^{2x}(11-7D)\frac{1}{(11-7D)(11+7D)}\sin x$$

$$=e^{2x}(11-7D)\frac{1}{121-49D^2}\sin x=e^{2x}(11-7D)\frac{1}{121-49(-1^2)}\sin x=\frac{e^{2x}}{170}(11-7D)\sin x$$

$=(1/170)\times e^{2x}(11\sin x-7D\sin x)=(1/170)\times e^{2x}(11\sin x-7\cos x)$.

$\therefore$ Required solution is $\quad y=c_1e^{-2x}+c_2e^{-x}+(1/170)\,e^{2x}(11\sin x-7\cos x)$.

Ex. 7. *Solve* $(D^4-1)y=e^x\cos x$.

Sol. Here the auxiliary equation is $\quad D^4-1=0 \quad$ or $\quad (D^2-1)(D^2+1)=0$

so that $\quad D^2-1=0 \quad$ or $\quad D^2+1=0 \quad$ giving $\quad D=1,-1,\pm i$.

$\therefore$ C.F. $=c_1e^x+c_2e^{-x}+c_3\cos x+c_4\sin x$, c_1, c_2, c_3, c_4 being arbitrary constants.

and $$\text{P.I.}=\frac{1}{D^4-1}e^x\cos x=e^x\frac{1}{(D+1)^4-1}\cos x=e^x\frac{1}{(D^4+4D^3+6D^2+4D+1)-1}\cos x$$

$$=e^x\frac{1}{(D^2)^2+4D\cdot D^2+6D^2+4D}\cos x=e^x\frac{1}{(-1^2)^2+4D(-1^2)+6(-1^2)+4D}e^x=-\frac{1}{5}e^x\cos x$$

$\therefore$ Required solution is $\quad y=c_1e^x+c_2e^{-x}+c_3\cos x+c_4\sin x-(1/5)\times e^x\cos x$.

Ex. 8. *Solve* $(D^3-D^2+3D+5)y=e^x\cos x$. **[Delhi Maths (G) 1995]**

Sol. Here the auxiliary equation is $\quad D^3-D^2+3D+5=0$

or $\quad (D+1)(D^2-2D+5)=0 \quad$ so that $\quad D=-1, 1\pm 2i$.

$\therefore$ C.F. $=c_1e^{-x}+e^x(c_2\cos 2x+c_3\sin 2x)$, c_1, c_2, c_3 being arbitrary constants

$$\text{P.I.}=\frac{1}{D^3-D^2+3D+5}e^x\cos x=e^x\frac{1}{(D+1)^3-(D+1)^2+3(D+1)+5}\cos x$$

$$=e^x\frac{1}{D^3-D^2+4D+7}\cos x=e^x\frac{1}{D(-1^2)-(-1^2)+4D+7}\cos x$$

$$=e^x\frac{1}{3D+5}\cos x=e^x(3D-5)\frac{1}{(3D-5)(3D+5)}\cos x$$

$$=e^x(3D-5)\frac{1}{9D^2-25}\cos x=e^x(3D-5)\frac{1}{9(-1^2)-25}\cos x$$

$=-(1/34)\times e^x(3D\cos x-5\cos x)=(1/34)\times e^x(3\sin x+5\cos x)$.

$\therefore$ Required solution is $y=c_1e^{-x}+e^x(c_2\cos 2x+c_3\sin 2x)+(1/34)\times e^x(3\sin x+5\cos x)$.

Ex. 9 (*a*). *Solve* $(D^2-1)y=\cosh x\cos x$. **[Rohilkhand 1994]**

Sol. Given $\quad (D^2-1)y=\cosh x\cos x=(1/2)\times(e^x+e^{-x})\cos x$.

or $\quad (D^2-1)y=(1/2)\times e^x\cos x+(1/2)\times e^{-x}\cos x \quad$ ... (1)

Here auxiliary equation is $D^2-1=0$, giving $D=1,-1$. So $\quad$ C.F. $=c_1e^x+c_2e^{-x}$.

P.I. corresponding to $(1/2)\times e^x\cos x$

$$=\frac{1}{2}\frac{1}{D^2-1}e^x\cos x=\frac{e^x}{2}\frac{1}{(D+1)^2-1}\cos x=\frac{e^x}{2}\frac{1}{D^2+2D}\cos x$$

$$=\frac{e^x}{2}\frac{1}{-1^2+2D}\cos x=\frac{e^x}{2}(2D+1)\cdot\frac{1}{(2D+1)(2D-1)}\cos x$$

$$=\frac{e^x}{2}(2D+1)\cdot\frac{1}{4D^2-1}\cos x=\frac{e^x}{2}(2D+1)\frac{1}{-4-1}\cos x=-\frac{e^x}{10}(2D+1)\cos x$$

$=-(1/10)\times e^x(2D\cos x+\cos x)=-(1/10)\times e^x(-2\sin x+\cos x)$

P.I. corresponding to $(1/2)\times e^{-x}\cos x$.

$$=\frac{1}{2}\frac{1}{D^2-1}e^{-x}\cos x=\frac{1}{2}e^{-x}\frac{1}{(D-1)^2-1}\cos x=\frac{e^{-x}}{2}\frac{1}{D^2-2D}\cos x$$

$$=\frac{e^{x}}{2}\frac{1}{(-1^2-2D)}\cos x=-\frac{e^{-x}}{2}(2D-1)\frac{1}{(2D-1)(2D+1)}\cos x$$

$$=-\frac{e^{-x}}{2}(2D-1)\frac{1}{4D^2-1}\cos x=-\frac{e^{-x}}{2}(2D-1)\frac{1}{4(-1^2)-1}\cos x$$

$=(1/10)\times e^{-x}(2D-1)\cos x=(1/10)\times e^{-x}(-2\sin x-\cos x)$.

$\therefore$ Solution is $y=c_1e^x+c_2e^{-x}-(1/10)\times e^x(-2\sin x+\cos x)+(1/10)\times e^{-x}(-2\sin x-\cos x)$

or $\quad y=c_1e^x+c_2e^{-x}+(2/5)\times\sin x\,\{(e^x-e^{-x})/2\}-(1/5)\times\cos x\,\{(e^x+e^{-x})/2\}$

or $\quad y=c_1e^x+c_2e^{-x}+(2/5)\times\sin x\sinh x-(1/5)\times\cos x\cosh x$.

Ex. 9 (*b*). *Solve* $(D^2-1)y=\cosh x\cos x+a^x$.

Sol. As in Ex. 19 (*a*), find C.F. and P.I. corresponding to $\cosh x\cos x$.

Now, P.I. corresponding to a^x.

$$=\frac{1}{D^2-1}a^x=\frac{1}{D^2-1}e^{x\log a}=\frac{1}{(\log a)^2-1}e^{x\log a}=\frac{1}{(\log a)^2-1}a^x$$

$\therefore$ solution is $\quad y=c_1e^x+c_2e^{-x}+(2/5)\sinh x\sin x-(1/5)\cosh x\cos x+a^x/\{(\log a)^2-1\}$

Ex. 10. *Solve* $(D^4+D^2+1)y=e^{-x/2}\cos(x\sqrt{3}/2)$ **[I.A.S. 1993]**

Sol. Given $\quad (D^4+D^2+1)y=e^{-x/2}\cos(x\sqrt{3}/2)$. $\quad$ ... (1)

Here auxiliary equation is $\quad D^4+D^2+1=0 \quad$ or $\quad (D^2+1)^2-D^2=0$

or $\quad (D^2+1+D)(D^2+1-D)=0 \Rightarrow D^2+D+1=0 \quad$ or $\quad D^2-D+1=0$,

Solving these, $\quad D=-(1/2)\pm i(\sqrt{3}/2), \quad (1/2)\pm i(\sqrt{3}/2)$.

$\therefore$ C.F. $=e^{-x/2}[c_1\cos(x\sqrt{3}/2)+c_2\sin(x\sqrt{3}/2)]+e^{x/2}[c_3\cos(x\sqrt{3}/2)+c_4\sin(x\sqrt{3}/2)]$,

c_1, c_2, c_3 and c_4 being arbitrary constants

$$\text{P.I.}=\frac{1}{D^4+D^2+1}e^{-x/2}\cos(x\sqrt{3}/2)=e^{-x/2}\frac{1}{(D-1/2)^4+(D-1/2)^2+1}\cos(x\sqrt{3}/2)$$

$$=e^{-x/2}\frac{1}{D^4-4D^3\cdot(1/2)+6D^2(1/2)^2-4D(1/2)^3+(1/2)^4+D^2-D+(1/4)+1}\cos\frac{x\sqrt{3}}{2}$$

$$=e^{-x/2}\frac{1}{D^4-2D^3+(5/2)D^2-(3/2)D+(21/16)}\cos\frac{x\sqrt{3}}{2}.$$

$$= e^{-x/2}\frac{1}{[(D^2+(3/4)][D^2-2D+(7/4)]}\cos\frac{x\sqrt{3}}{2}$$

[Since denominator is zero when $D^2 = -3/4$ so $D^2 + (3/4)$ must be a factor of the denominator]

$$= e^{-x/2}\frac{1}{D^2+(3/4)}\cdot\frac{1}{(-3/4)-2D+(7/4)}\cos\frac{x\sqrt{3}}{2} = e^{-x/2}\frac{1}{D^2+(3/4)}\frac{1}{1-2D}\cos\frac{x\sqrt{3}}{2}$$

$$= e^{-x/2}\frac{1}{D^2+(3/4)}(1+2D)\cdot\frac{1}{(1+2D)(1-2D)}\cos\frac{x\sqrt{3}}{2}$$

$$= e^{-x/2}\frac{1}{D^2+(3/4)}(1+2D)\frac{1}{1-4D^2}\cos\frac{x\sqrt{3}}{2} = e^{-x/2}\frac{1}{D^2+(3/4)}(1+2D)\frac{1}{1-4\times(-3/4)}\cos\frac{x\sqrt{3}}{2}$$

$$= \frac{1}{4}e^{-x/2}\frac{1}{D^2+(3/4)}(1+2D)\cos\frac{x\sqrt{3}}{2} = \frac{1}{4}e^{-x/2}\frac{1}{D^2+(\sqrt{3}/2)^2}\left[\cos\frac{x\sqrt{3}}{2} - \sqrt{3}\sin\frac{x\sqrt{3}}{2}\right]$$

$$= \frac{1}{4}e^{-x/2}\left[\frac{1}{D^2+(\sqrt{3}/2)^2}\cos\frac{x\sqrt{3}}{2} - \sqrt{3}\frac{1}{D^2+(\sqrt{3}/2)^2}\sin\frac{x\sqrt{3}}{2}\right]$$

$$= \frac{1}{4}e^{-x/2}\left[\frac{x}{2\times(\sqrt{3}/2)}\sin\frac{x\sqrt{3}}{2} + \sqrt{3}\cdot\frac{x}{2\times(\sqrt{3}/2)}\cos\frac{x\sqrt{3}}{2}\right]$$, using results (4) and (5) of Art. 5.16

$$= (x/4\sqrt{3})\times e^{-x/2}\{\sin(x\sqrt{3}/2) + \sqrt{3}\cos(x\sqrt{3}/2)\}.$$

$\therefore$ The required solution is $y = e^{x/2}[c_1\cos(x\sqrt{3}/2) + c_2\sin(x\sqrt{3}/2)] + e^{x/2}[c_3\cos(x\sqrt{3}/2)$
$+ c_4\sin(x\sqrt{3}/2)] + (x/4\sqrt{3})\times e^{-x/2}[\sin(x\sqrt{3}/2) + \sqrt{3}\cos(x\sqrt{3}/2)]$

Ex. 11. *Solve* $(d^4y/dx^4) + 6(d^3y/dx^3) + 11(d^2y/dx^2) + 6(dy/dx) = 20e^{-2x}\sin x.$ **[I.A.S. 1997]**

Sol. Re - writing the given equation, $(D^4 + 6D^3 + 11D^2 + 6D)y = 20e^{-2x}\sin x.$ ...(1)

Its auxiliary equation is $D^4 + 6D^3 + 11D^2 + 6D = 0$ or $D(D+1)(D+2)(D+3) = 0$

Solving it, we get $D = 0, -1, -2, -3.$

$\therefore$ C.F. $= c_1e^{0x} + c_2e^{-x} + c_3e^{-2x} + c_4e^{-3x}$, c_1, c_2, c_3 and c_4 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^4+6D^3+11D^2+6D}20e^{-2x}\sin x$$

$$= 20e^{-2x}\frac{1}{(D-2)^4+6(D-2)^3+11(D-2)^2+6(D-2)}\sin x$$

$$= 20e^{-2x}\frac{1}{D^4 - 8D^3+24D^2+32D+16+6(D^3-6D^2+12D-8)+11(D^2-4D+4)+6(D-2)}\sin$$

$$= 20e^{-2x}\frac{1}{D^4-2D^3-D^2+2D}\sin x = 20e^{-2x}\frac{1}{(D^2)^2-2D(D^2)-D^2+2D}\sin x$$

$$= 20e^{-2x}\frac{1}{(-1^2)^2-2(-1^2)D-(-1^2)+2D}\sin x = 20e^{-2x}\frac{1}{2(1+2D)}\sin x$$

$$= 10e^{-2x}(1-2D)\frac{1}{1-4D^2}\sin x = 10e^{-2x}(1-2D)\frac{1}{1-4(-1^2)}\sin x$$

$$= 2e^{-2x}(1-2D)\sin x = 2e^{-2x}(\sin x - 2\cos x)$$

The required solution is $y = c_1 + c_2e^{-x} + c_3e^{-2x} + c_4e^{-3x} - 2e^{-2x}(\sin x - 2\cos x)$

Ex. 12. *Solve* $(D^2 + 2D + 1)y = xe^x\sin x.$ **[Agra 1996, Lucknow 1996, Purvanchal 1998]**

Sol. Here the auxiliary equation is $D^2 - 2D + 1 = 0$ so that $D = 1, 1.$

$\therefore$ C.F. $= (c_1 + c_2x)e^x$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{(D-1)^2}e^x(x\sin x) = e^x\frac{1}{(D+1-1)^2}x\sin x = e^x\frac{1}{D^2}x\sin x$$

$$= e^x\frac{1}{D}\int x\sin x\,dx = e^x\frac{1}{D}\left[-x\cos x - \int 1\cdot(-\cos x)dx\right]$$, integrating by parts

$$= e^x\frac{1}{D}[-x\cos x + \sin x] = e^x\int(\sin x - x\cos x)dx$$

$$= e^x\left[\int\sin x\,dx - \int x\cos dx\right] = e^x\left[-\cos x - \{x\sin x - \int 1\cdot\sin x\,dx\}\right]$$

$$= e^x(-\cos x - x\sin x - \cos x) = -e^x(x\sin x + 2\cos x).$$

$\therefore$ The required solution is $y = (c_1 + c_2x)e^x - e^x(x\sin x + 2\cos x).$

Ex. 13(*a*) Solve $(D^2 - 4D + 4)y = e^{2x}\sin 2x$ **[Purvanchal 2007]**

Sol. Given $(D^2 - 4D + 4)y = e^{2x}\sin 2x,\ D \equiv d/dx$ ... (1)

Its auxiliary equation is $D^2 - 4D + 4 = 0$ or $(D-2)^2 = 0 \Rightarrow 0 = 2, 2.$

$\therefore$ C.F. $= (c_1 + c_2x)e^{2x}$, c_1, c_2 being arbitrary constants

$$\text{P.I.} = \frac{1}{D^2-4D+4}e^{2x}\sin x = \frac{1}{(D-2)^2}e^{2x}\sin x = e^{2x}\frac{1}{(D+2-2)^2}\sin x$$

$$= e^{2x}\frac{1}{D^2}\sin x = e^{2x}\frac{1}{D}\int\sin x\,dx = e^{2x}\frac{1}{D}(-\cos x) = -e^{2x}\int\cos x\,dx = -e^{2x}\sin x$$

$\therefore$ Required solution is $y = (c_1 + c_2x)e^{2x} - e^{2x}\sin x$

Ex. 13(*b*) *Solve* $(D^2 - 4D + 4)y = 8x^2e^{2x}\sin 2x.$ **[Kanpur 1997, Lucknow 1993, 97]**

Sol. Here the auxiliary equation is $D^2 - 4D + 4 = 0$ giving $D = 2, 2.$

$\therefore$ C.F. $= (c_1 + c_2x)e^{2x}$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{(D-2)^2}8x^2e^{2x}\sin 2x = 8e^{2x}\frac{1}{(D+2-2)^2}x^2\sin 2x = 8e^{2x}\frac{1}{D^2}x^2\sin 2x$$

$$= 8e^{2x}\frac{1}{D}\int x^2\sin 2x\,dx = 8e^{2x}\frac{1}{D}\left[x^2\left(-\frac{\cos 2x}{2}\right) - \int(2x)\left(-\frac{\cos 2x}{2}\right)dx\right]$$, integrating by parts

$$= 8e^{2x}\frac{1}{D}\left[-\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx\right] = 8e^{2x}\frac{1}{D}\left[-\frac{1}{2}x^2\cos 2x + x\left(\frac{\sin 2x}{2}\right) - \int 1\cdot\frac{\sin 2x}{2}dx\right]$$

$$= 8e^{2x}\frac{1}{D}\left[-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right] = 8e^{2x}\int(-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x)dx$$

$$= 8e^{2x}[-\frac{1}{2}\int x^2\cos 2x\,dx + \frac{1}{2}\int x\sin 2x\,dx + \frac{1}{4}\int\cos 2x\,dx]$$

$$= 8e^{2x}[-\frac{1}{2}\{x^2(\frac{1}{2}\sin 2x) - \int 2x(\frac{1}{2}\sin 2x)dx\} + \frac{1}{2}\int x\sin 2x\,dx + \frac{1}{8}\sin 2x]$$

$$= 8e^{2x}[-\frac{1}{4}x^2\sin 2x\,dx + \frac{1}{2}\int x\sin 2x\,dx + \frac{1}{2}\int x\sin 2x\,dx + (1/8)\times\sin 2x]$$

$$= 8e^{2x}[-\frac{1}{4}x^2 \sin 2x + \int x \sin 2x\, dx + \frac{1}{8}\sin 2x]$$

$$= 8e^{2x}\left[-\frac{1}{4}x^2 \sin 2x + x(-\frac{1}{2}\cos 2x) - \int 1\cdot(-\frac{1}{2}\cos 2x)\, dx + \frac{1}{8}\sin 2x\right]$$

$$= 8e^{2x}[-\frac{1}{4}x^2 \sin 2x - \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + \frac{1}{8}\sin 2x]$$

$$= 8e^{2x}[-\frac{1}{4}x^2 \sin 2x - \frac{1}{2}x\cos 2x + \frac{3}{8}\sin 2x]$$

$\therefore$ The required solution is $\quad y = (c_1 + c_2 x)\, e^{2x} + e^{2x}\,(3 \sin 2x - 4x \cos 2x - 2x^2 \sin 2x)$.

Ex. 14. *Solve* $(D^3 + 1)\, y = e^{2x} \sin x + e^{x/2} \sin\, (\sqrt{3}x/2)$.

Sol. Given $\quad (D^3 + 1)\, y = e^{2x} \sin x + e^{x/2} \sin\, (\sqrt{3}x/2)$.

The auxiliary equation is $\quad D^3 + 1 = 0, \quad$ or $\quad (D + 1)\,(D^2 - D + 1) = 0, \quad$ ... (1)

giving $\quad D = -1,\ \{1 \pm (1-4)^{1/2}\}/2 \quad$ or $\quad D = -1,\ 1/2 \pm i(\sqrt{3}/2)$

$\therefore \quad$ C.F. $= c_1 e^{-x} + e^{x/2}\,[c_2 \cos\,(x\sqrt{3}/2) + c_3 \sin\,(x\sqrt{3}/2)]$,

c_1, c_2 and c_3 being arbitrary constants.

P.I. corresponding to $e^{2x} \sin x$.

$$= \frac{1}{D^3+1} e^{2x} \sin x = e^{2x} \frac{1}{(D+2)^3+1} \sin x = e^{2x} \frac{1}{D^3+3D^2\cdot 2+3D\cdot 2^2+2^3+1} \sin x$$

$$= e^{2x} \frac{1}{D^3+6D^2+12D+9} \sin x = e^{2x} \frac{1}{-D-6+12D+9} \sin x$$

$$= e^{2x} \frac{1}{11D+3} \sin x = e^{2x}(11D-3) \frac{1}{121D^2-9} \sin x = e^{2x} \frac{(11D-3)}{-121-9} \sin x$$

$$= -(1/130) \times e^{2x}\,(11 \cos x - 3 \sin x).$$

P.I. corresponding to $e^{x/2} \sin\,(x\sqrt{3}/2)$

$$= \frac{1}{D^3+1} e^{x/2} \sin\left(\frac{x\sqrt{3}}{2}\right) = e^{x/2} \frac{1}{(D+1/2)^3+1} \sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= e^{x/2} \frac{1}{D^3+3D^2\cdot(1/2)+3D\cdot(1/2)^2+(1/2)^3+1} \sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= e^{x/2} \frac{1}{D^3+(3/2)\,D^2+(3/4)\,D+(9/8)} \sin\left(\frac{x\sqrt{3}}{2}\right) = e^{x/2} \frac{1}{[D^2+(3/4)][D+(3/2)]} \sin\left(\frac{x\sqrt{3}}{2}\right)$$

[As denominator is zero when $D^2 = -3/4$, factorize denominator]

$$= e^{x/2} \frac{1}{D^2+(3/4)} \left(D - \frac{3}{2}\right) \frac{1}{D^2-(9/4)} \sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= e^{x/2} \frac{1}{D^2+(3/4)} \left(D - \frac{3}{2}\right) \frac{1}{(-3/4)-(9/4)} \sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= -\frac{e^{x/2}}{3} \frac{1}{D^2+(3/4)} \left(D - \frac{3}{2}\right) \sin\left(\frac{x\sqrt{3}}{2}\right) = -\frac{e^{x/2}}{3} \frac{1}{D^2+(3/4)} \left[\frac{\sqrt{3}}{2} \cos\left(\frac{x\sqrt{3}}{2}\right) - \frac{3}{2} \sin\left(\frac{x\sqrt{3}}{2}\right)\right]$$

$$= -\frac{e^{x/2}}{6} \left[\sqrt{3} \frac{1}{D^2+(\sqrt{3}/2)^2} \cos\left(\frac{x\sqrt{3}}{2}\right) - 3 \frac{1}{D^2+(\sqrt{3}/2)^2} \sin\left(\frac{x\sqrt{3}}{2}\right)\right]$$

$$= -\frac{e^{x/2}}{6} \left[\sqrt{3} \frac{x}{2(\sqrt{3}/2)} \sin\left(\frac{x\sqrt{3}}{2}\right) + 3 \cdot \frac{1}{2(\sqrt{3}/2)} \cos\left(\frac{x\sqrt{3}}{2}\right)\right]$$

$$= -\frac{xe^{x/2}}{6} \left[\sin\left(\frac{x\sqrt{3}}{2}\right) + \sqrt{3} \cos\left(\frac{x\sqrt{3}}{2}\right)\right].$$

$\therefore$ Required solution is $\quad y = c_1 e^{-x} + e^{x/2}\,[c_2 \cos\,(x\sqrt{3}/2) + c_2 \sin\,(x\sqrt{3}/2)]$
$\quad - (x/6) \times e^{x/2}\,[\sin\,(x\sqrt{3}/2) + \sqrt{3} \cos\,(x\sqrt{3}/2)]$.

Ex. 15. *Solve* $(d^2y/dx^2) - 5\,(dy/dx) + 6y = e^{4x}\,(x^2 + 9)$. **[I.A.S. 1998]**

Sol. Given $\quad (D^2 - 5D + 6)\, y = e^{4x}\,(x^2 + 9)$, where $\quad D \equiv d/dx \quad$ ... (1)

Its auxiliary equation is $\quad D^2 - 5D + 6 = 0 \quad$ so that $\quad D = 2, 3$.

Hence, here $\quad$ C.F. $= c_1 e^{2x} + c_2 e^{3x}$, c_1 and c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2-5D+6} e^{4x}(x^2+9) = e^{4x} \frac{1}{(D+4)^2-(D+4)+6}(x^2+9) = e^{4x} \frac{1}{D^2+3D+2}(x^2+9)$$

$$= \frac{e^{4x}}{2} \frac{1}{1+D^2/2+3D/2}\,(x^2+9) = \frac{1}{2} e^{4x}\{1+(D^2/2+3D/2)\}^{-1}$$

$$= (1/2) \times e^{4x}\left\{1-(D^2/2+3D/2)+(D^2/2+3D/2)^2 \ldots\ldots\right\}(x^2+9)$$

$$= (1/2) \times e^{4x}\,[1 - (D^2/2) - (3D/2) + 9D^2/4 + \ldots]\,(x^2 + 9)$$

$$= (1/2) \times e^{4x}\,\{(x^2 + 9) - (3/2)\, D\,(x^2 + 9) + (7/4)\, D^2\,(x^2 + 9)]$$

$$= (1/2) \times e^{4x}\,\{x^2 + 9 - (3/2)\,(2x) + (7/4)\,(2)\}$$

$$= (1/2) \times e^{4x}\,(x^2 - 3x + 25/2) = (1/4)\, e^{4x}\,(2x^2 - 6x + 25)$$

Hence the required solution is $\quad y = c_1 e^{2x} + c_2 e^{3x} + (1/4)\, e^{4x}\,(2x^2 - 6x + 25)$.

Ex. 16. *Solve* $d^2y/dx^2 - 4\,(dy/dx) - 5y = xe^{-x}$, *given that* $y = 0$ *and* $dy/dx = 0$ *when* $x = 0$.

Sol. Re-writting the given equation, $\quad (D^2 - 4D - 5)\, y = xe^{-x} \quad$ where $\quad D = d/dx$ ... (1)

Also, given that $\quad y = 0, \quad$ when $\quad x = 0 \quad$ ... (2)

and $\quad dy/dx = 0, \quad$ when $\quad x = 0 \quad$ ... (3)

The auxiliary equation of (1) is $D^2 - 4D - 5 = 0 \quad$ or $\quad (D - 5)\,(D + 1) = 0 \Rightarrow D = 5, -1$.

So C.F. $= c_1 e^{5x} + c_2 e^{-x}$, c_1, c_2 being arbitrary constants

and $\quad$ P.I. $= \dfrac{1}{D^2-4D-5} xe^{-x} = e^{-x} \dfrac{1}{(D-1)^2-4(D-1)-5} x$

$$= e^{-x} \frac{1}{D^2-6D} x = -\frac{1}{6} e^{-x} \frac{1}{D(1-D/6)} x = -\frac{1}{6} e^{-x} \frac{1}{D}\left(1-\frac{D}{6}\right)^{-1} x = -\frac{1}{6} e^{-x} \frac{1}{D}\left(1+\frac{D}{6}+\ldots\right) x$$

$$= -\frac{1}{6} e^{-x} \frac{1}{D}\left(x+\frac{1}{6}\right) = -\frac{1}{6} e^{-x}\left(\frac{x^2}{2}+\frac{x}{6}\right)$$

$\therefore$ The solution of (1) is $\quad y = c_1 e^{5x} + c_2 e^{-x} - (1/12)\, e^{-x}(x^2 + x/3) \quad$ ... (4)

Putting $x = 0$, $y = 0$ (due to (2)), in (4), we get $\qquad c_1 + c_2 = 0.$ ... (5)

From (4), $$\frac{dy}{dx} = 5c_1e^{5x} - c_2e^{-x} + \frac{1}{12}e^{-x}\left(x^2 + \frac{x}{3}\right) - \frac{1}{12}e^{-x}\left(2x + \frac{1}{3}\right) \qquad \text{... (6)}$$

Putting $x = 0$, $dy/dx = 0$ (due to (3)) in (6), we get $\qquad 5c_1 - c_2 - (1/36) = 0.$ ... (7)

Solving (5) and (7), $c_1 = 1/216$ and $c_2 = -(1/216)$. With these values, (4) reduces to

$$y = (1/216) \times (e^{5x} - e^{-x}) - (1/12) \times e^{-x}(x^2 + x/3).$$

Exercise 5(F)

Solve the following differential equations:

1. $(D-2)^3 y = xe^{2x}$. **Ans.** $y = (c_1 + c_2x + c_3x^2)e^{2x} + (x^4/24) \times e^{2x}$
2. $(D+1)^3 y = x^2e^{-x}$ **Ans.** $y = (c_1 + c_2x + c_3x^2)e^{-x} + (x^5/60) \times e^{-x}$
3. $(D^2 - 2D + 5)y = xe^x$ **[Madras 2012]** **Ans.** $y = e^{x}(c_1 \cos 2x + c_2 \sin 2x) + (1/16) \times e^x(2x - 1)$
4. $(D^2 - 1)y = e^x \cos x$. **Ans.** $y = c_1e^x + c_2e^{-x} + (1/5)e^x(2 \sin x - \cos x)$
5. $(D^2 - 4D + 1)y = e^{2x} \sin x$. **Ans.** $y = c_1e^{2x} \cosh(x\sqrt{3} + c_2) - (1/7)e^{2x} \sin 3x$
6. $(D^2 - 2D + 5)y = e^{2x} \sin x$ **[Agra 1996, Rohilkhand 1996]** **Ans.** $y = e^x(c_1 \cos 2x + c_2 \sin 2x) - (1/10)e^{2x}(\cos x - 2 \sin x)$
7. $(D^2 - 2D + 4)y = e^x \cos x$ **[Agra 1995; Calcutta 1996, Delhi Math(H) 1995, Kanpur 1996, S.V. University (A.P.) 1997]** **Ans.** $y = e^x[c_1 \cos(x\sqrt{3}) + c_2 \sin(x\sqrt{3})] + (1/2)e^x \cos x$
8. $D^2y = e^x \cos x$. **Ans.** $y = c_1x + c_2 + (1/2)e^x \sin x$
9. $(D^2 + 4D - 12)y = (x-1)e^{2x}$ **[Bangalore 1996; Lucknow 1997, 98]** **Ans.** $y = c_1e^{2x} + c_2e^{-6x} + (1/64)e^{2x}(4x^2 - 9x)$
10. $(D^3 - 7D - 6)y = (x+1)e^{2x}$ **[Delhi Maths (G) 2002, Kanpur 1996]** **Ans.** $y = c_1e^{x} + c_2e^{-2x} + c_3e^{3x} - (1/144)e^{2x}(12x + 17)$
11. $(D^3 - D - 6)y = (x+1)e^{2x}$ **Ans.** $y = c_1e^{2x} + c_2e^{-x}\cos(x\sqrt{2} + c_3) + (1/242)(10x + 11x^2)$
12. $(D^3 - 3D^2 + 3D - 1)y = (x+1)e^x$ **Ans.** $y = (c_1 + c_2x + c_3x^2)e^x + (1/24)(x+1)^4e^x$
13. $(D^2 - 2D + 6)y = e^x \cos x$ **[Agra 1996; Meerut 1999]** **Ans.** $y = e^x(c_1 \cos x\sqrt{5} + c_2 \sin x\sqrt{5}) + (1/4) \times e^x \cos x$
14. $(D^2 - 2D + 4)y = e^x \sin x$ **Ans.** $y = e^x\{c_1 \cos(x\sqrt{3}) + c_2 \sin(x\sqrt{3})\} + (1/2)e^x \sin x$
15. $(D^2 - 4D + 1)y = e^{2x} \sin 2x$. **Ans.** $y = e^{2x}\{c_1 \cosh(x\sqrt{3}) + c_2 \sinh(x\sqrt{3})\} + (1/7)e^{2x} \sin 2x$
16. $(D^2 - 2D + 4)y = e^{2x} \cos x$. **Ans.** $y = e^x\{c_1 \cos(x\sqrt{3}) + c_2 \sin(x\sqrt{3})\} + (1/13)e^{2x}(2 \sin x + 3 \cos x)$
17. $(D^2 + 1)y = xe^{2x}$. **Ans.** $y = c_1 \cos x + c_2 \sin x + (1/25)(5x - 4)e^{2x}$
18. $(D^2 + 2)y = x^2e^{3x}$. **[Delhi Maths (G) 1996]** **Ans.** $y = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}) - (1/11)\{x^2 - (12/11)x - (50/121)\}e^{3x}$
19. $(D^2 - 1)y = e^x(1 + x^2)$. **Ans.** $y = c_1e^x + c_2e^{-x} + (1/12)e^x(2x^3 - 3x^2 + 9x)$
20. $(D^2 - 1)y = \sinh x \cosh x \cos x$ **Ans.** $y = c_1e^x + c_2e^{-x} + (1/20) \times (\cos x \sinh 2x - 2 \sin x \cosh 2x)$

5.22 Short method of finding P.I. when $X = xV$, where V is any function of x.

For this purpose, we have the following theorem

Theorem. To prove that $$\frac{1}{f(D)}xV = x\frac{1}{f(D)}V + \left(\frac{d}{dD}f(D)\right)V$$

Proof. Let U be a function x. If u and v be functions of x, then by Leibnitz's theorem,

$$D^n(uv) = D^nu \cdot v + {}^nc_1 D^{n-1}u \cdot Dv + {}^nc_2 D^{n-2}u \cdot D^2v + \dots u \cdot D^nv. \qquad \text{... (1)}$$

Using Leibnitz's theorem (1), we have

$$D^n(xU) = D^n(Ux) = D^nU \cdot x + {}^nc_1 D^{n-1}U \cdot 1 = x D^nU + n D^{n-1}U$$

or $$D^n(xU) = x D^nU + \frac{d}{dD}(D^n)U, \qquad \text{as} \qquad \frac{d}{dD}D^n = nD^{n-1} \qquad \text{... (2)}$$

Since $f(D)$ is a polynomial in D, from (2) we have

$$f(D)(xU) = xf(D)U + \left(\frac{d}{dD}f(D)\right)U. \qquad \text{... (3)}$$

Putting $f(D)U = V$, we have

$$\frac{1}{f(D)}(f(D)U) = \frac{1}{f(D)}V \qquad \text{so that} \qquad U = \frac{1}{f(D)}V$$

Moreover, since U is function of x, so is V.

$\therefore$ From (3), $$f(D)\left(x\frac{1}{f(D)}V\right) = xV + \left(\frac{d}{dD}f(D)\right)\frac{1}{f(D)}V. \qquad \text{... (4)}$$

Operating on both sides of (4) by $1/f(D)$, we have

$$x\frac{1}{f(D)}V = \frac{1}{f(D)}(xV) + \frac{f'(D)}{[f(D)]^2}V, \qquad \text{where} \qquad f'(D) = \frac{df(D)}{dD}$$

or $$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V \qquad \text{... (5A)}$$

or $$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V + \left(\frac{d}{dD}\frac{1}{f(D)}\right)V \qquad \text{... (5B)}$$

Working rule of finding P.I. when $X = xV$. We shall apply the formula (5B) to compute $\{1/f(D)\}(xV)$. Proceeding by repated application of the above formula, $\{1/f(D)\}(x^mV)$ can be evaluated, if m is a positive integer. We shall apply the above formula (5A) when V is of the form $\sin ax$ or $\cos ax$. In practice the above formula should not be used when $V = \sin ax$ or $\cos ax$ and $f(-a^2) = 0$ i.e., $f(D^2)$ vanishes by putting $-a^2$ for D^2. In such situations we shall apply the following alternative method. This alternative method can also be applied even when $f(-a^2) \neq 0$.

Alternative Working rule for finding P.I. when $X = x^m \sin ax$ or $x^m \cos ax$.

(*i*) $$\text{P.I} = \frac{1}{f(D)}x^m \cos ax = \text{Real part of } \frac{1}{f(D)}x^m(\cos ax + i \sin ax)$$

$$= \text{R.P. of } \frac{1}{f(D)}x^m e^{aix},$$ by Euler's theorem, where R.P. stands for real part

(*ii*) $$\text{P.I} = \frac{1}{f(D)}x^m \sin ax = \text{Imaginary part of } \frac{1}{f(D)}x^m(\cos ax + i \sin ax)$$

$$= \text{I.P. of } \frac{1}{f(D)}x^m e^{aix},$$ by Euler's theorem, where I.P. stands for imaginary part.

5.23 Solved examples based an Art. 5.22

Ex. 1. *Solve* $(D^2+9)=x \sin x$

Sol. Method I. Here auxilary equation $D^2+9=0 \Rightarrow D=\pm 3i.$

$\therefore$ C.F. $= c_1 \cos 3x + c_2 \sin 3x$, c_1, c_2 being arbitrary constants.

$$\text{P.I} = \frac{1}{D^2+9} x \sin x = \text{I.P. of } \frac{1}{D^2+9} x e^{ix}, \text{ where I.P. stands for imaginary part}$$

$$= \text{I.P. of } e^{ix} \frac{1}{(D+i)^2+9} x, \qquad \text{as} \quad \frac{1}{f(D)} V e^{ax} = e^{ax} \frac{1}{f(D+a)} V$$

$$= \text{I.P. of } e^{ix} \frac{1}{D^2+2iD+8} x = \text{I.P. of } e^{ix} \frac{1}{8[1+(1/8)(D^2+2iD)]} x$$

$$= \text{I.P. of } \frac{e^{ix}}{8}\left[1+\left(\frac{iD}{4}+\frac{D^2}{8}\right)\right]^{-1} x = \text{I.P. of } \frac{e^{ix}}{8}\left[1-\frac{iD}{4}+\dots\right] x$$

$= \text{I.P. of } (1/8) \times (\cos x + i \sin x)(x - i/4) = (1/8) \times \{x \sin x - (1/4) \cos x\}.$

$\therefore$ Required solution is $\quad y = c_1 \cos 3x + c_2 \sin 3x + (1/8)\, x \sin x - (1/32) \cos x.$

Method II. As in Method I, $\quad$ C.F. $= c_1 \cos 3x + c_2 \sin 3x.$

$$\text{P.I.} = \frac{1}{D^2+9} x \sin x = x \frac{1}{D^2+9} \sin x + \left[\frac{d}{dD}\frac{1}{D^2+9}\right] \sin x$$

$$\left[\because \frac{1}{f(D)} xV = x \frac{1}{f(D)} V + \left\{\frac{d}{dD}\frac{1}{f(D)}\right\} V\right]$$

$$= x \frac{1}{D^2+9} \sin x - \frac{2D}{(D^2+9)^2} \sin x = x \frac{1}{-1^2+9} \sin x - 2D \frac{1}{(-1^2+9)^2} \sin x$$

$= (x/8) \times \sin x - (1/32) \times D \sin x = (x/8) \times \sin x - (1/32) \times \cos x$

$\therefore$ Required solution is $\quad y = c_1 \cos 3x + c_2 \sin 3x + (1/8)\, x \sin x - (1/32) \cos x.$

Ex. 2. *Solve (a)* $(D^2+2D+1)\, y = x \cos x$ **[Agra 1995]**

(b) $(D^2+2D+1)\, y = x \sin x$ **[IAS 1998, Delhi Maths (G) 1995]**

Sol. *(a)* Here the auxiliary equation is $D^2+2D+1=0$ so that $D=-1,-1.$

$\therefore$ C.F. $= (c_1+c_2 x) e^{-x}$, c_1, c_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D+1)^2} x \cos x = \text{R.P. of } \frac{1}{(D+1)^2} x e^{ix}, \text{ where R.P. stands for real part}$$

$$= \text{R.P. of } e^{ix} \frac{1}{[(D+i)+1)]^2} x = \text{R.P. of } \frac{e^{ix}}{(1+i)^2}\left(1+\frac{D}{1+i}\right)^{-2}$$

$$= \text{R.P. of } \frac{e^{ix}}{2i}\left(1-\frac{2D}{1+i}+\dots\right) x = \text{R.P. of } \frac{e^{ix}}{2i}\left(x-\frac{2}{1+i}\right)$$

$$= \text{R.P. of } i\frac{e^{ix}}{2i^2}\left[x-\frac{2(1-i)}{(1-i)(1+i)}\right] = \text{R.P. of } \frac{ie^{ix}}{(-2)}\left[x-\frac{2(1-i)}{1+1}\right]$$

$= \text{R.P. of } (-i/2) \times (\cos x + i \sin x) \{(x-1)+i\}$

$= \text{R.P. of } (-1/2) \times (i \cos x - \sin x) \{(x-1)+i\} \quad \dots (1)$

P.I. $= (-1/2) \times [-\sin x \cdot (x-1) - \cos x] = (1/2) \times [(x-1) \sin x + \cos x]$

$\therefore$ Solution is $\quad y = (c_1 + c_2 x) e^{-x} + (1/2) \times [(x-1) \sin x + \cos x].$

(b) Proceed as in part *(a)*. As before $\quad$ C.F. $= (c_1 + c_2 x) e^{-x}$

$$\text{P.I.} = \frac{1}{(D+1)^2} x \sin x = \text{I.P. of } \frac{1}{(D+1)^2} x e^{ix}, \text{ where I.P. stands for imaginary part}$$

$= \text{I.P. of } (-1/2) \times (i \cos x - \sin x) [(x-1)+i],$

[Proceeding as in part *(a)* upto equation (1) replacing R.P. by I.P.]

$= (-1/2) \times \{(x-1) \cos x - \sin x\} = (1/2) \times \{\sin x - (x-1) \cos x\}.$

$\therefore$ The required solution is $\quad y = (c_1 + c_2 x) e^{-x} + (1/2) \times \{\sin x - (x-1) \cos x\}.$

Ex. 3. *Solve* $(D^2-2D+1)\, y = x \sin x.$

[Agra 1996, Kanpur 1994, Allahabad 1998, Meerut 1993, Ravishankar 1994]

Sol. Here the auxiliary equation $D^2-2D+1=0$ so that $D=1, 1.$

$\therefore$ C.F. $= (c_1 + c_2 x) e^{x}$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2-2D+1} x \sin x = x \frac{1}{D^2-2D+1} \sin x + \left(\frac{d}{dD}\frac{1}{D^2-2D+1}\right) \sin x$$

$$= x \frac{1}{-1^2-2D+1} \sin x \frac{(2D-2)}{(D^2-2D+1)^2} \sin x = -\frac{x}{2}\frac{1}{D} \sin x - 2(D-1) \frac{1}{(-1^2-2D+1)^2} \sin x$$

$$= \frac{x}{2} \cos x - \frac{1}{2}(D-1)\frac{1}{4D^2} \sin x = \frac{x}{2} \cos x - \frac{1}{2}(D-1)\frac{1}{D}\int \sin x \, dx$$

$$= \frac{x}{2} \cos x - \frac{1}{2}(D-1)\frac{1}{D}(-\cos x) = \frac{x}{2} \cos x - \frac{1}{2}(D-1)\int(-\cos x)\, dx$$

$= (x/2) \cos x + (1/2)(D-1) \sin x = (x/2) \cos x + (1/2)(\cos x - \sin x)$

$\therefore$ Required solution is $\quad y = (c_1 + c_2 x) e^{x} + (1/2)(x \cos x + \cos x - \sin x).$

Ex. 4. *Solve* $(D^2+1)\, y = x^2 \sin 2x.$ **[Kanpur 1995 Delhi Maths (H) 2000]**

Sol. Here the auxiliary equation is $\quad D^2+1=0 \quad$ so that $\quad D = \pm i.$

$\therefore$ C.F. $= c_1 \cos x + c_2 \sin x$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2+1} x^2 \sin 2x = \text{I.P. of } \frac{1}{D^2+1} x^2 e^{2ix}, \text{ where I.P. stands for imaginary part}$$

$$= \text{I.P. of } e^{2ix} \frac{1}{(D+2i)^2+1} x^2 = \text{I.P. of } e^{2ix} \frac{1}{D^2+4iD-3} x^2, \text{as } i^2=-1$$

$$= \text{I.P. of } \frac{e^{2ix}}{-3} \frac{1}{\{1-(4iD+D^2)/3\}} x^2 = \text{I.P. of } \frac{e^{2ix}}{-3}\left[1-\left(\frac{4iD}{3}+\frac{D^2}{3}\right)\right]^{-1} x^2$$

$$= \text{I.P. of } \frac{e^{2ix}}{-3}\left[1+\left(\frac{4iD}{3}+\frac{D^2}{3}\right)+\left(\frac{4iD}{3}+\frac{D^2}{3}\right)^2+\dots\right] x^2$$

$$= \text{I.P. of } \frac{e^{2ix}}{-3}\left[1+\frac{4iD}{3}+\frac{D^2}{3}-\frac{16D^2}{9}+\dots\right] x^2$$

$$= \text{I.P. of } \frac{e^{2ix}}{-3}\left[1+\frac{4iD}{3}-\frac{13D^2}{9}+\dots\right] x^2 = \text{I.P. of } \frac{e^{2ix}}{-3}\left[x^2+\left(\frac{4i}{3}\times 2x\right)-\left(\frac{13}{9}\times 2\right)\right]$$

$= \text{I.P. of } (-1/3)(\cos 2x + i \sin 2x)\{x^2 + (8/3)\, ix - (26/9)\}$

$= -(1/3)\,[(x^2 - 26/9)\sin 2x + (8/3)\, x \cos 2x]$

$\therefore$ Solution is $\quad y = c_1 \cos x + c_2 \sin x - (1/3)\,[(x^2 - 26/9)\sin 2x + (8/3)\, x\cos 2x]$.

Ex. 5. *Solve* $(D^4 + 2D^2 + 1)\, y = x^2 \cos x$. **[Purvanchal 2007; Agra 1995, Meeruth 1997, Kumaun 1995, Kanpur 1998, Lucknow 1998, Delhi Maths (H) 2004]**

Sol. Here the auxiliary equation is $D^4 + 2D^2 + 1 = 0$ or $(D^2 + 1)^2 = 0$ so that $D = \pm i, \pm i$.

$\therefore$ C.F. $= (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$, c_1, c_2, c_3, c_4 being arbitrary constants.

$$\text{P.I.} = \frac{1}{(D^2+1)^2}x^2\cos x = \text{R.P. of } \frac{1}{(D^2+1)^2}x^2 e^{ix}, \text{ where R.P. stands for real part}$$

$$= \text{R.P. of } e^{ix}\frac{1}{\{(D+i)^2+1\}^2}x^2 = \text{R.P. of } e^{ix}\frac{1}{(D^2+2Di)^2}x^2$$

$$= \text{R.P. of } e^{ix}\frac{1}{(2Di)^2(1+D/2i)^2}x^2 = \text{R.P. of } \frac{e^{ix}}{-4D^2}\left(1+\frac{D}{2i}\right)^{-2}x^2$$

$$= \text{R.P. of } \frac{e^{ix}}{-4D^2}\left(1-\frac{iD}{2}\right)^{-2}x^2 = \text{R.P. of } \frac{e^{ix}}{-4D^2}\left(1+iD+\frac{3i^2D^2}{4}+\dots\right)x^2$$

$$= \text{R.P. of } \frac{e^{ix}}{-4D^2}\left(1+iD-\frac{3D^2}{4}\right)x^2 = \text{R.P. of } \frac{e^{ix}}{-4D^2}\left[x^2+2ix-\frac{3}{4}\cdot 2\right]$$

$$= \text{R.P. of } \frac{e^{ix}}{-4}\frac{1}{D}\left[\frac{x^3}{3}+ix^2-\frac{3}{2}x\right] = \text{R.P. of } \frac{e^{ix}}{-4}\left[\frac{x^4}{12}+\frac{ix^3}{3}-\frac{3x^2}{4}\right]$$

$= \text{R.P. of } (-1/4)\times(\cos x + i\sin x)\,\{(x^4/12) - (3x^2/4) + i\,(x^3/3)\}$

$= -(1/4)\times[(x^4/12 - 3x^2/4)\cos x - (x^3/3)\sin x]$

$\therefore$ Solution is $\; y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x - (1/4)\times[(x^4/12 - 3x^2/4)\cos x - (x^3/3)\times \sin x]$.

Ex. 6. *Solve* $(D^4 - 1)\, y = x \sin x$. **[GATE 2002; Meerut 2000]**

Sol. Here the auxiliary equation is $\quad D^4 - 1 = \Rightarrow D = 1, -1 \pm i$.

$\therefore$ C.F. $= c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$, c's being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^4-1}x\sin x = \text{I.P. of } \frac{1}{D^4-1}xe^{ix}, \text{ where I.P. stand for imaginary part}$$

$$= \text{I.P. of } e^{ix}\frac{1}{(D+i)^4-1}x = \text{I.P. of } e^{ix}\frac{1}{D^4+4D^3i+6D^2i^2+4Di^3+i^4-1}x$$

$$= \text{I.P of } e^{ix}\frac{1}{D^4+4iD^3-6D^2-4iD}x = \text{I.P. of } \frac{e^{ix}}{-4iD}\frac{1}{[1+3D/2i-D^2-D^3/4i]}x$$

$$= \text{I.P. of } \frac{ie^{ix}}{4D}\left[1+\left(-\frac{3Di}{2}-D^2+\frac{iD^3}{4}\right)\right]^{-1}x$$

$$= \text{I.P. of } \frac{ie^{ix}}{4D}\left[1+\left(\frac{3Di}{2}+D^2-\frac{iD^3}{4}\right)+\dots\right]x = \text{I.P. of } \frac{ie^{ix}}{4}\frac{1}{D}\left(x+\frac{3i}{2}\right)$$

$= \text{I.P. of } (1/4)\times ie^{ix}\,\{(x^2/2) + (3/2)\, ix\} = \text{I.P. of } (1/8)\times(\cos x + i\sin x)\,(ix^2 - 3x)$

Thus, $\quad$ P.I. $= (1/8)\times(x^2\cos x - 3x\sin x)$

$\therefore$ Solution is $\quad y = c_1 e^x + c_2 e^{-x} + c_3\cos x + c_4 \sin x + (1/8)\times(x^2\cos x - 3x\sin x)$.

Ex. 7. *Solve* $(D^2 + 1)^2 = 24x\cos x$ *given that* $y = Dy = D^2y = 0$ *and* $D^3y = 12$ *when* $x = 0$. **[I.A.S. 2001]**

Sol. Given $\quad (D^2+1)^2\, y = 24\, x\cos x \quad$ ... (1)

with $\quad y = 0, \quad y' = 0, \quad y'' = 0 \quad$ and $\quad y''' = 12 \quad$ when $\quad x = 0$.

The auxiliary equation is $\quad (D^2+1)^2 = 0 \quad$ giving $\quad D = \pm i$ (twice) $\quad$ ... (2)

$\therefore$ C.F. $= (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$, c_1, c_2, c_3, c_4 being arbitrary constants ... (3)

$$\text{P.I.} = \frac{1}{(D^2+1)^2}24\,x\cos x = \text{Real part of } \frac{1}{(D^2+1)^2}24xe^{ix} \qquad \dots (4)$$

$$\text{Now, } \frac{1}{(D^2+1)^2}24xe^{ix} = 24e^{ix}\frac{1}{\{(D+i)^2+1\}^2}x = 24e^{ix}\frac{1}{(D^2+2Di)^2}x$$

$$= 24e^{ix}\frac{1}{(2Di)^2}\left(1+\frac{D}{2i}\right)^{-2}x = \frac{24e^{ix}}{(-4D^2)}\left(1-\frac{D}{i}+\dots\dots\right)x$$

$$= -6e^{ix}\frac{1}{D^2}(x+i) = -6e^{ix}\left(\frac{x^3}{6}+\frac{ix^2}{2}\right) \qquad \dots (5)$$

Using (5), from (4) we, have

P.I. = Real part of $(\cos x + \sin x)\,(-x^3 + 3i\, x^2) = 3x^2\sin x - x^3\cos x$

Hence the general solution of (1) is $\quad y =$ C.F. + P.I. $\quad$ *i.e.*

$$y = c_1\cos x + c_3\sin x + x\,(c_2\cos x + c_4\sin x) + 3x^2\sin x - x^3\cos x \qquad \dots (6)$$

Putting $x = 0$, in (6) and noting that $y = 0$ when $x = 0$, we get $c_1 = 0$. Putting $c_1 = 0$ in (6),

$$y = c_3\sin x + x\,(c_2\cos x + c_4\sin x) + 3x^2\sin x - x^3\cos x \qquad \dots (7)$$

Differentiating (7) w.r.t. 'x', we get $\quad y' = c_3\cos x + c_2\cos x + c_4\sin x$

$$+ x\,(-c_2\sin x + c_4\cos x) + 6x\sin x + 3x^2\cos x - (3x^2\cos x - x^3\sin x) \qquad \dots (8)$$

Putting $x = 0$ in (8) and noting that $y' = 0$ when $x = 0$, we get

$$0 = c_3 + c_2 \qquad \text{so that} \qquad c_3 = -c_2 \qquad \dots (9)$$

$\therefore$ (8) gives $\quad y' = c_4\sin x + x\,(-c_2\sin x + c_4\cos x) + 6\,x\sin x + x^3\sin x \quad$ ... (10)

Differentiating (10) w.r.t. 'x', we get $\quad y'' = c_4\cos x - c_2\sin x + c_4\cos x$

$$+ x\,(-c_2\cos x - c_4\sin x) + 6\sin x + 6x\cos x + 3x^2\sin x + x^3\cos x \qquad \dots (11)$$

Putting $x = 0$ in (11) and noting that $y'' = 0$ when $x = 0$ (11) gives $c_4 = 0$. Putting $c_4 = 0$ in (11),

$$y'' = -c_2\sin x - c_2\, x\cos x + 6\sin x + 6x\cos x + 3x^2\sin x + x^3\cos x \qquad \dots (12)$$

Differentiating (12) w.r.t. 'x', we get $\quad y''' = -c_2\cos x - c_2(\cos x - x\sin x) + 6\cos x$

$$+ 6\,(\cos x - x\sin x) + 3\,(2x\sin x + x^2\cos x) + 3x^2\cos x - x^3\sin x \qquad \dots (13)$$

Putting $x = 0$ in (13) and noting that $y''' = 12$ when $x = 0$, (13) reduces to

$12 = -2\, c_2 + 12 \quad$ so that $\quad c_2 = 0. \quad$ So by (9), $\quad c_3 = 0$.

Thus, finally, $c_1 = c_2 = c_3 = c_4 = 0$ and so (6) reduces to

$y = 3x^2\sin x - x^3\cos x$. which is the required solution.

Exercise 5(G)

Solve the following differential equations :

1. (*a*) $(D^2 + 4)\, y = x\sin x$ **[Delhi Maths (G) 1994]**

Ans. $y = c_1\cos 2x + c_2\sin 2x + (x/3)\cdot\sin x - (2/9)\cos x$

(b) $(D^2 + m^2)y = x \cos mx$ **Ans.** $y = c_1 \cos mx + c_2 \sin mx + (x/4m^2) \cos mx + (x^2/4m) \sin mx$ **[Delhi Maths (G) 1996]**

2. $(D^2 - 5D + 6)y = x \sin 3x$ **Ans.** $y = c_1 e^{2x} + c_2 e^{3x} - \{(78x + 40) \sin 3x - (123 - 390x) \cos 3x\}/6084$

3. $(D^2 + D)y = x \cos x$ **Ans.** $y = c_1 + c_2 e^{-x} + (1/2) \times (\sin x - \cos x) + (1/2) \times x \sin x$

4. $(D^2 - 1)y = x^2 \cos x$ [Delhi:Maths(H) 2007] **Ans.** $y = c_1 e^x + c_2 e^{-x} + (1/2) \times (1 - x^2) \cos x + x \sin x$

5. $(D^2 - 1)y = x^2 \sin x$ **Ans.** $y = c_1 e^x + c_2 e^{-x} - x \cos x - (1/2) \times (x^2 - 1) \sin x$

6. $(D^2 + 4)y = x \sin^2 x$ **Ans.** $y = c_1 \cos 2x + c_2 \sin 2x + (x/8) \times (1 + 2 \sin 2x)$

5.24 More about particular integral of $f(D)\, y = X$

We now consider examples in which X is sum of two or more special functions of x considered sperarately. In such cases we obtain P.I. corresponding each function separately and then add these to get the required P.I. of a differential equation.

5.25 Solved Examples based on Art. 5.24 and Miscellaneous examples

Ex. 1. *Solve* $(D^2 - 4D + 4)\,y = x^2 + e^x + \sin 2x$. **[Delhi B.A. (Prog) II 2010]**

Sol. Given $(D^2 - 4D + 4)\,y = x^2 + e^x + \sin 2x$. ... (1)

Here the auxiliary equation is $D^2 - 4D + 4 = 0$ so that $D = 2, 2$.

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{2x}$, c_1, c_2 being arbitrary constants.

P.I. corresponding to x^2

$$= \frac{1}{D^2 - 4D + 4} x^2 = \frac{1}{(2-D)^2} x^2 = \frac{1}{4[1 - (D/2)]^2} x^2 = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2} x^2$$

$$= \frac{1}{4}\left[1 + D + \frac{(-2)(-3)}{1 \cdot 2} \cdot \frac{D^2}{4} + \ldots\right] x^2 = \frac{1}{4}\left(1 + D + \frac{3}{4} D^2 + \ldots\right) x^2 = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

P.I. corresponding to $e^x = \dfrac{1}{D^2 - 4D + 4} e^x = \dfrac{1}{1 - 4 + 4} e^x = e^x$

and P.I. Corresponding to $\sin 2x$

$$= \frac{1}{D^2 - 4D + 4} \sin 2x = \frac{1}{-2^2 - 4D + 4} \sin 2x = -\frac{1}{4} \cdot \frac{1}{D} \sin 2x = \frac{1}{8} \cos 2x$$

$\therefore$ Required solution is $y = (c_1 + c_2 x)\, e^{2x} + (1/8) \times (2x^2 + 4x + 3) + (1/8) \times \cos 2x$.

Ex. 2. *Solve* $(D^2 - 1)\, y = x\, e^x + \cos^2 x$. **[Delhi Maths 2006, I.A.S. 1992]**

Sol. Given: $(D^2 - 1)\, y = x\, e^x + (1/2) \times (1 + \cos 2x)$. ... (1)

The auxiliary equation is $D^2 - 1 = 0$ so that $D = \pm 1$.

So C.F. $= c_1 e^x + c_2 e^{-x}$, c_1, c_2 being arbitrary constants.

P.I. corresponding to xe^x

$$= \frac{1}{D^2 - 1} x e^x = e^x \frac{1}{(D+1)^2 - 1} x = e^x \frac{1}{D^2 + 2D} x = \frac{e^x}{2} \frac{1}{D(1 + D/2)} x = \frac{e^x}{2} \cdot \frac{1}{D}\left(1 + \frac{D}{2}\right)^{-1} x$$

$$= \frac{e^x}{2} \cdot \frac{1}{D}\left(1 - \frac{D}{2} + \ldots\right) x = \frac{e^x}{2} \cdot \frac{1}{D}\left(x - \frac{1}{2}\right) = \frac{1}{2} e^x \left(\frac{x^2}{2} - \frac{x}{2}\right) = \frac{1}{4} e^x \,(x^2 - x)$$

P.I. corresponding to $\dfrac{1}{2} = \dfrac{1}{2} \dfrac{1}{D^2 - 1} e^{0 \cdot x} = \dfrac{1}{2} \dfrac{1}{0^2 - 1} e^{0 \cdot x} = -\dfrac{1}{2}$.

P.I. corresponding to $\dfrac{1}{2} \cos 2x = \dfrac{1}{2} \dfrac{1}{D^2 - 1} \cos 2x = \dfrac{1}{2} \dfrac{1}{-2^2 - 1} \cos 2x = -\dfrac{1}{10} \cos 2x$.

$\therefore$ The required solution $y = c_1 e^x + c_2 e^{-x} + (1/4) \times e^x (x^2 - x) - (1/2) - (1/10) \times \cos 2x$.

Ex. 3. *Solve* $(D^4 - 4D^2 - 5)\, y = e^x\, (x + \cos x)$ **[I.A.S. 2004]**

Sol. Here auxiliary equation is $D^4 - 4D^2 - 5 = 0$ or $(D^2 - 5)(D^2 + 1) = 0$

giving $D^2 = 5$ or -1 so that $D = \pm\sqrt{5}, \pm i$

$\therefore$ C.F. $= c_1 \cosh x\sqrt{5} + c_2 \sinh x\sqrt{5} + c_3 \cos x + c_4 \sin x$

P.I. corresponding to $x\, e^x$

$$= \frac{1}{D^4 - 4D^2 - 5} x e^x = e^x \frac{1}{(D+1)^4 - 4(D+1)^2 - 5} x = e^x \frac{1}{D^4 + 4D^3 + 2D^2 - 4D - 8} x$$

$$= -\frac{e^x}{8} \frac{1}{1 + D/2 - D^2/4 - D^3/2 - D^4/8} x = -\frac{e^x}{8}\left(1 + \frac{D}{2} - \frac{D^2}{4} - \frac{D^3}{2} - \frac{D^4}{8}\right)^{-1} x$$

$= -(e^x/8) \times \{1 - (D/2 - D^2/4 - D^3/2 - D^4/8) + \ldots\}\, x = -(e^x/8) \times (x - 1/2) = -(e^x/16) \times (2x - 1)$

P.I. corresponding to $e^x \cos x = \dfrac{1}{D^4 - 4D^2 - 5} e^x \cos x = e^x \dfrac{1}{(D+1)^4 - 4(D+1)^2 - 5} \cos x$

$$= e^x \frac{1}{D^4 + 4D^3 + 2D^2 - 4D - 8} \cos x = e^x \frac{1}{(D^2)^2 + 4D^2 D + 2D^2 - 4D - 8} \cos x$$

$$= e^x \frac{1}{(-1^2)^2 + 4(-1^2)\, D + 2(-1^2) - 4D - 8} \cos x$$

$$= -e^x \frac{1}{9 + 8D} \cos x = -e^x \frac{9 - 8D}{(9 + 8D)(9 - 8D)} \cos x$$

$$= -e^x \frac{9 - 8D}{81 - 64D^2} \cos x = -e^x \frac{9 - 8D}{81 - 64 \times (-1^2)} \cos x = -\frac{1}{145} e^x\, (9 \cos x + 8 \sin x)$$

$\therefore$ Required solution is $y = c_1 \cosh x\sqrt{5} + c_2 \sinh x\sqrt{5} + c_3 \cos x + c_4 \sin x - (e^x/16) \times (2x - 1) - (1/145)\, e^x\, (9 \cos x + 8 \sin x)$, c_1, c_2, c_3, c_4 being arbitrary constants.

Ex. 4. *Solve* $(d^3y/dx^3) - 3\,(d^2y/dx^2) + 4\,(dy/dx) - 2y = e^x + \cos x$. **[I.A.S. 1999]**

Sol. Given $(D^3 - 3D^2 + 4D - 2)\, y = e^x + \cos x$, where $D \equiv d/dx$ ... (1)

Its auxiliary equation is $D^3 - 3D^2 + 4D - 2 = 0$ or $D^2(D - 1) - 2D\,(D - 1) + 2(D - 1) = 0$

or $(D - 1)(D^2 - 2D + 2) = 0$ giving $D = 1$, $(2 \pm \sqrt{4 - 8})/2$, *i.e.*, $D = 1, 1 \pm i$.

$\therefore$ C.F. $= c_1\, e^x + e^x\, (c_2 \cos x + c_3 \sin x)$, c_1, c_2, c_3 being arbitrary constants

P.I. corresponding to $e^x = \dfrac{1}{D^3 - 3D^2 + 4D - 2} e^x = \dfrac{1}{(D-1)(D^2 - 2D + 2)} e^x$

$$= \frac{1}{D-1} \frac{1}{1 - 2 + 2} e^x = \frac{1}{D-1} e^x \cdot 1 = e^x \frac{1}{(D+1) - 1} \cdot 1 = e^x \frac{1}{D} \cdot 1 = xe^x$$

P.I. corresponding to $\cos x = \dfrac{1}{D^3 - 3D^2 + 4D - 2} \cos x = \dfrac{1}{D^2 \cdot D - 3D^2 + 4D - 1} \cos x$

$$= \frac{1}{(-1^2)D - 3(-1^2) + 4D - 2}\cos x = \frac{1}{3D+1}\cos x = (3D-1)\frac{1}{9D^2-1}\cos x$$

$$= (3D-1)\frac{1}{9(-1^2)-1}\cos x = -\frac{1}{10}(3D\cos x - \cos x) = -\frac{1}{10}(-3\sin x - \cos x)$$

$\therefore$ Required solution is $\quad y = e^x(c_1 + c_2 \cos x + c_3 \sin x) + x e^x + (3 \sin x + \cos x)/10$

Ex. 5. *Solve* $(D^2 + a^2)\, y = \sin ax + x e^{2x}$ **[Delhi Maths (G) 1999]**

Sol. Auxiliary equation $\quad D^2 + a^2 = 0 \quad$ gives $\quad D^2 = -a^2 \quad$ or $\quad D = \pm ia.$

$\therefore$ C.F. $= c_1 \cos ax + c_2 \sin ax$, c_1, c_2 being arbitrary constants

P.I. corresponding to $\sin ax = \dfrac{1}{D^2 + a^2}\sin ax = -\dfrac{x}{2a}\cos ax$

P.I. corresponding to $e^{2x} x$

$$= \frac{1}{D^2+a^2}e^{2x}x = e^{2x}\frac{1}{(D+2a)^2+a^2}x = e^{2x}\frac{1}{D^2+4Da+4+a^2}x$$

$$= \frac{e^{2x}}{4+a^2}\left[1+\frac{D^2+4D}{4+a^2}\right]^{-1}x = \frac{e^{2x}}{4+a^2}\left[1-\frac{D^2+4D}{4+a^2}+\ldots\right]x$$

$$= \frac{e^{2x}}{4+a^2}\left(x - \frac{4}{4+a^2}\right) = \frac{e^{2x}}{(4+a^2)^2}\{x(4+a^2)-4\}$$

$\therefore$ Solution is $\quad y = c_1 \cos x + c_2 \sin ax - (x/2a)\cos ax + e^{2x}(4 + a^2)^{-2}(4x + x a^2 - 4)$

Ex. 6. *Solve* $(D^2 - 6D + 8)\, y = (e^{2x} - 1)^2 + \sin 3x.$ **[Delhi Maths (G) 2000]**

Sol. The auxiliary equation is $\quad D^2 - 6D + 8 = 0 \quad$ giving $\quad D = 4, 2.$

$\therefore$ C.F. $= c_1 e^{2x} + c_2 e^{4x}$, $\quad c_1, c_2$ being arbitrary constants

P.I. corresponding to $(e^{2x} - 1)^2 = \dfrac{1}{(D-4)(D-2)}(e^{4x} - 2e^{2x} + 1)$

$$= \frac{1}{(D-4)(D-2)}e^{4x} - 2\frac{1}{(D-2)(D-4)}e^{2x} + \frac{1}{(D-4)(D-2)}e^{0.x}$$

$$= \frac{1}{D-4}\frac{1}{(4-2)}e^{4x} - 2\frac{1}{(D-2)(2-4)}e^{2x} + \frac{1}{(0-4)(0-2)}e^{0.x} = \frac{1}{2}x e^{2x} + x e^{2x} + \frac{1}{8}$$

P.I. corresponding to $\sin 3x = \dfrac{1}{D^2-6D+8}\sin 3x = \dfrac{1}{-3^2-6D+8}\sin 3x$

$$= -\frac{1}{6D+1}\sin 3x = -\frac{6D-1}{36D^2-1}\sin 3x = -\frac{(6D-1)\sin 3x}{36(-3^2)-1} = \frac{1}{325}(18\cos 3x - \sin 3x)$$

$\therefore$ Solution is $\quad y = c_1 e^{2x} + c_2 e^{4x} + (x/2)\, e^{4x} + x e^{2x} + 1/8 + (1/325)(18 \cos 3x - \sin 3x)$

Ex. 7(a). *Solve* $(D^4 + 4)\, y = e^x + x^2$, where $D \equiv d/dx$. **[Delhi Maths (Prog) 2007]**

Sol. Auxiliary equation of the given equation is $\quad D^4 + 4 = 0$

or $\quad (D^2 + 2)^2 - (2D)^2 = 0 \quad$ or $\quad (D^2 + 2D + 2)(D^2 - 2D + 2) = 0 \;\Rightarrow\; D = 1 \pm i, -1 \pm i$

$\therefore$ C.F. $= e^x(c_1 \cos x + c_2 \sin x) + e^{-x}(c_3 \cos x + c_4 \sin x)$, c_1, c_2, c_3, c_4 being arbitrary constants

P.I. corresponding to $\quad e^x = \dfrac{1}{D^4+4}e^x = \dfrac{1}{1^4+4}e^x = \dfrac{1}{5}e^x$

and P.I. corresponding to $x^2 = \dfrac{1}{D^4+4}x^2 = \dfrac{1}{4(1+D^4/4)}x^2 = \dfrac{1}{4}\left(1+\dfrac{D^4}{4}\right)^{-1}x^2$

$$= (1/4) \times (1 - D^4/4 + \ldots)\, x^2 = x^2/4$$

$\therefore$ Required solution is $\quad y = e^x(c_1 \cos x + c_2 \sin x) + e^{-x}(c_3 \cos x + c_4 \sin x) + (1/5) \times e^x + x^2/4$

Ex. 7(b). *Solve* $(D^2 + 2)\, y = x^2 e^{3x} + e^x \cos 2x$ **[B.Sc. (Prog) II 2012]**

Sol. Here auxiliary equation is $\quad D^2 + 2 = 0 \quad$ giving $\quad D = \pm i\sqrt{2}$

$\therefore$ C.F. $= c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2})$, c_1, c_2 being arbitrary constants

P.I. corresponding to $x^2 e^{3x}$

$$= \frac{1}{D^2+2}x^2 e^{3x} = e^{3x}\frac{1}{(D+3)^2+2}x^2 = e^{3x}\frac{1}{D^2+6D+11}x^2 = e^{3x}\frac{1}{11(1+6D/11+D^2/11)}x^2$$

$$= \frac{e^{3x}}{11}\left\{1+\left(\frac{(6D+D^2)}{11}\right)\right\}^{-1}x^2 = \frac{e^{3x}}{11}\left\{1-\frac{6D+D^2}{11}+\left(\frac{6D+D^2}{11}\right)^2-\ldots\ldots\right\}x^2$$

$$= \frac{e^{3x}}{11}\left(1-\frac{6D}{11}-\frac{D^2}{11}+\frac{36D^2}{121}+\ldots\right)x^2 = \frac{e^{3x}}{11}\left(1-\frac{6D}{11}+\frac{25D^2}{121}+\ldots\right)x^2 = \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right)$$

$$= e^{3x}(121x^2 - 132x + 50)/(11)^3$$

P.I. corresponding to $e^x \cos 2x$

$$= \frac{1}{D^2+2}e^x \cos 2x = e^x\frac{1}{(D+1)^2+2}\cos 2x = e^x\frac{1}{D^2+2D+3}\cos 2x$$

$$= e^x\frac{1}{-2^2+2D+3}\cos 2x = e^x\frac{1}{2D-1}\cos 2x = e^x\frac{2D+1}{4D^2-1}\cos 2x$$

$$= e^x\frac{2D+1}{4\times(-2^2)-1}\cos 2x = -\frac{e^x}{17}(2D+1)\cos 2x = -\frac{e^x}{17}(-4\sin 2x + \cos 2x)$$

$\therefore$ The required solution is $\quad y = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2})$

$$+ \{1/(11)^3\} \times e^{3x}(121x^2 - 132x + 50) - (1/17) \times e^x(\cos 2x - 4 \sin 2x).$$

Ex. 8. *Solve* $(D^4 + D^2 + 1)\, y = ax^2 + be^{-x} \sin 2x.$ **[Ravishankar 1994]**

Sol. Given $\quad (D^4 + D^2 + 1)\, y = ax^2 + be^{-x} \sin 2x. \quad \ldots (1)$

A.E. is $\quad D^4 + D^2 + 1 = 0 \quad$ or $\quad (D^2 + 1)^2 - D^2 = 0 \quad$ or $\quad (D^2 + D + 1)(D^2 - D + 1) = 0$

so that $\quad D^2 + D + 1 = 0 \quad$ or $\quad D^2 - D + 1 = 0, \quad$ giving $\quad D = (-1 \pm i\sqrt{3})/2,\ (1 \pm i\sqrt{3}/2).$

$\therefore$ C.F. $= e^{-x/2}[c_1 \cos(x\sqrt{3}/2) + c_2 \sin(x\sqrt{3}/2)] + e^{x/2}[c_3 \cos(x\sqrt{3}/2) + c_4 \sin(x\sqrt{3}/2)]$,

c_1, c_2, c_3 and c_4 being arbitrary constants

Now, P.I. corresponding to ax^2

$$= a\frac{1}{D^4+D^2+1}x^2 = a[1+(D^4+D^2)]^{-1}x^2 = a[1 - D^2 + \ldots]\, x^2 = a(x^2 - 2).$$

Next, P.I. corresponding to $be^{-x} \sin 2x$

$$= b\frac{1}{D^4+D^2+1}e^{-x}\sin 2x = be^{-x}\frac{1}{(D-1)^4+(D-1)^2+1}\sin 2x$$

$$= be^{-x}\frac{1}{D^4-4D^3+6D^2-4D+1+D^2-2D+1+1}\sin 2x$$

$$= be^{-x}\frac{1}{(D^2)^2-4D(D^2)+7D^2-6D+3}\sin 2x$$

$$= be^{-x}\frac{1}{(-2^2)^2-4D(-2^2)+7(-2^2)-6D+3}\sin 2x$$

$$= be^{-x}\frac{1}{10D-9}\sin 2x = be^{-x}(10D+9)\cdot\frac{1}{100D^2-81}\sin 2x$$

$$= be^{-x}(10D+9)\frac{1}{(100)\times(-2^2)-81}\sin 2x$$

$$= -be^{-x}(10D+9)\frac{1}{481}\sin 2x = -\frac{be^{-x}}{481}(20\cos 2x+9\sin 2x).$$

$\therefore$ Required solution is $y = e^{x/2}\,[c_1 \cos(x\sqrt{3}/2) + c_2 \sin(x\sqrt{3}/2)] + e^{x/2}\,[c_3 \cos(x\sqrt{3}/2)$
$+ c_4 \sin(x\sqrt{3}/2)] + a\,(x_2 - 2) - (b/481)\,e^{x}\,(20 \cos 2x + 9 \sin 2x).$

Ex. 9. *Solve* $(D-1)^2 (D^2+1)^2 y = \sin^2 (x/2) + e^x + x.$

Sol. Re-writing, $(D-1)^2 (D^2+1)^2 y = (1-\cos x)/2 + e^x + x.$

Here auxiliary equation $(D-1)^2 (D^2+1)^2 = 0$ gives $D = 1, 1, \pm i$ (twice).

So C.F. $= (c_1 + c_2 x)\, e^x + (c_3 + c_4 x) \cos x + (c_5 + c_6 x) \sin x,$

where c_1, c_2, c_3, c_4, c_5 and c_6 are arbitrary constants.

$\therefore$ P.I. corresponding to $\dfrac{1}{2} = \dfrac{1}{2}\dfrac{1}{(D-1)^2(D^2+1)^2}e^{0x} = \dfrac{1}{2}\dfrac{1}{(0-1)^2(0+1)^2}e^{0x} = \dfrac{1}{2},$

P.I. corresponding to $(-1/2)\times\cos x$

$$= -\frac{1}{2}\frac{1}{(D^2+1)^2(D^2-2D+1)}\cos x = -\frac{1}{2}\frac{1}{(D^2+1)^2(-1^2-2D+1)}\cos x$$

$$= \frac{1}{4}\frac{1}{(D^2+1)^2}\frac{1}{D}\cos x = \frac{1}{4}\frac{1}{(D^2+1)^2}\sin x = \text{Imaginary part of } \frac{1}{4}\frac{1}{(D^2+1)^2}e^{ix} \quad \text{... (1)}$$

Now, $\dfrac{1}{(D^2+1)^2}e^{ix}\cdot 1 = e^{ix}\dfrac{1}{[(D+i)^2+1]^2}\cdot 1 = e^{ix}\dfrac{1}{(D^2+2iD)^2}\cdot 1 = e^{ix}\dfrac{1}{(2iD)^2(1+D/2i)^2}e^{0x}$

$$= -\frac{1}{4}e^{ix}\frac{1}{D^2}\frac{1}{(1+0)^2}e^{0x} = -\frac{1}{4}e^{ix}\frac{1}{D^2}\cdot 1 = -\frac{1}{4}e^{ix}\left(\frac{x^2}{2}\right) = -\frac{1}{8}x^2(\cos x + i\sin x). \quad \text{... (2)}$$

Using (1) and (2), P.I. corresponding to $(-1/2)\times\cos x = -(x^2/32)\times\sin x$

Again, P.I. corresponding to e^x

$$= \frac{1}{(D-1)^2(D^2+1)^2}e^x = \frac{1}{(D-1)^2(1+1)^2}e^x = \frac{1}{4}\frac{1}{(D-1)^2}e^x = \frac{1}{4}\frac{x^2}{2!}e^x = \left(\frac{x^2}{8}\right)e^x.$$

Finally, P.I. corresponding to x.

$$= \frac{1}{(D-1)^2(D^2+1)^2}x = (1-D)^{-2}(1+D^2)^{-2}x = (1+2D+\ldots)(1+\ldots)x = (1+2D+\ldots)x = x+2.$$

Required solution is $y = (c_1 + c_2 x)\, e^x + (c_3 + c_4 x) \cos x + (c_5 + c_6 x) \sin x$
$+ (1/2) - (x^2/32) \sin x + (x^2/8)\, e^x + x + 2.$

Ex. 10. *solve* $(D^5 - D)\, y = 12\, e^x + 8 \sin x - 2x.$

Sol. A.E. $D\,(D^4-1) = 0 \Rightarrow D\,(D^2-1)(D^2+1) = 0 \Rightarrow D = 0, 1, -1, \pm i.$

$\therefore$ C.F. $= c_1 + c_2 e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x$, $c_1, \ldots, c_5$ being arbitrary constants

Now P.I. corresponding to $12e^x$

$$= 12\frac{1}{(D-1)D(D+1)(D^2+1)}e^x = 12\frac{1}{(D-1)1\cdot(1+1)(1+1)}e^x = 3\frac{1}{D-1}e^x = 3\cdot\frac{x}{1!}e^x = 3xe^x.$$

P.I. corresponding to $8 \sin x$ is

$$= 8\frac{1}{(D^2+1)D(D^2-1)}\sin x = 8\frac{1}{(D^2+1)D(-1^2-1)}\sin x = -4\frac{1}{(D^2+1)}\left[\frac{1}{D}\sin x\right]$$

$$= 4\frac{1}{D^2+1}\cos x = 4\left(\frac{x}{2\times 1}\sin x\right) = 2x\sin x \qquad \left[\because \frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin x\right]$$

P.I. corresponding to $(-2x)$ is

$$= -2\frac{1}{D(D^2-1)(D^2+1)}x = 2\frac{1}{D(1-D^2)(1+D^2)}x = 2\frac{1}{D}(1-D^2)^{-1}(1+D^2)^{-1}x$$

$$= 2\frac{1}{D}(1+D^2+\ldots)(1-D^2+\ldots)x = 2\frac{1}{D}(1+D^2-D^2+\ldots)x = 2\frac{1}{D}x = 2\left(\frac{x^2}{2}\right) = x^2.$$

$\therefore$ Solution is $y = c_1 + c_2 e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x + 3\,xe^x + 2x \sin x + x^2.$

Exercise 5(H)

1. $d^3y/dx^3 + d^2y/dx^2 + dy/dx = e^{2x} + x^2 + x.$ **[Meerut 2009; Lucknow 1994, 98]**
   Ans. $y = c_1 + (c_2 + c_3 x)\, e^{-x} + (1/18)\times e^{2x} + (1/3)\times x^3 - (3/2)\times x^2 + 4x$
2. $(D^2 - 4D + 4)\, y = \sin 2x + x^2.$ **[G.N.D.U. Amritsar 2011]**
   Ans. $y = (c_1 + c_2 x)\, e^{2x} + (3 \sin 2x + 8 \cos 2x)/25 + (2x^2 + 4x + 3)/8$
3. $(D^2 + 4)\, y = e^x + \sin 2x.$ **[Allahabad 1994 ; Agra 2005 ; Rohilkhand 1994]**
   Ans. $y = c_1 \cos 2x + c_2 \sin 2x + (1/5)\times e^x - (1/4)\times x \cos 2x$
4. $(D^4 + 2D^3 - 3D^2)\, y = 3e^{2x} + 4 \sin x.$ **[Kanpur 1993]**
   Ans. $y = c_1 + c_2 x + c_3 e^x + c_4 e^{-3x} + (3/20)\times e^{2x} + (2/5)\times(2 \sin x + \cos x)$
5. $(D^2 + D - 2)\, y = x + \sin x.$ **[Guwahati 1998; Meerut 1998 ; Delhi 2007, 09 ; Utkal 2003]**
   Ans. $y = c_1 e^x + c_2 e^{-2x} - (x/2) - (1/4) - (1/10)\times(\cos x + 3 \sin x)$
6. (i) $D^3 - 3D^2 + 3D - 1)\, y = x\,e^{-x} + e^x$. **Ans.** $y = (c_1 + c_2 x + c_3 x^2)\, e^x - (1/16)\times(2x+3)\, e^{-x} + (x^3/6)\, e^x$
   (ii) $(D^3 - 3D^2 + 3D - 1)\, y = x\,e^x + e^x.$ **Ans.** $y = e^x\,[c_1 + c_2 x + c_3 x^2 + (1/6)\times x^3 + (1/24)\times x^4]$
7. $(D^2 + 5D + 6)\, y = e^{-2x} + 5 \sin 4x.$ **Ans.** $y = c_1 e^{-3x} + c_2 e^{-2x} + xe^{-2x} - (1/10)\times(\sin 4x + 2 \cos 4x)$
8. $(D^2 + 1)\, y = e^{-x} + \cos x.$ **Ans.** $y = c_1 \cos x + c_2 \sin x + (1/2)\times e^{-x} + (1/2)\times x \sin x$
9. $(D^2 + 4)\, y = \sin^2 x.$ **Ans.** $y = c_1 \cos 2x + c_2 \sin 2x + (1/8)\times(9 - x \sin 2x)$
10. $(D^2 + 1)\, y = e^{-x} + \cos x + x^3 + e^x \sin x.$
    Ans. $y = c_1 \cos x + c_2 \sin x + (1/2)\times e^{-x} + (1/2)\times x \sin x + x^3 - 6x - (1/5)\times e^x\ (2 \cos x - \sin x)$
11. $(D^2 + 1)\, y = \cos^2 (x/2).$ **Ans.** $y = c_1 \cos x + c_2 \sin x + (1/2) + (1/4)\times x \sin x$
12. $(D^2 + 4)\, y = x^2 + 3 \sin x$ **Ans.** $y = c_1 \cos 2x + c_2 \sin 2x + (2x - 1)/8 + \sin x.$
13. $(D^2 + 4)\, y = \sin 2x + x^2.$ **Ans.** $y = c_1 \cos 2x + c_2 \sin 2x - (1/4)\times x \cos 2x + (1/8)\times(2x^2 - 1)$
14. $(2D^2 - D - 6)\, y = e^{-(3x/2)} + \sin x$ **[Pune 2010]**
    Ans. $y = c_1 e^{2x} + c_2 e^{-(3x/2)} + (\cos x - 8 \sin x)/65 - (x/7)\times e^{-(3x/2)}$