Example 5. *The voltage V and the current i at a distance x from the sending end of the transmission line satisfy the equations*

$$-\frac{dV}{dx} = Ri \text{ and } -\frac{di}{dx} = GV, \quad \text{where R and G are constants.}$$

If $V = V_0$ *at the sending end* $(x = 0)$ *and* $V = 0$ *at the receiving end* $(x = l)$ *; show that*

$$V = V_0 \left[\frac{\sinh n(l - x)}{\sinh nl}\right], \text{ where } n^2 = RG.$$

Sol.
$$-\frac{di}{dx} = \text{GV}$$

$$\Rightarrow \quad -\frac{d}{dx}\left(-\frac{1}{\text{R}}\frac{d\text{V}}{dx}\right) = \text{GV}$$

$$\Rightarrow \quad \frac{d^2\text{V}}{dx^2} - \text{RGV} = 0$$

Auxiliary equation is

$$m^2 - \text{RG} = 0$$

$$\Rightarrow \quad m^2 - n^2 = 0 \qquad (\because \ \text{RG} = n^2)$$

$$\Rightarrow \quad m = \pm n$$

$$\therefore \quad \text{C.F.} = c_1 e^{nx} + c_2 e^{-nx}$$

$$\text{P.I.} = 0$$

$$\therefore \quad \text{V} = c_1 e^{nx} + c_2 e^{-nx} \qquad ...(1)$$

At $x = 0, \ \text{V} = \text{V}_0 \quad \therefore \quad \text{V}_0 = c_1 + c_2$ ...(2)

At $x = l, \ \text{V} = 0 \quad \therefore \quad 0 = c_1 e^{nl} + c_2 e^{-nl}$ ...(3)

Solving (2) and (3), we get

$$c_1 = \frac{\text{V}_0}{1 - e^{2nl}}, \ c_2 = \frac{-\text{V}_0 e^{2nl}}{1 - e^{2nl}}$$

From (1),
$$\text{V} = \frac{\text{V}_0}{1 - e^{2nl}}\left[e^{nx} - e^{2nl - nx}\right]$$

$$= \frac{\text{V}_0 [e^{(nl - nx)} - e^{-(nl - nx)}]}{e^{nl} - e^{-nl}} = \text{V}_0 \left\{\frac{\sinh n(l - x)}{\sinh nl}\right\}.$$

Example 6. *The differential equation* $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + n^2 x = 0$ *; (k < n) represents the damped harmonic oscillations of a particle. Solve this equation and show that the ratio of the amplitude of any oscillation to that of the preceding one is constant i.e., its amplitude form a G.P.*

Sol. We have,

$$(\text{D}^2 + 2k\text{D} + n^2)\, x = 0\,; \quad \text{D} \equiv \frac{d}{dt} \qquad ...(1)$$

Auxiliary equation is

$$m^2 + 2km + n^2 = 0$$

$$m = -k \pm \sqrt{k^2 - n^2} = -k \pm \sqrt{n^2 - k^2}\, i \qquad | \because \ k < n$$

$\therefore$ $$\text{C.F.} = e^{-kt}\left(c_1 \cos \sqrt{n^2 - k^2}\, t + c_2 \sin \sqrt{n^2 - k^2}\, t\right)$$

Also, $$\text{P.I} = 0$$

Hence $$x = e^{-kt}\left(c_1 \cos \sqrt{n^2 - k^2}\, t + c_2 \sin \sqrt{n^2 - k^2}\, t\right) \qquad ...(2)$$

which is the reqd. solution.

At time t, amplitude $= e^{-kt}$

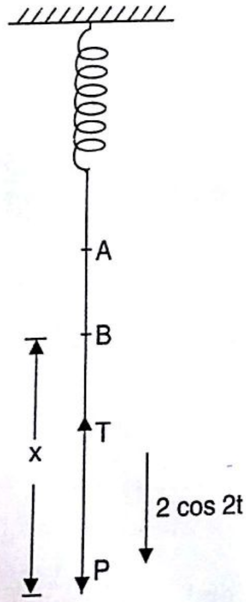
At time $t + 1$, amplitude $= e^{-k(t+1)}$

At time $t + 2$, amplitude $= e^{-k(t+2)}$ and so on.

Ratio of amplitude of any oscillation to the preceding one $= e^{-k}$ (a constant)

Hence the amplitudes of successive vibrations are in G.P.

Example 7. *A spring of negligible weight which stretches 1 inch under tension of 2 lb is fixed at one end and is attached to a weight of ω lb at the other. It is found that resonance occurs when an axial periodic force 2 cos 2t lb acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by x = ct sin 2t and find the values of ω and c.*



Sol. When a weight of 2 lb is attached to A, spring stretches by $\frac{1}{12}$ ft.

$\therefore$ $$2 = k \cdot \frac{1}{12} \quad \Rightarrow \quad k = 24 \text{ lb/ft.}$$

Let B be the equation position of the weight w attached to A then,

$$\omega = k \times AB \quad \Rightarrow \quad AB = \frac{\omega}{24} \text{ ft.}$$

At any time t, Let the weight be at P where BP $= x$

Tension at P, $$T_P = k \times AP = 24\left(\frac{\omega}{24} + x\right) = \omega + 24x$$

Its equation of motion is

$$\frac{\omega}{g}\frac{d^2x}{dt^2} = -T + \omega + 2\cos 2t = -\omega - 24x + \omega + 2\cos 2t$$

$$\Rightarrow \omega \frac{d^2x}{dt^2} + 24gx = 2g\cos 2t \qquad ...(1)$$

The phenomenon of resonance occurs when the period of free oscillations is equal to the period of forced oscillations.

From (1), $$\frac{d^2x}{dt^2} + \mu^2 x = \frac{2g}{\omega}\cos 2t \qquad \left| \text{ where } \mu^2 = \frac{24g}{\omega}\right.$$

$\therefore$ The period of free oscillations is $\frac{2\pi}{\mu}$ and the period of the force $\left(\frac{2g}{\omega}\right)\cos 2t$ is π.

$\therefore$ $$\frac{2\pi}{\mu} = \pi \quad \Rightarrow \quad \mu = 2$$

Hence $$4 = \frac{24g}{\omega} \Rightarrow \omega = 6g \qquad ...(2)$$

Again from (1), $$\frac{d^2x}{dt^2} + 4x = \frac{1}{3}\cos 2t \qquad ...(3) \mid \because \omega = 6g \text{ and } \mu = 2$$

we know that the free oscillations are given by the C.F. and the forced oscillations are given by P.I. Thus, when the free oscillations have died out, the forced oscillations are given by the P.I. of (3)

$$\therefore \quad \text{P.I.} = \frac{1}{3}\cdot\left(\frac{1}{D^2+4}\cos 2t\right) = \frac{1}{3}t\cdot\frac{1}{2D}\cos 2t = \frac{t}{12}\sin 2t$$

Hence, $$c = \frac{1}{12}.$$

Example 8. *A mass M suspended from the end of a helical spring is subjected to a periodic force $f = F\sin\omega t$ in the direction of its length. The force f is measured positive vertically downwards and at zero time M is at rest. If the spring stiffness is S, prove that the displacement of M at time t from the commencement of motion is given by* $x = \frac{F}{M(p^2-\omega^2)}\left(\sin\omega t - \frac{\omega}{p}\sin pt\right)$, *where* $p^2 = \frac{S}{M}$ *and damping effects are neglected.*

Sol. Let x be the displacement from the equilibrium position at any time t, then the equation of motion of the mass M is

$$M\frac{d^2x}{dt^2} = -Sx + F\sin\omega t$$

or $$\frac{d^2x}{dt^2} + \frac{S}{M}x = \frac{F}{M}\sin\omega t$$

or $$\frac{d^2x}{dt^2} + p^2x = \frac{F}{M}\sin\omega t \qquad ...(1) \quad \left(\because \frac{S}{M} = p^2\right)$$

Its A.E. is $m^2 + p^2 = 0 \Rightarrow m = \pm ip$

$$\therefore \quad \text{C.F.} = c_1\cos pt + c_2\sin pt$$

and $$\text{P.I.} = \frac{F}{M}\cdot\frac{1}{D^2+p^2}\sin\omega t = \frac{F}{M}\cdot\frac{1}{p^2-\omega^2}\sin\omega t$$

$\therefore$ Complete solution of equation (1) is

$$x = c_1\cos pt + c_2\sin pt + \frac{F}{M}\cdot\frac{1}{p^2-\omega^2}\sin\omega t \qquad ...(2)$$

Initially, when $t = 0, x = 0 \quad \therefore \quad c_1 = 0$

Differentiating (2) w.r.t. t,

$$\frac{dx}{dt} = -pc_1\sin pt + pc_2\cos pt + \frac{F}{M}\cdot\frac{\omega}{p^2-\omega^2}\cos\omega t$$

Since $\frac{dx}{dt} = 0$, when $t = 0$

$$\therefore \quad pc_2 + \frac{F}{M}\cdot\frac{\omega}{p^2-\omega^2} = 0$$

or $$c_2 = -\frac{\omega}{p} \cdot \frac{F}{M(p^2 - \omega^2)}.$$

Substituting the values of c_1 and c_2 in (2), the displacement of the mass at any time t is given by

$$x = -\frac{\omega}{p} \cdot \frac{F}{M(p^2 - \omega^2)} \sin pt + \frac{F}{M} \cdot \frac{1}{p^2 - \omega^2} \sin \omega t$$

or $$x = \frac{F}{M(p^2 - \omega^2)} \left(\sin \omega t - \frac{\omega}{p} \sin pt \right).$$

Example 9. *A spring which stretches by an amount e under a force mk^2e is suspended from a support P and has a mass m at its lower end. At time $t = 0$, the mass is at rest in its equilibrium position at a point A below P. A vertical oscillation is now given to the support P such that at any time $t(> 0)$ its displacement below its initial position is a sin nt. Show that the displacement x of the mass below A is given by $\frac{d^2x}{dt^2} + k^2x = k^2 a \sin nt$. Hence show that if $n \neq k$, the displacement is given by $x = \frac{ka}{k^2 - n^2}(k \sin nt - n \sin kt)$. What happens, when $n = k$?*

Sol. If λ is the restoring force, then $mk^2e = \lambda e$ so that $\lambda = mk^2$

If e is the elongation produced by the mass m hanging in equilibrium, then $mg = \lambda e$.

Due to vertical oscillation of the support, if at any time t (> 0) the mass is at B, where AB = x, then the stretch of the spring is $(e + x - a \sin nt)$

The equation of motion is

$$m \frac{d^2x}{dt^2} = mg - \lambda(e + x - a \sin nt)$$

$$= -\lambda x + \lambda a \sin nt \qquad (\because \; mg = \lambda e)$$

$$= -mk^2x + mk^2 a \sin nt$$

or $$\frac{d^2x}{dt^2} + k^2x = k^2 a \sin nt \qquad ...(1)$$

Its A.E. is $m^2 + k^2 = 0 \quad \Rightarrow \quad m = \pm ik$

$$\text{C.F.} = c_1 \cos kt + c_2 \sin kt$$

$$\text{P.I.} = \frac{1}{D^2 + k^2} k^2 a \sin nt$$

Case I. When $n \neq k$

$$\text{P.I.} = k^2 a \cdot \frac{1}{k^2 - n^2} \sin nt$$

$\therefore$ The complete solution of (1) is

$$x = c_1 \cos kt + c_2 \sin kt + \frac{k^2 a}{k^2 - n^2} \sin nt \qquad ...(2)$$

Initially, when $t = 0$, $x = 0 \Rightarrow c_1 = 0$

Differentiating (2) w.r.t. t,

$$\frac{dx}{dt} = -kc_1 \sin kt + kc_2 \cos kt + \frac{k^2an}{k^2-n^2} \cos nt$$

Since $\frac{dx}{dt} = 0$, when $t = 0$, we have

$$kc_2 + \frac{k^2an}{k^2-n^2} = 0 \quad \text{or} \quad c_2 = -\frac{kan}{k^2-n^2}$$

Substituting the values of c_1 and c_2 in (2), the displacement of the mass at any time t is given by

$$x = -\frac{kan}{k^2-n^2} \sin kt + \frac{k^2a}{k^2-n^2} \sin nt$$

or

$$x = \frac{ka}{k^2-n^2} (k \sin nt - n \sin kt).$$

Case II. When $n = k$

$$\text{P.I.} = n^2a \cdot \frac{1}{D^2+a^2} \sin nt = n^2a \cdot t \cdot \frac{1}{2D} \sin nt = -\frac{nat}{2} \cos nt$$

$\therefore$ Complete solution of (1) is

$$x = c_1 \cos nt + c_2 \sin nt - \frac{nat}{2} \cos nt \text{ (since } k = n) \qquad \ldots(3)$$

Initially, when $t = 0, x = 0 \Rightarrow c_1 = 0$

Differentiating (3) w.r.t. t,

$$\frac{dx}{dt} = -nc_1 \sin nt + nc_2 \cos nt - \frac{na}{2} (\cos nt - nt \sin nt)$$

Since $\frac{dx}{dt} = 0$, when $t = 0$, we have $nc_2 - \frac{na}{2} = 0 \quad \therefore \quad c_2 = \frac{a}{2}$

Substituting the values of c_1 and c_2 in (3), the displacement of the mass at any time t is given by

$$x = \frac{a}{2} \sin nt - \frac{na}{2} \cos nt$$

or

$$x = \frac{a}{2} (\sin nt - nt \cos nt) = \frac{ar}{2} \sin (nt - \phi),$$

where $r = \sqrt{1+n^2t^2}$, $\phi = \tan^{-1} nt$

The amplitude $\frac{ar}{2} = \frac{a}{2}\sqrt{1+n^2t^2}$ involves t and increases with t.

Thus the phenomenon of resonance occurs.