Differential Equation

Linear Differential Equation with constant coefficient:

$$\left[\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots a_n\right] y = X$$

where $a \cdots a_n$ = constant

X = function of x or constant

$D = \frac{d}{dx}$ (Differential operator).

$Dy = \frac{dy}{dx}$.

$D^2 y = \frac{d^2 y}{dx^2}$

$D^n y - \frac{d^n y}{dx^n}$

$$\left(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots a_n\right] y = X$$

$f(D)\, y = X$ } Linear Differential Eq^n with constant Coefficient

y = General Solution = Complimentary function + particular integral.

(Right Side)

y $X = 0$

y = General Solution = Complimentary function.

$f(D)\, y = 0$

$f(D)\; = 0$

is Auxiliary equation

(1) if Auxillary equation have different roots $m_1, m_2 \dots m_n$

then

Complimentary function $= c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots c_n e^{m_n x}$

(2) if Auxillary equation has two equal roots

$$m_1 = m_2 = m$$

then

Complimentary function $= (c_1 + c_2 x)\, e^{mx}$

(3) if 3 roots are equal

$$m_1 = m_2 = m_3 = m$$

then

Complimentary function $= (c_1 + c_2 x + c_3 x^2)\, e^{mx}$

(4) if auxillary equation has complex roots

i.e. $\alpha \pm i\beta$

then Complimentary ~~equation~~ function

$$= e^{\alpha x}\left[c_1 \cos \beta x + c_2 \sin \beta x\right]$$

(5) If complex root are repeating.
i.e. $\alpha \pm i\beta$, $\alpha \pm i\beta$.

then $CF = e^{\alpha x}[(C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x]$

Q Solve the following differential equation

(1) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

$(D^2 + 5D + 6) = 0$

$D = -2, -3$

$CF = C_1 e^{-2x} + C_2 e^{-3x}$

(2) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$

$D^3 - 3D^2 + 4 = 0$

$D+1 \overline{) D^3 - 3D^2 + 4} \; (D^2 - 4D + 4$
$D^3 + D^2$
$(-) \quad (-)$
$-4D^2 + 4$
$-4D^2 - 4D$
$+4D + 4$

~~$(D+1)(D-4)$~~
D^2

$D = -1, 2, 2$

$CF = C_1 e^{-x} + (C_2 + C_3 x)e^{2x}$

Q $(D^3+a^3)y=0$

$D = \quad D^3+a^3=0$

$D=-a, \quad \frac{+a \pm i\sqrt{3}a}{2}$

$y = CF = c_1 e^{-ax} + e^{+a/2\, x}\left[c_2 \cos \frac{\sqrt{3}a}{2}x + c_3 \sin \frac{\sqrt{3}a}{2}x\right]$

Q $(D^4+m^4)y=0$.

$D^4+m^4=0$.

$\cancel{D^4=-m^4}$
$\cancel{D^4=i^2 m^4}$
$\cancel{D^2=\pm i m^2}$

$x^4=-1$
$x^4=t^2$
$t^2=-1$
$t^2+1=0$
$\frac{\pm 2i}{2} = \pm i$

$D^4=-m^4$
$D=m(-1)^{1/4}$

\# $F(D)\,y = X$

(1) $X = e^{ax}$

Particular Integral $= \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ $\quad f(a) \neq 0$

if $f(a)=0$

$\frac{1}{f(D)} e^{ax} = \frac{x\, e^{ax}}{f'(a)}$. $\quad f'(a) \neq 0$.

$\frac{1}{D}x = \int x\,dx$

Eg ① $(D^2+2D+3)y = e^{2x}$

$PI = \frac{1}{D^2+2D+3} e^{2x}$

$= \frac{1}{11} e^{2x}$ $\qquad \underline{D=2}$

② $(D^2-2D+1)y = e^x$

$PI = \frac{1}{\underline{D^2-2D+1}} e^x \qquad D=1$

$\to 0$

$PI = \frac{xe^x}{2D-2} \qquad D=1 \to 0$

$PI = \frac{x^2e^x}{2}$

② $x = \sin ax$ or $\cos ax$ or $\sin(ax+b)$ or $\cos(ax+b)$

$PI = \frac{1}{f(D^2)} \sin ax$

$PI = \frac{1}{f(-a^2)} \sin ax \qquad f(-a^2) \neq 0$

$\rightarrow$ if $f(-a^2) = 0$

$PI = \frac{x \sin ax}{f'(D^2)}$

Eg $(D^2+2D+3)y = \sin 2x$

$PI = \frac{1}{D^2+2D+3} \sin 2x$

$= \frac{1}{-4+2D+3} \sin 2x$ $\quad D^2 = -(2)^2$

$= \frac{1}{2D-1} \sin 2x$

$= \frac{2D+1}{4D^2-1} \sin 2x$ $\quad D^2 = -(2)^2$

$= \frac{-1}{17} (2D+1) \sin 2x$ $\quad D = \frac{d}{dx}$

$= \frac{-1}{17} [2D \sin 2x + \sin 2x]$ $\quad D \sin 2x = \frac{d}{dx} \sin 2x$

$= \frac{-1}{17} [4 \cos 2x + \sin 2x]$

(3) $x = x^m$

$PI = \frac{1}{f(D)} x^m = \frac{1}{[1 \pm f(D)]} x^m = [1 \pm f(D)]^{-1} x^m$

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Eg $(D^2+2D+3)y = x^2$.

$PI = \frac{1}{D^2+2D+3} x^2$

$PI = \frac{1}{3\left[1+\frac{1}{3}(D^2+2D)\right]} x^2$

$PI = \frac{1}{3}\left[1+\frac{1}{3}D^2+2D\right]^{-1} x^2$ By Binomial.

$= \frac{1}{3}\left(1 - \frac{1}{3}(D^2+2D) + \frac{1}{9}(D^2+2D)^2\right) x^2$

$= \frac{1}{3}\left[x^2 - \frac{1}{3}(2+4x] + \frac{1}{9}8\right]$

opening the bracket

D^2 of $x^2 = 2$

D of $x^2 = 2x$

D^3, D^4 of $x^2 = 0$

Properties

$\frac{1}{D}x = \int x\,dx$

$\frac{1}{D-m}x = e^{mx}\int xe^{-mx}dx$

$y = \frac{1}{D-m}x$

$(D-m)y = x$

$ye^{-mx} = \int xe^{-mx}dx$

$y = e^{mx}\int xe^{-mx}dx.$

Ⓢ $(D^2+3D+2)y = \cos 3x$

$PI = \dfrac{1}{D^2+3D+2} \cos 3x$

$PI = \dfrac{1}{-9+3D+2} \cos 3x$

$PI = \dfrac{1}{3D-7} \cos 3x$

$PI = \dfrac{(3D+7)}{9D^2-49} \cos 3x$

$PI = \dfrac{(3D+7)}{-81-49} \cos 3x$

$PI = -\dfrac{1}{130} (3D+7) \cos 3x$

$PI = -\dfrac{1}{130} \left[3 \dfrac{d}{dx} \cos 3x + 7 \cos 3x \right]$

$PI = -\dfrac{1}{130} \left[-9 \sin 3x + 7 \cos 3x \right]$.

Ⓘ D^3, D^2 D

$\Downarrow$

replace by $-a^2$

Proof of $F(D)\,y = X,\ X = e^{ax}$

$e^{ax} = e^{ax} \quad \times a_n$
$De^{ax} = ae^{ax} \quad \times a_{n-1}$
$D^2e^{ax} = a^2e^{ax} \quad \times a_{n-2}$
$\vdots \qquad \vdots$

$D^n e^{ax} = a^n e^{ax} \quad \times 1$

$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots a_n)e^{ax} =$

$(a^n + a_1 a^{n-1} + \cdots + a_n)\, e^{ax}$

$f(D)e^{ax} = f(a)\, e^{ax}$

$e^{ax} = \dfrac{f(a)\, e^{ax}}{f(D)}$

$\dfrac{1}{f(a)} e^{ax} = \dfrac{1}{f(D)} e^{ax}$

F when at $D = a$, $f(a) = 0$.

$D - a$ is factor

$f(D) = (D-a)\, F(D)$.

$\dfrac{1}{f(D)} e^{ax} = \dfrac{1}{D-a}\, \dfrac{1}{F(D)}\, e^{ax}$

$= \dfrac{1}{F(a)}\, \dfrac{1}{(D-a)}\, e^{ax}$

$= \dfrac{1}{F(a)}\, e^{ax} \int e^{ax}\, e^{-ax}\, dx$ (using property 2)

$$\frac{1}{f(D)} e^{ax} = \frac{1}{F(a)} x e^{ax}$$

$$f(D) = (D-a) F(D)$$

$$f'(D) = F(D) + (D-a) F'(D)$$

$$D = a$$

$$f'(a) = F(a)$$

$$\boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{f'(a)} x e^{ax}}$$

<u>Q</u> $(D^2+3D+2)y = x^2$

$$PI = \frac{1}{D^2+3D+2} x^2$$

$$PI = \frac{1}{2} \frac{1}{\left[1 + \frac{1}{2}(D^2+3D)\right]} x^2$$

$$PI = \frac{1}{2} \left[1 + \frac{1}{2}(D^2+3D)\right]^{-1} x^2$$

$$PI = \frac{1}{2} \left[1 - \frac{1}{2}(D^2+3D) + \frac{1}{4}(D^2+3D)^2\right] x^2$$

$PI = \frac{1}{2}\left[x^2 - \frac{1}{2}(2+6x) + \frac{1}{4}(18)\right]$

PI =

④ $X = e^{ax}V$, $V = \sin ax$ or $\cos ax$ or x^m

$PI = \frac{1}{f(D)} e^{ax} V$

$PI = e^{ax} \frac{1}{f(D+a)} V$

Q $(D^2+3D+2)y = x^2 e^x$

$PI = \frac{1}{D^2+3D+2} x^2 e^x$

$PI = e^x \frac{1}{(D+1)^2 + 3(D+1)+2} x^2$

$PI = e^x \left[\frac{1}{D^2+5D+6} x^2\right]$ → Same as previous method

⑤ $X = xV$, $V = \sin ax$ or $\cos ax$

$PI = \frac{1}{f(D)} xV = x \frac{1}{f(D)} V - \frac{f'(D)}{[f(D)]^2} V$

\# $f(D)y = e^{2x} + \sin 3x + e^{2x}x^2 + x^3$

$y = \frac{1}{f(D)} e^{2x} + \frac{1}{f(D)} \sin 3x + \frac{1}{f(D)} e^{2x}x^2 + \frac{1}{f(D)} x^3$

\# $\frac{1}{D^2-1} xe^x \sin x$

Imaginary part of $\frac{1}{D^2-1} xe^x e^{ix}$

$\frac{1}{D^2-1} x e^{(1+i)x}$

$a = 1+i$

answer expressed in $A + iB$

if $\cos$ is given A is answer

B is answer as $\sin$ is given

OR

$e^x \frac{1}{(D+1)^2-1} x\sin x$

now formula for xV

Q Solve the following Differential Equations

① $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}$

② $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = (1+e^x)^2$

③ $\frac{d^3y}{dx^3} + y = 4 + 2e^x$

④ $\frac{d^2y}{dx^2} + y = e^{2x} + \cosh 2x + x^3$

⑤ $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$

⑥ $\frac{d^2y}{dx^2} - y = xe^x \sin x$

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①

$(D^2 - 7D + 6)y = e^{2x}$

$D = 1, 6.$

$y = c_1e^x + c_2e^{6x}$

$y = \left(\frac{1}{D^2 - 7D + 6}\right)e^{2x}$

$y = \frac{-1}{4}e^{2x}$

② $(D^2 + D + 1)y = e^{0x} + e^{2x} + 2e^x$

$y = \frac{1}{D^2+D+1}e^{0x} + \frac{1}{D^2+D+1}e^{2x} + 2\frac{1}{D^2+D+1}e^x$

$y = 1 + \frac{1}{7}e^{2x} + \frac{2}{3}e^{x}$

(3) $(D^3+1)y = 4e^{0x} + 2e^{x}$

$y = 4\frac{1}{D^3+1}e^{0x} + 2\frac{1}{D^3+1}e^{x}$

$y = 4 + \cancel{\frac{2}{2}e^{x}}$ $\quad 4 + e^{x}$

(4) $\cancel{(D^2+1)y = e^{2x} + \frac{e^{2x}+e^{-2x}}{2} + x^3}$

(2) $D^2 + D + 1 = 0$

$D = \frac{-1 \pm \sqrt{1-4}}{2}$

$D = \frac{-1 \pm \sqrt{3}i}{2}$

$y = e^{-\frac{1}{2}x}\left[c_1 \cos \frac{\sqrt{3}x}{2} + c_2 \sin \frac{\sqrt{3}}{2}x\right]$.

(3) $D^3 + 1 = 0$

$$\begin{array}{r l}
 & D^2 - D + 1 \\
D+1 \,) & D^3 + 1 \\
 & D^3 + D^2 \\
 & (-) \quad (-) \\
\hline
 & -D^2 + 1 \\
 & -D^2 - D \\
 & (+) \quad (+) \\
\hline
 & D + 1
\end{array}$$

$D^2 - D + 1 = 0$

$D = \frac{1 \pm \sqrt{3}i}{2}$.

$y = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right]$

(4) $D^2 + 1 = 0$

$D = \frac{\pm\sqrt{-4}}{2}$

$D = \pm i$

$y = c_1 \cos x + c_2 \sin x$

$(D^2+1)y = e^{2x} + \frac{e^{2x}}{2} + \frac{e^{-2x}}{2} + x^3$

$y = \frac{3}{2}\left(\frac{1}{D^2+1} e^{2x}\right) + \frac{1}{2}\left(\frac{1}{D^2+1}\right)e^{-2x} + \frac{1}{D^2+1} x^3$

$y = \frac{3}{10} e^{2x} + \frac{1}{10} e^{-2x} + (1+D^2)^{-1} x^3$

$y = \frac{3}{10} e^{2x} + \frac{e^{-2x}}{10} + \left(1 - D^2 + \frac{1}{2} D^4 + R D^6\right) x^3$

$y = \frac{3}{10} e^{2x} + \frac{e^{-2x}}{10} + x^3 - 6x$

(5) $D^2 3 D + 2 = 0$

$D = 1, 2$

$y = c_1 e^{x} + c_2 e^{2x}$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \frac{n(n-1)(n-2)}{\lfloor 3}x^3 + \cdots$

$(D^2-3D+2)y = xe^{2x} + \sin 2x$

$y = \frac{1}{D^2-3D+2} xe^{2x} + \frac{1}{D^2-3D+2} \sin 2x$

$y = e^{3x}\left[\frac{1}{(D+3)^2 - 3(D+3)+2]} x - \frac{1}{3D+2}\sin 2x\right]$

$y = e^{3x}\left[\frac{1}{D^2+3D+2} x - \frac{3D-2}{9D^2-4}\sin 2x\right]$

$y = \frac{e^{2x}}{2}\left[\frac{1}{2}\left[1+\frac{1}{2}(D^2+3D)\right]^{-1} x + \frac{(3D-2)\sin 2x}{40}\right]$

$y = e^{2x}\left[\frac{1}{2}\left[1-\frac{1}{2}(D^2+3D)x\right] + \frac{1}{40}(3D-2)\sin 2x\right]$

$y = e^{3x}\left[\frac{1}{2}\left[x - \frac{3}{2}\right] + \frac{1}{40}\left[6\cos 2x - 2\sin 2x\right]\right]$

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$D^2 - 1 = 0$

$D^2 = 1$

$D = \pm 1$

$y = c_1 e^x + c_2 e^{-x}$

$(D^2-1)y = xe^x \sin x$

$y = xe^x \frac{1}{2} x \sin x$

Q2

$$y = \frac{1}{D^2-1} e^x x \sin x$$

$$y = e^x \frac{1}{(D+1)^2-1} (x \sin x)$$

$$y = e^x \frac{1}{D^2+2D} x \sin x$$

$$y = e^x \left[x \frac{1}{D^2+2D} \sin x - \frac{2D+2}{(D^2+2D)^2} \sin x \right]$$

$$y = e^x \left[x \frac{1}{2D-1} \sin x - \frac{2D+2}{(2D-1)^2} \sin x \right]$$

$$y = e^x \left[x \frac{2D+1}{4D^2-1} \sin x - \frac{2D+2}{4D^2+1-4D} \sin x \right]$$

$$y = e^x \left[-\frac{x}{5} [2\cos x + \sin x] - \frac{2D+2}{4D+3} \sin x \right]$$

$$y = e^x \left[-\frac{x}{5} (2\cos x + \sin x) + \frac{(2D+2)(4D-3)}{16D^2-9} \sin x \right]$$

$$y = e^x \left[-\frac{x}{5} (2\cos x + \sin x) \mp \frac{1}{25} [8D^2+2D-6) \sin x \right]$$

$$y = e^x \left[-\frac{x}{5} (2\cos x + \sin x) - \frac{1}{25} [14 \sin x + 2\cos x] \right]$$

$$y = e^x \left[-\frac{x}{5} (2\cos x + \sin x) - \frac{1}{25} [2\cos x - 14 \sin x] \right]$$

Q $(D^2+a^2)y = \tan ax$

$D^2+a^2=0$

$D = \pm ia$

$y = C_1\cos ax + C_2 \sin ax$

$PI = \frac{1}{D^2+a^2} \tan ax$

$= \frac{1}{D^2 - i^2a^2} \tan ax$

$= \frac{1}{(D-ia)(D+ia)} \tan ax$

$= \frac{1}{2ia}\left(\frac{1}{D-ia} - \frac{1}{D+ia}\right) \tan ax$

$\frac{1}{D-ia} \tan ax \longleftrightarrow$ Replace $i \to -i$ to find $\left(\frac{1}{D+ia}\right) \tan ax$

$\frac{1}{D-m} X = e^{mx}\int X e^{-mx}\,dx$

$e^{iax}\int \tan ax\, e^{-iax}\,dx$

$= e^{iax}\int \tan ax\,(\cos ax - i\sin ax)\,dx$

$= e^{iax}\int \left(\sin ax - i\frac{\sin^2 ax}{\cos ax}\right)dx$

$\hookrightarrow$ integrate & find answer

$$= e^{iax}\int\left(\sin ax - i\left(\frac{1-\cos^2 ax}{\cos ax}\right)\right)dx$$

$$= e^{iax}\int \sin ax - i(\sec ax) + i\cos ax\, dx$$

$$= e^{iax}\left[-\frac{\cos ax}{a} - \frac{i}{a}\log[\sec ax+\tan ax] + \frac{i\sin ax}{a}\right]$$

$$= \cancel{e^{iax}}\ \frac{e^{iax}}{a}\left[-(\cos ax - i\sin ax) - i\log(\sec ax+\tan ax)\right]$$

$\cancel{\frac{1}{D+ia}\tan x}$

$$= -\frac{e^{iax}}{a}\left[e^{-iax} + i\log[\sec ax+\tan ax]\right]$$

$$= -\frac{1}{a}\left[1 + ie^{iax}\log[\sec ax+\tan ax]\right]$$

$$\frac{1}{D+ia}\tan ax = -\frac{1}{a}\left[1 - ie^{-iax}\log[\sec ax+\tan ax]\right]$$

$$PI = \frac{-1}{2ia^2}\left(-\frac{a}{a}[1+ie^{iax}\log[\sec ax+\tan ax]] - [1+ie^{-iax}\log[\sec ax+\tan ax]]\right)$$

$PI = -\cancel{\frac{1}{a^2}e^{x}}$

Q $(D^2+a^2)y = \sec ax$

$D^2+a^2=0$

$D = \pm ia$

$y_{CF} = C_1\cos ax + C_2\sin ax$.

$PI = \frac{1}{(D^2+a^2)}\sec ax$

$PI = \frac{1}{(D+ia)(D-ia)}\sec ax$

$PI = \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]\sec ax$

$$\frac{1}{D-ia}\sec ax = e^{iax}\int \sec ax\, e^{-iax}\,dx$$

$$= e^{iax}\int \sec ax\left[\cos ax - i\sin ax\right]dx$$

$$= e^{iax}\int 1 - i\,\cancel{\sin ax}\tan ax\,]\,dx$$

$$= e^{iax}\left[x - \frac{i\,\cancel{\sec}}{a}\log(\sec ax)\right]$$

$$\frac{1}{D+ia}\sec ax = e^{-iax}\left[x + \frac{i}{a}\log(\sec ax)\right].$$

HOMOGENOUS LINEAR DIFFERENTIAL EQUATION.

OR

CAUCHY EULER'S EQUATION.

Of the form,

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots a_n y = X$$

X is function of x or constant.

$x = e^z$

$z = \log x$

$\frac{dz}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$x\frac{dy}{dx} = \frac{dy}{dz}$$

$$xD = D_1$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\frac{dy}{dz}\frac{dz}{dx}. \qquad \frac{dz}{dx} = \frac{1}{x},$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}$$

$x^2 D^2 = D_1^2 - D_1 = D_1(D_1 - 1)$

Illy,

$x^3 D^3 = D_1(D_1-1)(D_1-2)$

;

;

$x^n D^n = D_1(D_1-1)(D_1-2) + \cdots (D_1 - n+1)$.

Eq

$(x^n D^n + a_1 x^{n-1} D^{n-1} + \cdots a_n) y = X$.

putting all value,

$$\underbrace{f(D_1) y = f(z)}$$

Linear differential Equation with constant coefficient.

- $D_1 = 1, 2$.

$CF = C_1 e^z + C_2 e^{2z}$

$CF = C_1 x + C_2 x^2$.

Q $(x^2 D^2 + x D + 2) y = \cancel{e^{2x}} \; x^2$

Let $x = e^z \Rightarrow z = \log x$

$$\frac{dz}{dx} = \frac{1}{x}$$

$x^2 D^2 = D_1(D_1 - 1)$

$x D = D_1$

where $D = \frac{d}{dx}$, $D_1 = \frac{d}{dz}$

$(D_1(D_1-1)+D_1+2)y = e^{2z}$

$(D_1^2+2)y = e^{2z}$

$D_1^2+2=0$

$D_1 = \pm\sqrt{2}\,i$

$CF = e^{0z}(c_1\cos\sqrt{2}z + c_2\sin\sqrt{2}z)$

$CF = c_1\cos\sqrt{2}\log x + c_2\sin\sqrt{2}\log x$

$PI = \dfrac{e^{2z}}{D_1^2+2}$

$PI = \dfrac{e^{2z}}{6}$

$PI = \dfrac{1}{6}x^2$

Q $(x^2D^2-2)y = x^2+\dfrac{1}{x}$.

$(D_1(D_1-1)-2)y = e^{2z}+e^{-z}$

$(D_1^2-D_1-2)y = e^{2z}+e^{-z}$

$D_1^2-D-2=0$

$D_1 = \dfrac{1\pm\sqrt{9}}{2} = 2, -1$

$CF = c_1x^2 + \dfrac{c_2}{x}$.

(1) $x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$

(2) $(x^2 D^2 + 4xD + 2)y = x + \sin x.$

(3) $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$

(1)

$(x^3D^3 + 2x^2D^2 - xD + 1)y = \frac{1}{x}$

$(D_1(D_1-1)(D_1-2) + 2D_1(D_1-1) - D_1 + 1)y = e^{-2}$

$(D_1^3 - 2D_1^2 - D_1^2 + 2D_1 + 2D_1^2 \cancel{-2D_1} - D_1 + 1)y = e^{-2}$

$\cancel{D_1^3 - 5D_1^2 + 4D_1}$

$(D_1^3 - D_1^2 - D_1 + 1)y = e^{-2}$

$D_1^3 - D_1^2 - D_1 + 1 = 0$

$D_1 - 1 \,)\, D_1^3 - D_1^2 - D_1 + 1 \,(\, D_1^2 - 1$

$D_1^3 - D_1^2$

$(-) \quad (+)$

$-D_1 + 1$

$D_1(D_1^2 - 1)^2 = 0$

$\cancel{D_1 = 0, \; D_1 = 1, -1}$ $\quad D_1 = 1, 1, -1$

$CF = (C_1 + C_2 \log x)x + C_3 x$

$PI = \frac{1}{D_1^3 - D_1^2 - D_1 + 1} e^{-z}$

$z = \log x$

$x = e^z$

$PI = \frac{z e^{-z}}{3D_1^2 - 2D_1 - 1}$ $\quad 3+2-1$

$= \frac{z e^{-z}}{4}$

$= \frac{\cancel{x} \log x}{4} = + \frac{\log x}{4x}.$

② $(x^2D^2 + 4xD + 2)y = x + \sin x.$

$(D_1(D_1 - 1) + 4D_1 + 2)y = e^z + \sin e^z$

$(D_1^2 + 3D_1 + 2)y = e^z + \sin e^z$

$(D_1 + 1)(D_1 + 2) = 0$

$D_1 = -1, -2$

$CF = \frac{C_1}{x} + \frac{C_2}{x^2}$

$PI = \frac{e^z}{D_1^2 + 3D_1 + 2} + \frac{\sin e^z}{(D_1 + 1)(D_1 + 2)}.$

$PI = \quad \mathrm{I} \qquad \mathrm{II}$

$$PI_1 = \frac{e^z}{6} = \frac{x}{6}$$

$$PI_2 = \frac{(D_1+2)-(D_1+1)}{(D_1+1)(D_1+2)} \sin e^z$$

$$PI_2 = \left(\frac{1}{D_1+1} - \frac{1}{D_1+2}\right) \sin e^z$$

$$PI_2 = \underbrace{\frac{\sin e^z}{D_1+1}}_{1} - \frac{\sin e^z}{D_1+2}$$

$$\frac{\sin e^z}{D_1+1} = e^{-z}\int \sin e^z \, e^z \, dz$$

$$= -e^{-z} \cos e^z$$

$$\frac{\sin e^z}{D_1+2} = e^{-2z}\int \sin e^z \, e^{2z} dz$$

$$= e^{-2z}\int t \sin t \, dt$$

$$= e^{-2z}$$

$$\int t \sin t \, dt$$

$$-t\cos t + \int \cos t \, dt$$

$$-t\cos t + \sin t$$

$$\frac{\sin e^z}{D_1+2} = e^{-2z}\left(-e^z \cos e^z + \sin e^z\right).$$

$$PI_2 = \cancel{-e^{-z}\cos e^{z}} + \cancel{e^{-z}\cos e^{z}} - \cancel{e^{z}\sin e^{-2z}}\; e^{-2z}\sin e^{z}$$

$$PI_2 = -e^{-2z}\sin e^{z}$$

$$PI = \frac{x}{6} - \frac{1}{x^2}\sin x.$$

$x = e^{z}$

(3) $(x^2D^2 - xD - 3)y = x^2\log x$

$$(D_1(D_1-1) - D_1 - 3)y = ze^{2z}$$

$$(D_1^2 - 2D_1 - 3)y = ze^{2z}.$$

$$(D_1-3)(D_1+1) = 0.$$

$$D_1 = 3,\ D_1 = -1$$

$$CF = c_1x^3 + \frac{c_2}{x}.$$

Equation reducable to homogenous form.

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots a_n y = x$$

$$a + bx = t$$

$$\frac{dt}{dx} = b$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = b\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = b\frac{d}{dx}\frac{dy}{dt} = \cancel{\frac{d^2y}{}}\ b\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = b^2\frac{d^2y}{dt^2}$$

$$\vdots$$

$$D^n = b^n D_1^n$$

$$b^n t^n \frac{d^n y}{dt^n} + a_1 b^{n-1} t^{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = g(t)$$

$$t^n \frac{d^n y}{dt^n} + \left(\frac{a_1}{b}\right) t^{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + \frac{a_n}{b^n} y = \frac{1}{b^n} g(t)$$

II) Homogenous linear differential equation.

Q (1) $(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x$.

(2) $(1+2x)^2 \frac{d^2y}{dx^2} - 6(1+2x)\frac{dy}{dx} + 16y = 8(1+2x)^2$

Ans 1

$x+a = t$

$\frac{dt}{dx} = 1$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$

$t^2 \frac{d^2y}{dt^2} - 4t\frac{dy}{dt} + 6y = t-a$.

$t = e^z$

$(D_1(D_1-1) - 4D_1 + 6)y = e^z - a$

$(D_1^2 - 5D_1 + 6)\, y = e^z - a$.

$D_1 = 2, 3$

$CF = C_1 e^{2z} + C_2 e^{3z}$

$= C_1 t^2 + C_2 t^3$

$= C_1 (x+a)^2 + C_2 (x+a)^3$

$PS = \frac{e^z}{D_1^2 - 5D_1 + 6} - a\,\frac{e^{0z}}{D_1^2 - 5D_1 + 6}$

$$PI = \frac{e^{z}}{2} - \frac{a}{6}$$

$$PI = \frac{x+a}{2} - \frac{a}{6}$$

$$PI = \frac{3x+3a-a}{6} = \frac{3x+2a}{6}$$

Que 2 $\quad (1+2x)^2 \frac{d^2y}{dx^2} - 6(1+2x)\frac{dy}{dx} + 16y = 8(1+2x)^2$

$$1+2x = t$$

$$\frac{dt}{dx} = 2$$

$$\frac{dy}{dx} = 2\frac{dy}{dt}$$

$$4t^2\frac{d^2y}{dt^2} - 12t\frac{dy}{dt} + 16y = 8t^2$$

$$t^2\frac{d^2y}{dt^2} - 3t\frac{dy}{dt} + 4y = 2t^2$$

$$t = e^z$$

$$(D_1(D_1-1) - 3D_1 + 4)\, y = 2e^{2z}$$

$$(D_1^2 - 4D_1 + 4)\, y = e^{2z}$$

$$D_1 = 2, 2.$$

$$CF = (C_1 + C_2 z)\, e^{2z}$$

$CF = (C_1 + C_2 \log t)\, t^2$

$CF = (C_1 + C_2 (1+2x))\, (1+2x)^2$

$$PI = \frac{2e^{2z}}{D_1^2 - 4D_1 + 4}$$

$$PI = \frac{2z\,e^{2z}}{2D_1 - 4}$$

$$PI = \frac{\cancel{2} z^2 e^{2z}}{\cancel{2}} = z^2 e^{2z}$$

$PI = (\log t)^2\, t^2$

$PI = [\log(1+2x)]^2\, (1+2x)^2.$

LINEAR DIFFERENTIAL EQUATION OF 2nd ORDER

$y'' + Py' + Qy = R.$

Coefficients are not constant.

(1) Complete solution of this differential equation in terms of known integral belonging to Complementry function

$y = uv$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

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$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

Putting in the DE

$$\left(u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}\right) + P\left(u\frac{du}{dx} + v\frac{du}{dx}\right) + Quv = R$$

$$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} + v\underbrace{\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)}_{0} = R$$

u belongs to complimentary function;

$$CF = \frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

u will satisfy it

So $\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0$

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\frac{dv}{dx} = z$$

$$\frac{d^2v}{dx^2} = \frac{dz}{dx}$$

$$\frac{dz}{dx} + \left(P + \frac{2}{u}\frac{du}{dx}\right)z = \frac{R}{u}$$] Linear differential Equat

Integrating factor $= e^{\int}$

I Integrating factor $= e^{\int \left(P + \frac{2}{u}\frac{du}{dx}\right) dx}$

$$= u^2 e^{\int P dx}$$

$$Z u^2 e^{\int P dx} = \int \frac{R}{u} u^2 e^{\int P dx} dx + C_1$$

$$\frac{dv}{dx} = \frac{e^{-\int P dx}}{u^2} \int R u \, e^{\int P dx} dx + \frac{C_1}{u^2} e^{-\int P dx}.$$

[v by integrating]

To find u:

$$y'' + Py' + Qy = R$$

Auxillary Equation: $y'' + Py' + Qy = 0$.

1) $y = e^{ax}$ $\quad a^2 + Pa + Q = 0$.

$a = 1$ = $1 + P + Q = 0$ (if true, e^x is solution)

$a = -1$ = $1 - P + Q = 0$ (if true, e^{-x} is solution)

2) $y = x^m$.

$$m(m-1)x^{m-2} + Pmx^{m-1} + Qx^m = 0.$$

$$m(m-1) + P(mx) + Qx^2 = 0.$$

$m = 1$

= $P + Qx = 0$. (if true : $y = x$ is solution)

Q Solve

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} - y = e^x. \quad (1)$$

$$\frac{d^2y}{dx^2} + \left(\frac{1}{x} - 1\right)\frac{dy}{dx} - \frac{y}{x} = \frac{1}{x}e^x$$

$$y'' + Py' + Qy = R.$$

$$P = \frac{1}{x} - 1$$

$$Q = -\frac{1}{x}$$

$$R = \frac{e^x}{x}$$

Since, $1 + P + Q = 1 + \frac{1}{x} - 1 - \frac{1}{x} = 0$

Hence $u = e^x$.

Let $y = uv$ be complete solution of (1).

1st reduces to

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + \left[\frac{1}{x} - 1 + \frac{2}{e^x}e^x\right]\frac{dv}{dx} = \frac{e^x}{xe^{+x}}$$

$$\frac{d^2v}{dx^2} + \left(\frac{1}{x} + 1\right)\frac{dv}{dx} = \frac{1}{x}.$$

Let $\frac{dv}{dx} = z \Rightarrow \frac{d^2v}{dx^2} = \frac{dz}{dx}$,

$$\frac{dz}{dx} + \left(1+\frac{1}{x}\right)z = \frac{1}{x}$$

Which is linear differential equation

$$IF = e^{\int 1+\frac{1}{x}dx}$$

$$IF = e^{x+\ln x}$$

$$IF = e^x e^{\ln x}$$

$$IF = xe^x$$

$$zxe^x = \int e^x \frac{1}{x} xe^x dx + c_1$$

$$zxe^x = e^x + c_1$$

$$\frac{dv}{dx} = z = \frac{1}{x} + \frac{c_1 e^{-x}}{x}$$

$$v = \log x + c_1 \int \frac{e^{-x}}{x} dx + c_2$$

$$y = uv$$

$$y = e^x\left[\log x + c_1 \int \frac{e^{-x}}{x} dx + c_2\right].$$

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Q Solve

$$x\frac{d^2y}{dx^2} - 2(x+1)\frac{dy}{dx} + (x+2)y = (x-2)e^x$$

$$\frac{d^2y}{dx^2} - 2\left(1+\frac{1}{x}\right)\frac{dy}{dx} + \left(1+\frac{2}{x}\right)y = \left(1-\frac{2}{x}\right)e^x$$

$\cancel{P = -2(1}$ $\quad P = -2 - \frac{2}{x}$

$$Q = 1 + \frac{2}{x}$$

$$R = \left(1-\frac{2}{x}\right)e^x$$

$[a^2 + Pa + Q = 0]$

$$1 + P + Q = 1 - 2 - \frac{2}{x} + 1 + \frac{2}{x} = 0$$

So $u = e^x$ is

let uv be complete solution.

(1) reduce is

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}.$$

$$\frac{d^2v}{dx^2} + \left[-2 - \frac{2}{x} + 2\right]\frac{dv}{dx} = 1 \cancel{e^{-x}} - \frac{2}{x}$$

$$\frac{d^2v}{dx^2} - \frac{2}{x}\frac{dv}{dx} = e^{-x} - \frac{2}{x}$$

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$\frac{dv}{dx} = dx\ z$

$\frac{d^2v}{dx^2} = \frac{dz}{dx}$

$\frac{dz}{dx} - \frac{2}{x} z = -1\cancel{e^{-x}} - \frac{2}{x}$.

$IF = e^{\int -\frac{2}{x} dx}$

$IF = \frac{1}{x^2}$

$\cancel{\frac{z}{x^2} = \int \left(1 e^{-x} - \frac{2}{x}\right) \frac{1}{x^2} dx + c_1}$

$\cancel{\frac{z}{x^2} = \int e^{-x}}$

$\frac{z}{x^2} = \int \left(1 - \frac{2}{x}\right) \frac{1}{x^2} dx + c_1$

$\frac{z}{x^2} = \int \frac{1}{x^2} - \frac{2}{x^3} dx + c_1$

$\frac{x^{-3+1}}{-3+1}$

$\cancel{\frac{z}{x^2} = \frac{-2}{x^3} + \frac{6}{x^4} + c_1}$

$z = \frac{-2}{x} + \frac{6}{x^2} + c_1 x^2$

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$$V = \frac{1}{x^2} - \frac{12}{x^3} + C_1\frac{x^3}{3} + C_2$$

$$\frac{Z}{x^2} = \int$$

$$\frac{Z}{x^2} = \frac{-1}{x} + \frac{1}{x^2} + C_1$$

$$\frac{dv}{dx} = Z = -x + 1 + C_1 x^2$$

$$V = \frac{-x^2}{2} + x + C_1\frac{x^3}{3} + C_2$$

$$y = UV = e^x\left(-\frac{x^2}{2} + x + C_1\frac{x^3}{3} + C_2\right)$$

Q $x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = 0$

$$y'' - \left(2 - \frac{1}{x}\right)y' + \left(1 - \frac{1}{x}\right)y = 0.$$

$P = -2 + \frac{1}{x}.$

$Q = 1 - \frac{1}{x}.$

$R = 0$

$P + Q + 1 = 0.$

$u = e^x$

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$x^{-1} \quad \frac{x^{-1+1}}{-1+1}$

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + \left[-2 + \frac{1}{x} + 2\right]\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx} = 0.$$

$$\frac{dz}{dx} + \frac{z}{x} = 0$$

$$dz = -\frac{z}{x}dx$$

$$\int\frac{dz}{z} = -\int\frac{dx}{x}$$

$$\log z = \log\frac{c}{x}$$

$$z = \frac{c_1}{x}$$

$$V = c_1\log x + c_2$$

$$y = e^x(c_1\log x + c_2).$$

$$\cancel{z = -\log x}$$

$$\cancel{\frac{dz}{dx} = -\frac{1}{x}}$$

$$\cancel{dz = \frac{-dx}{x}}$$

$$\cancel{z = \int\frac{dx}{}}$$

$$\int\frac{dz}{z} = -\int\frac{dx}{x}$$

$$\log z = -\log x + c_1$$

$$z = \frac{c_1}{x} \cancel{+ c_1}$$

$$V = \cancel{-\frac{1}{x^2} + c_1x + c_2}$$

$$V = \log x + c_1x + c_2$$

$$y = e^x(\log x + c_1x + c_2)$$

Q $x^2 y_2 - 2x(1+x) y_1 + 2(1+x) y = x^3$

$y_2 - 2\left(\frac{1}{x}+1\right) y_1 + 2\left(\frac{1}{x^2}+\frac{1}{x}\right) y = x.$

$P = -\frac{2}{x} - 2$

$Q = \frac{2}{x^2} + \frac{2}{x}$

$R = x$

$P + Qx = 0$

$u = x.$

$\frac{d^2v}{dx^2} + \left(-\frac{2}{x} - 2 + \frac{2}{x}\right)\frac{dv}{dx} = 1$

$\frac{d^2v}{dx} - 2\frac{dv}{dx} = 1$ $\qquad \frac{dz}{dx} - 2z = 1$

$\cancel{\frac{d^2v}{dx^2} - 2d}$

$IF = e^{-2x}.$

dz

$ze^{-2x} = \int e^{-2x} dx + C_1$

$z = -\frac{1}{2} + C_1 e^{2x}.$

$v = -\frac{x}{2} + \frac{C_1 e^{2x}}{2} + C_2$

$y = uv$

$$y = -\frac{x^2}{2} + c_1 \frac{x e^{2x}}{2} + c_2 x$$

2 Removal of 1st Derivative - Reduce to Normal form

1) Obtain a suitable substitution for the dependent variable which transform the equation.

$y'' + Py' + Qy = R$ into normal form i.e from which the 1st Derivative is absent

OR

2) Reduce the DE $y'' + Py' + Qy = R$, where P, Q, R are functions of x to the form

$$\frac{d^2v}{dx^2} + Iv = S$$

where $I = Q - \frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx}$.

$$S = \frac{R}{u}.$$

$$u = e^{-\frac{1}{2}\int P\,dx}.$$

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• $y'' + Py' + Qy = R$

$y = uv$

$$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} + v\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) = R$$

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} + \frac{v}{u}\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) = R/u \quad \text{--- (1)}$$

$$P + \frac{2}{u}\frac{du}{dx} = 0.$$

$$\frac{1}{u}du = -\frac{1}{2}P\,dx$$

$$\log u = -\frac{1}{2}\int P\,dx$$

$$u = e^{-\frac{1}{2}\int P\,dx}.$$

$$\frac{du}{dx} = -\frac{1}{2}Pu$$

$$\frac{d^2u}{dx^2} = -\frac{1}{2}\left[P\frac{du}{dx} + u\frac{dP}{dx}\right]$$

$$\frac{d^2u}{dx^2} = -\frac{1}{2}\left(-\frac{1}{2}P^2u + u\frac{dP}{dx}\right)$$

$$\frac{d^2u}{dx^2} = \frac{P^2u}{4} - \frac{u}{2}\frac{dP}{dx}.$$

Putting values in ①.

$$\frac{d^2v}{dx^2} + \frac{v}{u}\left(\frac{P^2u}{4} - \frac{u}{2}\frac{dP}{dx} - \frac{P^2}{2}u + Qu\right) = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + v\left(\frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{2} + Q\right) = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + \underbrace{\left(Q - \frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx}\right)}v = \frac{R}{u}$$

$\Downarrow$

Constant : Linear DE with constant coeff

$\frac{1}{x^2}$: $\left(x^2\frac{d^2v}{dx}\right)$ Homogenous linear Differential Equation

Q Solve: $y'' - \frac{2}{x}y' + \left(1 + \frac{2}{x^2}\right)y = xe^x$

$P = -\frac{2}{x}$

$Q = 1 + \frac{2}{x^2}$

$R = xe^x$

To remove $\frac{dy}{dx}$, $u = e^{-\frac{1}{2}\int P\,dx}$.

$$u = e^{-\frac{1}{2}\int -\frac{2}{x}dx}$$

$$u = e^{\int \frac{1}{x}dx} = x.$$

Let $y = uv$ be complete solution.

DE reduces to

$$\frac{d^2v}{dx^2} + \left(Q - \frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx}\right)v = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + \left[1 + \frac{2}{x^2} - \frac{4}{4x^2} - \frac{1}{2}\left(\frac{2}{x^2}\right)\right]v = \frac{xe^x}{x}$$

$$\frac{d^2v}{dx^2} + v = e^x.$$

$$(D^2+1)v = e^x.$$

$$D^2+1=0$$
$$D = \pm i$$

$$CF = C_1\cos x + C_2 \sin x.$$

$$PI = \frac{1}{D^2+1}e^x = \frac{e^x}{2}.$$

$$v = C_1\cos x + C_2\sin x + \frac{e^x}{2}.$$

$$y = uv = C_1 x\cos x + C_2 x\sin x + \frac{xe^x}{2}$$

Q Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-3)y = e^{x^2}$

$P = -4x$
$Q = 4x^2 - 3$
$R = e^{x^2}$

$y = uv$

for removal of 1^{st} derivative:

$u = e^{-\frac{1}{2}\int P dx}$

$u = e^{+\frac{1}{2}\int 4x dx}$

$u = e^{+x^2}$

DE reduces to

$\frac{d^2v}{dx^2} + \left(Q - \frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx}\right)v = \frac{R}{u}.$

$\frac{d^2v}{dx^2} + \left[\cancel{4x^2} - 3 - \cancel{4x^2} - \frac{1}{2}(-4)\right]v = \frac{e^{x^2}}{e^{+x^2}}$

$\frac{d^2v}{dx^2} - V = 1$

$(D^2-1)v = 1$

$D = \pm 1.$

$CF = \cancel{C_1e^{v} + C_2e^{v}} \quad C_1e^{x} + C_2e^{-x}$

$$PI = \frac{e^{0x}}{D^2-1} = -1$$

$$v = c_1 e^{x} + c_2 e^{-x} - 1$$

$$y = e^{x^2}\left[c_1 e^{x} + c_2 e^{-x} - 1\right].$$

Q $x^2(\log x)^2 \frac{d^2y}{dx^2} - 2x(\log x)\frac{dy}{dx} + \left[2+\log x - 2(\log x)^2\right]$

$$y = x^2(\log x)^3$$

$$\frac{d^2y}{dx^2} - \frac{2}{x\log x}\frac{dy}{dx} + \left[\frac{2+\log x - x(\log x)^2}{x^2(\log x)^2}\right]y = y = \log x$$

here $u = e^{-\frac{1}{2}\int P\,dx}$

$$u = e^{-\frac{1}{2}\int \frac{-2}{x\log x}dx}$$

$$u = e^{\int \frac{1}{x\log x}dx}$$

$\log x = t$

$\frac{1}{x}dx = dt$

$$u = e^{\log t}$$

$$u = t = \log x.$$

Let uv be complete solution

DE reduces to:

$$\frac{d^2v}{dx^2} + \left(\frac{2+\log x - 2(\log x)^2}{x^2(\log x)^2} - \frac{1}{x^2(\log x)^2} - \frac{1}{2} \right.$$

$$\left. \frac{d}{dx} \frac{-2}{x\log x} \right) v = \frac{x\log x}{\log x}$$

$$\frac{d^2v}{dx^2} + \left(\cancel{\frac{2}{x^2(\log x)^2}} + \cancel{\frac{1}{x^2(\log x)}} - \frac{2}{x^2} - \cancel{\frac{1}{x^2(\log x)^2}} + \right.$$

$$\left. - \cancel{\frac{1}{x^2}\frac{1}{(\log x)^2}} - \cancel{\frac{1}{x^2\log x}} \right) v = 1$$

$$\frac{d^2v}{dx^2} - \frac{2v}{x^2} = 1$$

$$x^2\frac{d^2v}{dx^2} - 2v = x^2$$

$$x = e^z$$

$$[D(D-1)-2]v = e^{2z}$$

$$v = \frac{1}{D^2-D-2} e^{2z}$$

CF = $D^2 - D - 2 = 0$.

$$\frac{1 \pm \sqrt{1+8}}{2} = \frac{1\pm 3}{2} = \underline{\underline{2, -1}}$$

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$CF = c_1 e^{2z} + c_2 e^{-z}$

$CF = c_1 x^2 + \frac{c_2}{x}$.

$PI = \frac{1}{D^2 - D - 2} e^{2z}$

$PI = \frac{z e^{2z}}{2D - 1}$

$PI = \frac{z e^{2z}}{3}$.

$PI = \frac{x^2 \log x}{3}$.

$V = c_1 x^2 + \frac{c_2}{x} + \frac{x^2 \log x}{3}$.

$y = (\log x)\left(c_1 x^2 + \frac{c_2}{x} + \frac{x^2 \log x}{3} \right)$.

Solve $\frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + (4x^2 - 1) y = -3e^{x^2} \sin 2x$

Let $u = e^{-\frac{1}{2}\int P\,dx} = u = e^{-\frac{1}{2}\int -4x\,dx} = e^{2x^2}$.

$u = e^{x^2}$.

The Eq of reduces to

$$\frac{d^2v}{dx^2} + \left[4x^2 + 1 - \frac{4x^2}{0} + \frac{1}{2}4\right]v = -3\sin 2x$$

$$\frac{d^2v}{dx^2} + v = 3\sin 2x$$

$(D^2+1)v = -3\sin 2x$

$D = \pm i$

$CF = c_1\cos x + c_2\sin x$

$PI = \frac{-3\sin 2x}{D^2+1}$

$PI = \frac{-3\sin 2x}{-3}$

$PI = \sin 2x$

$v = c_1\cos x + c_2\sin x + \sin 2x$

$y = e^{x^2}\left[c_1\cos x + c_2\sin x + \sin 2x\right]$

Q $y'' - 2\tan x\, y' + 5y = \sec x\, e^x$

Q $x^2y'' - 2x(3x-2)y' + 3x(3x-4)y = e^{3x}$

$''$

3) Transformation of Equation by changing the independent variable

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dz}\frac{dz}{dx}\right).$$

$$\frac{d^2y}{dx^2} = \frac{d}{dz}\left(\frac{dy}{dz}\frac{dz}{dx}\right)\frac{dz}{dx}.$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\left(\frac{d}{dz}\frac{dz}{dx}\right)\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$$

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2} + P\frac{dy}{dz}\frac{dz}{dx} + Qy = R$$

P_1 Q_1 R_1

$$\frac{d^2y}{dz^2} + \frac{\left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz} + \frac{Q}{\left(\frac{dz}{dx^2}\right)^2}y = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

Question can be solved only when P_1 & constant.

Q. Solve:

$$x^6y'' + 3x^5y' - a^2y = \frac{1}{x^2}$$

$$y'' + \frac{3}{x}y' - \frac{a^2}{x^6}y = \frac{1}{x^8}$$

$$P = \frac{3}{x}, \quad Q = \frac{a^2}{x^6}, \quad R = \frac{1}{x^8}.$$

choose z $\left(\frac{dz}{dx}\right)^2 = \frac{a^2}{x^6}$. (only +ve).

$$\frac{dz}{dx} = \frac{a}{x^3}$$

$\frac{x^{-3+1}}{-3+1}$

$$z = \frac{-a}{2x^2}$$

with this change, eq ① reduces to

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

x^{-3}

$-3x^{-4}$

when $P_1 = \dfrac{d^2y}{dz^2}$ $\dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}$

$$P_1 = \frac{-\frac{3a}{x^4} + \frac{Pa}{x^3}}{\frac{a^2}{x^6}} = \frac{-\frac{3a}{x^4} + \frac{3a}{x^4}}{\frac{a^2}{x^7}}$$

$$P_1 = 0$$

$$Q_1 = \frac{\frac{a^2}{x^6}}{\frac{a^2}{x^2}} = 1$$

$$R_1 = \frac{1}{x^8} \frac{x^6}{a^2} = \frac{1}{a^2x^2}$$

$$\frac{d^2y}{dz^2} + y = \frac{1}{a^2x^2}$$

$$\cancel{(D^2+1)y = \frac{-1}{a^2}\frac{\alpha}{2z}} =$$

$$\frac{d^2y}{dz^2} + y = \frac{-1}{a^2}\frac{2z}{a} = \frac{-2z}{a^3}$$

$$(D^2+1)y = \frac{-2z}{a^3}$$

$D = \pm i$

$CF = C_1 \cos z + C_2 \sin z$

$PI = \frac{-2}{a^3} \frac{1}{D^2+1} z$

$PI = \frac{-2}{a^3} [1 + D^2]^{-1} z$

$PI = -\frac{2}{a^3} [1 - D^2] z$.

$PI = -\frac{2}{a^3} [z]$

$y = C_1 \cos z + C_2 \sin z - \frac{2z}{a^3}$

$y = C_1 \cos\left(\frac{+a}{2x^2}\right) \mp C_2 \sin\left(\frac{+a}{2x^2}\right) - \frac{1}{a^2 x^2}$.

Q

$xy'' - y' + 4x^3 y = x^5$

$y'' - \frac{1}{x} y' + 4x^2 y = x^4$.

$P = -\frac{1}{x}, \quad Q = 4x^2, \quad R = x^4$

$\left(\frac{dz}{dx}\right)^2 = 4x^2$

$$\frac{dz}{dx} = 2x$$

$$z = x^2$$

PE reduces to

$$\frac{d^2y}{dz^2} + \left(\frac{2 + \frac{2x}{x}}{4x^2}\right) y' + y = \frac{x^2}{4}$$

$$\frac{d^2y}{dz^2} + y = \frac{x^2}{4}$$

$$(D^2+1)y = \frac{z}{4}$$

$$D = \pm i$$

$$CF = C_1 \cos z + C_2 \sin z$$

$$PI = \frac{1}{4}\left(\frac{1}{D^2+1}\right) z$$

$$PI = \frac{1}{4}\left[1+D^2\right]^{-1} z$$

$$PI = \frac{1}{4}\left[1-D^2\right] z$$

$$PI = \frac{z}{4}$$

$$PI = \frac{x^2}{4}$$

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$y = c_1 \cos x^2 + c_2 \sin x^2 + \frac{x^2}{4}$.

Q. $\cos x\, y'' + \sin x\, y' - 2\cos^3 x\, y = 2\cos^5 x$

$y'' + \tan x\, y' - 2\cos^2 x\, y = 2\cos^4 x$.

$\left(\frac{dz}{dx}\right)^2 = 2\cos^2 x$.

$\frac{dz}{dx} = \sqrt{2\cos x} \cdot \sqrt{2}\cos x$.

$\frac{1 - 2\sin x}{2\sqrt{2\cos x}}$

$z = \sqrt{2}\int (\cos x)^{1/2}\, dx$.

$\frac{\sin x}{\sqrt{2\cos x}}$

$\int \left\{ \begin{array}{l} \cos x = t^2 \\ -\sin x\, dx = 2t\, dt \end{array} \right.$

$\frac{d^2y}{dz^2} + \frac{\frac{-\sin x}{\sqrt{2\cos x}} + \tan x\sqrt{2\cos x}}{2\cos x} =$

$z = \sqrt{2}\sin x$

$z^2 = 2\sin^2 x$

$\frac{z^2}{2} = \sin^2 x$

$\frac{d^2y}{dz^2} + \left(\frac{-\sqrt{2}\sin x + \tan x\sqrt{2}\cos x}{2\cos^2 x}\right)\frac{dz}{dx}$

$y = \cos^2 x$

$$\frac{d^2y}{dz^2} - y = \cos^2 x.$$

$$(D^2-1)\,y = \cos^2 x$$

$$D = \pm 1.$$

$$CF = \cancel{C_1\cos} \; C_1e^{z} + C_2e^{-z}$$

$$CF = C_1 e^{\sqrt{2}\sin x} + C_2 e^{-\sqrt{2}\sin x}.$$

$$PI = \left(\frac{1}{D^2-1}\right)\left(1 - \frac{z^2}{2}\right)$$

$$PI = -1 - \frac{1}{2}\,\frac{1}{D^2-1}\,z^2$$

$$PI = -1 + \frac{1}{2}\left(1-D^2\right)^{-1} z^2$$

$$PI = -1 + \frac{1}{2}\left[1 + D^2 - kD^4\right] z^2$$

$$PI = -1 + \frac{1}{2}\left(z^2 + 2\right)$$

$$PI = \frac{z^2}{2}.$$

$$PI = \frac{\sin^2 x}{2}.$$

$$y = C_1 e^{\sqrt{2}\sin x} + C_2 e^{-\sqrt{2}\sin x} + \frac{\sin^2 x}{2}.$$

Q. $y'' + (3\sin x - \cot x)y' + 2\sin^2 x\, y = e^{-\cos x}\sin^3 x$

Q. $(1+x^2)y'' + 2x(1+x^2)y' + 4y = 0$

Q. $xy'' - y' - 4x^2 y = 8x^3 \sin x^2$

Q. $y'' - \cot x\, y' - (\sin^2 x)y = \cos x - \cos^3 x$.

THEOREM

If $y = y_1(x)$ and $y = y_2(x)$ are two solutions of the equation $y'' + Py' - Qy = 0$ where P and Q are continuous function of x, then show that

$$y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} = ce^{-\int P dx}$$

$$\frac{d^2 y_1}{dx^2} - P\frac{d}{dx}y_1 - Qy_1 = 0 \quad -(1) \quad \times y_2$$

$-$

$$\frac{d^2 y_2}{dx^2} - P\frac{dy_2}{dx} - Qy_2 = 0 \quad -(2) \quad \times y_1$$

$$y_1 \frac{d^2 y_2}{dx^2} - y_2 \frac{d^2 y_1}{dx^2} + P\Big[\underbrace{y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx}}_{v}\Big] = 0$$

$$v = y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx}$$

$$\frac{dv}{dx} = y_1 \frac{d^2 y_2}{dx^2} - y_2 \frac{d^2 y_1}{dx^2}$$

$$\frac{dv}{dx} + Pv = 0.$$

$$IF = e^{\int P dx}$$

$$v e^{\int P dx} = c$$

$$v = c e^{-\int P dx}$$

$$y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} = c e^{-\int P dx}$$

(4) Variation of Parameter

$$y'' + Py' + Qy = R$$

Let $y = au + bv$ be CF. a, b are constants.

u, v are function of x.

$$\left.\begin{aligned} u'' + Pu' + Qu = 0 \\ v'' + Pv' + Qv = 0 \end{aligned}\right\}$$ as u, v are part of CF, it will satisfy Auxillary Eq^n

Let $y = Au + Bv$ be complete solution, such that $A_1 u + B_1 v = 0$. (A, B are function of x)

$$y_1 = Au_1 + Bv_1 + \cancel{A_1 u + B_1 v}^{\,0} = Au_1 + Bv_1$$

$$y_2 = Au_2 + Bv_2 + A_1 u_1 + B_1 v_1$$

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Put y_2, y_1, in, y in DE

$$Au_2 + Bv_2 + A_1u_1 + B_1v_1 + P(Au_1 + Bv_1) + Q(Au + Bv) = R$$

$$A\underbrace{[u_2 + Pu_1 + Qu]}_{=0} + B\underbrace{(v_2 + Pv_1 + Qv)}_{=0} + A_1u_1 + B_1v_1 = R$$

$$A_1u_1 + B_1v_1 = R$$

$$A_1u_1 + B_1v_1 - R = 0$$

& $\quad A_1u + B_1v = 0$

$$\frac{A_1}{+vR} = \frac{B_1}{-uR} = \frac{1}{u_1v - uv_1}$$

$$\frac{dA}{dx} = A_1 = \frac{vR}{u_1v - uv_1} \Rightarrow A = f(x) + c_1$$

$$\frac{dB}{dx} = B_1 = \frac{uR}{uv_1 - u_1v} \Rightarrow B = g(x) + c_2$$

$y = \underline{\underline{Au + Bv}}$ (Put values of A & B & get the answer)

Q Apply the method of variation of parameters to solve

$$y_2 + 4y = 4\tan 2x.$$

$D = \pm 2i \qquad (D^2 + 1 = 0)$

C.F. $\quad y = a\cos 2x + b\sin 2x$

Let $y = A\cos 2x + B\sin 2x$ be compl. solution such that

$$A_1 u + B_1 v = 0$$
$$A_1 \cos 2x + B_1 \sin 2x = 0 \quad \text{--- (1)}$$

by putting y, y_1, y_2 in DE, It reduces to

$$A_1 u_1 + B_1 v_1 = R \longrightarrow$$

$$-2A_1 \sin 2x + 2B_1 \cos 2x = 4\tan 2x$$

$$B_1 \cos 2x - A_1 \sin 2x = 2\tan 2x \quad \text{--- (2)}$$

from (1) & (2)

$$A_1 = \frac{\sin 2x (4\tan 2x)}{(-2\sin 2x)\sin 2x) - 2(\cos 2x)\cos 2x)}$$

$$A_1 = \frac{-2\sin 2x \tan 2x}{\sin^2 2x + \cos^2 2x}$$

$$A_1 = -2\sin 2x \tan 2x .$$

$$A = -2\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$B_1 = \frac{2\tan 2x\, \cancel{\sin 2x} \cos 2x}{1} = 2\sin 2x .$$

$$B = 2\int \sin 2x\,dx$$

$$B = -\cos 2x + C_2$$

$$A = -2\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$A = -2\int \frac{1-\cos^2 2x}{\cos 2x}\,dx$$

$$A = -2\int \sec 2x - \cos 2x\,dx$$

$$A = -\log[\sec 2x + \tan 2x] + \sin 2x + C_2$$

$$y = [\sin 2x - \log[\sec 2x + \tan 2x] + C_2]\cos 2x$$
$$(-\cos 2x + C_1)\sin 2x$$

$$y = C_2\cos 2x + C_1\sin 2x - \cos 2x\log[\sec 2x + \tan 2x]$$
$$+ \cos 2x\sin 2x - \cos 2x\sin 2x$$

$$y = C_2\cos 2x + C_1\sin 2x - \cos 2x\log(\sec 2x + \tan 2x)$$

Q Apply the method of variation of parameter

$$y_2 - y = \frac{2}{1+e^x}$$

$$(D^2 - 1) = 0$$
$$D = \pm 1$$

$$CF = ae^{x} + be^{-x}$$

let $y = Ae^{x} + Be^{-x}$ be complete solution

such that $A_1 e^{x} + B_1 e^{-x} = 0$.

$$\Rightarrow \quad A_1 e^{x} - B_1 e^{-x} = \frac{2}{1+e^{x}}$$

$$\frac{-A_1}{+\frac{2e^{-x}}{1+e^{x}}} = \frac{+B_1}{\frac{2e^{x}}{1+e^{x}}} = \frac{1}{-2}.$$

$$A_1 = \frac{e^{-x}}{1+e^{x}}.$$

$$B_1 = \frac{-e^{x}}{1+e^{x}}.$$

$$Aq- \quad A = \int \frac{e^{-x}}{1+e^{x}}\,dx$$

$$e^{-x} = t$$

$$-e^{-x}\,dx = dt$$

$$A = -\int \frac{1}{1+\frac{1}{t}}\,dt = -\int \frac{t+1-1}{t+1}\,dt.$$

$$= -\int 1 - \frac{1}{t+1}\,dt$$

$$A = -e^{-x} + \log(e^{-x}+1] + c_1$$

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$B = -\int \frac{e^x}{1+e^x}\,dx$

$e^x = t$

$B = -\log(1+e^x) + C_2$

$y = [-e^{-x} \cancel{+} \log(e^{-x}+1) + C_1] e^x$

$+ [-\log(1+e^x) + C_2] e^{-x}$

$\cancel{y_p} = \quad y = C_1 e^x + C_2 e^{-x} - 1 + e^x \log(1+e^{-x})$

$- e^{-x}(\log(1+e^x))$

Q $x^2 y_2 + x y_1 - y = x^2 e^x$

$[D(D-1) + D - 1] = 0$

$D^2 - 1 = 0$

$D = \pm 1.$

$CF = C_1 e^z + C_2 e^{-z} = C_1 x + \frac{C_2}{x}.$

Q verify that e^x and x are solution of the homogeneous eq^n corresponding to

$$(1-x)y_2 + xy_1 - y = 2(x-1)^2 e^{-x}$$

$$0 < x < 1$$

then find its general solution

solution

$$W = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} \neq 0$$

derivative of soln

linearly independent so we can take these solution

Q Apply the methods of variation of parameter to solve the eq^n.

$$\cancel{(x+2)y_2} \quad (x+2)y_2 - (2x+5)y_1 + 2y = (x-1)e^x$$

Q Using method of variation of parameter, solve DE,

$$(x-1)D^2y + xDy + y = (x-1)^2$$

5. Solution by Operators.

Q Solve:

$xy'' + (1-x)y' - y = e^x$

So. $[xD^2 + (1-x)D - 1]y = e^x$

$[xD^2 + D - xD - 1]\ y = e^x$

$\{D[xD + 1] - 1[xD+1]\}\ y = e^x$

$[(xD+1)(D-1)]\ y = e^x$

Solve to check whether differential equation is tommia or not

$(xD+1)(D-1)$ 2. $(D-1)(xD-1)$

$xD^2 - xD + D - 1$ $(D)^2 + xD^2 + D - xD - 1$

$\underbrace{xD^2 + (1-x)D - 1}$ ✗

✓

$(xD+1)\underbrace{(D-1)y}_{v} = e^x$

$(xD+1)v = e^x$

$x\dfrac{dv}{dx} + v = e^x$

$\dfrac{dv}{dx} + \dfrac{1}{x}v = \dfrac{1}{x}e^x$

$IF = x$

$vx = e^{x} + c_1$

$(D-1)y = \frac{e^{x}}{x} + \frac{c_1}{x}$

$\frac{dy}{dx} - y = \frac{e^{x}}{x} + \frac{c_1}{x}$ $\Big\}$ solve to get y

$IF = \cancel{x} \cdot e^{-x}$

$\cancel{xy = e^{x} + c_1 x + c_2}$

$\cancel{y = \frac{e^{x}}{x} + c_1 + \frac{c_2}{x}}$

$ye^{-x} = \log x + \int \frac{c_1 e^{-x}}{x} dx + c_2$

$y = e^{x}\log x + e^{x}c_1 \int \frac{e^{-x}}{x} dx + c_2 e^{x}$

Q Solve:

$((x+2)D^2 - (2x+5)D + 2)y = (x+1)e^{x}$

Q Solve:

$xy'' + (x-2)y' - 2y = x^3$

$((x+2)D^2 - 2(x+2)D - D + 2)y = (x+1)e^{x}$

$\{(x+2)D[D-2] - (D-2)\}y = (x+1)e^{x}$

$$[(x+2)D-1](D-2)]\underbrace{y}_{v} = (x+1)e^{x}$$

$D-$

$$[(x+2)D-1]v = (x+1)e^{x}$$

$$(x+2)\frac{dv}{dx} - v = (x+1)e^{x}$$

$$\frac{dv}{dx} - \frac{1}{x+2}v = \left[\frac{x+1}{x+2}\right]e^{x}$$

$$e^{-} \quad IF = e^{-\int \frac{1}{x+2}dx}$$

$$IF = e^{-\log(x+2)}$$

$$IF = \frac{1}{x+2}$$

$$\frac{v}{x+2} = \int \frac{x+1}{(x+2)^2} e^{x}$$

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To Solve DE : (FROBENIUS METHOD)

$$f(x)y'' + g(x)y' + r(x)y = 0 \quad \text{—①}$$

Let $y = \sum_{m=0}^{\infty} C_m x^{m+k}$ —② $C_0 \neq 0$ be series solution of eq ①

$y' = \sum C_m (m+k) x^{m+k-1}$

$y'' = \sum C_m (m+k)(m+k-1) x^{m+k-2}$ } after substituting in ①

Eq ① reduces to identity

Equate to zero the coefficient of smallest power of x in identity, we get Quadratic Equation in k. It is called Indicial Eqn

Solve the Indicial Eq and following cases arrive

1. Roots unequal & not differing by integer

2. Roots unequal, differing by integer & making coefficient of y indeterminate

3. Roots of Indicial equation unequal, differing by integer & making coefficient of y infinite

(4) Roots of eq^n equal

Equate to zero the coefficient of general power eg x^{m+k}, x^{m+k-1} whichever is lowest in the identity, we get recurrence relation because it connects the coefficients, C_m, C_{m-1} or C_m, C_{m-1}

If the recurrence relation connects C_m, C_{m-2} then in general C_1 can be obtained by equating to zero the coefficient of next higher power (than already used to get indicial eq^n)

After getting various eq^n coefficient, the solution can be obtained by substituting these coeff in Eq 2

Type-I Roots of indicial eq^n are unequal & not differing by integer

Q Solve by series

$$9x(1-x)y'' - 12y' + 4y = 0$$

Let $y = \Sigma C_m x^{m+k}$, $C_0 \neq 0$

$$y' = \Sigma C_m (m+k)\, x^{m+k-1}$$

$$y'' = \Sigma C_m (m+k)(m+k-1)\, x^{m+k-2}$$

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$9x(1-x)\sum C_m (m+k)(m+k-1)x^{m+k-2} - 12\sum C_m (m+k)$
$x^{m+k-1} + 4\sum C_m x^{m+k} = 0$

$9\sum C_m (m+k)(m+k-1)\, x^{m+k-1} - 9\sum C_m (m+k)(m+k-1)x^{m+k}$

$-12\sum C_m (m+k)\, x^{m+k-1} + 4\sum C_m x^{m+k} = 0$

$3\sum C_m (m+k)(3m+3k-7)x^{m+k-1} - \sum C_m [9(m+k)^2 - 9(m+k) - 4]$
$x^{m+k} = 0$

$3\sum C_m (m+k)(3m+3k-7)x^{m+k-1} - \sum C_m [9(m+k)^2 - 12(m+k) - 3$
$(m+k) - 4]x^{m+k} = 0$

$3\sum C_m (m+k)(3m+3k-7)x^{m+k-1} - \sum C_m (3m+3k-4)(3m+3k+1)$
$x^{m+k} = 0$

Coeff of $x^{k-1} = 0$

$3C_0 k(3k-7) = 0$ Indicial Eqn

$k = 0, 7/3$

(both are different & difference is not integer)

Coeff of $x^{m+k-1} = 0$

$3C_m (m+k)(3m+3k-7) - C_{m-1}(3m+3k-7)(3m+3k-2) = 0$

$C_m = \left(\dfrac{3m+3k-2}{3(m+k)}\right) C_{m-1}$ recurrence relation for $K=0$

$$C_m = \frac{3m-2}{3m} C_{m-1}$$

$$C_1 = \frac{1}{3} C_0$$

$$C_2 = \frac{4}{6} C_1 = \frac{2}{9} C_0$$

$$C_3 = \frac{7}{9} C_2 = \frac{14}{81} C_0$$

$$y = \sum C_m x^{m+k}$$

$$y = x^k \left[\sum C_m x^m\right]$$

$k=0$

$$y = \sum C_m x^m$$

$$y = C_0 + C_1 x + C_2 x^2 + \cdots$$

$$y = \underbrace{C_0}_{a} \underbrace{\left[1 + \frac{x}{3} + \frac{2}{9}x^2 + \frac{14}{81}x^3 + \cdots\right]}_{u}$$

$$y = au$$

Simillary for $k = 7/3$,

we get $y = bv$

<u>Complete solution is</u>

$$y = au + bv$$

Q Solve:

$$x^2y'' + xy' + (x^2-n^2)y = 0$$

in series taking $2n$ as non-integral

Let

$$y = \sum_{k=0}^{\infty} C_m x^{m+k}$$

$$y' = \sum_{k=0}^{\infty} C_m (m+k) x^{m+k-1}$$

$$y'' = \sum_{k=0}^{\infty} C_m (m+k)(m+k-1) x^{m+k-2}$$

$$x^2 \sum_{k=0}^{\infty} C_m (m+k)(m+k-1) x^{m+k-2} + x \sum C_m (m+k) x^{m+k-1}$$

$$+ (x^2-n^2) \sum C_m x^{m+k} = 0$$

$$\sum_{k=0}^{\infty} C_m (m+k)(m+k-1) x^{m+k} + x \sum C_m (m+n) x^{m+k}$$

$$+ x \sum C_m x^{m+k+2} - n^2 \sum C_m x^{m+k} = 0$$

$$\sum_{k=0}^{\infty} \quad \sum C_m \left[(m+k)^2 \cancel{- (m+k) + (m+k)} - n^2 \right] x^{m+k}$$

$$+ \sum C_m x^{m+k+2} = 0$$

$$\sum C_m \left[(m+k)^2 - n^2 \right] x^{m+k} + \sum C_m x^{m+k+2} = 0$$

coef of $x^k = 0$.

$C_0 k^2 = 0$

$C_0 (k^2 - n^2) = 0$

$k = \pm n$.

coef. of $x^{m+k} = 0$.

$C_m (m+k-n)(m+k+n) + C_{m-2} = 0$.

$$\boxed{C_m = \frac{-1}{(m+k-n)(m+k+n)} C_{m-2}}$$

coef of $x^{k+1} = 0$.

$C_1 (1+k-n)(1+k+n) = 0$.

$C_1 = \frac{0}{(1+k-n)(1+k+n)} = 0$

as $1+k-n \neq 0$
$1+k+n \neq 0$
for $k \in \{n, -n\}$.

$C_1 = C_3 = C_5 = \cdots = 0$.

$\boxed{\text{for } k = n}$

$m = 2$

$C_2 = \frac{-1}{2(2+2n)} C_0$.

$C_2 = \frac{-1}{4(n+1)} C_0$

$$C_4 = \frac{-1}{4(4+2n)} C_2$$

$$C_4 = \frac{1}{8(n+2)} \frac{1}{4(n+1)} C_0$$

$$C_4 = \frac{1}{32(n+1)(n+2)} C_0$$

32
12
64
32
384

$$C_6 = \frac{-1}{6(6+2n)} C_4$$

$$C_6 = \frac{-1}{384(n+1)(n+2)(n+3)} C_0$$

$$y = \sum C_m x^{m+n}$$

$$y = x^n \sum C_m x^m$$

$$y = x^n (C_0 + C_1 x + C_2 x^2 + \cdots)$$

$$y = x^n \left(C_0 - \frac{C_0 x^2}{4(n+1)} + \frac{C_0 x^4}{32(n+1)(n+2)} - \frac{C_0 x^6}{384(n+1)(n+2)(n+3)}\right)$$

$$y = x^n C_0 \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{32(n+1)(n+2)} - \frac{x^6}{384(n+1)(n+2)(n+3)} + \cdots\right]$$

$$y = a x^n \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{32(n+1)(n+2)} - \frac{x^6}{384(n+1)(n+2)(n+3)} + \cdots\right]$$

Case: Roots unequal differing by an integer & making a coeff of y indeterminate:

Rule: If indicial equation have 2 unequal roots k_1, k_2 & one of the coeff of y becomes indeterminate when $k = k_2$, then complete solution is given by $k = k_2$ in the y from lower of auxiliary equation. By putting $k = k_1$ in y gives a numerical multiple of one of the series in k_2 hence we reject it because solution should be linearly independent.

Q $(1-x^2)y'' - xy' + 4y = 0$

$$y = \sum C_m x^{m+k}$$

$$y' = \sum C_m (m+k) x^{m+k-1}$$

$$y'' = \sum C_m (m+k)(m+k-1) x^{m+k-2}$$

$$(1-x^2) \sum C_m (m+k)(m+k-1) x^{m+k-2} - x \sum C_m (m+k) x^{m+k-1}$$

$$+ 4 \sum C_m x^{m+k} = 0$$

$$\sum C_m (m+k)(m+k-1) x^{m+k-2} + \sum C_m \left[-(m+k)^2 - (m+k) - (m+k) + 4 \right] x^{m+k} = 0$$

$$\sum C_m (m+k)(m+k-1) x^{m+k-2} + \sum C_m \left[4 - (m+k)^2 \right] x^{m+k} = 0$$

coeff of $x^{k-2} = 0$

$c_0 k(k-1) = 0$

$k = 0, k = 1$.

coeff of $x^{m+k-2} = 0$

$c_m (m+k)(m+k-1) - c_{m-2}(m+k)(m+k-4) = 0$

$c_m = \frac{m+k-4}{m+k-1} c_{m-2}$

coeff of $x^{k-1} = 0$

$\cancel{c_1 (m+1)(m) = 0}$.

$c_1 (k+1)(k) = 0$

$c_1 = \frac{0}{k(1+k)}$.

for $k=0$, $c_1 = \frac{0}{0}$. (indeterminant)

for $k=0$

$c_m = \frac{m-4}{m-1} c_{m-2}$

$c_2 = -2c_0$.

$c_4 = 0, = c_6 = c_8 = \dots$

$c_3 = -\frac{1}{2} c_1$

$c_5 = \frac{1}{4} c_3$

$c_5 = -\frac{1}{8} c_1$

$y = x^0 (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots)$

$y = c_0 + c_1 x - 2 c_0 x^2 - \frac{c_1}{2} x^3 + 0 - \frac{c_1}{8} x^5 + \cdots$

$y = c_0 (1 - 2x^2) + c_1 (x - \frac{1}{2} x^3 - \frac{1}{8} x^5 + \cdots)$

$y = au + bv$.

Q Solve

$(1-x^2) y'' - 2x y' + n(n+1) y = 0$.

$y = \Sigma c_m x^{m+k}$

$(1-x^2) \Sigma c_m (m+k)(m+k-1) x^{m+k-2} - 2x \Sigma c_m (m+k) x^{m+k-1}$

$+ n(n+1) \Sigma c_m x^{m+k} = 0$.

$\Sigma c_m (m+k)(m+k-1) x^{m+k-2} + \Sigma c_m \left[-\{(m+k)^2 - (m+k)\} - 2(m+k) \right.$

$\left. + n(n+1) \right] x^{m+k} = 0$

$$\sum C_m (m+k)(m+k-1) x^{m+k-2} + \sum C_m \left[(m+k)^2 + (m+k) - n(n+1) \right] x^{m+k} = 0$$

Coeff of $x^{k-2} = 0$.

$C_0 \, k(k-1) = 0$

$k = 0, \; k = 1.$

Coeff of $x^{m+k-2} = 0$

$$C_m (m+k)(m+k-1) + C_{m-2} \left[(m+k-2)^2 + (m+k-2) - n(n+1) \right] = 0$$

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Type-3 Roots of indicial eqⁿ unequal differing by a integer and making coefficient of y infinite

If indicial eqⁿ has 2 unequal roots $k_1 < k_2$ & if some of the coeff of y becomes ∞ when $k = k_2$, we modify y by replacing c_0 to $d_0 (k - k_2)$. We then obtain 2 independent solution by putting $k = k_2$ in the modified form of y and by putting $k = k_1$ in $\frac{\partial y}{\partial k}$.

The result of putting $k = k_1$ in y gives a numerical multiple of that obtained by putting $k = k_2$. we reject the solution obtained by putting $k = k_1$

Q. Solve in series:

$$x(1-x)y'' - 3xy' - y = 0$$

Ans $y = \sum c_m x^{m+k}$, $y' = \sum c_m (m+k) x^{m+k-1}$,

$$y'' = \sum c_m (m+k)(m+k-1) x^{m+k-2}$$

$$x(1-x) \sum c_m (m+k)(m+k-1) x^{m+k-2} - 3x \sum c_m (m+k) x^{m+k-1} - \sum c_m x^{m+k} = 0$$

$$\sum c_m (m+k)(m+k-1) x^{m+k-1} - 3 \sum c_m (m+k) x^{m+k} - \sum c_m (m+k)(m+k-1) x^{m+k} - \sum c_m x^{m+k} = 0$$

$$\sum c_m (m+k)(m+k-1) x^{m+k-1} - \sum c_m \left(3(m+k) + (m+k)^2 - (m+k) + 1 \right) x^{m+k} = 0$$

$$\sum C_m (m+k)(m+k-1) x^{m+k-1} - \sum C_m (m+k+1)^2 x^{m+k} = 0$$

coeff of $x^{k-1} = 0$; $m=0$.

$C_0 k(k-1) = 0$

$k = 0, 1$.

coeff of $x^{m+k-1} = 0$

$$C_m \cancel{(m+k)}(m+k-1) - C_{m-1} \cancel{(m+k)}^2 = 0.$$

$$C_m = \left(\frac{m+k}{m+k-1}\right) C_{m-1}$$

$$C_1 = \frac{k+1}{k} C_0$$

$$C_2 = \frac{2+k}{k} C_0.$$

$$C_3 = \frac{3+k}{k} C_0.$$

$$y = C_0 x^k \left[1 + \left(\frac{k+1}{k}\right) x + \frac{2+k}{k} x^2 + \cdots \right]$$

$k=0$, then terms is infinite.

$$C_0 = d_0 (k-0) = d_0 k$$

$$y = d_0 x^k \left[k + (1+k) x + (2+k) x^2 + \cdots \right]$$

$$y = d_0 \left[x + 2x^2 + 3x^3 + \cdots \right] \quad \text{at } k=0$$

$\underset{u}{a}$

$$\frac{\partial y}{\partial k} = d_0 x^k \log x \left[x + 2x^2 + 3x^3 + - - - \right] + d_0 x^k \left[x + 4x + 9x^2 + - - - \right]$$

at $k = 0$,

∂

$$\frac{\partial y}{\partial k} = d_0 x^k \log x \left[k + (1+k)x + (1+k)x^2 + - - - \right] + d_0 x^k \left[1 + x + x^2 + - - - \right]$$

$$\frac{\partial y}{\partial k} = d_0 \log x \left[x + 2x^2 + - - - \right] + \underbrace{d_0 \left[1 + x + x^2 + - - - \right]}_{bv}$$

$\frac{\partial y}{\partial k}$

$$y = au + bv$$

Q $\quad x^2 y'' + x y' + (x^2 - 4) y = 0$

$$x^2 \sum C_m (m+k)(m+k-1) x^{m+k-2} + x \sum C_m (m+k) x^{m+k-1} + (x^2 - 4) \sum C_m x^{m+k}$$

$$\sum C_m \left[(m+k)^2 - \cancel{(m+k)} + \cancel{(m+k)} - 4 \right] x^{m+k} + \sum C_m x^{m+k+2}$$

$\sum c_m \left[(m+k+2)(m+k-2) x^{m+k} + \sum c_m x^{m+k+2} \right.$

coeff of $x^k = 0$

$c_0 (k+2)(k-2) = 0$

$k = \pm 2$

coeff of $x^{m+k} = 0$

$c_m (m+k+2)(m+k-2) + c_{m-2} = 0$

$$c_m = \frac{-1\, c_{m-2}}{(m+k+2)(m+k-2)}$$

coeff of $x^{k+1} = 0$

$\cancel{c_0 [c(k+k)^2 + 4]}$

$c_1 [k+3][k-1] = 0$

$$c_1 = \frac{0}{(k-1)(k+3)}$$

$0 \neq 0$

$c_1 = 0$

$c_1 = c_3 = c_5 = c_7 = \cdots = 0$

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$$C_2 = \frac{-1}{K(K+4)} C_0$$

$$C_4 = \frac{1}{K(K+2)(K+4)(K+6)} C_0$$

$$y = C_0 x^K \left[1 - \frac{1}{K(K+4)} x^2 + \frac{1}{\underbrace{K(K+2)(K+4)(K+6)}_{\infty}} x^4 + \cdots \right]$$

at $K = -2$

$$C_0 = d_0(K+2)$$

$$y = d_0 x^K \left[K+2 - \frac{K+2}{K(K+4)} x^2 + \frac{1}{K(K+4)(K+6)} x^4 + \cdots \right]$$

at $K = -2$

$$y = d_0 x^{-2} \left[\frac{1}{(-2)(2)4} x^4 + \cdots \right] \Big\} u$$

$$\frac{\partial y}{\partial K} = d_0 x^K \log x \left[K+2 - \frac{K+2}{K(K+4)} x^2 + \frac{1}{K(K+4)(K+6)} x^4 + \cdots \right]$$

$$+ d_0 x^K \left[1 - \frac{K+2}{K(K+4)} \left[\frac{1}{K+2} - \frac{1}{K} - \frac{1}{K+4} \right] x^2 + \left(\frac{1}{K(K+4)(K+6)} \right.\right.$$

$$\left[-\frac{1}{k}-\frac{1}{k+4}-\frac{1}{k+6}\right]x^4 + \ldots$$

at $k=-2$

$$\frac{\partial y}{\partial x} = d_0 x^{-2}\log x \left[\frac{1}{(-2)(2)(4)}x^4 + \ldots\right]$$

$$+ \underbrace{d_0 x^k}_{bv}\left[1+\frac{1}{(-2)(2)(4)}\left[\cancel{\frac{1}{2}}-\cancel{\frac{1}{2}}-\frac{1}{4}\right]x^4+\ldots\right]$$

$$y = au + bv$$

Type-4 Roots of indizial Eq^n Equal

If indizial eq^n has 2 equal roots, $k_1 = k_2 = k$, we obtain 2 independent solution by substituting this value of k in y & $\frac{\partial y}{\partial k}$.

Q Solve in series:

$$xy'' + y' + xy = 0$$

$$x\sum c_m(m+k)(m+k-1)x^{m+k-2} + \sum c_m(m+k)x^{m+k}$$

$$+ x\sum c_m x^{m+k} = 0$$

$$\sum c_m(m+k)^2 x^{m+k-1} + \sum c_m x^{m+k+1} = 0$$

coeff of $x^{k-1} = 0$

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$k^2=0; \; k=0,0$

Coeff of $x^{m+k-1} = 0$

$c_m (m+k)^2 + c_{m-2} = 0$

$c_m = -\frac{c_{m-2}}{(m+k)^2}$

Coeff of $x^k = 0$.

$c_1 (k+1)^2 = 0$

$c_1 = 0$

$c_1 = c_3 = c_5 = c_7 = 0$.

$c_2 = \frac{-c_0}{(k+2)^2}$

$c_4 = \frac{c_0}{(k+2)^2 (k+4)^2}$

$$y = c_0 x^k \left[1 - \frac{1}{(k+2)^2} x^2 + \frac{1}{(k+2)^2 (k+4)^2} x^4 + \cdots \right]$$

$k=0$

$$y = c_0 \left[1 - \frac{1}{2^2} x^2 + \frac{1}{2^2 4^2} x^4 + \cdots \right]$$

$y = au$.

$$\frac{\partial y}{\partial x} = C_0 x^k \log x \left[1 - \frac{1}{(2+k)^2} x^2 + \frac{1}{(2+k)^2 (4+k)^2} x^4 + \cdots \right]$$

$$+ C_0 x^k \left[\frac{-1}{(2+k)^2} \left[-\frac{2}{(2+k)} \right] x^2 + \frac{1}{(2+k)^2 (4+k)^2} \left[\frac{-2}{2+k} \frac{-2}{4+k} \right] - \cdots \right]$$

$k = 0$

$= 0$

$$\frac{\partial y}{\partial x} = C_0 \log x \left[1 - \frac{1}{2^2} x^2 + \frac{1}{2^2 4^2} x^4 + \cdots \right]$$

$$\underbrace{+ C_0 \left[\frac{-1}{2^2} \left[-\frac{2}{2} \right] x^2 + \frac{1}{2^2 4^2} \left[-\frac{2}{2} - \frac{2}{4} \right] + \cdots \right]}_{bv}$$

$= 0$

$y = au + bv$

Q $(x - x^2) y'' + (1 - x) y' - y = 0$

$(x - x^2) \sum C_m (m+k)(m+k-1) x^{m+k-2} + ($

$(1-x) \sum C_m (m+k) x^{m+k-1} - \sum C_m x^{m+k} = 0$

$= \sum C_m (m+k)^2 x^{m+k-1} - \sum C_m \left((m+k)^2 - (m+k) + (m+k) + 1 \right] x^{m+k} = 0$

$\sum C_m (m+k)^2 x^{m+k-1} - \sum C_m \left((m+k)^2 + 1 \right] x^{m+k} = 0$

coeff of $x^{k-1} = 0$

$k = 0, 0$

Coeff of $x^{m+k-1} = 0$

$$C_m(m+k)^2 - C_{m-1}\left[(m+k-1)^2+1\right] = 0$$

$$C_m = \frac{(m+k-1)^2+1}{(m+k)^2} C_{m-1}$$

$$C_1 = \frac{k^2+1}{(k+1)^2} C_0$$

$$C_2 = \left[\frac{(k+1)^2+1}{(k+2)^2}\right] \frac{k^2+1}{(k+1)^2} C_0$$

$$y = C_0 x^k \left[1 + \frac{k^2+1}{(k+1)^2} x + \left(\frac{(k+1)^2+1}{(k+2)^2}\right)\frac{(k^2+1)}{(k+1)^2} - - \right]$$

at $k = 0$

$$y = C_0 \left[1 + \frac{1}{1^2} x + \frac{2}{4} x^2 + - - \right]$$

$y = au$

$$\frac{\partial y}{\partial k} = ?$$

$$\frac{\partial y}{\partial k} = C_0 x^k \log x \left[1 + \frac{k^2+1}{(k+1)^2} x + \cdots \right]$$

$$+ C_0 x^k \left[\frac{k^2+1}{(k+1)^2} \left[\frac{2k}{k^2+1} - \frac{2}{k+1} \right] x + \left[\frac{(k+1)^2+1}{(k+2)^2} \right] \cdot \right.$$

$$\left[\frac{k^2+1}{(k+1)^2} \right] \left[\frac{2(k+1)}{(k+1)^2+1} + \frac{2k}{k^2+1} - \frac{2}{k+2} \right.$$

$$\left. - \frac{2}{k+1} \right] x^2 = 0$$

$k = 0$

$$\frac{\partial y}{\partial k} = C_0 \log x \left[1 + x + \frac{2}{4} x^2 + \cdots \right]$$

$$+ C_0 \left\{ -2x + \left(\frac{2}{4} \right) \left[1 - 1 - 2 \right] x^2 = 0 \right.$$

bv

$y = au + bv$

Power Series

· Instead of $y = \sum c_m x^{m+k} \Rightarrow \sum c_m x^m$

Q use power series to solve:

$$(1-x^2)\,y''+2y=0$$

$$y=\sum C_m x^m$$

$$y'=\sum C_m\, m\, x^{m-1}$$

$$y''=\sum C_m\, m(m-1)\, x^{m-2}$$

$$\Rightarrow (1-x^2)\sum C_m\, m(m-1)\, x^{m-2}+2\sum C_m x^m=0$$

$$\sum C_m[m(m-1)]x^{m-2}-\sum C_m[m(m-1)-2]x^m=0$$

$$\sum C_m m(m-1)x^{m-2}-\sum C_m[m^2-m-2]x^m=0$$

$$\sum C_m\, m(m-1)x^{m-2}-\sum C_m[m-2][m+1]x^m=0$$

Coeff of $x^{m-2}=0$.

$$C_m\, m\,\cancel{(m-1)}-C_{m-2}(m-4)\,\cancel{(m-1)}=0$$

$$\cancel{C_m=\frac{m}{m-4}C_{m-2}}$$

$$C_m=\frac{m-4}{m}C_{m-2} \quad -(1)$$

Coeff of $x^{m-1}=0$.

$$C_{m+1}(m+1)\,\cancel{m}-C_{m-1}(m-3)\,\cancel{m}=0$$

$$C_{m+1} = \frac{m-3}{m+1} C_{m-1}$$

$$C_3 = \frac{-1}{3} C_1$$

$$C_5 = \frac{1}{5} C_3 = \frac{-1}{15} C_1$$

From ①:

$$C_2 = -C_0$$

$$C_4 = 0$$

$$C_4 = C_6 = C_8 = C_{10} = \cdots = 0$$

$$y = \underbrace{C_0\left[1 - x^2\right]}_{au} + \underbrace{C_1\left[1 - \frac{x}{3} - \frac{x^2}{15} + \cdots\right]}_{bv}$$

$$y = au + bv$$

Q $(1-x^2)y'' + 2y = 0 \qquad y(0) = 4, \; y'(0) = 5$

NOTE:

If $y(1)$, $y'(1)$ are given, Solve in power of $x-1$

for solving in power of $x-1$

we write
$$y = \sum C_m (x-1)^m .$$

take, $x-1 = t$.

$$dx = dt$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} .$$

now, we ~~ha~~ will have DE in (y, t)

Solve as $y = \sum C_m t^m$, Similar to (y, x).

& finally put $t = x-1$

Q1 Find the power series solution in power of $x-1$ of DE

$$xy'' + y' + 2y = 0$$

$$y(1) = 2, \quad y'(1) = 4.$$

Q2 Solve $y'' + (x-1)y' + y = 0$. in power of ~~$x-2$~~ $x-2$.

Q3 Find power series solution of initial Value problem:

$$(1-x^2)\,y'' + 2y = 0, \quad y(2) = 4, \; y'(2) = 5.$$

(in power of $x-2$).