for the integration being performed along the horizontal strip PQ (y being held constant) and the integral in the outer rectangle stands for sliding the strip PQ from AC to BD ($y=c$ to $y=d$) to cover the entire region $ABDC$.

(*iii*) The region of integration R is bounded by the straight lines $x=a$, $x=b$ and $y=c$ and $y=d$. In this case it is immaterial whether $f(x, y)$ is integrated first with respect to x or y, the result being same in both the cases.

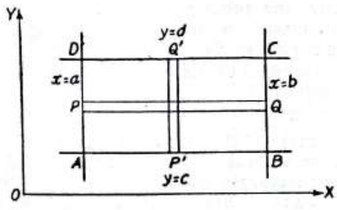

Fig. 4·4.

If the region of integration is not one of these types, it can be split into more regions, each of which being one of the above types.

Example 1. *Evaluate*

$$\int_0^1 \int_{x^2}^{2-x} xy\, dxdy. \qquad (P.U., 1978)$$

Sol. Since the limits of y are functions of x, the integration will first be performed w.r.t. y (treating x as a constant).

$$I=\int_0^1 \left[\frac{xy^2}{2}\right]_{x^2}^{2-x} dx=\int_0^1 \frac{x}{2}[(2-x)^2-x^4]\, dx,$$

$$=\frac{1}{2}\int_0^1 (4x-4x^2+x^3-x^5)dx,$$

$$=\frac{1}{2}\left[2x^2-\frac{4}{3}x^3+\frac{x^4}{4}-\frac{x^6}{6}\right]_0^1=\frac{3}{8}.$$

Example 2. *Evaluate* $\iint (x^2+y^2)\, dxdy$ *over the region in the positive quadrant for which* $x+y\leqslant 1$.

Sol. The region of integration R is the area bounded by the straight line $x+y=1$, and the coordinate axes *i.e.* it is bounded by the lines,

$x=0$, $x=1-y$, $y=0$, and $y=1$.

Integrating first over the horizontal strip PQ *i.e.* w.r.t. x from the point $P(x=0)$ to $Q(x=1-y)$ and then w.r.t. y from $y=0$ to $y=1$, we get

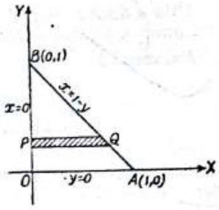

Fig. 4·5

$$\iint_R (x^2+y^2)dxdy$$

$$=\int_0^1 \int_0^{1-y} (x^2+y^2)dx\; dy,$$

$$=\int_0^1 \left[\frac{x^3}{3}+xy^2\right]_0^{1-y} dy=\int_0^1 \left[\frac{(1-y)^3}{3}+(1-y)y^2\right]dy,$$

$$=\left[-\frac{(1-y)^4}{12}+\frac{y^3}{3}-\frac{y^4}{4}\right]_0^1,$$

$$=\left(\frac{1}{3}-\frac{1}{4}\right)-\left(-\frac{1}{12}\right)=\frac{1}{6}.$$

Strip ∥ to axes
पहले होगा

Aliter. The integral may first be integrated along the vertical strip PQ *i.e.* w.r.t. y from the point $P(y=0)$ to $Q(y=1-x)$ and then with respect to x from $x=0$ to $x=1$.

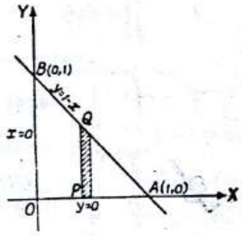

Fig. 4·6.

$$\therefore \quad \iint_R (x^2+y^2)dxdy$$

$$=\int_0^1 \int_0^{1-x} (x^2+y^2)dydx,$$

$$=\int_0^1 \left[x^2y+\frac{y^3}{3}\right]_0^{1-x} dx,$$

$$=\int_0^1 \left[x^2(1-x)+\frac{(1-x)^3}{3}\right]dx,$$

$$=\left[\frac{x^3}{3}-\frac{x^4}{4}-\frac{(1-x)^4}{12}\right]_0^1=\left(\frac{1}{3}-\frac{1}{4}\right)-\left(-\frac{1}{12}\right)=\frac{1}{6}.$$

Thus a double integral may be evaluated in any order, provided the limits are suitably changed.

Example 3. *Evaluate*

$$\int_0^b \int_0^a \frac{dx\,dy}{\sqrt{(a^2-x^2)(b^2-y^2)}}.$$

Sol. Since the limits are constant, we may integrate it in any order. Integrating w.r.t. x first, we get

$$\int_0^b \int_0^a \frac{dx\,dy}{\sqrt{(a^2-x^2)(b^2-y^2)}}=\int_0^b \frac{1}{\sqrt{b^2-y^2}}\cdot\left[\sin^{-1}\frac{x}{a}\right]_0^a dy,$$

$$=\int_0^b \frac{1}{\sqrt{b^2-y^2}}\cdot\frac{\pi}{2}\cdot dy=\frac{\pi}{2}\int_0^b \frac{1}{\sqrt{b^2-y^2}}\,dy=\frac{\pi}{2}\left[\sin^{-1}\frac{y}{b}\right]_0^b$$

$$=\frac{\pi}{2}\cdot\frac{\pi}{2}=\frac{\pi^2}{4}.$$

Exercise 4 (*a*)

Evaluate the following integrals.

1. $\int_0^2 \int_0^1 x^2\,dxdy.$

2. $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\,dy\,dx.$

3. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{(1+x^2+y^2)}.$ (*A.M.I.E. 1977*)

4. $\int_0^{2a} \int_{y^2/4a}^{3a-y} (x^2+y^2)\,dxdy.$

5. $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dxdy.$

6. Show that $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\,dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\,dx.$ (*Kurukshetra, 1975*)

7. Evaluate $\iint_A xy\,dxdy$, where A is the domain bounded by x-axis, ordinate $x=2a$ and the curve $x^2=4ay$. (*Punjab, 1976*)

8. Evaluate $\iint (x+y)^2\,dxdy$, over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ (*A.M.I.E. 1976*)

9. Evaluate $\iint xy\,(x+y)\,dxdy$ over the area between $y=x^2$ and $y=x$.

10. Evaluate $\iint_S \sqrt{xy-y^2}\,dxdy$, where S is a triangle with vertices (0, 0), (10, 1) and (1, 1).

11. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy$ and hence show that $\int_0^\infty e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}.$

12. Show that $\iint_R e^{ax+by}\,dxdy=\frac{1}{ab}$, where R is the area of the triangle bounded by the lines

$$x=0,\ y=0,\ ax+by=1\ (a>0,\ b>0).$$

4·5. Change of Variables (*Double Integrals*)

Some times change of variables greatly facilitates the evaluation of a double integral. Let us consinder the integral

$$\iint_R (x, y)dA, \qquad ..(i)$$

which can be written as the iterated integral $\iint_R (x, y)\,dxdy.$

Now, let it be required to change the variables x, y in the above integral to a new set of variables u, v, which are connected by the relations

$$x=\phi\,(u, v),\ y=\psi(u, v).$$

Then $$\boxed{\iint_R f(x, y)\,dxdy=\iint_R f[\phi(u, v), \psi(u, v)]J\,du\,dv,^*}$$

*The proof of this theorem is beyond the scope of this book.

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Jacobian

where $J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$

For example if, we want to change from cartesian to polar coordinates, then

$$x=r\cos\theta,\ y=r\sin\theta.$$

Thus $$\iint_R f(x,y)\,dxdy=\iint_R f(r\cos\theta, r\sin\theta)\,J\,drd\theta, \quad ...(ii)$$

where $$J=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix}=\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}=r(\cos^2\theta+\sin^2\theta)=r.$$

Therefore, from result (*ii*), we have

$$\boxed{\iint_R f(x,y)\,dxdy=\iint_R f(r\cos\theta, r\sin\theta)\,r\,drd\theta.}$$

The above result is also evident from the figure 4·7, because the elementary area $dxdy$ can be replaced by $r\,d\theta\,.\,dr$. *i.e.* $r\,drd\theta$.

Hence the result.

The limits of integration in the two cases will be different. The new limits can be ascertained from the fact that the region of integration is the same.

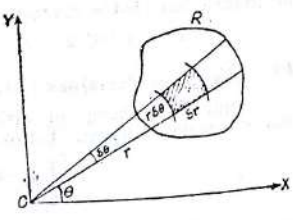


Fig. 4·7.

4·6. Evaluation of Double Integral (*Polar Coordinates*)

The double integral

$$\iint_R f(r,\theta)\,dA$$

can be expressed as the iterated integral

$$\iint_R f(r,\theta)\,drd\theta.$$

Let R be the region bounded by the curves $AB[r=f_1(\theta)]$, and CD $[r=f_2(\theta)]$ and the straight lines $\theta=\alpha$ and $\theta=\beta$. Then

$$\iint_R f(r,\theta)\,drd\theta=\int_{\theta=\alpha}^{\theta=\beta}\int_{r=f_1(\theta)}^{r=f_2(\theta)} f(r,\theta)\,drd\theta. \quad ...(i)$$

The integral (*i*) is evaluated first w.r.t. r (treating θ as a constant) between the limits $r=f_1(\theta)$ and $r=f_2(\theta)$. The resultant integrand, a function of θ, is then integrated w.r.t. θ from $\theta=\alpha$ to $\theta=\beta$, *i.e.*

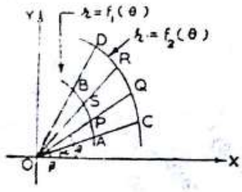


Fig. 4·8.

$$\int_\alpha^\beta \boxed{\int_{r=f_1(\theta)}^{r=f_2(\theta)} f(r,\theta)\,dr}\,d\theta, \quad ...(ii)$$

the integration being carried from inner to outer rectangle where θ is held constant while integrating

$$\int_{r=f_1(\theta)}^{r=f_2(\theta)} f(r,\theta)\,dr.$$

Geomertically, the process of integration is illustrated in the figure 4·8. Let AB and CD be the continuous curves $r=f_1(\theta)$ and $r=f_2(\theta)$, respectively. Take a wedge $PQRS$ of angular thickness $\Delta\theta$. The integral in the inner rectangle in (*ii*) means that the integration is performed along the radial strip PQ (*i.e.* r varies, θ being held constant) and the integral in the outer rectangle means rotation of this strip from AC $(\theta=\alpha)$ to BD $(\theta=\beta)$, to cover the entire region of integration $ACDB$.

Example 1. *Evaluate the integral*

$$\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2\,dr\,d\theta.$$

Show the region of integration also.

Sol. Since the limits of r, are functions of θ, we shall first integrate the given integral w.r.t. r, between the limits $r=a(1-\cos\theta)$ and $r=a$.

$$\text{Thus, } \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2\,dr\,d\theta=\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_{a(1-\cos\theta)}^{a} d\theta,$$

$$=\frac{1}{3}\int_0^{\pi/2}\left[a^3-a^3(1-\cos\theta)^3\right]d\theta,$$

$$=\frac{a^3}{3}\int_0^{\pi/2}\left(3\cos\theta-3\cos^2\theta+\cos^3\theta\right)d\theta,$$

$1-(1-\cos\theta)^3$

$1-[1-\cos^3\theta-3\cos\theta+3\cos^2\theta)$

$\cos^3\theta+3\cos\theta-3\cos^2\theta$

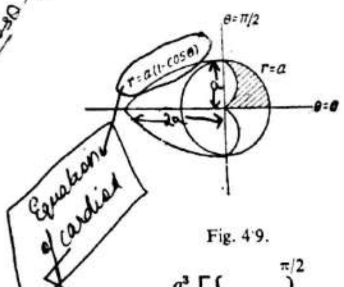


Fig. 4·9.

$$=\frac{a^3}{3}\left[\left\{3\sin\theta\right\}_0^{\pi/2}-3\cdot\frac{1}{2}\cdot\frac{\pi}{2}+\frac{2}{3\cdot1}\right],$$

$$\frac{a^3}{3}\left[\left(3\sin\frac{\pi}{2}-3\sin 0\right)-\frac{3\pi}{4}+\frac{2}{3}\right],$$

$$=\frac{a^3}{3}\left[3-\frac{3\pi}{4}+\frac{2}{3}\right]=\frac{a^3}{36}(44-9\pi).$$

The region of integration is shown in the figure.

Example 2. *Evaluate* $\iint r^3\,dr\,d\theta$, *over the area included between the circles* $r=2a\cos\theta$ *and* $r=2b\cos\theta$ $(b<a.)$

Sol. The given circles are shown in the figure. Let the region between the two circles be denoted by R. In the region R, variation of r is from $2b\cos\theta$ to $2a\cos\theta$ and θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

$\cos 0°=1$ so $r=2a$, $\cos 90°=0$ so $r=0$

Thus the given integral

$$\iint_R r^3\,dr\,d\theta=\int_{-\pi/2}^{\pi/2}\int_{2b\cos\theta}^{2a\cos\theta} r^3\,dr\,d\theta,$$

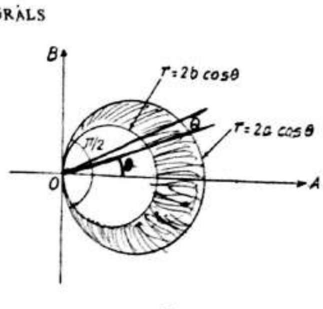


Fig. 4·10.

$$=\int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_{2b\cos\theta}^{2a\cos\theta}d\theta,$$

$$=\frac{1}{4}\int_{-\pi/2}^{\pi/2}\left[(2a\cos\theta)^4-(2b\cos\theta)^4\right]d\theta,$$

$$=4(a^4-b^4)\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta,$$

$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ odd even property as $f(x)=f(-x)$

$$=8(a^4-b^4)\int_0^{\pi/2}\cos^4\theta\,d\theta,$$

$$=8(a^4-b^4)\ \frac{3\cdot1}{4\cdot2}\cdot\frac{\pi}{2}=\frac{3}{2}\pi(a^4-b^4).$$

Example 3. *Evaluate* $I=\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} x^2\,dy\,dx$, *by changing to polar coordinates.*

Sol. The limits for y are from 0 to $\sqrt{2ax-x^2}$ and those for x are from 0 to $2a$. Thus

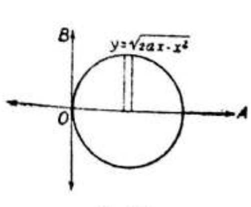


Fig. 4·11.

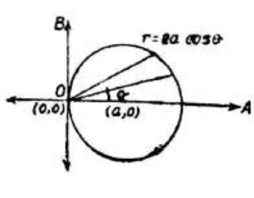


Fig. 4·12.

$y=0,\ y=\sqrt{2ax-x^2}$

or $y=0,\ y^2=2ax-x^2$ *i.e.* $x^2+y^2=2ax$.

In polar coordinates, we have $r^2=2ar\cos\theta$ *i.e.* $r=0$, $r=2a\cos\theta$. Hence the limits of integration for r are from 0 to $2a\cos\theta$. Also limits for x are from 0 to $2a$, that correspond to $\theta=\pi/2$ to $\theta=0$.

Now changing into polar coordinates by substituting $x=r\cos\theta$, $y=r\sin\theta$, $dydx=r\,drd\theta$, in the given integral, we get

$x = r\cos\theta$
$y = r\sin\theta$
$r = 2a r\cos\theta$

$$I=\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\cos^2\theta\, r\, drd\theta,$$

$$=\int_0^{\pi/2}\int_0^{2a\cos\theta} r^3\cos^2\theta\, drd\theta,$$

$$=\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2a\cos\theta}\cos^2\theta\, d\theta,$$

$$=4a^4\int_0^{\pi/2}\cos^6\theta\, d\theta=4a^4\ \frac{5\cdot3\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2}=\frac{5}{8}\pi a^4.$$

Exercise 4 (*b*)

Evaluate the following integrals.

1. $$\int_a^b\int_\alpha^\beta r^2\sin\theta\, d\theta dr.$$

$r^3/3 \quad (\cos\alpha-\cos\beta)(b^3-a^3)/3$

2. $\iint r^3\, drd\theta$; over the area included between the circles $r=a\sin\theta$ and $r=b\sin\theta$.

3. $$\int_0^{\pi}\int_0^{\infty} e^{-br^2}\ r\,drd\theta.$$

4. $$\int_0^{\pi/2}\int_{a(1+\cos\theta)}^{a} r\,drd\theta.$$

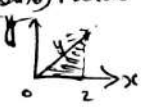
5. $$\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\ d\theta dr.$$

6. Show that $\iint r^2\sin\theta\, drd\theta=\frac{2a^3}{3}$, where R is the region bounded by the semicircle $r=2a\cos\theta$, above the initial line.

7. Integrate $r\sin\theta$ over the area of the cardiod $r=a(1-\cos\theta)$, above the inital line.

Evaluate the following by changing into polar coordinates.

8. $$\int_0^2\int_0^x y\,dydx.$$ $\frac{4}{3}$ $\quad \int_0^{\pi/4}\int_0^{2\sec\theta}(r\sin\theta)\, r\,dr\,d\theta$



9. $$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx.$$ πa^2

4·7. Change of Order of Integration

In a double integral with constants limits, the order of integration is immaterial except when the function has discontinuties with in or on the boundary of the region of integration. However, this is not so if the limits of integration are variable. In such a case, sometimes it is required to change the order of integration, while evaluating a double integral. While changing the order of integration, the limits also change. The new limits can be obtained by geometrical considerations and hence a clear sketch of the region of integration is necessary. Sometime, this region may have to be split into many regions and the new limits obtained for each region. Consider the integral

$$\int_a^b\int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\,dydx$$

As already shown in § 4·2, in this case the summation is done first along the vertical strips and then all these strips from $x=a$ to $x=b$ are added. When the order of integration is changed the summation is done first along the horizontal strips and then these are added so as to cover the entire region of integration. Let the new limits be $x=\phi_1(y)$, $x=\phi_2(y)$, $y=c$ and $y=d$. Then the above integral is written as

$$\int_c^d\int_{\phi_1(y)}^{\phi_2(y)} f(x,y)\,dxdy.$$

4.8. Rules for Change of Order of Integration

(*i*) Mark the region of integration by drawing the curves

$$y=f_1(x),\ y=f_2(x),\ x=a \text{ and } x=b.$$

(*ii*) Mark the points of intersection (if any) in the region of integration, where any two or more of the aforesaid curves meet. Through these points draw straight lines parallel to the x-axis, dividing the region in different parts.

(*iii*) In one of the parts obtained in step (*ii*), draw an elementary strip parallel to x-axis, obtain the values of x in terms of y at the extremeties of the elementary strip. These values give the limits for x in terms of y. The constant limits for y are obtained by the extension of that part.

(*iv*) Repeat the above procedure for other parts of the region.

Similarly, the order of integration in the integral

$$\int_a^b \int_{x=f_1(y)}^{x=f_2(y)} f(x, y)dy\, dx,$$

can be changed.

Example 1. *Change the order of integration in the following integral and evaluate*

$$\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} 1\, dy\, dx.$$

Sol. In the above integral, the limits of y vary from $y=\dfrac{x^2}{4a}$ to $y=2\sqrt{ax}$ and those of x from 0 to $4a$. Here the integration is first performed w.r.t. y *i.e.* along the vertical strip PQ which extends from a point P on the parabola $y=\dfrac{x^2}{4a}$, to the point Q on the parabola $y^2=4ax$ $(y=2\sqrt{ax})$.

Then the strip slides from the point O to R $(4a, 4a)$, covering the whole region of integration $OQ'R\,P'O$.

On chaning the order of integration, the integration is first performed along the horizontal strip $P'Q'$, which extends from $P'[x=\dfrac{y^2}{4a}]$ to Q' $[x=\sqrt{2ay}]$. Then this strip slides from O to R $(4a, 4a)$, covering the whole region of

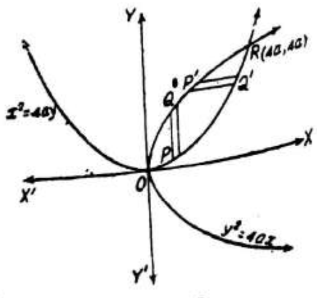


Fig. 4.12.

$$\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} 1\cdot dy\, dx = \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} 1\cdot dxdy.$$

Now,

$$\int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dxdy = \int_0^{4a} [x]_{y^2/4a}^{2\sqrt{ay}} dy,$$

$$=\int_0^{4a}\left(2\sqrt{a\,y}-\frac{y^2}{4a}\right)dy,$$

$$=\left[2a^{1/2}\frac{y^{3/2}}{3/2}-\frac{y^3}{12a}\right]_0^{4a}$$

$$=\left[\frac{4}{3}a^{1/2}\,y^{3/2}-\frac{y^3}{12a}\right]_0^{4a}=\frac{4}{3}a^{1/2}\,8a^{3/2}-\frac{64a^2}{12}=\frac{16}{3}a^2.$$

Example 2. *Evaluate the following integral by changing the order of integration in*

$$\int_0^a \int_{x^2/a}^{2a-x} xy\, dy\, dx.$$

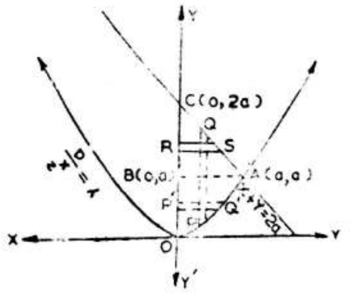


Fig. 4.13

Sol. In the above integral the limits of y vary from $y=\dfrac{x^2}{a}$ to $y=2a-x$ and those of x from 0 to a. This indicates that the region of integration is bounded by the parabola $y=\dfrac{x^2}{a}$ and the straight lines $x+y=2a$ and $x=0$, as shown in the figure 4.13.

Here the integration is first performed w.r.t. y, along the vertical strip PQ, which extends from a point P on the parabola $y=\frac{x^2}{a}$ to the point Q on the straight line $x+y=2a$. Then this strip slides from the point O to $A(a, a)$ covering the whole region of integration $OACO$. On changing the order of integration, the integration is first performed along a horizontal strip. Now, from the figure, it is quite evident that there can't be a horizontal strip whose extremeties fall on the parabola $x^2=ay$ and the straight line $x+y=2a$.

Since the parabola meets the straight line $x+y=2a$ at the point $A(a, a)$ we draw a straight line parallel to x-axis, meeting the y-axis at the point B, splitting the region $OACO$ in two parts OAB and ABC. On changing the order of integration, the integration is performed first w.r.t. x along a horizontal strip and then sliding this strip in such a manner, so as to cover the entire region of integration. For the part OAB, let us consider the horizontal strip $P'Q'$, which extends from the point $P'[x=0]$ to the point $Q'[x=\sqrt{ay}]$. Then this strip slides from O to $A(a, a)$, covering the part $OABO$.

Similarly for the part ABC, the horizontal strips RS extends from $R(x=0)$ to $S(x=2a-y)$, which then slides from $B(0, a)$ to $C(0, 2a)$. Thus, we have

$$\int_0^a \int_{x^2/a}^{2a-x} xy\, dy\, dx=\int_0^a \int_0^{\sqrt{ay}} xy\, dxdy+\int_a^{2a} \int_0^{2a-y} xy\, dxdy,$$

$$=\int_0^a \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} y\, dy+\int_a^{2a} \left[\frac{x^2}{2}\right]_0^{2a-y} y\, dy,$$

$$=\tfrac{1}{2}\int_0^a ay^2\, dy+\tfrac{1}{2}\int_a^{2a} (2a-y)^2\, y\, dy,$$

$$=\tfrac{1}{2}..\left[\frac{y^3}{3}\right]_0^a+\tfrac{1}{2}\int_a^{2a} (4a^2y-4ay^2+y^3)dy,$$

$$=\frac{1}{6}a^4+\frac{1}{2}\left[2a^2y^2-\frac{4ay^3}{3}+\frac{y^4}{4}\right]_a^{2a},$$

$$=\frac{1}{6}a^4+\frac{1}{2}\left(6a^4-\frac{28a^4}{3}+\frac{15a^4}{4}\right),$$

$$=\frac{3a^4}{8}.$$

Example 3. *Change the order of integration in the following integral*

$$\int_0^b \int_0^{(a/b)\sqrt{b^2-y^2}} xy\, dxdy$$

and hence evaluate. (*P.U. 1980*)

Sol. In the above integral the limits of x vary from $x=0$ to $x=\frac{a}{b}\sqrt{b^2-y^2}$ $\left(i.e.\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right)$ and those of y from 0 to b. This indicates that the region of integration is bounded by the first quadrant of the ellipse $OABO$ and the coordinate axes, as shown in the figure 4·14.

$\boxed{I=Q}$
$bx=a\sqrt{b^2-y^2}$
$b^2x^2=a^2b^2-a^2y^2$
$a^2y^2+b^2x^2=a^2b^2$
$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$

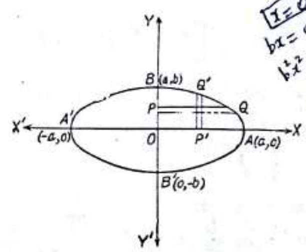


Fig. 4·14.

Here the integration is first performed w.r.t x along the horizontal strip PQ which extends from the point $P(x=0)$ to the point Q on the ellipse $\left(x=\frac{a}{b}\sqrt{b^2-y^2}\right)$. Then this strip slides from the point O to the point $B(o, b)$, covering the region of integration $OABO$.

On changing the order of integration, the integration is to be performed first along the vertical strip $P'Q'$ which extends from P' $(y=0)$ to Q' $\left(y=\frac{b}{a}\sqrt{a^2-x^2}\right)$, from the equation of the ellipse.

Then this strip slides from O to $A(a, 0)$, covering the whole region of integration. Thus

$$\int_0^b \int_0^{(a/b)\sqrt{b^2-y^2}} xy\, dxdy=\int_0^a \int_0^{(b/a)\sqrt{a^2-x^2}} xy\, dy\, dx.$$

$$=\int_0^a x\left[\frac{y^2}{2}\right]_0^{(b/a)\sqrt{a^2-x^2}} dx,$$

$$=\frac{1}{2}\int_0^a x\cdot\frac{b^2}{a^2}(a^2-x^2)\,dx,$$

$$=\frac{1}{2}\cdot\frac{b^2}{a^2}\int_0^a (a^2x-x^3)\,dx,$$

$$=\frac{1}{2}\cdot\frac{b^2}{a^2}\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_0^a,$$

$$=\frac{1}{2}\cdot\frac{b^2}{a^2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right]=\frac{1}{8}\,a^2b^2.$$

Example 4. *Change the order of integration in the integral*

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)\,dr\,d\theta.$$

Sol. In the above integral, the limits of r vary from $r=0$ to $r=2a\cos\theta$ and those of θ from 0 to $\frac{\pi}{2}$.

Thus the region of integration is bounded by the circle $r=2a\cos\theta$, in the first quadrant, as shown in the figure.

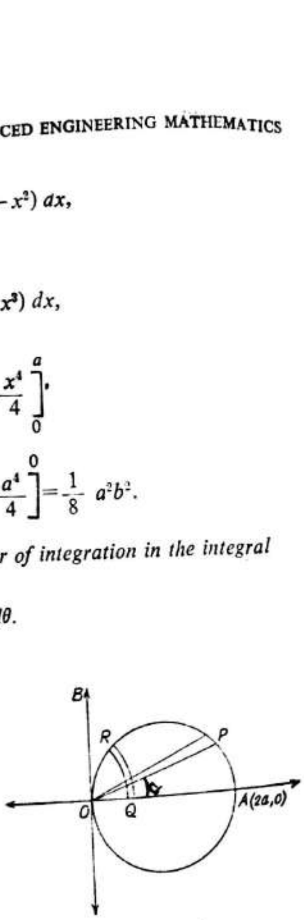


Fig. 4·15.

Here, the integration is first performed along the radial strip OP, which extends from the point O $(r=0)$ to the point $P(r=2a\cos\theta)$. Then this strip rotates from $\theta=0$ to $\theta=\frac{\pi}{2}$ i.e. from the straight line OA to the line OB, to cover the region of integration $OAPRO$.

On changing the order of integration the integration is to be performed first along the circular strip QR which extends from the point O $(\theta=0)$ to the point $R\left(\theta=\cos^{-1}\frac{r}{2a}\right)$. Then r varies from O to A $(2a, o)$ to cover the entire region of integration. Thus

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)\,dr\,d\theta=\int_0^{2a}\int_0^{\cos^{-1}r/2a} f(r,\theta)\,d\theta\,dr.$$

Exercise 4 (*c*)

Change the order of integration in the following integrals.

1. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$

2. $\displaystyle\int_0^2\int_0^x \phi(x,y)\,dy\,dx.$

3. $\displaystyle\int_0^{2a}\int_{y^2/4a}^{3a-y} f(x,y)\,dx\,dy$

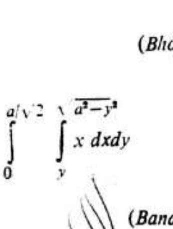
4. $\displaystyle\int_0^a\int_x^a f(x,y)\,dy\,dx.$ 5. $\displaystyle\int_0^2\int_{x-2}^{\sqrt{(x-1)(2-x)}} \phi(x,y)\,dy\,dx.$

Evaluate the following integrals by changing the order of integration.

6. $\displaystyle\int_0^1\int_x^{\sqrt{2-x^2}}\frac{x\,dy\,dx}{\sqrt{x^2+y^2}}.$ (*Bhopal, 1975*)

7. $\displaystyle\int_0^a\int_{x/a}^{\sqrt{x/a}}(x^2+y^2)\,dy\,dx.$ 8. $\displaystyle\int_0^{a/\sqrt{2}}\int_y^{\sqrt{a^2-y^2}} x\,dx\,dy$

9. $\displaystyle\int_0^a\int_y^a\frac{x\,dx\,dy}{x^2+y^2}$ (*Banaras, 1979*)



10. $\displaystyle\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\frac{\phi'(y)\,(x^2+y^2)x\,dx\,dy}{\sqrt{4a^2x^2-(x^2+y^2)^2}}$ (*Delhi, 1982*)

11. $\displaystyle\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$

12. $\displaystyle\int_0^\infty\int_x^\infty\frac{e^{-y}}{y}\,dy\,dx$ $=1$ (*A.M.I.E. 1977*)

$y=x$ L.L
$y=\infty$ U.P
$x=0$ L.L
$x=\infty$ U.L

$\boxed{\displaystyle\int_0^\infty\int_0^y \frac{e^{-y}}{y}\,dx\,dy}$

$e^{-y}\Big|_0^\infty = 0+1 = 1$

4·9. Triple Integral

Consider a region V of a three dimensional space and a function $f(x, y, z)$, which is continuous at every point of this region.

Divide the region V into subregions of respective volumes $\triangle V_1, \triangle V_2, \ldots, \triangle V_n$. Further, let $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots\ldots, (x_n, y_n, z_n)$ be the arbitrarly chosen points in these regions respectively. Now consider the sum

$$f(x_1, y_1, z_1)\triangle V_1+f(x_2, y_2, z_2)\triangle V_2+\ldots+f(x_n, y_n, z_n)\triangle V_n$$

$$=\sum_{r=1}^{n} f(x_r, y_r, z_r)\triangle V_r \qquad \ldots(i)$$

Let the number of subregions increase indefinitely such that the largest dimension of the subregion $\triangle V_r \to 0$. Then the limit of the sum (*i*) is called the **Triple Integral** of $f(x, y, z)$ over the region V and is denoted by

$$\boxed{\iiint_V f(x, y, z)\, dV}$$

In other words

$$\lim_{\substack{n\to\infty \\ \triangle V_r\to 0}} \sum_{r=1}^{n} f(x_r, y_r, z_r)\triangle V_r=\iiint_V f(x, y, z)\, dV.$$

4·10. Evaluation of Triple Integral

For the purpose of evaluation the triple integral

$$\iiint_V f(x, y, z)\, dV,$$

is expressed as an iterated integral

$$\boxed{\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y, z)\, dx dy\, dz.}$$

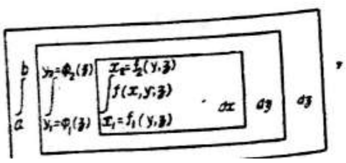
The above integration is performed in three stages, the order of integration depending upon the limits.

(*i*) Let the limits x_1 and x_2 be functions of y, z ; y_1, y_2 functions of z and z_1, z_2 be constants, *i.e.*

$$x_1=f_1(y, z),\ x_2=f_2(y, z),\ y_1=\phi_1(z),\ y_2=\phi_2(z)$$

and $\qquad z=a,\ z=b.$

Then the above integral is evaluated as follows,

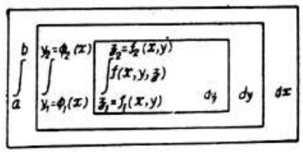


which shows that first $f(x, y, z)$ is integrated w.r.t. x keeping y and z constant between the limits x_1 and x_2. The resultant integrand then a function of y, z is integrated w.r.t. y keeping z, as constant between the limits y_1 and y_2. Finally, this integral is evaluated w.r.t. z between the limits $z=a$ to $z=b$. Thus the order of integration is from the innermost rectangle to the outermost rectangle.

(*ii*) Let the limits of z_1, z_2 be functions of x, y ; y_1, y_2 be function of x and x_1, x_2 be constants *i.e.* $\quad z_1=f_1(x,y),\ z_2=f_2(x, y)$

$$y_1=\phi_1(x),\ y_2=\phi_2(x) \qquad \text{and} \qquad x_1=a \text{ and } x_2=b.$$

Then the above integral is evaluated as follows.



The order of integration being from innermost to outermost rectangle.

(*iii*) If x_1, x_2, y_1, y_2 and z_1, z_2 are all constants, then

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y, z)\, dxdydz=\int_{x_1}^{x_2}\int_{z_1}^{z_2}\int_{y_1}^{y_2} f(x, y, z)\, dy\, dz\, dx$$

$$=\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\, dz\, dy\, dx.$$

Thus in this case the order of integration can be changed by suitable change in the limits.

The definite integrals of functions of two or more variables are termed as the multiple integrals.

Example 1. *Evaluate the integral*

$$\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx. \qquad \textbf{(P.U. 1980)}$$

Sol. Let $\quad I=\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx,$

$$=\int_0^{\log 2}\int_0^{x} e^{x+y}\Big[e^{z}\Big]_0^{x+\log y} dy\, dx,$$

$$=\int_0^{\log 2}\int_0^{x} e^{x+y}.\,(e^{x+\log y}-1)\, dydx,$$

$$=\int_0^{\log 2}\int_0^{x} [e^{2x}.\, y\, e^{y}-e^{x}.\, e^{y}]\, dydx,$$

$$=\int_0^{\log 2}\Big[e^{2x}.\Big\{y.\, e^{y}-e^{y}\Big\}_0^{x}-e^{x}.\Big\{e^{y}\Big\}_0^{x}\Big]$$

$$=\int_0^{\log 2}[e^{2x}(xe^{x}-e^{x}+1)-e^{x}(e^{x}-1)]dx,$$

$$=\int_0^{\log 2}(xe^{3x}-e^{2x}+e^{x})dx,$$

$$=\int_0^{\log 2}[(x-1)e^{3x}+e^{x}]dx,$$

$$=\Big[(x-1).\,\frac{e^{3x}}{3}-\frac{e^{3x}}{9}+e^{x}\Big]_0^{\log 2}$$

$$=(\log 2-1).\,\frac{e^{3\log 2}}{3}-\frac{e^{3\log 2}}{9}+e^{\log 2}-\frac{5}{9},$$

$$=(\log 2-1).\,\frac{e^{\log 2^3}}{3}-\frac{e^{\log 2^3}}{9}+e^{\log 2}-\frac{5}{9},$$

$$=\frac{8}{3}(\log 2-1)-\frac{8}{9}+2-\frac{5}{9},$$

$$=\frac{8}{3}\log 2-\frac{19}{9}\,.$$

Imp

Example 2. *Evaluate the integral*

$$\iiint\frac{dxdydz}{\sqrt{a^2-x^2-y^2-z^2}},$$

taken throughout the volume of the sphere $x^2+y^2+z^2=a^2$.

Sol. Here the region of integration V is the sphere

$$x^2+y^2+z^2=a^2. \qquad \ldots(i)$$

We can express any of the three variables x, y, z in terms of the other two, say

$$z=\pm\sqrt{a^2-x^2-y^2}.$$

Hence on the surface of the sphere, z varies from $-\sqrt{a^2-x^2-y^2}$ to $\sqrt{a^2-x^2-y^2}$.

Projection of the sphere (i) on the xy-plane is the circle

$$x^2+y^2=a^2.$$

On this circles y varies from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$ and x from $-a$ to a.

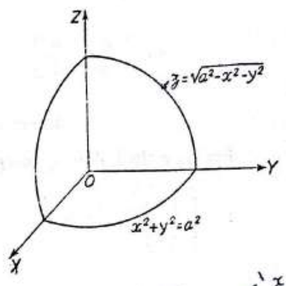


Fig. 4·16.

$\int\frac{1}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}$

Let $$I=\iiint_V\frac{dxdydz}{\sqrt{a^2-x^2-y^2-z^2}},$$

$$=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}\frac{dzdydx}{\sqrt{a^2-x^2-y^2-z^2}},$$

(Note the change in order)

$$=8\int_0^{a}\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}\frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}},$$

$$=8\int_0^{a}\int_0^{\sqrt{a^2-x^2}}\Big[\sin^{-1}\frac{z}{\sqrt{a^2-x^2-y^2}}\Big]_0^{\sqrt{a^2-x^2-y^2}} dydx,$$

$$=8\int_0^{a}\int_0^{\sqrt{a^2-x^2}}[\sin^{-1}(1)-\sin^{-1}(0)]\,dy\,dx,$$

$$=8\int_0^{a}\int_0^{\sqrt{a^2-x^2}}\frac{\pi}{2}dydx=4\pi\int_0^{a}\Big[y\Big]_0^{\sqrt{a^2-x^2}}dx,$$

$$=4\pi\int_0^{a}\sqrt{a^2-x^2}\,dx,$$

$$=4\pi\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a,$$

$$=4\pi\left[\frac{a^2}{2}\cdot\frac{\pi}{2}\right]=\pi^2a^2.$$

Exercise 4 (*d*)

Evaluate the following integrals.

1. $\displaystyle\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx.$

2. $\displaystyle\int_0^1\int_{y^2}^1\int_0^{1-x} x\,dzdxdy.$ (*Mysore, 1979*)

3. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{1-x^2-y^2-z^2}}\,dz\,dy\,dx.$ (*P.U. 1980*) $\pi^2/8$

4. Evaluate $\iiint (x+y+z)\,dxdydz$, over the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$. $1/8$

5. $\displaystyle\int_2^4\int_0^x\int_0^{\sqrt{x+y}} z\,dzdydx.$

6. $\displaystyle\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dzdrd\theta.$

4·11. Change of Variables in a Triple Integral

Triple integral in

(*i*) **Cylinderical Coordinates**. A triple integral may frequently be evaluated easily by use of cylinderical coordinaets. The cylinderical coordinates (r, ϕ, z) of a point $P(x, y, z)$ are determined as follows. Draw a perpendicular PQ from the point P on the xy-plane. Then

$$OQ=r,\ \angle XOQ=\phi \text{ and } PQ=z.$$

$x = r\cos\phi$
$y = r\sin\phi$
z

From the figure, the relation between the cartesian coordinates (x, y, z) and (r, ϕ, z) of the point P can readily be found as under

$$x=r\cos\phi,\quad y=r\sin\phi.$$

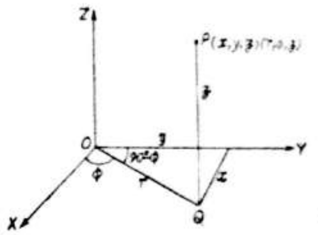


Fig. 4·17.

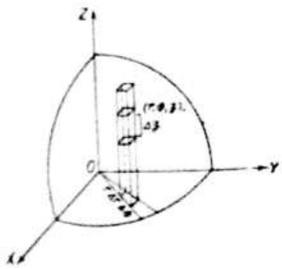


Fig. 4·18.

Let us consider the triple integral

$$\iiint_V f(x, y, z)\,dV,$$

which is expressed as

$$\iiint_V f(x, y, z)\,dxdydz.$$

Then, $$\iiint_V f(x, y, z)\,dxdydz=\iiint_V f(r, \phi, z)\,J.\,drd\phi\,dz,$$

where
$$J=\frac{\partial(x, y, z)}{\partial(r, \phi, z)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r}\\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi}\\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z}\end{vmatrix},$$

$$=\begin{vmatrix}\cos\phi & \sin\phi & 0\\ -r\sin\phi & r\cos\phi & 0\\ 0 & 0 & 1\end{vmatrix},$$

$$=r(\cos^2\theta+\sin^2\phi)=r.$$

Hence $$\boxed{\iiint_V f(x, y, z)\,dxdy\,dz=\iiint_V f(r, \phi, z)\,rdrd\phi dz.}$$

or $$A=\int_a^b \int_{y=f_1(x)}^{y=f_2(x)} dy\,dx,$$

where A is the area of the region of integration bounded by the curves $y=f_1(x)$, $y=f_2(x)$ and the lines $x=a$ and $x=b$.

Similarly, if the region of integration R is bounded by the curves $x=f_1(y)$, $x=f_2(y)$ and the lines $y=c$ and $y=d$, we obtain

$$\boxed{A=\int_c^d \int_{x=f_1(y)}^{x=f_2(y)} dx\,dy.}$$

(b) **Polar coordinates.** The double integral in polar coordinates is (§ 4·6)

$$\iint_R f(r,\theta)\,dA=\int_\alpha^\beta \int_{r=f_1(\theta)}^{r=f_2(\theta)} f(r,\theta)\,r\,dr\,d\theta.$$

Putting $f(r,\theta)=1$, in above, we have

$$\iint_R dA=\int_\alpha^\beta \int_{r=f_1(\theta)}^{r=f_2(\theta)} r\,dr\,d\theta,$$

or $$\boxed{A=\int_\alpha^\beta \int_{r=f_1(\theta)}^{r=f_2(\theta)} r\,dr\,d\theta,}$$

where A is the area of the region R, bounded by the curves $r=f_1(\theta)$, $r=f_2(\theta)$ and the lines $\theta=\alpha$ and $\theta=\beta$.

Example 1. *Find by double integration, the smaller of the areas bounded by the circle $x^2+y^2=9$ and the straight line $x+y=3$.*

Sol. The equation of the circle is

$$x^2+y^2=9, \qquad \ldots(i)$$

and the line is

$$x+y=3 \qquad \ldots(ii)$$

From the equations (i) and (ii), on eliminating y, we have

$$x^2+(3-x)^2=9,$$

or $$2x^2-6x=0,$$

or $$2x(x-3)=0,$$

$$\therefore \quad x=0, \quad x=3,$$

which gives $y=3$ and $y=0$.

Thus the points of intersection of curves (i) and (ii) are $A(3, 0)$ and $B(0, 3)$.

So the required area is the

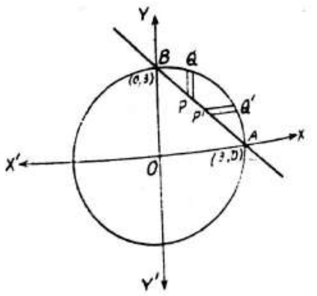


Fig. 4·21.

area lying between the curves

$$y=3-x,\ y^2=9-x^2,\ x=0 \text{ and } x=3.$$

Integrating along a vertical strip PQ, y varies from $P(y=3-x)$ to $Q\,[y=\sqrt{9-x^2}]$ and x from $A(x=0)$ to $B(x=3)$.

Thus, $$A=\int_0^3 \int_{3-x}^{\sqrt{9-x^2}} dy\,dx.$$

$$=\int_0^3 \Big[\, y \,\Big]_{3-x}^{\sqrt{9-x^2}} dx=\int_0^3 \Big[\sqrt{9-x^2}-(3-x)\Big]dx$$

$$=\left[\frac{x\sqrt{9-x^2}}{2}+\frac{9}{2}\sin^{-1}\frac{x}{3}-3x+\frac{x^2}{2}\right]_0^3$$

$$=\frac{9}{2}\sin^{-1}(1)-9+\frac{9}{2}=\frac{9}{4}\ (\pi-2).$$

Note. The area can be calculated by integrating along the horizontal strip $P'Q'$ also.

Example 2. *Find by double integration the area of the quadrant of the ellipse*

$$4x^2+9y^2=36.$$

Sol. The equation of the ellipse is

$$\frac{x^2}{9}+\frac{y^2}{4}=1.$$

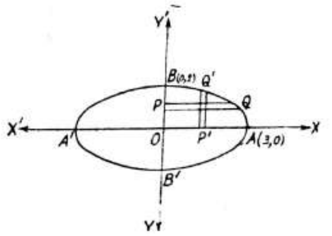


Fig. 4·22.

Integrating along the horizontal strip PQ, x varies from $P(x=0)$ to $Q\left(x=\frac{3}{2}\sqrt{4-y^2}\right)$ and y from $O(y=0)$ to $B(y=2)$.

Let A be the required area. Then

$$A=\int_0^2\int_0^{(3/2)\sqrt{4-y^2}} dxdy\ ,$$

$$=\int_0^2\Big[\ x\ \Big]_0^{(3/2)\sqrt{4-y^2}} dy\ ,$$

$r=a(1-\cos\theta)$
$r=a(1+\cos\theta)$

$$=\frac{3}{2}\int_0^2\sqrt{4-y^2}\ dy,$$

$$=\frac{3}{2}\left[\frac{y\sqrt{4-y^2}}{2}+2\sin^{-1}\frac{y}{2}\right]_0^2,$$

$$=\frac{3}{2}\cdot 2\sin^{-1}(1)=\frac{3\pi}{2}.$$

Example 3. *Find by double integration the area out side the circle $r=a$ and inside the cardiod $r=a(1+\cos\theta)$.*

Sol. The required area A is shown as shaded. Integrating along the radial strip PQ, r varies from $P(r=a)$ to $Q[r=a(1+\cos\theta)]$ and above the initial line OX, θ varies from 0 to π. Hence

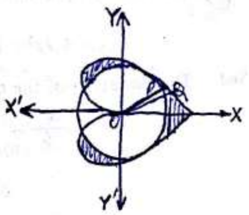


Fig. 4·23.

$$A=2\int_0^{\pi}\int_a^{a(1+\cos\theta)} r\ drd\theta,$$

$$=2\int_0^{\pi}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta,$$

$$=\int_0^{\pi}[a^2(1+\cos\theta)^2-a^2]\ d\theta,$$

$$=a^2\int_0^{\pi}(2\cos\theta+\cos^2\theta)d\theta,$$

$$=a^2\int_0^{\pi}\left(2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta,$$

$$=\frac{a^2}{2}\int_0^{\pi}(1+4\cos\theta+\cos 2\theta)\ d\theta$$

$$=\frac{a^2}{2}\left[\ \theta+4\sin\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi}=\frac{\pi a^2}{2}.$$

2. Volume of Solids.

(*a*) **Volume as a double integral.** (*Cartesian coordinates*). Consider a surface given by the equation

$$z=f(x,y), \qquad ...(i)$$

whose orthogonal projection on the xy-plane be the closed curve,

$$\phi(x,y)=0. \qquad ...(ii)$$

In three dimensions the equation $\phi(x,y)=0$, represents a cylinder with generators parallel to z-axis and base as the curve given by the equation (*ii*). Let V be the volume of this cylinder enclosed by the surface (*i*).

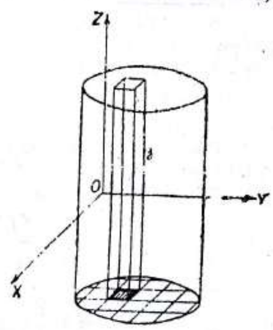


Fig. 4·24.

Let A be the area of the region bounded by the closed curve (*ii*). Divide A into elementary rectangles of area $\delta x\delta y$, by drawing lines parallel to OX and OY. On each of the rectangles erect prisms having their lengths parallel to z-axis.

The volume V is composed of these prisms, each having elementry volume $z\delta x\delta y$.

$$\therefore \qquad V=\lim_{\substack{\delta x\to 0\\ \delta y\to 0}}\sum\sum z\,\delta x\delta y=\boxed{\iint_A z\ dxdy}$$

or

$$\boxed{V=\iint_A f(x,y)\ dxdy}$$

(*b*) **Volume in cylinderical coordinates.** Let the equation of the surface be $z=f(r,\theta)$.

Then dividing the area A into elementary areas $rdrd\theta$, and proceeding as above, we get

$$\boxed{V=\iint_A z\ rdrd\theta=\iint_A f(r,\theta)\ rdrd\theta.}$$

IMP

Example 1. *Prove that the volume enclosed between the cylinders*

$$x^2+y^2=2ax \text{ and } z^2=2ax \text{ is } \frac{128a^3}{15} \quad \text{(Delhi, 1984)}$$

Sol. The required volume V is that volume of the cylinder

$$x^2+y^2=2ax, \qquad \ldots(i)$$

which in bounded by the cylinder

$$z^2=2ax. \qquad \ldots(ii)$$

bcos circle + parabola cut in 3D

Only half of the volume is shown in the figure.

as complete is now shown

Now, from the figure, it is evident that $z=\sqrt{2ax}$ is to be evaluated over the circle

$$x^2+y^2=2ax.$$

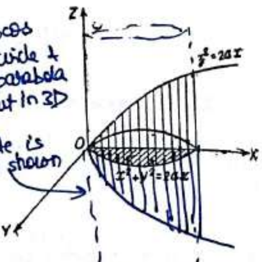


Fig. 4·25.

Thus on this circle, y varies from $-\sqrt{2ax-x^2}$ to $\sqrt{2ax-x^2}$ and x varies from 0 to $2a$. Hence

$$V=2\int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} z\,dydx, \quad ?$$

$$=4\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} \sqrt{2ax}\,dydx,$$

$$=4\sqrt{2a}\int_0^{2a}\sqrt{x}\,.\Big[\,y\,\Big]_0^{\sqrt{2ax-x^2}} dx\,,$$

$$=4\sqrt{2a}\int_0^{2a}\sqrt{x}.\ \sqrt{2ax-x^2}\,dx.$$

Put $\boxed{x=2a\sin^2\theta,}\ dx=4a\sin\theta\cos\theta\,d\theta$, such that when

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$x=2a,\ \theta=\frac{\pi}{2}$ and when $x=0,\ \theta=0.$

$$\therefore \quad V=4\sqrt{2a}\int_0^{\pi/2}\sqrt{2a\sin^2\theta}.\ \sqrt{4a^2\sin^2\theta-4a^2\sin^4\theta}.\ 4a\sin\theta\cos\theta\,d\theta,$$

$$=64a^3\int_0^{\pi/2}\sin\theta.\sin\theta\cos\theta.\sin\theta\cos\theta\,d\theta,$$

$$=64a^3\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta,$$

$$=64a^3.\frac{2}{5.3.1}=\frac{128a^3}{15}$$

IMP

Example 2. *Find the volume bounded by the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Sol. The section of the ellipsoid by the xy-plane $(z=0)$ is the ellipse,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad \ldots(i)$$

Only half the volume is shown in the figure.

as z is taken as only +ve.

From the equation of the ellipsoid, we have

$$z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}.$$

(taking +ve value only)

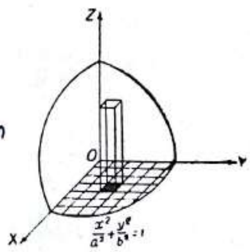


Fig. 4·26.

Thus, from the figure it is evident, that z is to be integrated over the ellipse given by the equation (i).

On this ellipse, y varies from

$$-b\sqrt{1-\frac{x^2}{a^2}} \text{ to } b\sqrt{1-\frac{x^2}{a^2}}$$

and x from $-a$ to a.

Thus V the required volume is given by

$$V=2\int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} z\,dydx\,,$$

as only half vol.

$$=2\int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dydx,$$

$$=8c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\sqrt{\left(1-\frac{x^2}{a^2}\right)-\frac{y^2}{b^2}}\,dydx,$$

$$=\frac{8c}{b}\int_0^a\int_0^t\sqrt{t^2-y^2}\,dydx, \text{ where } t=b\sqrt{1-\frac{x^2}{a^2}}$$

only for simplifying

$$=\frac{8c}{b}\int_0^a\left[\frac{y\sqrt{t^2-y^2}}{2}+\frac{t^2}{2}\sin^{-1}\frac{y}{t}\right]_0^t dx,$$

$$=\frac{8c}{b}\int_0^a\left[\frac{t^2}{2}\sin^{-1}(1)\right]dx,$$

$$=\frac{8c}{2b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)\cdot\frac{\pi}{2}\,dx,$$

$$=2\pi bc\int_0^a\left(1-\frac{x^2}{a^2}\right)dx=2\pi bc\left[x-\frac{x^3}{3a^2}\right]_0^a,$$

$$=2\pi bc\left[a-\frac{a}{3}\right]=\frac{4}{3}\pi abc.$$

Example 3. *Find the volume of the region bounded by the cylinder $x^2+y^2=4$ and the cone $x^2+y^2=z^2\tan^2\alpha$. $(0<\alpha<\pi)$.*

Sol. The equation of the cone is

$$x^2+y^2=z^2\tan^2\alpha. \qquad ...(i)$$

$\therefore$

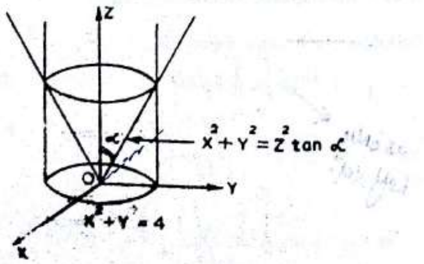
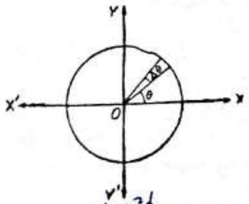


Fig. 4·27.

as θ is integrated from 0 to π/2

Only one eighth of the volume is shown in the figure.

From the figure, it is evident that the required volume can be obtained by integrating z over the circle $x^2+y^2=4$.



Changing to polar coordinates, by substituting

$x=r\cos\theta$, $y=r\sin\theta$, we get

$z=r\cot^2\alpha$.

The equation of the circle becomes $r=2$.

$r^2 = z^2 \tan^2\alpha$
$z = r^2 \cot^2\alpha$

The required volume

$$=8\int_0^{\pi/2}\int_0^2 zr\,dr\,d\theta,$$

$$=8\int_0^{\pi/2}\int_0^2 r\cot^2\alpha\,.\,r\,dr\,d\theta,$$

$$=8\,.\cot^2\alpha\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^2 d\theta,$$

$$=\frac{8}{3}\cot^2\alpha\,.\int_0^{\pi/2}8\,.\,d\theta,$$

$$=\frac{64}{3}\cot^2\alpha\,.\,\frac{\pi}{2}=\frac{32}{3}\pi\cot^2\alpha.$$

Example 4. *Find the volume under the paraboloid*

$$x^2+y^2+z=16,$$

over the rectangle $x=\pm a$, $y=\pm b$, in the xy-plane,

(where $a, b<4$)

Sol. The equation of the paraboloid is

$$x^2+y^2+z=16,$$

or $$z=16-x^2-y^2. \qquad (i)$$

Its section by the xy-plane $(z=0)$ is the circle

$$x^2+y^2=16. \qquad ..(ii)$$

Only one fourth of the volume is shown in the figure.

because z is +ve so it wd not range in −ve z axis dir.

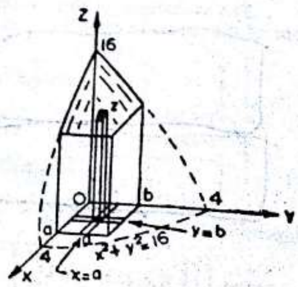


Fig. 4·28.

The required volume V is obtained by integrating z over this rectangle $x=\pm a$, $y=\pm b$.

$$\because \quad V=4\int_0^a\int_0^b z\,dy\,dx,$$

$$=4\int_0^a\int_0^b (16-x^2-y^2)\,dy\,dx,$$

$$=4\int_0^a\left[\left(16y-x^2y-\frac{y^3}{3}\right)\right]_0^b dx,$$

$$=4\int_b^a\left(16\,b-x^2b-\frac{b^3}{3}\right)dx,$$

$$=4\left[16\,bx-\frac{x^3}{3}\,b-\frac{b^3x}{3}\right]_0^a,$$

$$=4\left[16\,ab-\frac{a^3b}{3}-\frac{ab^3}{3}\right],$$

$$=\frac{4}{3}\,ab\,(48-a^2-b^2).$$

(c) (i) **Volume as a triple integral.** (*Cartesian coordinates*) The triple integral of a function $f(x, y, z)$, which is continuous in a three dimensional region having volume V, has been defined as

$$\iiint_V f(x, y, z)\,dV.$$

The evaluation of this integral is achieved by expressing it as one of the iterated integrals

$$\boxed{\int_a^b\int_{y=\varphi_1(z)}^{y=\varphi_2(z)}\int_{x=f_1(y,z)}^{x=f_2(y,z)} f(x, y, z)\,dx\,dy\,dz}\ \boxed{\int_a^b\int_{y=\varphi_1(x)}^{y=\varphi_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x, y, z)\,dz\,dy\,dx}$$

or

$$\boxed{\int_a^b\int_{z=\varphi_1(x)}^{z=\varphi_2(x)}\int_{y=f_1(z,x)}^{y=f_2(z,x)} f(x, y, z)\,dy\,dz\,dx,}$$

depending upon the nature of the region V.

$$\text{Let } \iiint_V f(x, y, z)\,dV=\int_a^b\int^{y=\varphi_2(z)}\int^{x=f_2(y,z)} f(x, y, z)\,dxdydz$$

If $f(x, y, z)=1$, then we have

$$\iiint_V dV=\int_a^b\int_{y=\varphi_1(z)}^{y=\varphi_2(z)}\int_{x=f_1(y,z)}^{x=f_2(y,z)} dxdydz,$$

or

$$\boxed{V=\int_a^b\int_{y=\varphi_1(z)}^{y=\varphi_2(z)}\int_{x=f_1(y,z)}^{x=f_2(y,z)} dxdydz.}$$

The order of integration in the above integral may be changed, with a suitable change in limits.

(*ii*) **Volume in Cylinderical and Polar Spherical Cooordinates.** Consider the triple integral in cylinderical coordinates.

$$\iiint_V f(r, \phi, z)\,dV=\iiint_V f(r, \phi, z)\,r\,drd\varphi dz$$

Let $f(r, \phi, z)=1$, then

$$\iiint_V dV=\iiint_V r\,drd\phi dz,$$

or $\boxed{V=\iiint_V r\,drd\varphi dz,}$ gives the volume in cylinderical coordinates.

Similarly, The volume V in polar spherical coordinates can be expressed as

$$\boxed{V=\iiint_V r^2\sin\theta\,drd\theta d\varphi.}$$

Example 1. *Find the volume bounded by the xy plane, the cylinder $x^2+y^2=1$ and the plane $x+y+z=3$.* **(*Mysore, 1979*)**

Sol. The section of the plane

$$x+y+z=3 \qquad ...(i)$$

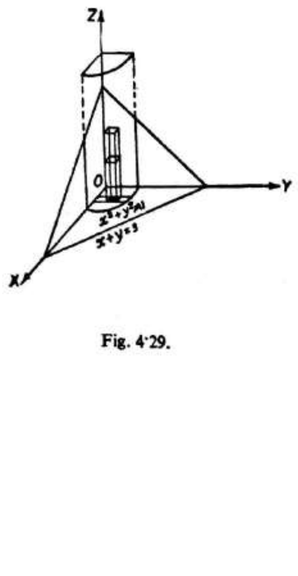


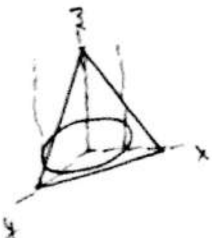
Fig. 4·29.

by the xy-plane $(z=0)$ is the straight line

$$x+y=3$$

Let V be the volume of the solid figure which is bounded laterally by the cylinder $x^2+y^2=1$ and on the top by the plane $x+y+z=3$.

Thus limits of z are from 0 (on the xy-plane) to $3-x-y$ and x or y vary on the circle $x^2+y^2=1$ (in the xy-plane).



$$\therefore \quad V=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{3-x-y} dz\, dy\, dx,$$

$$=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\Big[z\Big]_{0}^{3-x-y} dy\, dx,$$

$$=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(3-x-y)\, dy\, dx,$$

$$=\int_{-1}^{1}\left(3y-xy-\frac{y^2}{2}\right)_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx,$$

$$=\int_{-1}^{1}\left(6\sqrt{1-x^2}-2x\sqrt{1-x^2}\right) dx.$$

Put $x=\sin\theta$,
then $dx=\cos\theta\, d\theta$.

Here θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

$$\therefore \quad V=\int_{-\pi/2}^{\pi/2}\left(6\sqrt{1-\sin^2\theta}-2\sin\theta\sqrt{1-\sin^2\theta}\right)\cos\theta\, d\theta,$$

$$=\int_{-\pi/2}^{\pi/2}(6\cos^2\theta-2\sin\theta\cos^2\theta)\, d\theta,$$

$$=12\int_{0}^{\pi/2}\cos^2\theta\, d\theta-\int_{-\pi/2}^{\pi/2}2\sin\theta\cos^2\theta\, d\theta,$$

$$=12\cdot\frac{1}{2}\cdot\frac{\pi}{2}+2\left[\frac{\cos^3\theta}{3}\right]_{-\pi/2}^{\pi/2}=3\pi.$$

Example 2. *Find the volume common to the cylinders* $x^2+y^2=a^2$ *and* $x^2+z^2=a^2$. (*A.M.I.E. 1978*)

Sol. The sections of the cylinders

$$x^2+y^2=a^2 \quad \ldots(i)$$

and

$$x^2+z^2=a^2 \quad \ldots(ii)$$

are the circles $x^2+y^2=a^2$ and $x^2+z^2=a^2$, in their respective planes.

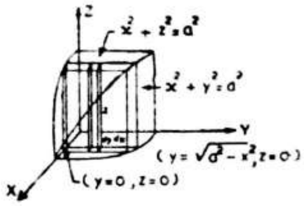


Fig. 4·30.

In the figure, only one-eigth (in first octant) of the volume is shown.

Now, in the common region z varies from 0 to $\sqrt{a^2-x^2}$ and x and y vary on the circle

$$x^2+y^2=a^2.$$

If V is the required volume, then

$$V=8\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}}\int_{0}^{\sqrt{a^2-x^2}} dz\, dy\, dx,$$

$$=8\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}}\Big[z\Big]_{0}^{\sqrt{a^2-x^2}} dy\, dx,$$

$$=8\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2}\, dy\, dx,$$

$$=8\int_{0}^{a}\sqrt{a^2-x^2}\Big[y\Big]_{0}^{\sqrt{a^2-x^2}} dx,$$

$$=8\int_0^a \sqrt{a^2-x^2}\left(\sqrt{a^2-x^2}\right)dx=8\int_0^a (a^2-x^2)\,dx,$$

$$=8\left[a^2x-\frac{x^3}{3}\right]_0^a=8\left(a^3-\frac{a^3}{3}\right)=\frac{16a^3}{3}.$$

Example 3. *Find the volume of the region bounded by the paraboloid $az=x^2+y^2$ and the cylinder $x^2+y^2=R^2$.*

Sol. The volume common to the paraboloid and the cylinder can be very easily found by use of cylinderical coordinates. Transforming the given equations to the polar form, by substituting

$$x=r\cos\phi,\ y=r\sin\phi,$$

we get the equation of the cylinder $x^2+y^2=R^2$ as $r=R$ and that of the paraboloid as

$$az=r^2 \text{ or } z=\frac{r^2}{a}.$$

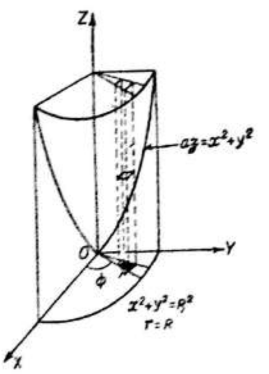


Fig. 4·31.

... of the common volume is shown.

$$=4\int_0^{\pi/2}\int_0^R r\cdot\Big[z\Big]_0^{r^2/a}drd\phi,$$

$$=4\int_0^{\pi/2}\int_0^R r\cdot\left(\frac{r^2}{a}\right)drd\phi,$$

$$=\frac{4}{a}\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^R d\phi,$$

$$=\frac{R^4}{a}\Big[\phi\Big]_0^{\pi/2}=\frac{\pi R^4}{2a}.$$

Example 4. *Find the volume of the region bounded above by the sphere $x^2+y^2+z^2=a^2$ and below by the cone $x^2+y^2=z^2$.*

Sol. The equation of the sphere is

$$x^2+y^2+z^2=a^2 \qquad \ldots(i)$$

and that of the cone is

$$x^2+y^2=z^2. \qquad \ldots(ii)$$

In polar spherical coordinates, we have

$$x=r\sin\theta\cos\phi,$$
$$y=r\sin\theta\sin\phi,\ z=r\cos\theta$$

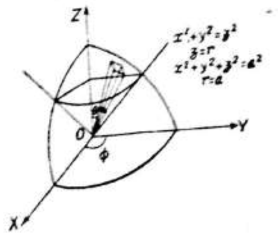


Fig. 4·32.

... (i) and (ii) in polar spherical coordinates

$$V=4\int_0^{\pi/2}\int_0^{\pi/4}\int_0^a r^2\sin\theta\,dr\,d\theta\,d\phi,$$ [§ 4·12, 2 (b) ii]

$$=4\int_0^{\pi/2}\int_0^{\pi/4}\left[\frac{r^3}{3}\right]_0^a\sin\theta\,d\theta\,d\phi,$$

$$=\frac{4}{3}a^3\int_0^{\pi/2}\Big[-\cos\theta\Big]_0^{\pi/4}d\phi,$$

$$=\frac{4}{3}a^3\left(1-\frac{1}{\sqrt{2}}\right)\int_0^{\pi/2}d\phi,$$

$$=\frac{4}{3}a^3\left(1-\frac{1}{\sqrt{2}}\right)\Big[\phi\Big]_0^{\pi/2},$$

$$=\frac{2}{3}\pi a^3\left(1-\frac{1}{\sqrt{2}}\right).$$

4·13. Volume of Revolution

Let $P(x, y)$ be a point in a plane area of the region R. Consider an elementary area $\delta x\delta y$ of this region, at the point P.

If this elementary area is revolved about the x-axis, it will generate a ring of radius y. The elementary volume δV of the ring is given by

$$\delta V=2\pi y\,.\,\delta y\delta x.$$

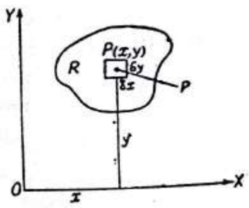


Fig. 4 33.

$$\therefore\quad V=2\pi\iint_R y\,dydx, \qquad ...(i)$$

which is the required volume of revolution as the region R revolves about x-axis.

Similarly, if this region revolves about y-axis, then the required volume V is given by

$$V=2\pi\iint_R x\,dxdy \qquad ...(ii)$$

The equivalent of the formulae (i) and (ii) in polar coordinates can be obtained by substituting

$$x=r\cos\theta,\ y=r\sin\theta,\ \text{and}\ dydx=r\,drd\theta.$$

Thus in polar coordinates, the volume of revolution of the region R, about the initial line and the line through the pole and perpendicular to intial line are respectively

$$2\pi\iint_R r^2\sin\theta\,drd\theta,\quad 2\pi\iint_R r^2\cos\theta\,drd\theta.$$

Example 1. *Find by double integration the volume of the solid generated by revolving the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *about the x-axis.*

Sol. Here we shall consider the revolution of the only upper half of the ellipse about the x-axis, because volumes generated by upper and lower halves overlap. If V is the required volume, then

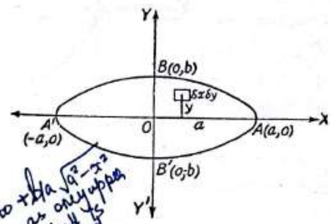


Fig 4·34.

$$V=\int_{-a}^{a}\int_0^{(b/a)\sqrt{a^2-x^2}}2\pi y\,dy\,dx,$$

0 to +b/a $\sqrt{a^2-x^2}$ as only upper half is considered

$$=2\pi\int_{-a}^{a}\left[\frac{y^2}{2}\right]_0^{(b/a)\sqrt{a^2-x^2}}dx,$$

$$=\pi\int_{-a}^{a}\frac{b^2}{a^2}(a^2-x^2)\,dx,$$

$$=\frac{2\pi b^2}{a^2}\int_0^a(a^2-x^2)\,dx=\frac{2\pi b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a,$$

$$=\frac{2\pi b^2}{a^2}\left[a^3-\frac{a^3}{3}\right]=\frac{4\pi ab^2}{3}$$

Example 2. *The area bounded by the parabola $y^2=4x$, and the straight lines $x=1$ and $y=0$, in the first quadrant is revolved about the line $y=2$. Find by double integration the volume of the solid generated.*

equate it to zero.

Sol. The line $y=2$, meets the parabola $y^2=4x$ at the point $L\,(1, 2)$.

2-y

$$\text{The required volume}=\int_0^1\int_0^{2\sqrt{x}}2\pi(2-y)\,\cancel{dxdy}\ dy\,dx,$$

$$=2\pi\int_0^1\left(2y-\frac{y^2}{2}\right)_0^{2\sqrt{x}}dx,$$

$$=2\pi\int_0^1\left(4\sqrt{x}-2x\right)dx,$$

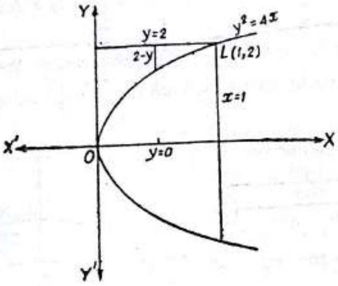


Fig. 4·35.

$$=2\pi\left[\frac{8}{3}x^{3/2}-x^2\right]_0^1=2\pi\left[\frac{8}{3}-1\right]=\frac{10\pi}{3}.$$

Example 3. *By using double integral, show that the volume generated by revolution of the cardiod $r=a(1+\cos\theta)$ about the initial line is $\frac{8}{3}\pi a^3$.*

Sol. We shall consider only the upper half of the cardiod, because lower half generates the same volume.

The required volume

$$=\int_0^{\pi}\int_0^{a(1+\cos\theta)}2\pi r^2\sin\theta\,dr\,d\theta,$$

$$=2\pi\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)}\sin\theta\,d\theta,$$

$$=\frac{2\pi}{3}a^3\int_0^{\pi}(1+\cos\theta)^3\sin\theta\,d\theta,$$

$$=\frac{2\pi a^3}{3}\left[-\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi},$$

$$=-\frac{2\pi a^3}{3}\left[0-\frac{(2)^4}{4}\right]=\frac{8\pi a^3}{3}.$$

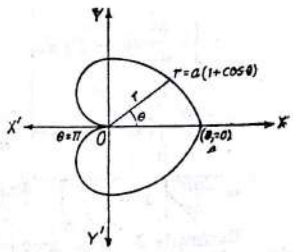


Fig. 4·36.

Exercise 4 (e)

1. Find by double integration, the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

2. Show by double integration that the area between the parabolas $y^2=4ax$ and $x^2=4ay$ is $\frac{16}{3}a^2$. *(Roorkee, 1977)*

3. Find the area lying between the parabola $y=4x-x^2$ and the line $y-x=0$. *(Ranchi, 1976)*

4. Find by double integration the area of the region enclosed by the curves $\sqrt{x}+\sqrt{y}=\sqrt{a}$ and $x+y=a$. *(Punjab, 1979)*

5. Compute the area bounded by the ellipse $(y-x)^2+x^2=1$.

6. Changing to polar coordinates, find the area bounded by the curves $x^2+y^2=2x$, $x^2+y^2=4x$, $y=x$ and $y=0$.

7. Find the area bounded by the circles $r=2\sin\theta$, $r=4\sin\theta$.

8. Find by double integration the area of one loop of the leminscate of Bernoulli $r^2=a^2\cos 2\theta$.

9. Find by double integration the volume of the sphere $x^2+y^2+z^2=4$.

10. Find the volume enclosed by the paraboloid $x^2+y^2=2az$ and the cylinder $x^2+y^2=2ax$.

11. Find the volume bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. *(Gorakhpur, 1975)*

12. Find the volume of the solid bounded by the paraboloid $y^2+z^2=4x$ and the plane $x=5$.

13. Compute the volume of the region bounded by the surface $z=4-x^2-y^2$ and the xy-plane.

14. Find the volume cut off from the sphere $x^2+y^2+z^2=a^2$ by the cylinder $x^2+y^2=ax$. *(Bombay, 1976)*

15. Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$. *(Delhi, 1975)*

16. Find the volume of the cylinder $x^2+y^2=2ax$ which is intercepted by the planes $z=x\tan\alpha$, $z=x\tan\beta$.

17. Find the volume of the cylinderical column standing on the area common to the parabolas $y^2=x$, $x=y^2$ and cut off by the surface $z=12+y-x^2$.

18. Find the volume bounded by the paraboloid $x^2+y^2=4az$ and the plane $x+y+z=a$.

19. A triangular prism is formed by the planes whose equations are $ay=bx$, $y=0$ and $x=a$. Prove that the volume of this

Now $$P_{xy}=\int_0^{2b}\int_0^{2a} \rho\, xy\, dxdy,$$

$$=\rho\int_0^{2b}\left[\frac{x^2}{2}\right]_0^{2a} ydy,$$

$$=2a^2\rho\int_0^{2b} ydy=2a^2\rho\left[\frac{y^2}{2}\right]_0^{2b}=4a^2b^2\rho.$$

Hence, $$\tan 2\theta=\frac{2P_{xy}}{I_y-I_x}=\frac{2\,.\,4a^2b^2\rho}{\left(\frac{16}{3}a^3b-\frac{16}{3}ab^3\right)\rho},$$

$$=\frac{3ab}{2(a^2-b^2)},$$

or $$\theta=\frac{1}{2}\tan^{-1}\left\{\frac{3ab}{2(a^2-b^2)}\right\}.$$

Exercise 4 (*f*)

1. Find the mass of an elliptic plate $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if the surface density at any point $P(x, y)$ on it is μxy.

2. A plate has for its edge the curve $y=e^x$, the line $x=1$ and the coordinate axes. If the density varies as the square of the distance from the origin, find the mass of the plate.

3. A thin plate is bounded by an arc of the parabola $y=2x-x^2$ in the interval $0\leqslant x\leqslant 2$ and the x-axis. Show that its mass is $\frac{26}{3}-\frac{15}{2}\log 3$, if the density at a point is $\frac{(1-y)}{(1+x)}$.

4. Find the mass of the area bounded by the curves $y^2=x$ and $y=x^3$, if $\rho=\mu(x^2+y^2)$.

5. Find the mass of the lamina which is in the form of Lemniscate of Bernoulli $r^2=a^2\cos 2\theta$, the density at a point being proportional to the square of its distance from the pole.

6. Determine the mass of a slab in the shape of a circle of radius a, if the density at any point P on it, is proportional to OP, where is O is a point on the axis of the cylinder.

7. Find the mass of the solid bounded by the planes $y=0$, $z=0$ and $z=h$ $(h>0)$ and the cylinder $x^2+y^2=a^2$ $(x>0)$, if $\rho=kyz$.

8. Find the mass of a rectangular parallelopiped bounded by the planes $x=0$, $x=a$, $y=0$, $y=b$ and $z=0$, $z=c$, if the desity ρ at a point $P(x, y, z)$ is given by $\rho=x+y+z$.

9. Find the mass of the solid bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and the coordinate planes, where the density at any point $P(x, y, z)$ is $k\,xyz$.

10. Find the coordinates of the centre of gravity a lamina in the shape of a quadrant of the curve $x=a\cos^3 t$, $y=b\sin^3 t$, the density being given by $\rho=kxy$.

11. Find by double integration the centre of gravity of the area of the cardiod $r=a(1+\cos\theta)$.

12. Find the centroid of the area of the loop of the leminscate $r^2=a^2\cos 2\theta$.

13. Find the position of the centre of gravity of the area enclosed by the curves $x^2+y^2-2ax=0$ and $x^2+y^2-2bx=0$, on the positive side of the x-axis.

14. Find the centre of gravity of the uniform solid in the form of $$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}=1,$$ contained in the first octant.

15. Find the C.G. of the volume cut off from the cylinder $x^2+y^2=2ax$ by the planes $z=mx$ and $z=nx$.

16. Find the centroid of the region R bounded by the parabolic cylinder $z=4-x^2$, and the planes $x=0$, $y=0$, $y=6$, $z=0$, assuming the density to be constant.

17. Find the moment of inertia for the area of the parabola $y^2=4ax$, bounded by x-axis and the latus rectum about the x-axis, assuming the surface density at each point varying as the cube of abscissa.

18. Find the moment of inertia of the area bounded by the circle $r=2a\cos\theta$ with respect to a line perpendicular to its plane through the pole.

19. Compute the moment of inertia for the area of the cardiod $r=a(1-\cos\theta)$ relative to the pole.

20. Find the moment of inertia of a sphere of radius r and mass m about a diameter, if the density is constant.

21. Compute the moment of inertia of a right circular cone whose altitude is h and base radius r. about (*i*) the axis of symmetry (*ii*) the diameter of the base.

22. A lamina has the shape of a right angled triangle OAB, where $\angle AOB=90°$, $OA=b$, $OB=a$, and its density at any point is equal to the distance of the point from the side OA. Using double integrals, find the moment of inertia of the lamina about the sides OA and OB.

23. A horizontal boiler has a flat bottom and its ends are plane and semicircular. If it is just full of water, show that the depth of centre of pressure of either end is $0{\cdot}7\times$ total depth (approximately).

24. A rectangular lamina of length $2a$ and breadth $2b$ is completely immersed in a vertical plane in a fluid so that it centre is at depth h and the side $2a$ makes an angle α with the horizontal. Find the coordinates of the centre of pressure.

25. If an area is bounded by two concentric semicircles with their common bounding diameter in a free surface, prove that the depth of the centre of pressure is $\frac{3}{16}\pi\frac{(a+b)(a^2+b^2)}{a^2+ab+b^2}$.

26. Find the product of inertia of an equilateral triangle about two perpendicular axes in its plane at a vertex, one of the axes being along a side.

27. Show that the principal axes at the node of a half loop of the leminscate $r^2=a^2\cos 2\theta$ are inclined to the initial line at angles $\frac{1}{2}\tan^{-1}\frac{1}{2}$ and $\frac{\pi}{2}+\frac{1}{2}\tan^{-1}\frac{1}{2}$.

28. A uniform lamina is in the form of an arc of the parabola $y^2=4ax$ bounded by a double ordinate at a distance h from the vertex. If $h=\frac{a\sqrt{7}}{3}(\sqrt{7}+4)$, show that the principal axes at the end of a latus rectum are the tangent and normal at that point.

29. Find the product of inertia of a quadrant of an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and show that at the centre the principal axes are inclined at an angle of $\frac{1}{2}\tan^{-1}\left(\frac{4}{\pi}\frac{ab}{a^2-b^2}\right)$ to the axes.

30. Show that the principal axes at the origin of the triangle enclosed by $x=0$, $y=0$, $\frac{x}{a}+\frac{y}{b}=1$ are inclined at an angle α and $\frac{\pi}{2}+\alpha$ to the x-axis

where $\alpha=\frac{1}{2}\tan^{-1}\left(\frac{4ab}{a^2-b^2}\right)$.

Exercise 3 (*i*) (Pages 268—269)

1. $-a^3$.
2. only min. value ; -8.
3. Maximum at (0, 0).
4. $\frac{3\sqrt{3}}{8}$; $\frac{-3\sqrt{3}}{8}$.
5. Max. at $x=\frac{1}{2}$, $y=\frac{1}{3}$.
7. $m^m n^n p^p \left(\frac{a}{m+n+p}\right)^{m+n+p}$
8. $\frac{l^2}{au-1}+\frac{m^2}{bu-1}+\frac{n^2}{cu-1}=0$; where $u=x^2+y^2+z^2$.
Here maxima and minima of the distance of the origin from the points of intersection of central conicoid
$ax^2+by^2+cz^2=1$, by the central plane $lx+my+nz=0$.
9. Each of the two variables is equal to $\frac{4}{3}$, the third being $\frac{7}{3}$. Max. value $\frac{7}{3}$. Each of the two variables is equal to 2, the third being equal to 1, min. value 4.
10. $\frac{\Sigma(l-l')^2}{\Sigma(mn'-m'n)^2}$.
13. A cube.
14. 50 units.
15. Into equal parts.
17. $a=\frac{\sqrt{5}}{2}b$, $h=\frac{1}{2}b$.
18. $I_1 : I_2 : I_3 = \frac{1}{R_1} : \frac{1}{R_2} : \frac{1}{R_3}$.
19. $\left(\frac{1}{2}, \frac{1}{4}\right)$; $\left(\frac{11}{8}, \frac{-5}{8}\right)$; $\frac{7\sqrt{2}}{8}$.

Exercise 3 (*j*) (Page 273)

2. $\log\left(\frac{b}{a}\right)$; $(a>0, b>0)$.
5. $\sqrt{\frac{\pi}{2}}$, $\sqrt{\frac{\pi}{2}}$.
4. $-\frac{1}{(n+1)^2}$.
6. $\frac{\pi(a^2+b^2)}{4a^3b^3}$.

Exercise 4 (*a*) (Pages 280—281)

1. $\frac{2}{3}$.
2. $\frac{1}{3}a^3$.
3. $\frac{\pi}{4}\log(1+\sqrt{2})$.
4. $\frac{314}{35}a^4$.
5. $\frac{\pi a^3}{6}$.
7. $\frac{a^4}{3}$.
8. $\frac{\pi}{4}ab(a^2+b^2)$.
9. $\frac{3}{56}$.
10. 6.

Exercise 4 (*b*) (Pages 286—287)

1. $\left(\frac{b^3-a^3}{3}\right)(\cos\alpha-\cos\beta)$.
2. $\frac{3(a^4-b^4)\pi}{64}$.

3. $\frac{\pi}{2b}$.
4. $\frac{a^2(8+\pi)}{8}$.
5. $\frac{a^3}{18}\cdot(3\pi-4)$.
7. $\frac{4}{3}a^3$.
8. $\frac{4}{3}$.
9. πa^2

Exercise 4 (*c*) (Page 293)

1. $\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy$.
2. $\int_0^2\int_y^2 \varphi(x, y)\,dx\,dy$.
3. $\int_0^a\int_0^{2\sqrt{ax}} f(x, y)\,dy\,dx+\int_a^{3a}\int_0^{3a-x} f(x, y)\,dy\,dx$.
4. $\int_0^a\int_0^y f(x, y)\,dx\,dy$.
5. $\int_{-2}^{0}\int_{\frac{3-\sqrt{1-4y}}{2}}^{y+2}\varphi(x, y)\,dx\,dy+\int_0^{\frac{1}{4}}\int_{\frac{3-\sqrt{1-4y}}{2}}^{\frac{3+\sqrt{1-4y}}{2}}\varphi(x, y)\,dx\,dy$
6. $1-\frac{1}{2}\sqrt{2}$.
7. $\frac{a}{140}(5a^2+7)$.
8. $\frac{a^3\sqrt{2}}{6}$.
9. $\frac{\pi a}{4}$.
10. $\pi a^2[\varphi(a)-\varphi(0)]$.
11. $\frac{\pi}{4}$.
12. 1.

Exercise 4 (*d*) (Page 298)

1. $\frac{e^{4a}}{8}-\frac{3}{4}e^{2a}+e^a-\frac{3}{8}$.
2. $\frac{4}{35}$.
3. $\frac{\pi^2}{8}$.
4. $\frac{1}{8}$.
5. 16.
6. $\frac{a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)$.

Exercise 4 (*e*) (Pages 319—320)

1. πab.
3. $\frac{9}{2}$.

4. $\frac{a^2}{3}$. 5. π.

6. $3\left(\frac{\pi}{4}+\frac{1}{2}\right)$. 7. 3π.

8. $\frac{a^2}{2}$. 9. $\frac{32}{3}\pi$.

10. $3\pi a^3$. 11. $\frac{abc}{6}$.

12. 50π. 13. 8π.

14. $\frac{4}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right)$. 15. 16π.

16. $\pi a^3(\tan\alpha-\tan\beta)$. 17. $\frac{569}{140}$.

18. $18\pi a^3$. 20. $\frac{2}{3}\pi a^3$.

21. $2\pi^2 a^3$. 22. $\frac{8\pi a^3}{3}$.

Exercise 4 (*f*) (Pages 342—345)

1. $\frac{\mu^2 a^2 b^2}{2}$. 2. $2{\cdot}84\,k$. 3. $\frac{23\mu}{105}$. 5. $\frac{k\,\pi a^4}{8}$.

6. $\mu\pi a$. 7. $\frac{1}{3}ka^3h^2$. 8. $\frac{abc}{2}(a+b+c)$.

9. $\frac{1}{48}ka^2b^2c^2$. 10. $\left(\frac{128a}{429}, \frac{128b}{429}\right)$.

11. $\left(\frac{5a}{6}, 0\right)$ 12. $\frac{\pi a}{4}$

14. $\left(\frac{21a}{128}, \frac{21b}{128}, \frac{21c}{128}\right)$. 15. $\left[\frac{5a}{4}, 0, \frac{5a}{8}(m+n)\right]$

16. $\left(\frac{3}{4}, 3, \frac{8}{5}\right)$ 17. $\frac{16}{33}ka^7$. 18. $\frac{3}{2}\pi a^4$.

19. $\frac{35\pi a^4}{16}$. 20. $\frac{2}{5}mr^2$. 21. (*i*) $\frac{\pi}{10}hr^4$.

(*ii*) $\frac{\pi hr^2}{60}(2h^2+3r^2)$. 22. $\frac{a^3b}{12}, \frac{a^2b^2}{24}$

24. $\left(-\frac{a^2}{3h}\sin\alpha, -\frac{b^2}{3h}\cos\alpha\right)$. 26. a^4.

Exercise 5 (*a*) (Pages 353—355)

1. $\vec{b}-\vec{a}$; $-\vec{a}$; $-\vec{b}$; $\vec{a}-\vec{b}$. 2. $\frac{1}{3}(\vec{a}+\vec{b}+\vec{c})$.

5. (*i*) $\frac{7}{3}(i+2j+2k)$. (*ii*) $-\frac{15}{2\sqrt{11}}(6i+2j+2k)$.

(*iii*) $-\sqrt{\frac{22}{3}}(2i-j+k)$. 6. $\sqrt{6}$.

11. $\frac{a}{\sqrt{2}}(4i+3j+5k)$; $5a$.

Exercise 5 (*b*)

1. (*i*) 0 (*ii*) $8i+11j-5k$ (*iii*) $\pm\frac{(6i+11j-5k)}{\sqrt{210}}$.

2. $\lambda=\frac{1}{2}(\pm\sqrt{73}-11)$. 7. $-\frac{1}{2}(i+j-\sqrt{2}k)$.

8. $\frac{1}{7}(i+2j+3k)$. 14. $\frac{7}{2}\sqrt{3}$.

15. (*i*) $5\sqrt{17}$. (*ii*) $5\sqrt{3}$. 18. 17 ; $-24i+13j+$

19. 48 units. 20. $\frac{102\sqrt{13}}{11}$.

21. $9(12i-10j+7k)$; $9\sqrt{293}$

Exercise 5 (*c*) (Pages 372—374)

1. -120 ; $-10(6i+4j-5k)$. 2. $\lambda=-4$.

Exercise 6 (*a*) (Pages 380—383)

3. $\frac{ti+2j+(2t-3)k}{\sqrt{(5t^2-12t+13)}}$; $\frac{1}{3}(2i+2j+k)$.

4. $\cos^{-1}\left(\frac{9}{17}\right)=58°\ 2'$. 5. $\frac{8\sqrt{14}}{7}$; $-\frac{\sqrt{14}}{7}$.

6. $\sqrt{37}$; $5\sqrt{13}$. 7. 16 ; $2\sqrt{73}$.

8. $\alpha=\pm\frac{1}{\sqrt{6}}$.

Exercise 6 (*b*) (Pages 394—395)

3. $\frac{1}{3}\left(1-\frac{1}{r^3}\right)$. 4. $f(x, y, z)=x^2yz^3+20$.

5. $\frac{-4i+9j+6k}{\sqrt{133}}$.

6. (*i*) $-3\frac{2}{3}$, (*ii*) $\frac{14}{3}$, (*iii*) $-14\frac{2}{3}$, (*iv*) 144, (*v*) $\frac{15}{\sqrt{17}}$