SET OPERATIONS

Solved Problems 1.6 to 1.8 refer to the universal set $U = \{1, 2, \ldots, 9\}$ and the sets

$$A = \{1, 2, 3, 4, 5\}, \quad C = \{5, 6, 7, 8, 9\}, \quad E = \{2, 4, 6, 8\}$$
$$B = \{4, 5, 6, 7\}, \quad D = \{1, 3, 5, 7, 9\}, \quad F = \{1, 5, 9\}$$

1.6 Find:

(a) $A \cup B$ and $A \cap B$ (b) $B \cup D$ and $B \cap D$ (c) $A \cup C$ and $A \cap C$
(d) $D \cup E$ and $D \cap E$ (e) $E \cup E$ and $E \cap E$ (f) $D \cup F$ and $D \cap F$

Recall that the union $X \cup Y$ consists of those elements in either X or Y (or both), and that the intersection $X \cap Y$ consists of those elements in both X and Y.

(a) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$ $\quad A \cap B = \{4, 5\}$
(b) $B \cup D = \{1, 3, 4, 5, 6, 7, 9\}$ $\quad B \cap D = \{5, 7\}$
(c) $A \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = U$ $\quad A \cap C = \{5\}$
(d) $D \cup E = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = U$ $\quad D \cap E = \varnothing$
(e) $E \cup E = \{2, 4, 6, 8\} = E$ $\quad E \cap E = \{2, 4, 6, 8\} = E$
(f) $D \cup F = \{1, 3, 5, 7, 9\} = D$ $\quad D \cap F = \{1, 5, 9\} = F$

Observe that $F \subseteq D$; so by Theorem 1.2 we must hav $D \cup F = D$ and $D \cap F = F$.

1.7 Find: (a) A^c, B^c, D^c, E^c; (b) $A \setminus B$, $B \setminus A$, $D \setminus E$, $F \setminus D$; (c) $A \oplus B$, $C \oplus D$, $E \oplus F$.

Recall that:

Therefore:
(a) $A^c = \{6, 7, 8, 9\}$; $B^c = \{1, 2, 3, 8, 9\}$; $D^c = \{2, 4, 6, 8\} = E$; $E^c = \{1, 3, 5, 7, 9\} = D$.
(b) $A \backslash B = \{1, 2, 3\}$; $B \backslash A = \{6, 7\}$; $D \backslash E = \{1, 3, 5, 7, 9\} = D$; $F \backslash D = \varnothing$.
(c) $A \oplus B = \{1, 2, 3, 6, 7\}$; $C \oplus D = \{1, 3, 8, 9\}$; $E \oplus F = \{2, 4, 6, 8, 1, 5, 9\} = E \cup F$.

1.8 Find: (a) $A \cap (B \cup E)$; (b) $(A \backslash E)^c$; (c) $(A \cap D) \backslash B$; (d) $(B \cap F) \cup (C \cap E)$.

(a) First compute $B \cup E = \{2, 4, 5, 6, 7, 8\}$. Then $A \cap (B \cup E) = \{2, 4, 5\}$.
(b) $A \backslash E = \{1, 3, 5\}$. Then $(A \backslash E)^c = \{2, 4, 6, 7, 8, 9\}$.
(c) $A \cap D = \{1, 3, 5\}$. Now $(A \cap D) \backslash B = \{1, 3\}$.
(d) $B \cap F = \{5\}$ and $C \cap E = \{6, 8\}$. So $(B \cap F) \cup (C \cap E) = \{5, 6, 8\}$.

1.9 Show that we can have $A \cap B = A \cap C$ without $B = C$.

Let $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{2, 4\}$. Then $A \cap B = \{2\}$ and $A \cap C = \{2\}$. Thus $A \cap B = A \cap C$ but $B \neq C$.

Fig. 1.22

1.12 Determine the validity of the following argument:

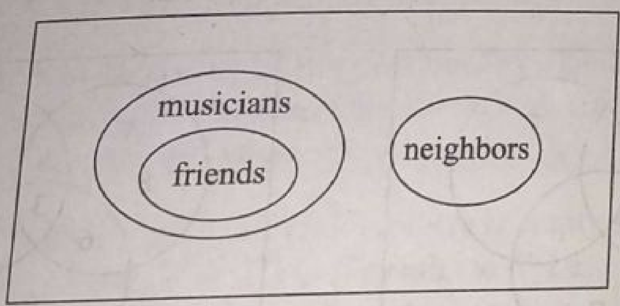
S_1: All my friends are musicians.

S_2: John is my friend.

S_3: None of my neighbors are musicians.

S: John is not my neighbor.

The premises S_1 and S_3 lead to the Venn diagram in Fig. 1.23. By S_2, John belongs to the set of friends which is disjoint from the set of neighbors. Thus S is a valid conclusion and so the argument is valid.



1.30 The 60000 fans who attended the homecoming football game brought up all the paraphernalia for their cars. Altogether 20000 bumper stickers, 36000 windows decals and 12000 key rings were sold. We know that 52000 fans bought at least 1 item and no one bought more than 1 of the given items. Also 6000 fans bought both decals and key rings, 9000 bought both decals and bumper stickers and 5000 bought both key rings and bumper stickers.

(a) How many fans bought all 3 items?

(b) How many fans bought exactly 1 item?

(c) Someone questioned the accuracy of total no. of purchasers is 52000. He claimed purchasers to be either 60000 or 44000. Dispel the claim.

1.31 Among 75 children who went to an amusement park, where they could ride on merry-go-round, roller coaster and ferris wheel. It is known that, 20 of them had taken all three rides and 55 had taken at least two of the 3 rides. Each ride costs Rs. 0.50 and total receipt of park is Rs. 70.

Determine the number of children who did not try any of the rides?

1.32 If number of students who got Grade A in first examination is equal to that of in second examination. If total number of students who got Grade A in exactly one examination is 40, and 4 students did not get Grade A in either examinations, determine the no. of students who got Grade A in first exam only, who got Grade A in second exam only and who got Grade A in both the exams?

Let us assume that,

1.36 Draw Venn diagrams showing

(a) $(A \cup B) \in (A \cup C)$ but $B \not\subseteq C$

(b) $(A \cap B) \in (A \cap C)$ but $B \not\subseteq C$

(c) $(A \cup B) = (A \cup C)$ but $B \neq C$

(d) $(A \cap B) = (A \cap C)$ but $B \neq C$

Statement	Reason
1. $(A \cup B) \cap (A \cup B^c) = A \cup (B \cap B^c)$	Distributive law
2. $B \cap B^c = \varnothing$	Complement law
3. $(A \cup B) \cap (A \cup B^c) = A \cup \varnothing$	Substitution
4. $A \cup \varnothing = A$	Identity law
5. $(A \cup B) \cap (A \cup B^c) = A$	Substitution

1.40 Prove: $(A \cup B) \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A)$. (Thus either one may be used to define $A \oplus B$.)

Using $X \backslash Y = X \cap Y^c$ and the laws in Table 1.1, including DeMorgan's laws, we obtain:

$$\begin{aligned}(A \cup B) \backslash (A \cap B) &= (A \cup B) \cap (A \cap B)^c = (A \cup B) \cap (A^c \cup B^c) \\ &= (A \cap A^c) \cup (A \cap B^c) \cup (B \cap A^c) \cup (B \cap B^c) \\ &= \varnothing \cup (A \cap B^c) \cup (B \cap A^c) \cup \varnothing \\ &= (A \cap B^c) \cup (B \cap A^c) = (A \backslash B \cup (B \backslash A).\end{aligned}$$

CLASSES OF SETS

Thus we se...

Miscellaneous Problems

1.46 Prove the proposition P that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$; that is,

$$P(n): 1 + 2 + 3 + \ldots + n = \frac{1}{2}\, n(n+1)$$

The proposition holds for $n = 1$ since

$$P(1) : 1 = \tfrac{1}{2}(1)(1+1)$$

Assuming $P(n)$ is true, we add $n + 1$ to both sides of $P(n)$, obtaining

$$\begin{aligned} 1 + 2 + 3 + \ldots + n + (n+1) &= \tfrac{1}{2}\, n(n+1) + (n+1) \\ &= \tfrac{1}{2}\,[n(n+1) + 2(n+1)] \\ &= \tfrac{1}{2}\,[(n+1)(n+2)] \end{aligned}$$

1.45 Find all partitions of $S = \{1, 2, 3\}$.

Note that each partition of S contains either 1, 2, or 3 cells. The partitions for each number of cells a as follows:

(1): $[S]$

(2): $[\{1\}, \{2, 3\}]$, $[\{2\}, \{1, 3\}]$, $[\{3\}, \{1, 2\}]$

(3): $[\{1\}, \{2\}, \{3\}]$

Thus we see that there are five different partitions of S.

1.21 Given $A = [\{a, b\}, \{c\}, \{d, e, f\}]$.

(a) State whether each of the following is true or false:

(i) $a \in A$, (ii) $\{c\} \subseteq A$, (iii) $\{d, e, f\} \in A$, (iv) $\{\{a, b\}\} \subseteq A$, (v) $\varnothing \subseteq A$.

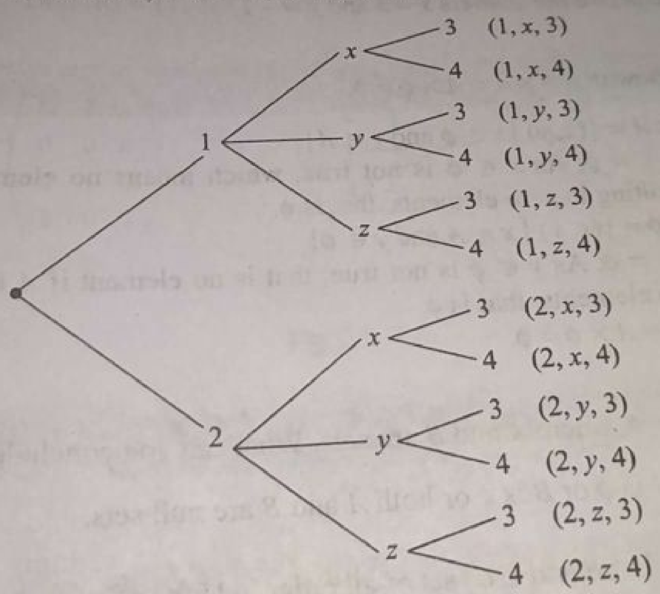
(b) Find the power set of A.

2.2 Given: $A = \{1, 2\}$, $B = \{x, y, z\}$, and $C = \{3, 4\}$. Find: $A \times B \times C$.

$A \times B \times C$ consists of all ordered triplets (a, b, c) where $a \in A$, $b \in B$, $c \in C$. These elements of $A \times B \times C$ can be systematically obtained by a so-called tree diagram (Fig. 2.7). The elements of $A \times B \times C$ are precisely the 12 ordered triplets to the right of the tree diagram.

Observe that $n(A) = n(B) = 3$, and $n(C) = 2$ and, as expected,

$$n(A \times B \times C) = 12 = n(A) \cdot n(B) \cdot n(C)$$



2.3 Let $A = \{1, 2\}$, $B = \{a, b, c\}$ and $C = \{c, d\}$. Find: $(A \times B) \cap (A \times C)$ and $(B \cap C)$.

We have

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$
$$A \times C = \{(1, c), (1, d), (2, c), (2, d)\}$$

Hence

$$(A \times B) \cap (A \times C) = \{(1, c), (2, c)\}$$

Since $B \cap C = \{c\}$,

$$A \times (B \cap C) = \{(1, c), (2, c)\}$$

Observe that $(A \times B) \cap (A \times C) = A \times (B \cap C)$. This is true for any sets A, B and C (see Problem 2.4).

2.10 Given $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Let R be the following relation from A to B:

$$R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$$

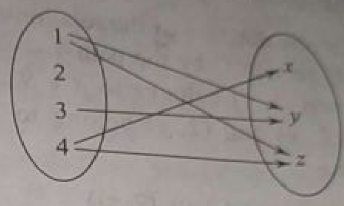
(a) Determine the matrix of the relation.
(b) Draw the arrow diagram of R.
(c) Find the inverse relation R^{-1} of R.
(d) Determine the domain and range of R.

... Fig. 2.7(b) we obtain the arrow diagram of R^{-1} ... Dom(R), consists of the first elements of the ordered pairs of R, and the range of R. Ran(R), consists of the second elements. Thus,

$$\mathrm{Dom}(R) = \{1, 3, 4\} \quad \text{and} \quad \mathrm{Ran}(R) = \{x, y, z\}$$

$$\begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

(a)



(b)

Fig. 2.8

2.11 Let $A = \{1, 2, 3, 4, 6\}$, and let R be the relation on A defined by "x divides y", written $x|y$. (Note $x|y$ iff there exists an integer z such that $xz = y$.)

(a) Write R as a set of ordered pairs.

(b) Draw its directed graph.

(c) Find the inverse relation R^{-1} of R. Can R^{-1} be described in words?

(a) Find those numbers in A divisible by 1, 2, 3, 4, and then 6. These are:

$$1|1, \quad 1|2, \quad 1|3, \quad 1|4, \quad 1|6, \quad 2|2, \quad 2|4, \quad 2|6, \quad 3|3, \quad 3|6, \quad 4|4, \quad 6|6$$

See Fig. 2.9.

(c) Reverse the ordered pairs of R to obtain R^{-1}:
$R^{-1} = \{(1, 1), (2, 1), (3, 1), (4, 1), (6, 1), (2, 2), (4, 2), (6, 2), (3, 3), (6, 3), (4, 4), (6, 6)\}$
R^{-1} can be described by the statement "x is a multiple of y".

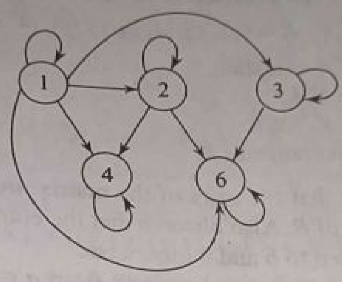


Fig. 2.9

2.12 Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and $C = \{x, y, z\}$. Consider the following relations R and S from A to B and from B to C, respectively.

$$R = \{(1, b), (2, a), (2, c)\} \quad \text{and} \quad S = \{(a, y), (b, x), (c, y), (c, z)\}$$

(a) Find the composition relation $R \circ S$.

(b) Find the matrices M_R, M_S, and $M_{R \circ S}$ of the respective relations R, S and $R \circ S$, and compare $M_{R \circ S}$ to the product $M_R M_S$.

(a) Draw the arrow diagram of the relations R and S as in Fig. 2.10. Observe that 1 in A is "connected" to x in C by the path $1 \rightarrow$

antisymmetric. The other three relations are antisymmetric.

2.16 Given: $A = \{1, 2, 3, 4\}$. Consider the following relation in A:

$$R = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)\}$$

(a) Draw its directed graph.
(b) Is R (i) reflexive, (ii) symmetric, (iii) transitive, or (iv) antisymmetric?
(c) Find $R^2 = R \circ R$.

(a) See Fig. 2.11.

(b) (i) R is not reflexive because $3 \in A$ but $3 \not R 3$, i.e. $(3, 3) \notin R$.

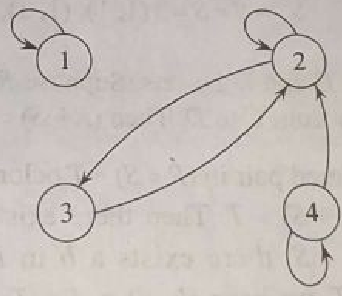
(ii) R is not symmetric because $4R2$ but $2 \not R 4$, i.e. $(4, 2) \in R$ but $(2, 4) \notin R$.

(iii) R is not transitive because $4R2$ and $2R3$ but $4 \not R 3$, i.e. $(4, 2) \in R$ and $(2, 3) \in R$ but $(4, 3) \notin R$.

(iv) R is not antisymmetric because $2R3$ and $3R2$ but $2 \neq 3$.

(c) For each pair $(a, b) \in R$, find all $(b, c) \in R$. Since $(a, c) \in R^2$,

$$R^2 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)\}$$



2.20 Consider a set $A = \{a, b, c\}$ and the relation R on A defined by

$$R = \{(a, a), (a, b), (b, c), (c, c)\}$$

Find (a) reflexive(R); (b) symmetric(R); and (c) transitive(R).

EQUIVALENCE RELATIONS AND PARTITIONS

2.21 Consider the $\mathbf{Z}$ of integers and an integer $m > 1$. We say that x is congruent to y modulo m, written

$$x \equiv y \pmod{m}$$

if $x - y$ is divisible by m. Show that this defines an equivalence relation on $\mathbf{Z}$.

We must show that the relation is reflexive, symmetric, and transitive.

(i) For any x in $\mathbf{Z}$ we have $x \equiv x \pmod{m}$ because $x - x = 0$ is divisible by m. Hence the relation is reflexive.

(ii) Suppose $x \equiv y \pmod{m}$, so $x - y$ is divisible by m. Then $-(x - y) = y - x$ is also divisible by m, so $y \equiv x \pmod{m}$. Thus the relation is symmetric.

6-3 3-6

(iii) Now suppose $x \equiv y \pmod{m}$ and $y \equiv z \pmod{m}$, so $x - y$ and $y - z$ are each divisible by m. Then the sum

$$(x - y) + (y - z) = x - z$$

is also divisible by m; hence $x \equiv z \pmod{m}$. Thus the relation is transitive.

Accordingly, the relation of congruence modulo m on $\mathbf{Z}$ in an equivalence relation.

Let A be a set of nonzero integers and let

$x \equiv z \pmod{m}$. Thus the relation is transitive. ... the relation of congruence modulo m on **Z** is an equivalence relation.

2.22 Let A be a set of nonzero integers and let $\approx$ be the relation on $A \times A$ defined by

$$(a, b) \approx (c, d) \quad \text{whenever} \quad ad = bc$$

Prove that $\approx$ is an equivalence relation.

We must show that $\approx$ is reflexive, symmetric, and transitive.

(i) *Reflexivity*: We have $(a, b) \approx (a, b)$ since $ab = ba$. Hence $\approx$ is reflexive.

(ii) *Symmetry*: Suppose $(a, b) \approx (c, d)$. Then $ad = bc$. Accordingly, $cb = da$ and hence $(c, d) \approx (a, b)$. Thus, $\approx$ is symmetric.

(iii) *Transitivity*: Suppose $(a, b) \approx (c, d)$ and $(c, d) \approx (e, f)$. Then $ad = bc$ and $cf = de$. Multiplying corresponding terms of the equations gives $(ad)(cf) = (bc)(de)$. Canceling $c \neq 0$ and $d \neq 0$ from both sides of the equation yields $af = be$, and hence $(a, b) \approx (e, f)$. Thus $\approx$ is transitive. Accordingly, $\approx$ is an equivalence relation.

2.23 Let R be the following equivalence relation on the set $A = \{1, 2, 3, 4, 5, 6\}$:

$$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$$

Find the partition of A induced by R, i.e., find the equivalence classes of R.

Those elements related to 1 are 1 and 5 hence

$$[1] = \{1, 5\}$$

... elements related to 2 are 2

… Let us consider that $(a, b) \in R$ and $(b, c) \in R$. Here if $(a, b) \in R$, a has to be male and if $(b, c) \in R$, b has to be male. Hence a and b both are male. Now if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$. R is not reflexive, not symmetric, not antisymmetric and is transitive. Hence R is neither equivalence nor partial ordering.

2.30 From following diagraphs, write relation as set of ordered pairs and check for equivalence or partial ordering.

(a) Here from Fig. 2.13 we get $R = \{(a, b), (b, a), (c, c), (c, d), (a, c), (b, d)\}$

R is not reflexive as $(b, b) \notin R$

R is not symmetric as $(a, c) \notin R$ but $(c, a) \notin R$

R is not antisymmetric as $(a, b) \notin R$ and $(b, a) \notin R$

R is not transitive as $(a, c) \notin R$ and $(c, d) \notin R$ but $(a, d) \notin R$.

Hence R is neither equivalence nor partial ordering relation.

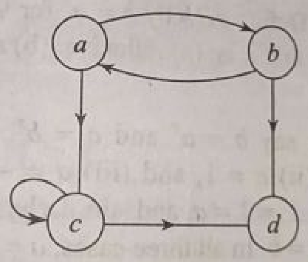


Fig. 2.13

(b) Here from Fig. 2.14 we get $R = \{(a, a), (b, b), (c, c), (a, c), (c, b), (b, a)\}$
R is reflexive.
R is not symmetric.
R is antisymmetric.
R is not transitive as (a, c) and $(c, b) \in R$ but $(a, b) \notin R$.
Hence R is neither equivalence nor partial ordering relation.

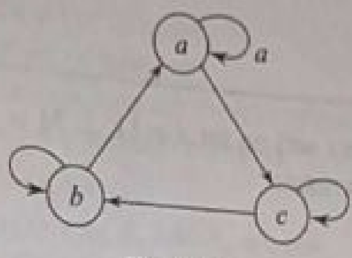
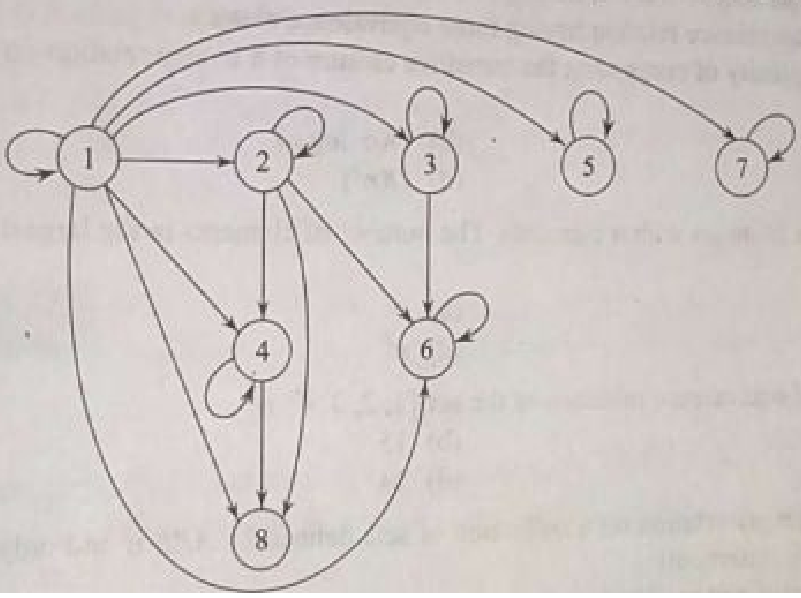


Fig. 2.14

2.31 Draw diagraph for relation R on A. (see Fig. 2.15)
$A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Let xRy whenever y is divisible by x. Is R equivalence relation? Is R Partial ordering?

R is reflexive, antisymmetric and transitive, hence R is not equivalence, is a partial ordering relation.



...isible by 3. Find the partition of S induced by ...

2.13 Let $A = \{1, 2, 3, ..., 9\}$ and let $\sim$ be the relation on $A \times A$ defined by

$$(a, b) \sim (c, d) \quad \text{if} \quad a + d = b + c$$

(a) Prove that $\sim$ is an equivalence relation.

(b) Find $[(2, 5)]$, i.e. the equivalence class of $(2, 5)$.

3.4 Sketch the graph of:

(a) $f(x) = x^2 + x - 6$

(b) $g(x) = x^3 - 3x^2 - x + 3$

3.5 Let the functions $f: A \to B$ and $g: B \to C$ be defined by Fig. 3.9. Find the composition function $g \circ f: A \to C$.

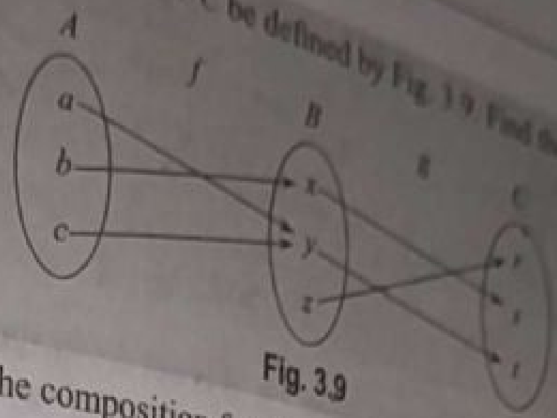


Fig. 3.9

We use the definition of the composition function to compute:

$$(g \circ f)(a) = g(f(a)) = g(y) = t$$
$$(g \circ f)(b) = g(f(b)) = g(x) = s$$
$$(g \circ f)(c) = g(f(c)) = g(y) = t$$

Note that we arrive at the same answer if we "follow the arrows" in the diagram:

$$a \to y \to t, \qquad b \to x \to s, \qquad c \to y \to t$$

3.6 Let the functions f and g be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Find the formula defining the composition function $g \circ f$.

Compute $g \circ f$ as follows: $(g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$.
Observe that the same answer can be found by writing

$$y = f(x) = 2x + 1 \qquad \text{and} \qquad z = g(y) = y^2 - 2$$

and then eliminating y from both equations:

$$z = y^2 - 2 = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$$

ONE-TO-ONE AND INVERTIBLE FUNCTIONS

3.10 Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x - 3$. Now f is one-to-one and onto; hence f has an inverse function f^{-1}. Find a formula for f^{-1}.

Let y be the image of x under the function f:

$$y = f(x) = 2x - 3$$

Consequently, x will be the image of y under the inverse function f^{-1}. Solve for x in terms of y in the above equation:

$$x = (y + 3)/2$$

Then $f^{-1}(y) = (y + 3)/2$. Replace y by x to obtain

$$f^{-1}(x) = \frac{x + 3}{2}$$

which is the formula for f^{-1} using the usual independent variable x.

3.22 Show that the function $f(x) = x^4$ and $g(x)\,x^{1/4}$ for $x \in R$ are inverses of one another. Here R is set of real number.

(a) It is not cold.
(b) It is cold and raining.
(c) It is cold or it is raining.
(d) It is raining or it is not cold.

4.2 Let p be "Erik reads *Newsweek*", let q be "Erik reads *The New Yorker*", and let r be "Erik reads *Time*". Write each of the following in symbolic form:

(a) Erik reads *Newsweek* or *The New Yorker*, but not *Time*.
(b) Erik reads *Newsweek* and *The New Yorker*, or he does not read *Newsweek* and *Time*.
(c) It is not true that Erik reads *Newsweek* but not *Time*.
(d) It is not true that Erik reads *Time* or *The New Yorker* but not *Newsweek*.

Use $\vee$ for "or", $\wedge$ for "and" (or, its logical equivalent, "but"), and $\neg$ for "not" (negation).

(a) $(p \vee q) \wedge \neg r$;
(b) $(p \wedge q) \vee \neg (p \wedge r)$;
(c) $\neg (p \wedge \neg r)$;
(d) $\neg [(r \vee q) \wedge \neg p]$.

ITH VALUES AND TRUTH TABLES

$\equiv F \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee F$

$\equiv (q \wedge \sim p) \vee (p \wedge \sim q)$

SOLVED PROBLEMS

PROPOSITIONS AND LOGICAL OPERATIONS

4.1 Let p be "It is cold" and let q be "It is raining". Give a simple verbal sentence which describes each of the following statements: (a) $\neg p$; (b) $p \wedge q$; (c) $p \vee q$; (d) $q \vee \neg p$.

In each case, translate $\wedge$, $\vee$, and $\sim$ to read "and", "or", and "It is false that" or "not", respectively, and then simplify the English sentence.

4.6 Show that the propositions $\neg(p \wedge q)$ and $\neg p \vee q$ are logically equivalent.

Construct the truth table for $\neg(p \wedge q)$ and $\neg p \vee \neg q$ as in Fig. 4.14. Since the truth tables are the same (both propositions are false in the first case and true in the other three cases), the propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent and we can write

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(a) $\neg(p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(b) $\neg p \vee \neg q$

Fig. 4.14

4.10 Consider the conditional proposition $p \to q$. The simple propositions $q \to p$, $\neg p \to \neg q$, and $\neg q \to \neg p$ are called, respectively, the *converse*, *inverse*, and *contrapositive* of the conditional $p \to q$. Which if any of these propositions are logically equivalent to $p \to q$?

Construct their truth tables as in Table 4.3. Only the contrapositive $\neg q \to \neg p$ is logically equiv to the original conditional proposition $p \to q$.

Table 4.3

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4.14 Prove that the following argument is valid: $p \to \neg q, r \to q, r \vdash \neg p$.

4.16 Determine the validity of the following argument:

If 7 is less than 4, then 7 is not a prime number.
7 is not less than 4.

---

7 is a prime number.

4.18 Determine the truth value of each of the following statements where $\mathbf{U} = \{1, 2, 3\}$ is the universal set: (a) $\exists x \, \forall y, x^2 < y + 1$; (b) $\forall x \, \exists y, x^2 + y^2 < 12$; (c) $\forall x \, \forall y, x^2 + y^2 < 12$.

(a) True. For if $x = 1$, then 1, 2 and 3 are all solutions to $1 < y + 1$.
(b) True. For each x_0, let $y = 1$; then $x_0^2 + 1 < 12$ is a true statement.
(c) False. For if $x_0 = 2$ and $y_0 = 3$, then $x_0^2 + y_0^2 < 12$ is not a true statement.

4.19 Negate each of the following statements:
(a) $\exists x \, \forall y, p(x, y)$; (b) $\forall x \, \forall y, p(x, y)$; (c) $\exists y \, \exists x \, \forall z, p(x, y, z)$.

Use $\neg \forall x \, p(x) \equiv \exists x \neg p(x)$ and $\neg \exists x \, p(x) \equiv \forall x \neg p(x)$:
(a) $\neg (\exists x \, \forall y, p(x, y)) \equiv \forall x \, \exists y \neg p(x, y)$.
(b) $\neg (\forall x \, \forall y, p(x, y)) \equiv \exists x \, \exists y \neg p(x, y)$.
(c) $\neg (\exists y \, \exists x \, \forall z, p(x, y, z)) \equiv \forall y \, \forall x \, \exists z \neg p(x, y, z)$.

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Show equivalence of the following

(a) $[d \rightarrow ((\sim a) \wedge b) \wedge c]$ and $\sim [(a \vee (\sim(b \wedge c))) \wedge d]$

Here LHS is

$$
\begin{aligned}
[d \rightarrow ((\sim a) \wedge b) \wedge c] &\leftrightarrow [(\sim d) \vee (((\sim a) \wedge b) \wedge c)] \\
&\leftrightarrow [(\sim d) \vee ((\sim a) \wedge (b \wedge c))] \\
&\leftrightarrow [(\sim d) \vee (\sim (a \vee (\sim (b \wedge c)))] \\
&\leftrightarrow \sim[(d \wedge (a \vee (\sim (b \wedge c))))] \\
&\leftrightarrow \sim[(a \vee (\sim (b \wedge c))) \wedge d] \\
&= \text{RHS}
\end{aligned}
$$

Hence are equivalent.

(b) $p \vee (q \vee r)$ and $(p \vee q) \wedge (p \vee r)$

Let us prove this by distribution law using truth Table 4.15.

Table 4.15

p	q	r	$(q \vee r)$	$p \vee (q \vee r)$	$(p \vee q)$	$(p \vee r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

nd th second by $\frac{11}{11}$ to obtain the

$\frac{1 \cdot 16!}{! \cdot 11!}$

$\frac{16!}{11!} = \frac{17!}{6! \cdot 11!} = \binom{17}{6}$

me denominator in both

$\frac{+1}{+1}$. Hence

6.8 There are four bus lines between A and B; and three bus lines between B and C. In how many ways can a man travel (a) by bus from A to C by way of B? (b) round-trip by bus from A to C by way of B? (c) round-trip by bus from A to C by way of B, if he does not want to use a bus line more than once?

(a) There are four ways to go from A to B and three ways to go from B to C, hence there are $4 \cdot 3 = 12$ ways to go from A to C by way of B.

(b) There are 12 ways to go from A to C by way of B, and 12 ways to return. Hence there are $12 \cdot 12 = 144$ ways to travel round-trip.

(c) The man will travel from A to B to C to B to A. Enter these letters with connecting arrows as follows:

$$A \longrightarrow B \longrightarrow C \longrightarrow B \longrightarrow A$$

The man can travel four ways from A to B and three ways from B to C, but he can only travel two ways from C to B and three ways from B to A since he does not want to use a bus line more than once. Enter these numbers above the corresponding arrows as folows:

$$A \xrightarrow{4} B \xrightarrow{3} C \xrightarrow{2} B \xrightarrow{3} A$$

Thus there are $4 \cdot 3 \cdot 2 \cdot 3 = 72$ ways to travel round-trip without using the same bus line more than once.

6.9 Suppose repetitions are not permitted. (a) How many three-digit numbers can be formed from the six digits 2, 3, 5, 6, 7 and 9? (b) How many of these numbers are less than 400? (c) How many are even?

In each case draw three boxes $\square\ \square\ \square$ to represent an arbitrary number, and then write in each box the number of digits that can be placed there.

(a) The box on the left can be filled in six ways; following this, the middle box can be filled in five ways; and, lastly, the box on the right can be filled in our ways; $\boxed{6}\,\boxed{5}\,\boxed{4}$. Thus there are $6 \cdot 5 \cdot 4 = 120$ numbers.

(b) The box on the left can be filled in only two ways, by 2 or 3, since each number must be less than 400; the middle box can be filled in five ways; and, lastly, the box on the right can be filled in four ways: $\boxed{2}\,\boxed{5}\,\boxed{4}$. Thus, there are $2 \cdot 5 \cdot 4 = 40$ numbers.

(c) The box on the right can be filled in only two ways, by 2 or 6, since the numbers must be even; the box on the left can then be filled in five ways; and lastly, the middle box can be filled in four ways: $\boxed{5}\,\boxed{4}\,\boxed{2}$. Thus, there are $5 \cdot 4 \cdot 2 = 0$ numbers.

6.10 Find the number of ways that a party of seven persons can arrange themselves: (a) in a row of seven chairs; (b) around a circular table.

(a) The seven persons can arrange themselves in a row in $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7!$ ways.

(b) One person can sit at any place in the circular table. The other six persons can then arrange themselves in $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!$ ways around the table.

This is an example of a *circular permutation*. In general, n objects can be arranged in a circle in $(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = (n-1)!$ ways.

(b) Total number of ways such that m is always immediate lef of $n = 5!$

6.19 If repetitions are not permitted, how many four digit numbers can be formed from digits 1, 2, 3, 7, 8 and 5.

(a) Total four digit numbrs $= {}^6P_4 = \frac{6!}{2!} = 360$

(b) Total number less than 5000

For number to be less than 5000, first position can be occupied by either of 1, 2 or 3,

Hence four digits number less 5000 $= 3 \times {}^5P_3 = 3 \times \frac{5!}{2!}$

$$= 3 \times 5 \times 4 \times 3$$
$$= 180$$

(c) Total four digit even numbers formed from digits 1, 2, 3, 7, 8 and 5 are

$$= 2 \times {}^5P_3 = 2 \times 5 \times 4 \times 3$$
$$= 120$$

(d) Total odd four digit numbers formed from these digits $= 4 \times {}^5P_3 = 240$

(e) Total number of numbers that contain both the digits 3 and 5.

Let us compute this using inclusion principle,

Let A denote set of numbers excluding 3.

Let B denote set of numbers excluding 5.

Let $A \cap B$ denote set of numbers excluding both 3 and 5

Total four digit numbers formed from digits 1, 2, 3, 5, 7 and 8 containing both 3 and 5 = (Total four digit numbers) – (Total four digit numbers excluding 3 or 5 or both)

$$= 360 - [|A \cup B|]$$
$$= 360 - [|A| + |B| - |A \cap B|]$$
$$= 360 - [{}^5P_4 + {}^5P_4 - {}^4P_4]$$
$$= 360 - [5! + 5! - 4!]$$
$$= 360 - [240 - 24]$$
$$= 360 - 216$$
$$= 144$$

6.25 There are 50 students in each of the senior and junior classes. Each class has 25 male and 25 female students. In how many ways can an eight students committee be formed so that there are four female and three juniors in the committee?

To find of number of ways of forming such a committee, we have to consider all cases.

Juniors		*Seniors*	
Female	*Male*	*Female*	*Male*
3	0	1	4
2	1	2	3
1	2	3	2
0	3	4	1

From above table, we can easily compute total possible ways of selecting eight students committee which includes 4 female and three juniors.

Total ways of selection

$$= 2 \times ({}^{25}C_3 \times {}^{25}C_1 \times {}^{25}C_4) + 2 \times ({}^{25}C_3 \times {}^{25}C_1 \times {}^{25}C_2 \times {}^{25}C_2)$$

These selections confirm that there are exactly four female and three juniors.

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6.34 Let L be a list (not necessarily in alphabetical order) of the 26 letters in the English alphabet (which consists of 5 vowels, A, E, I, O, U, and 21 consonants). (a) Show that L has a sublist consisting of four or more consecutive consonants. (b) Assuming L begins with a vowel, say A, show that L has a sublist consisting of five or more consecutive consonants.

(a) The five letters partition L into $n = 6$ sublists (pigeonholes) of consecutive consonants. Here $k + 1 = 4$ and so $k = 3$. Hence $nk + 1 = 6(3) + 1 = 19 < 21$. Hence some sublist has at least four consecutive consonants.

(b) Since L begins with a vowel, the remainder of the vowels partition L into $n = 5$ sublists. Here $k + 1 = 5$ and so $k = 4$. Hence $kn + 1 = 21$. Thus some sublist has at least five consecutive consonants.

6.35 Find the minimum number n of integers to be selected from $S = \{1, 2, ..., 9\}$ so that: (a) the sum of two of the n integers is even; (b) the difference of two of the n integers is 5.

(a) The sum of two even integers or of two odd integers is even. Consider the subsets $\{1, 3, 5, 7, 9\}$ and $\{2, 4, 6, 8\}$ of S as pigeonholes. Hence $n = 3$.

(b) Consider the five subsets $\{1, 6\}$, $\{2, 7\}$, $\{3, 8\}$, $\{4, 9\}$, $\{5\}$ of S as pigeonholes. Then $n = 6$ will guarantee that two integers will belong to one of the subsets and their difference will be 5.

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number of ord...

6.28 There are 12 students in a class. In how many ways can the 12 students take four different tests if three students are to take each test?

Method 1: We seek the number of ordered partitions of the 12 students into cells containing three students each. By Theorem 6.8, there are $\dfrac{12!}{3!\,3!\,3!\,3!} = 369\,600$ such partitions.

Method 2: There are $\binom{12}{3}$ ways to choose three students to take the first test; following this there are $\binom{9}{3}$ ways to choose three students to take the second test, and $\binom{6}{3}$ ways to choose three students to take the third test. The remaining students take the fourth test. Thus, altogether there are $\binom{12}{3}\binom{9}{3}\binom{6}{3} = (220)\,(84)\,(20) = 369\,600$ ways for the students to take the tests.

6.29 In how...

committees.

6.24 Out of 12 employees, a group of four trainees is to be sent for 'Software testing and QA' training of one month.

(a) In how many ways can the four employees be selected?

Selecting 4 persons of 2 = $^{12}C_4 = \dfrac{12!}{4! \times 8!}$

(b) What if there are two employees who refuse to go together for training?
Let A and B be the two employees who refuse to go together.
Total possible ways to select include

(i) Both A and B do not go = ${}^{10}C_4$

(ii) A is selected, hence B refuses = ${}^{10}C_3$

(iii) B is selected, hence A refuses = ${}^{10}C_3$

∴ Total ways of selection with this constraint = ${}^{10}C_4 + {}^{10}C_3 + {}^{10}C_3$

(c) There are two employees who want to go together that is either they both go or both do not go for training.
Let C and D be the two employees who want to go together that is either they both go or both not go for training = ${}^{10}C_4 + {}^{10}C_2$

(d) There are two employees who want to go together and there are two employees who refuse to go together.
Let A and B are two employees who refuse to go together and C and D are two who want to go together. Now let us consider the following cases

(i) A and B both do not go and C and D both go = ${}^{8}C_2$.

(ii) A and B and also C and D do not go = ${}^{8}C_4$.

(iii) C and D both and either of A or B go for training = $2 \times {}^{8}C_1$.

(iv) C and D both do not go for training and either of A or B go for training = $2 \times {}^{8}C_3$

∴ Total ways of selecting 4 of 12 with given constraints = ${}^{8}C_4 + {}^{8}C_2 + 2 \times {}^{8}C_1 + 2 \times {}^{8}C_3$

Let us multiply and

ORDERED AND UNORD

627 In how

6.38 Let A, B, C, D denote, respectively, art, biology, chemistry, and drama courses. Find the number of students in a dormitory given the data:

12 take A,	5 take A and B,	3 take A, B, C,
20 take B,	7 take A and C,	2 take A, B, D,
20 take C,	4 take A and D,	2 take B, C, D,
8 take D,	16 take B and C,	3 take A, C, D,
	4 take B and D,	2 take all four,
	3 take C and D,	71 take none.

Let T be the number of students who take at least one course. By the inclusion-exclusion principle (Theorem 6.7),

$$T = s_1 - s_2 + s_3 - s_4, \text{ where}$$

$$s_1 = 12 + 20 + 20 + 8 = 60, \qquad s_2 = 5 + 7 + 4 + 16 + 4 + 3 = 39$$

$$s_3 = 3 + 2 + 2 + 3 = 10, \qquad s_4 = 2$$

Thus $\qquad T = 29$, and $N = 71 + T = 100$.

girls are to sit together?

COMBINATIONS

6.11 A woman has 11 close friends.

(a) In How many ways can she invite five of them to dinner?

(b) In how many ways if two of the friends are married and will not attend separately?

(c) In how many ways if two of them are not on speaking terms and will not attend together?

6.12 A woman has 11 close friends of whom six are also women.

(a) In how many ways can she invite three or more to a party?

(b) In how many ways can she invite three or more of them if she wants the same number of men as women (including herself)?

(d) Th...

(e) Any set consis...

(f) Since 5 divides 15 which divides 30, the set is linearly ordered.

Let $A = \{1, 2, 3, 4, 5\}$ be ordered by the Hasse diagram in Fig. 14.9. Insert the correct symbol, $\prec$, $\succ$, or $\|$ (not comparable), between each pair of elements:

(a) 1 ___ 5; (b) 2 ___ 3; (c) 4 ___ 1; (d) 3 ___ 4.

(a) Since there is a "path" (edges slanting upward) from 5 to 3 to 1, 5 precedes 1; hence $1 > 5$.

(b) There is no path from 2 to 3, or vice versa; hence $2 \| 3$.

(c) There is a path from 4 to 2 to 1; hence $4 < 1$.

(d) Neiter $3 < 4$ nor $4 > 3$ hence $3 \| 4$.

Fig. 14.9

Consider the ordered set A in Fig. 14.9.

(a) Find all minimal and maximal elements of A.

(b) Does A have a first element or a last element?

...ents, neither is a last element

14.8 Consider the set $\{\mathbf{N} = (1, 2, 3, ...)\}$ of positive integers. Every number in N can be written uniquely as a product of a nonnegative power of 2 times an odd number. Suppose a and a' are positive integers such that

$$a = 2^r(2s+1) \qquad \text{and } a' = 2^{r'}(2s'+1)$$

where r and s are nonnegative integers. We define:

$$a < a' \qquad \text{if } r < r' \qquad \text{or if} \qquad r = r' \text{ but } s < s'$$

Insert the correct symbol, $<$ or $>$, between each of the following pairs of numbers:
(a) 5 ___ 14; (b) 6 ___ 9; (c) 3 ___ 20; (d) 14 ___ 21.

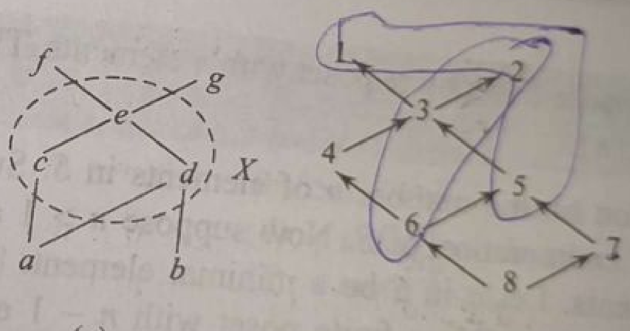
14.18 Let $S = \{a, b, c, d, e, f, g\}$ be ordered as in Fig. 14.14(a), and let $X = \{c, d, e\}$.

(a) Find the upper and lower bounds of X.

(b) Identify sup(X), the supremum of X, and inf(X), the infimum of X, if either exists.

The elements e, f, and g succeed every element of X; hence e, f, and g are the upper bounds of X. The element a precedes every element of X; hence it is the lower bound of X. Note that b is not a lower bound since b does not precede c; in fact, b and c are not comparable.

Since e precedes both f and g, we have $e = \sup(X)$. Likewise, since a precedes (trivially) every lower bound of X, we have $a = \inf(X)$. Note that sup(X) belongs to X but inf(X) does not belong to X.



(a) (b)

Fig. 14.14

14.35 Which of the partially ordered sets in Fig. 14.18 are lattices?

A partially ordered set is a lattice if and only if sup (x, y) and inf (x, y) exist for each pair x, y in the se
Only (c) is not a lattice since $\{a, b\}$ has three upper bounds, c, d and I, and no one of them preced
the other two, i.e., sup (a, b) does not exist.

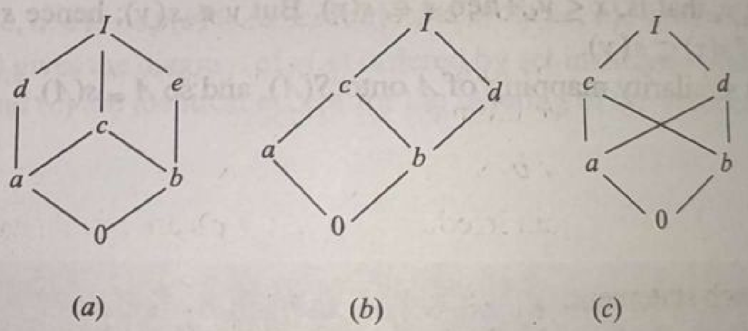


Fig. 14.18

length of longest chain is $n+1$.

Let us consider an example, Let $|A| = 3$. Let $A = \{a, b, c\}$.

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$

$$\text{Here } \phi \subseteq \{a\} \subseteq \{a, c\} \subseteq \{a, b, c\}$$

Here length of chian is $n + 1$.

14.42 Let R be the relation on A.

$A = \{2, 3, 4, 8, 12, 36, 48\}$

$R = \{(a, b) \mid a \text{ is divisor of } b\}$. Draw Hasse diagram.

Here $R = \{(2, 4), (2, 6), (2, 8), (2, 12),$
$(2, 36), (2, 48), (3, 6), (3, 12), (3, 36)$
$(3, 48), (4, 8), (4, 12), (4, 36),$
$(4, 48), (6, 12), (6, 36), (6, 48),$
$(8, 48,), (12, 36), (12, 48)\}$.

Hasse diagram for R is drown in Fig. 14.22

14.43 Show that set of all divisors of 70 form a lattice. Divisor of 70 =

Fig. 14.26

Fig. 14.27

(b) This (Fig. 14.27) is a lattice.

14.45 $X = \{2, 3, 6, 12, 24, 36\}$

R on $X = \{(x, y) \in R, x \text{ divides } y\}$

(a) Construct Hasse diagram

(b) Maximum and minimal element?

(c) Give chain and antichains

(d) Maximum length of chain?

(e) Is poset a lattice?

(a) Hasse diagram (Fig. 14.28)

(b) Maximum elements = $\{24, 36\}$

Minimal elements = $\{3, 2\}$

(c) Chains = $\{3, 6, 12, 24\}$, $\{3, 6, 12, 36\}$, $\{2, 6, 12, 24\}$ and $\{2, 6, 12, 36\}$

Antichains = $\{2, 3\}$ and $\{24, 36\}$

(d) Maximum length of chain 4.

(e) This is not a lattice as (2 and 3) has no lower bound and (24, 36) has no upper bound.

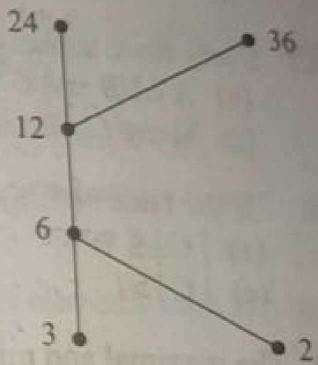


Fig. 14.28

14.44 Let $S = \{1, 2, 3, 4\}$. Using the notation $[12, 3, 4] \equiv [\{1, 2\}, \{3\}, \{4\}]$, three partitions of S are:

$$P_1 = [12, 3, 4], \quad P_2 = [12, 34], \quad P_3 = [13, 2, 4]$$

(a) Find the other nine partitions of S.

(b) Let L be the collection of 12 partitions S ordered by *refinment*, i.e., $P_i \le P_j$ if each cell of P_i is a subset of a cell of P_j. For example $P_1 \le P_2$, but P_2 and P_3 are noncomparable. Show that L is a bounded lattice and draw its diagram.

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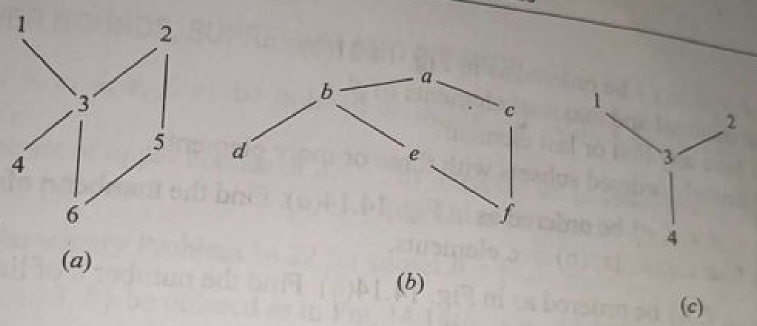

Fig. 14.20

14.2 Let $B = \{a, b, c, d, e, f\}$ be ordered as in Fig. 14.20(b).

(a) Find all minimal and maximal elements of B.

(b) Does B have a first or last element?

(c) List two and find the number of consistent enumerations of B into the set $\{1, 2, 3, 4, 5, 6\}$.

14.3 Let $C = \{1, 2, 3, 4\}$ be ordered as in Fig. 14.20(c). Let $L(C)$ denote the collection of all nonempty linearly ordered subsets of C ordered by set inclusion. Draw a diagram of $L(C)$.

14.4 Draw the diagrams of the partitions of m (see Example 14.14) where: (a) $m = 4$; (b) $m = 6$