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Stability radius for structured perturbations and the algebraic Riccati equation

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Abstract: In this paper we generalize the notion of stability radius introduced in [1] to allow for structured perturbations. We then relate the stability radius to the existence of Hermitian solutions of an algebraic Riccati equation and give some applications of this result.

Keywords: Perturbations, Riccati equation, Stability.

1. Introduction

In [1] we introduced the notions of real and complex stability radii of a stable matrix. Here we concentrate entirely on the complex stability radius but consider *structured perturbations*.

If $\dot{x} = Ax$ is the nominal system we assume that the perturbed system can be represented in the form

$$\dot{x} = (A + BDC)x \tag{1}$$

where D is an unknown disturbance matrix and B, C are known scaling matrices defining the 'structure' of the perturbation. Throughout the paper $A \in \mathbb{C}^{n\times n}$, $\sigma(A) \subset \mathbb{C}_-$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{m\times p}$. The matrices B, C may reflect, for instance, the possibility that not all of the elements of A are subject to perturbations. Consider, for example, the linear oscillator

$$\ddot{\xi} + a_1\dot{\xi} + a_2\xi = 0$$

or in state space form

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} x.$$

If we assume that a_1, a_2 are uncertain we can account for this by taking $B = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, $C = I_2$. Whereas if only a_2 is subject to perturbation we take $B = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, $C = [1 \;\; 0]$.

Formally, (1) may be interpreted as a closed loop system obtained by applying static linear output feedback (with unknown gain matrix D) to the system $\dot{x} = Ax$. However, we would like to emphasize that in this paper B, C do not represent input resp. output matrices but model the available knowledge about the structure and scale of uncertainty of the system parameters. Therefore, controllability and observability properties are not very natural assumptions for the triple (A, B, C) in our context.

The entries of the perturbed matrix $A + BDC$ depend affinely on the unknown entries d_{ij} of D. But not every affine perturbation of the entries of A can be represented in this way. More structural possibilities would be covered by considering

$$A + \sum_{i=1}^{l} B_i D_i C_i.$$

But a treatment of these more general perturbation structures is substantially more complicated and so we will not pursue this here.

We proceed as follows. In Section 2, we introduce the stability radius of a stable matrix when subjected to perturbations of fixed structure (B, C). We then derive a formula for this 'structured stability radius' and characterize it by various conditions.

In Section 3 we relate the stability radius to the existence of Hermitian solutions of a parametrized nonstandard algebraic Riccati equation. Solutions of general Riccati equations have been investigated by Brockett [3], Willems [2] and Popov [4]. However, their results are not directly applicable

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in our situation since we wish to avoid controllability assumptions. Fortunately, the main theorems remain valid without the assumption of controllability, if the system matrix A is stable.

In Section 4 we study the parameter dependence of the stabilizing solution P_ρ of the algebraic Riccati equation and analyze in particular, how the stability radius and the spectrum of the closed loop system change with the parameter ρ. This question is in a sense the complement of the cheap control problem.

In Section 5 we show how to determine a quadratic Liapunov function of maximal robustness for a linear system and describe how this function can be used to cope with nonlinear, time varying or even dynamic perturbations (i.e. perturbations 'with memory' resulting for example from neglected dynamics).

2. The structured stability radius — definitions and first characterizations

Generalizing (2) of [1] we define the stability radius of the stable matrix $A \in \mathbb{C}^{n\times n}$ with respect to perturbations of a given structure $(B, C) \in \mathbb{C}^{n\times m} \times \mathbb{C}^{p\times n}$, by

$$\begin{aligned} r_{\mathbb{C}} &= r_{\mathbb{C}}(A;\ B,\ C) \\ &= \inf\{\ \|D\|;\ \sigma(A+BDC)\cap \overline{\mathbb{C}}_+ \neq \emptyset\} \end{aligned} \tag{2}$$

where $\|D\|$ denotes the operator norm of the linear map $D: \mathbb{C}^p \to \mathbb{C}^m$ with respect to the Euclidian norms on $\mathbb{C}^m$, $\mathbb{C}^p$ and

$$\overline{\mathbb{C}}_+ = \{s \in \mathbb{C};\ \operatorname{Re} s \geq 0\}.$$

Clearly

$$r_{\mathbb{C}} = \inf\{\ \|D\|;\ \sigma(A+BDC)\cap \mathrm{i}\mathbb{R} \neq \emptyset\}. \tag{3}$$

Note that $m = p = n$, $B = C = I_n$ corresponds to the unstructured case studied in [1].

2.1. Proposition. *Let*

$$G(s) = C(sI-A)^{-1}B.$$

Then

$$r_{\mathbb{C}} = \begin{cases} \dfrac{1}{\max_{w\in\mathbb{R}} \|G(\mathrm{i}\omega)\|} & \text{if } G \neq 0, \\ \infty & \text{if } G \equiv 0. \end{cases} \tag{4}$$

Proof. First suppose that for some $D \in \mathbb{C}^{m\times p}$, $x \in \mathbb{C}^n$, $x \neq 0$, $\omega \in \mathbb{R}$,

$$(A+BDC)x = \mathrm{i}\omega x \tag{5}$$

then

$$x = (\mathrm{i}\omega I - A)^{-1}BDCx.$$

Since A is stable, $Cx \neq 0$. Set $u = Cx$, so

$$u = G(\mathrm{i}\omega)Du. \tag{6}$$

If $G \equiv 0$ this leads to a contradiction and so $r_{\mathbb{C}} = \infty$. If $G \not\equiv 0$ then (6) implies $\|G(\mathrm{i}\omega)\|\ \|D\| \geqslant 1$ and thus

$$r_{\mathbb{C}} \geq \frac{1}{\max_{\omega\in\mathbb{R}} \|G(\mathrm{i}\omega)\|}.$$

Suppose the maximum of $\omega \to \|G(\mathrm{i}\omega)\|$ occurs at ω_0, say. The singular value decomposition of $G(\mathrm{i}\omega_0)$ has the form

$$G(\mathrm{i}\omega_0) = \sum_{j=1}^{m} s_j u_j v_j^*$$

where $u_j \in \mathbb{C}^p$, $v_j \in \mathbb{C}^m$,

$$\|u_j\|_{\mathbb{C}^p} = \|v_j\|_{\mathbb{C}^m} = 1$$

and

$$s_1 = \|G(\mathrm{i}\omega_0)\| \geq s_2 \geq \cdots \geq s_m.$$

Let $D = s_1^{-1}v_1u_1^*$, then $G(\mathrm{i}\omega_0)Du_1 = u_1$, or

$$C(\mathrm{i}\omega_0 I - A)^{-1}BDu_1 = u_1.$$

If we set

$$x = (\mathrm{i}\omega_0 I - A)^{-1}BDu_1$$

then $Cx = u_1$, hence $x \neq 0$ and

$$x = (\mathrm{i}\omega_0 I - A)^{-1}BDCx$$

or

$$(\mathrm{i}\omega_0 I - A)x = BDCx.$$

So that

$$\sigma(A+BDC)\cap \overline{\mathbb{C}}_+ \neq \emptyset$$

and

$$\|D\| = \frac{1}{\|G(\mathrm{i}\omega_0)\|} = \frac{1}{\max_{\omega\in\mathbb{R}} \|G(\mathrm{i}\omega)\|}. \qquad \square$$

As a consequence of the above characterization of the structured stability radius, we have

$$r_{\mathbb{C}}(A;\ B,\ C)=r_{\mathbb{C}}(TAT^{-1};\ TB,\ CT^{-1}),$$
$$T\in \mathrm{Gl}_n(\mathbb{C}). \tag{7}$$

This may seem surprising since we showed in [1] that the stability radius $r_{\mathbb{C}}(A)=r_{\mathbb{C}}(A;\ I,\ I)$ can change substantially under similarity transformations on A. Note, however that in (7) the scaling matrices have also been transformed, whereas in [1] we considered $r_{\mathbb{C}}(TAT^{-1};\ I,\ I)$ and not $r_{\mathbb{C}}(TAT^{-1};\ T,\ T^{-1})$. Another interpretation of the stability radius which will be useful later can be obtained by considering the convolution operator

$$L:\quad L^2[0,\ \infty;\ \mathbb{C}^m]\to L^2[0,\ \infty;\ \mathbb{C}^p],$$
$$(Lv)(t)=\int_0^t C\,\mathrm{e}^{A(t-s)}Bv(s)\ \mathrm{d}s. \tag{8}$$

Note that $G(s)$ is the Laplace transform of the weighting function. We have as an extension of a result in [5].

2.2. Proposition.

$$\|L\|=\frac{1}{r_{\mathbb{C}}(A;\ B,\ C)}. \tag{9}$$

Proof. We denote the Fourier–Plancherel transform of $v(t)$ by $\hat{v}(\mathrm{i}\omega)$. Then

$$(\widehat{Lv})(\omega)=G(\mathrm{i}\omega)\hat{v}(\mathrm{i}\omega)$$

for $v\in L^2[0,\ \infty;\ \mathbb{C}^m]$ and by Parseval's Theorem

$$\|Lv\|^2_{L^2[0,\infty;\mathbb{C}^p]}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\|G(\mathrm{i}\omega)\hat{v}(\mathrm{i}\omega)\|^2_{\mathbb{C}^p}\ \mathrm{d}\omega$$
$$\le\frac{1}{r_{\mathbb{C}}^2}\|v\|^2_{L^2[0,\infty;\mathbb{C}^m]}.$$

Hence

$$\|L\|\le\frac{1}{r_{\mathbb{C}}}.$$

If $r_{\mathbb{C}}=\infty$ equation (9) follows directly from (2) since $G\equiv 0$ is equivalent to $L=0$. Now let $G\not\equiv 0$ and suppose the singular value decomposition of $G(\mathrm{i}\omega)$ is

$$G(\mathrm{i}\omega)=\sum_j s_j(\mathrm{i}\omega)u_j(\mathrm{i}\omega)v_j^*(\mathrm{i}\omega).$$

where $u_j(\mathrm{i}\omega)\in\mathbb{C}^p$, $v_j(\mathrm{i}\omega)\in\mathbb{C}^m$,

$$\|u_j(\mathrm{i}\omega)\|_{\mathbb{C}^p}=\|v_j(\mathrm{i}\omega)\|_{\mathbb{C}^m}=1$$

and

$$s_1(\mathrm{i}\omega)=\|G(\mathrm{i}\omega)\|\ge s_2(\mathrm{i}\omega)\ge\cdots\ge s_m(\mathrm{i}\omega).$$

Then $s_1(\mathrm{i}\omega)$, $u_1(\mathrm{i}\omega)$, $v_1(\mathrm{i}\omega)$ can be chosen to depend continuously on ω and for every ω such that $G(\mathrm{i}\omega)\ne 0$ there exists

$$D(\mathrm{i}\omega)=\frac{1}{s_1(\mathrm{i}\omega)}v_1(\mathrm{i}\omega)u_1^*(\mathrm{i}\omega),$$

satisfying

$$G(\mathrm{i}\omega)D(\mathrm{i}\omega)u_1(\mathrm{i}\omega)=u_1(\mathrm{i}\omega).$$

Suppose the maximum $\omega\mapsto\|G(\mathrm{i}\omega)\|$ occurs at ω_0. Let

$$\hat{v}_\varepsilon(\mathrm{i}\omega)=\gamma_\varepsilon(\mathrm{i}\omega)D(\mathrm{i}\omega)u_1(\mathrm{i}\omega),$$

where

$$\gamma_\varepsilon(\mathrm{i}\omega)=\begin{cases}\dfrac{1}{\varepsilon^{1/2}} & \text{if } \omega\in[\omega_0,\ \omega_0+\varepsilon],\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\|Lv_\varepsilon\|^2_{L^2[0,\infty;\mathbb{C}^p]}=\frac{1}{2\pi\varepsilon}\int_{\omega_0}^{\omega_0+\varepsilon}\mathrm{d}\omega=\frac{1}{2\pi},$$

$$\|v_\varepsilon\|^2_{L^2[0,\infty;\mathbb{C}^m]}=\frac{1}{2\pi\varepsilon}\int_{\omega_0}^{\omega_0+\varepsilon}\frac{1}{\|G(\mathrm{i}\omega)\|^2}\mathrm{d}\omega.$$

Hence

$$\|v_\varepsilon\|^2_{L^2[0,\infty;\mathbb{C}^m]}\to\frac{1}{2\pi}r_{\mathbb{C}}^2$$

and

$$\|Lv_\varepsilon\|^2_{L^2[0,\infty;\mathbb{C}^p]}\to\frac{1}{2\pi}$$

as $\varepsilon\downarrow 0$. Thus $\|L\|=1/r_{\mathbb{C}}$. □

The stability radius also plays a role in the following *optimization problem*.

Minimize $J(x_0,\ v)$

$$=\int_0^\infty[\|v(s)\|^2_{\mathbb{C}^m}-\rho\|y(s)\|^2_{\mathbb{C}^p}]\ \mathrm{d}s, \tag{10}$$

subject to

$$\begin{aligned} &v \in L^2[0, \infty; \mathbb{C}^m], \\ &\dot{x} = Ax + Bv, \quad x(0) = x_0, \qquad (11) \\ &y = Cx. \end{aligned}$$

We consider (10) as a parametrized optimization problem. If $\rho \le 0$ this is the usual linear quadratic regulator problem. In the case where $\rho > 0$ the state penalty is negative and we would expect the resulting feedback law to *deteriorate* the stability of A and this will in fact come out in the analysis (Proposition 4.1). The next proposition characterizes the set of parameter values ρ for which the optimal value starting at *zero* is nonnegative.

2.3. Proposition.

$$\begin{aligned} &\inf_{v \in L^2[0,\infty;\mathbb{C}^m]} J(0, v) = 0 \\ &\Leftrightarrow \rho \le r_{\mathbb{C}}^2 \\ &\Leftrightarrow I - \rho G^*(\mathrm{i}\omega) G(\mathrm{i}\omega) \ge 0 \quad \textit{for all } \omega \in \mathbb{R}. \end{aligned} \qquad (12)$$

Proof. We have

$$y(t) = C \int_0^t \mathrm{e}^{A(t-s)} Bv(s)\, \mathrm{d}s = (Lv)(t).$$

Hence

$$J(0, v) = \| v \|^2_{L^2[0,\infty;\mathbb{C}^m]} - \rho \| Lv \|^2_{L^2[0,\infty;\mathbb{C}^p]}.$$

For $\rho \le 0$, (12) is obvious and for $\rho > 0$ the first equivalence follows from Proposition 2.2. The second equivalence is a consequence of Parseval's Theorem which implies

$$J(0, v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle \hat{v}(\mathrm{i}\omega), (I - \rho G^*(\mathrm{i}\omega) G(\mathrm{i}\omega)) \hat{v}(\mathrm{i}\omega) \rangle_{\mathbb{C}^m}\, \mathrm{d}\omega. \quad \square$$

3. Characterization of $r_{\mathbb{C}}(A; B, C)$ via a parametrized algebraic Riccati equation

In this section we consider the parametrized algebraic Riccati equation

$$A^*P + PA - \rho C^*C - PBB^*P = 0. \qquad (\mathrm{ARE}_\rho)$$

We will say that a solution P of (ARE_ρ) is a stabilizing solution if $\sigma(A - BB^*P) \subset \mathbb{C}_-$. In the literature there are two basic approaches to solving general algebraic Riccati equations (see [2,3,4]). One is via the associated linear quadratic optimisation problem [2] and the other proceeds by direct analysis. Optimization techniques can only be used if the minimum costs are finite. To this end we have:

3.1. Proposition. *Suppose* $\sigma(A) \subset \mathbb{C}_-$, $r_{\mathbb{C}} < \infty$, *then for all* $\rho \in (-\infty, r_{\mathbb{C}}^2)$,

$$\left| \inf_{v \in L^2[0,\infty;\mathbb{C}^m]} J(x_0, v) \right| < \infty, \quad x_0 \in \mathbb{C}^n. \qquad (13)$$

Proof. Since $\sigma(A) \subset \mathbb{C}_-$ we need only consider the case $\rho \in (0, r_{\mathbb{C}}^2)$ and show

$$\inf_{v \in L^2[0,\infty;\mathbb{C}^m]} J(x_0, v) > -\infty, \quad x_0 \in \mathbb{C}^n.$$

We have

$$\begin{aligned} J(0, v) = &\int_0^\infty [\, \| v(t) \|^2_{\mathbb{C}^m} - r_{\mathbb{C}}^2 \| (Lv)(t) \|^2_{\mathbb{C}^p}]\, \mathrm{d}t \\ &+ (r_{\mathbb{C}}^2 - \rho) \int_0^\infty \| (Lv)(t) \|^2_{\mathbb{C}^p}\, \mathrm{d}t. \end{aligned}$$

Hence

$$J(0, v) \geqslant (r_{\mathbb{C}}^2 - \rho) \int_0^\infty \| (Lv)(t) \|^2_{\mathbb{C}^p}\, \mathrm{d}t.$$

But

$$\begin{aligned} J(x_0, v) = &-\rho \int_0^\infty \| C\, \mathrm{e}^{At} x_0 \|^2_{\mathbb{C}^p}\, \mathrm{d}t \\ &- 2\rho\, \mathrm{Re} \int_0^\infty \langle C\, \mathrm{e}^{At} x_0, (Lv)(t) \rangle\, \mathrm{d}t \\ &+ J(0, v) \end{aligned}$$

and since for $\alpha > 0$, $2ab \le a^2/\alpha + \alpha b^2$, we have

$$\begin{aligned} J(x_0, v) \geqslant &-\rho \left(1 + \frac{1}{\alpha}\right) \int_0^\infty \| C\, \mathrm{e}^{At} x_0 \|^2_{\mathbb{C}^p}\, \mathrm{d}t \\ &+ (r_{\mathbb{C}}^2 - \rho - \alpha) \int_0^\infty \| (Lv)(t) \|^2_{\mathbb{C}^p}\, \mathrm{d}t. \end{aligned}$$

For $\rho < r_{\mathbb{C}}^2$ we may choose α sufficiently small so that

$$J(x_0, v) \ge -\rho \left(1 + \frac{1}{\alpha}\right) \int_0^\infty \| C\, \mathrm{e}^{At} x_0 \|^2_{\mathbb{C}^p}\, \mathrm{d}t$$

from which the result follows since $\sigma(A) \subset \mathbb{C}_-$. $\square$

Using this proposition it is in fact possible — although by no means trivial — to show that there exists a unique stabilizing solution P_ρ of (ARE_ρ) for $\rho \in (-\infty, r_{\mathbb{C}}^2)$, without any controllability assumption. Moreover $\bar{v}(t) = -B^*P_\rho x(t)$ minimizes the performance index (10) subject to the dynamics (11) and

$$J(x_0, \bar{v}) = \langle x_0, P_\rho x_0 \rangle_{\mathbb{C}^n}.$$

However, we tried without success to extend the optimization approach to the case where $\rho = r_{\mathbb{C}}^2$, which is of special interest to us. The reason why optimization techniques fail for $\rho = r_{\mathbb{C}}^2$ is illustrated by the following simple but representative example.

3.2. Example. Consider the scalar equation

$$\dot{x} = -x + v, \quad x(0) = x_0$$

and

$$J(x_0, v) = \int_0^\infty [\,|v(t)|^2 - |x(t)|^2] \,\mathrm{d}t.$$

It is easy to see that $r_{\mathbb{C}} = 1$. ($\mathrm{ARE}_{r_{\mathbb{C}}^2}$) has the form $-2p - 1 - p^2 = 0$ and is solved by $p = -1$.

On the other hand,

$$\begin{aligned} &2\pi J(x_0, v) \\ &= \int_{-\infty}^{\infty} \left[\,|\hat{v}(\mathrm{i}\omega)|^2 - \frac{1}{\omega^2+1}|x_0 + \hat{v}(\mathrm{i}\omega)|^2\right] \mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} \left[\frac{\omega^2}{1+\omega^2}|\hat{v}(\mathrm{i}\omega)|^2 \right. \\ &\qquad \left. - \frac{1}{\omega^2+1}[2 \operatorname{Re} x_0\hat{v}(\mathrm{i}\omega) + |x_0|^2]\right] \mathrm{d}\omega. \end{aligned}$$

If we set

$$\hat{v}_n(\mathrm{i}\omega) = \begin{cases} \alpha n^{3/2} & \text{if } \omega \in \left[0, \dfrac{1}{n}\right], \\ 0 & \text{otherwise,} \end{cases}$$

then for large n,

$$2\pi J(x_0, v_n) \sim \tfrac{1}{3}\alpha^2 - \pi|x_0|^2 - 2\alpha n^{1/2} \operatorname{Re} x_0.$$

Choosing α such that $\alpha \operatorname{Re} x_0 > 0$ we see that $J(x_0, v_n) \to -\infty$ as $n \to \infty$. This shows that

$$\min_{v \in L^2} J(0, v) = 0$$

is *not* equivalent to

$$\inf_{v \in L^2} J(x_0, v) > -\infty$$

(even though (A, B) is controllable). Moreover ($\mathrm{ARE}_{r^2_{\mathbb{C}}}$) may have a solution P with $\sigma(A - BB^*P) \subset \bar{\mathbb{C}}_-$ although the associated optimal control problem (10), (11) does not have a solution.

In order to derive a comprehensive result including the case $\rho = r_{\mathbb{C}}^2$ we will now analyze (ARE_ρ) directly following Brockett's development in [3] and extend this theorem to not necessarily controllable systems.

3.3. Theorem. *Suppose $\sigma(A) \subset \mathbb{C}_-$, $r_{\mathbb{C}} < \infty$, $\rho \in (-\infty, r_{\mathbb{C}}^2)$. Then there exists a unique stabilizing solution P_ρ of* (ARE_ρ). *Moreover when $\rho = r_{\mathbb{C}}^2$ there exists a unique solution $P_{r_{\mathbb{C}}^2}$ of* ($\mathrm{ARE}_{r_{\mathbb{C}}^2}$) *having the property*

$$\sigma(A - BB^*P_{r_{\mathbb{C}}^2}) \subset \bar{\mathbb{C}}_-.$$

For all $\rho \in (-\infty, r_{\mathbb{C}}^2]$, $P_\rho = P_\rho^*$.

Proof. Choose $T \in \mathrm{Gl}_n(\mathbb{C})$ such that

$$TAT^{-1} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix},$$

$$TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \qquad CT^{-1} = [C_1 \quad C_2]$$

and (A_1, B_1) is controllable. Note that

$$C(\mathrm{i}\omega I - A)^{-1}B = C_1(\mathrm{i}\omega I - A_1)^{-1}B_1,$$

$\omega \in \mathbb{R}$. Multiplying (ARE_ρ) on the left by T^{-1*} and on the right by T^{-1} and setting

$$T^{-1*}PT^{-1} = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix},$$

gives

$$\begin{aligned} &\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} + \begin{bmatrix} A_1^* & 0 \\ A_2^* & A_3^* \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \\ &\quad -\rho\begin{bmatrix} C_1^*C_1 & C_1^*C_2 \\ C_2^*C_1 & C_2^*C_2 \end{bmatrix} \\ &\quad -\begin{bmatrix} P_1B_1B_1^*P_1 & P_1B_1B_1^*P_2 \\ P_3B_1B_1^*P_1 & P_3B_1B_1^*P_2 \end{bmatrix} = 0. \end{aligned}$$

So

$$P_1 A_1 + A_1^* P_1 - \rho C_1^* C_1 - P_1 B_1 B_1^* P_1 = 0, \tag{14a}$$

$$P_2 A_3 + (A_1 - B_1 B_1^* P_1^*)^* P_2 = -P_1 A_2 + \rho C_1^* C_2, \tag{14b}$$

$$P_3(A_1 - B_1 B_1^* P_1) + A_3^* P_3 = -A_2^* P_1 + \rho C_2^* C_1, \tag{14c}$$

$$P_4 A_3 + A_3^* P_4 = -P_3 A_2 - A_2^* P_2 + \rho C_2^* C_2 + P_3 B_1 B_1^* P_2. \tag{14d}$$

Since (A_1, B_1) is controllable there is a unique Hermitian solution $P_{1\rho}$ of (14a) with the property

$$\sigma(A_1 - B_1 B_1^* P_{1\rho}) \subset \mathbb{C}_-$$

if $\rho \in (-\infty, r_{\mathbb{C}}^2)$ and

$$\sigma(A_1 - B_1 B_1^*(P_1)_{r_{\mathbb{C}}^2}) \subset \overline{\mathbb{C}}_-$$

(see [3]). Replacing P_1 by $P_{1\rho}$ in the Liapunov equations (14b), (14c) and using the fact that $\sigma(A_3) \subset \mathbb{C}_-$ it follows (see [6]) that the solutions $P_{2\rho}$, $P_{3\rho}$ are uniquely determined by (14b), (14c). Moreover $P_{3\rho} = P_{2\rho}^*$. Substituting on the right hand side of (14d) we see that $P_{4\rho}$ is uniquely determined and $P_{4\rho}^* = P_{4\rho}$. Finally note

$$\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} [B_1^* \quad 0] \begin{bmatrix} P_{1\rho} & P_{2\rho} \\ P_{3\rho} & P_{4\rho} \end{bmatrix} = \begin{bmatrix} A_1 - B_1 B_1^* P_{1\rho} & A_2 \\ 0 & A_3 \end{bmatrix}. \quad \square$$

As a converse to this theorem we have:

3.4. Proposition. *Suppose $\sigma(A) = \mathbb{C}_-$. If there exists a solution of* (ARE$_\rho$) *with $P = P^*$, then necessarily $\rho \le r_{\mathbb{C}}^2$.*

Proof. If $P = P^*$ satisfies (ARE$_\rho$), then

$$(A - \mathrm{i}\omega I)^* P + P(A - \mathrm{i}\omega I) - \rho C^* C - PBB^* P = 0, \quad \omega \in \mathbb{R}.$$

So

$$0 \le \left(B^* P(A - \mathrm{i}\omega I)^{-1} B - I\right)^* \cdot \left(B^* P(A - \mathrm{i}\omega I)^{-1} B - I\right) = I - \rho G^*(\mathrm{i}\omega) G(\mathrm{i}\omega), \quad \omega \in \mathbb{R}.$$

Thus $\rho \le r_{\mathbb{C}}^2$ by (12). $\square$

Theorem 3.3 and Proposition 3.4 show that $r_{\mathbb{C}}^2$ can be characterized as the maximal parameter value ρ for which (ARE$_\rho$) has an Hermitian solution.

4. Parameter dependence of the solution P_ρ

In this section we examine the map $\rho \mapsto P_\rho$ on $(-\infty, r_{\mathbb{C}}^2]$.

4.1. Proposition. *Suppose $\sigma(A) \subset \mathbb{C}_-$ and set $A_\rho = A - BB^* P_\rho$.*

(a) *The maps $\rho \mapsto P_\rho$, $\rho \mapsto A_\rho$ are differentiable on $(-\infty, r_{\mathbb{C}}^2)$ and continuous on $(-\infty, r_{\mathbb{C}}^2]$.*

(b) $\rho_1 \le \rho_2 \le r_{\mathbb{C}}^2 \Rightarrow P_{\rho_1} \ge P_{\rho_2}$.

(c) *If $\rho_1 \le r_{\mathbb{C}}^2$, $\rho \le r_{\mathbb{C}}^2$, then $P_\rho - P_{\rho_1}$ is a solution of the Riccati equation*

$$A_{\rho_1}^* P + P A_{\rho_1} - (\rho - \rho_1) C^* C - PBB^* P = 0 \tag{15}$$

satisfying

$$\sigma\left(A_{\rho_1} - BB^*(P_\rho - P_{\rho_1})\right) \subset \overline{\mathbb{C}}_-.$$

(d) *If $\rho \le r_{\mathbb{C}}^2$, then*

$$r_{\mathbb{C}}^2(A_\rho; B, C) = r_{\mathbb{C}}^2(A; B, C) - \rho. \tag{16}$$

Proof. Assume $\rho_1 < r_{\mathbb{C}}^2$, then $\sigma(A_{\rho_1}) \subset \mathbb{C}_-$. Let P_ρ be the solution of (ARE$_\rho$), $\rho \le r_{\mathbb{C}}^2$, then

$$A_{\rho_1}^*(P - P_{\rho_1}) + (P - P_{\rho_1}) A_{\rho_1} - (\rho - \rho_1) C^* C - (P - P_{\rho_1}) BB^*(P - P_{\rho_1}) = 0. \tag{17}$$

Hence

$$P_\rho - P_{\rho_1} = -\int_0^\infty \mathrm{e}^{A_{\rho_1}^* t}\left[(\rho - \rho_1) C^* C + (P_\rho - P_{\rho_1}) BB^*(P_\rho - P_{\rho_1})\right] \mathrm{e}^{A_{\rho_1} t}\, \mathrm{d}t \le 0 \quad \text{if } \rho \ge \rho_1. \tag{18}$$

So P_ρ is decreasing on $(-\infty, r_{\mathbb{C}}^2]$ and bounded below by $P_{r_{\mathbb{C}}^2}$. It follows that P_ρ tends to a limit $\hat{P}$ as $\rho \uparrow r_{\mathbb{C}}^2$. Since $\hat{P}$ satisfies (ARE$_{r_{\mathbb{C}}^2}$) and $\sigma(A - BB^*\hat{P}) \subset \overline{C}_-$, we must have $\hat{P} = P_{r_{\mathbb{C}}^2}$ by the uniqueness statement in Theorem 3.3. This proves the continuity in (a). It then follows from (18) that the map $\rho \mapsto P_\rho$ is differentiable on $(-\infty, r_{\mathbb{C}}^2)$ and in fact

$$\frac{\mathrm{d}P_\rho}{\mathrm{d}\rho} = -\int_0^\infty \mathrm{e}^{A_\rho^* t} C^* C\, \mathrm{e}^{A_\rho t}\, \mathrm{d}t \le 0. \tag{19}$$

So we have proved (a), (b). Part (c) follows from (17) and the continuity of the map $\rho \mapsto P_\rho$ on $(-\infty, r_{\mathbb{C}}^2]$. It remains to prove (d). Since $P_{\rho_2} - P_{\rho_1}$ is an Hermitian solution of (15) by Proposition 3.4 we must have

$$\rho_2 - \rho_1 \le r_{\mathbb{C}}^2(A_{\rho_1};\ B,\ C)$$

for any $\rho_2 \le r_{\mathbb{C}}^2(A;\ B,\ C)$. Hence

$$r_{\mathbb{C}}^2(A;\ B,\ C) \le r_{\mathbb{C}}^2(A_\rho;\ B,\ C) + \rho,$$

$$\rho \le r_{\mathbb{C}}^2(A;\ B,\ C).$$

Now suppose that

$$r_{\mathbb{C}}^2(A;\ B,\ C) = r_{\mathbb{C}}^2(A_\rho;\ B,\ C) + \rho - \varepsilon$$

for some $\varepsilon > 0$, and $\rho < r_{\mathbb{C}}^2(A;\ B,\ C)$. Then by Theorem 3.3 there exists an Hermitian solution $\bar{P}$ of

$$A_\rho^* P + P A_\rho - \left(r_{\mathbb{C}}^2(A;\ B,\ C) - \rho + \tfrac{1}{2}\varepsilon\right)C^*C - PBB^*P = 0$$

with the property that $\sigma(A_\rho - BB^*\bar{P}) \subset \mathbb{C}_-$. But then it is easy to see that $P_\rho + \bar{P}$ is an Hermitian solution of

$$A^*P + PA - \left(r_{\mathbb{C}}^2(A;\ B,\ C) + \tfrac{1}{2}\varepsilon\right)C^*C - PBB^*P = 0$$

satisfying

$$\sigma\left(A - BB^*\left(P_\rho + \bar{P}\right)\right) = \sigma\left(A_\rho - BB^*\bar{P}\right) \subset \mathbb{C}_-.$$

This contradicts Proposition 3.4 and so

$$r_{\mathbb{C}}^2(A;\ B,\ C) = r_{\mathbb{C}}^2(A_\rho;\ B,\ C) + \rho,$$

$$\rho < r_{\mathbb{C}}^2(A;\ B,\ C).$$

It is clear that (16) holds when $\rho = r_{\mathbb{C}}^2$ since $r_{\mathbb{C}}^2(A_{r_{\mathbb{C}}^2};\ B,\ C) = 0$. □

4.2. Remark. (a) More general monotonicity results can be found in Wimmer [7].

(b) Employing the implicit function theorem as in Delchamps [8] one can show, in fact, that if it exists the stabilizing solution is an analytic function of A, B, C and ρ.

(c) It is easy to show that if (A, C) is observable then $P_\rho < 0$ (resp. $P_\rho > 0$) if $0 < \rho \le r_{\mathbb{C}}^2$ (resp. $\rho < 0$).

(d) As $\rho \to -\infty$ one obtains the so called cheap control problem. Thus the analysis of A_ρ as $\rho \to r_{\mathbb{C}}^2$ may be considered as a converse of the cheap control approach.

(e) By (16) there exists $x \in \mathbb{C}^n$, $\omega \in R$ such that

$$\left(A - BB^*P_{r_{\mathbb{C}}^2}\right)x = \mathrm{i}\omega x$$

and it is easy to show that

$$\| B^*P_{r_{\mathbb{C}}^2} x \|_{\mathbb{C}^n} = r_{\mathbb{C}} \| Cx \|_{\mathbb{C}^p}.$$

Now let

$$D = -B^*P_{r^2\mathbb{C}} x (Cx)^* / \| Cx \|_{\mathbb{C}^p}^2,$$

then $\| D \| = r_{\mathbb{C}}$ and

$$(A + BDC)x = Ax - BB^*P_{r_{\mathbb{C}}^2} x = \mathrm{i}\omega x.$$

Note that by (16), $r_{\mathbb{C}}^2(A_\rho;\ B,\ C) \to \infty$ as $\rho \to -\infty$. This might suggest that the feedback $u_\rho = -B^*P_\rho x$ will arbitrarily increase the stability radius. However B, C do not represent input and output operators so in general this control cannot be implemented. If the uncertain system (1) is controlled via a known input matrix $\tilde{B} \in \mathbb{R}^{n \times \tilde{m}}$

$$\dot{x} = (A + BDC)x + \tilde{B}u$$

one can formulate a state space version of the robustness optimization problem (cf. [9,10]) as follows:

Find a (minimal norm) state feedback matrix $F \in \mathbb{R}^{\tilde{m} \times n}$ which maximizes $r_{\mathbb{C}}(A + \tilde{B}F;\ B,\ C)$.

A similar formulation can be given for static or dynamic output feedback.

5. Liapunov functions for maximal robustness

One major source of difficulty in robust stability analysis is due to the fact that the set of (nicely) stable systems is not convex. However, the set of systems $\dot{x} = Ax$ for which a given positive definite quadratic form $V(x) = \langle x, Px \rangle$ is a Liapunov function turns out to be convex. Based on this observation Kiendl [11] has developed a method for checking the stability of parameter dependent matrices of the form

$$A(p) = A_0 + p_1 A_1 + \cdots + p_r A_r$$

where $A_k \in \mathbb{R}^{n \times n}$, $k = 1, \ldots, r$, and $p = (p_1, \ldots, p_r)$ is a real parameter vector in a multidimensional interval $a \le p \le b$. This shows that, given a nominal system $\dot{x} = Ax$ it may be of interest to construct a Liapunov function $V(x)$ of *maximal robustness*. By this we mean that V is a Liapunov function for all matrices $\tilde{A}$ in a ball with radius ρ around A, where ρ is the largest possible radius for which such a (quadratic) Liapunov function can be found. If we assume structured perturbations as in (1) it is clear that a joint Liapunov function can exist for the set $\{A + BDC;\ \|D\|^2 < \rho\}$ only if $\rho \le r_{\mathbb{C}}^2(A;\ B,\ C)$. The following proposition shows that the converse is true.

5.1. Proposition. *Suppose $\sigma(A) \subset \mathbb{C}_-$, $\rho \in (0, r_{\mathbb{C}}^2]$, and P_ρ is the solution of* (ARE_ρ) *with $\sigma(A - BB^*P_\rho) \subset \overline{\mathbb{C}}_-$. Then $V(x) = -\langle x, P_\rho x\rangle$ is a Liapunov function for all linear systems $\dot{x} = (A + BDC)x$ with $\|D\|^2 < \rho$.*

We do not prove this proposition since it is an immediate corollary of Proposition 5.2 where we analyze time varying and nonlinear perturbations. Consider

$$\dot{x} = Ax + BN(Cx, t), \quad x(0) = x_0, \tag{20}$$

where $N: \mathbb{R}^p \times \mathbb{R}_+ \to \mathbb{R}^n$ is continuously differentiable, and $N(0, t) \equiv 0$, $t \ge 0$, so that 0 is an equilibrium state of (20).

5.2. Proposition. *Suppose $\sigma(A) \subset \mathbb{C}_-$, $\rho \in (0, r_{\mathbb{C}}^2]$ and there exists $d = d(\rho) > 0$ such that*

$$-\langle x, P_\rho x\rangle < d \text{ and } Cx \ne 0$$
$$\Rightarrow \quad \|N(Cx, t)\|_{\mathbb{C}^m}^2 < \rho \|Cx\|_{\mathbb{C}^p}^2, \quad t \ge 0. \tag{21}$$

Then $\{x \in \mathbb{C}^n,\ -\langle x, P_\rho x\rangle < d\}$ is contained in the region of asymptotic stability of the origin for (20).

Proof. Consider $V(x) = -\langle x, P_\rho x\rangle$, then

$$\begin{aligned}\dot{V}(x) &= -\rho\|Cx\|_{\mathbb{C}^p}^2 - \|B^*P_\rho x\|^2 \\ &\quad -2\,\mathrm{Re}\langle B^*P_\rho x, N(Cx, t)\rangle_{\mathbb{C}^m} \\ &= -\|B^*P_\rho x + N(Cx, t)\|^2 \\ &\quad -\left[\rho\|Cx\|_{\mathbb{C}^p}^2 - \|N(Cx, t)\|_{\mathbb{C}^m}^2\right].\end{aligned}$$

Thus

$$\dot{V}(x) \le 0 \text{ whenever } -\langle x, P_\rho x\rangle < d. \tag{22}$$

Denote the solution of (20) by $\gamma(t, x_0)$ and let

$$G = \{x \in \mathbb{C}^n\colon -\langle x, P_\rho x\rangle < d\}.$$

Then if $x_0 \in G$ we have by (22) that $\gamma(t, x_0) \in G$ for all $t \ge 0$. Suppose

$$\gamma(t, x_0) = \gamma_1(t, x_0) + \gamma_2(t, x_0)$$

where

$$\gamma_1(t, x_0) \in (\ker P_\rho)^\perp, \quad \gamma_2(t, x_0) \in \ker P_\rho.$$

Then since P_ρ is a solution of (ARE_ρ) we see that $\gamma_2(t, x_0) \in \ker C$. Thus

$$x_0 \in G \ \Rightarrow\ \gamma_1(t, x_0) \in G, \ t \ge 0,$$

and so there exists $\overline{M}$ such that $\|\gamma_1(t, x_0)\| = \overline{M}$. But

$$\begin{aligned}\gamma(t, x_0) &= \gamma_1(t, x_0) + \gamma_2(t, x_0) \\ &= \mathrm{e}^{At}x_0 \\ &\quad + \int_0^t \mathrm{e}^{A(t-s)}BN(C\gamma(s, x_0), s)\,\mathrm{d}s.\end{aligned}$$

Since $\sigma(A) \subset \mathbb{C}_-$, there exist positive numbers M, ω such that

$$\|\mathrm{e}^{At}\| \le M\,\mathrm{e}^{-\omega t}, \quad t \ge 0.$$

So

$$\begin{aligned}\|\gamma(t, x_0)\| &\le M\,\mathrm{e}^{-\omega t}\|x_0\| \\ &\quad + \|B\|M\int_0^t \mathrm{e}^{-\omega(t-s)}\|N(C\gamma_1(s, x_0), s)\|\,\mathrm{d}s\end{aligned}$$

and hence

$$\begin{aligned}\|\gamma(t, x_0)\| &\le M\,\mathrm{e}^{-\omega t}\|x_0\| \\ &\quad + \|B\|M\overline{M}\sqrt{\rho}\,\|C\|\frac{1 - \mathrm{e}^{-\omega t}}{\omega}.\end{aligned}$$

Thus the orbit through x_0, $O(x_0)$, is relatively compact. We may now use La Salle's theorem [12] to conclude that

$$x_0 \in G \ \Rightarrow\ \gamma(t, x_0) \to S \quad \text{as } t \to \infty,$$

where S is the largest invariant set in $G \cap \dot{V}^{-1}(0)$. But

$$G \cap \dot{V}^{-1}(0) \subset \ker C,$$

so for any $\varepsilon > 0$, there exists T such that for all $t \ge T$, $\|C\gamma(t, x_0)\| < \varepsilon$. By (21),

$$\|N(C\gamma(t, x_0), t) < \sqrt{\rho}\,\varepsilon.$$

Now

$$\begin{aligned}\gamma(t, x_0) &= \mathrm{e}^{A(t-T)}x(T) \\ &\quad + \int_T^t \mathrm{e}^{A(t-s)}BN(C\gamma(s, x_0), s)\,\mathrm{d}s\end{aligned}$$

and so

$$\| \gamma(t, x_0) \| \leq M \,\mathrm{e}^{-\omega(t-T)} \| x(T) \| + \frac{M}{\omega}[1 - \mathrm{e}^{-\omega(t-T)}] \| B \| \sqrt{\rho}\, \varepsilon.$$

Thus

$$\gamma(t, x_0) \to \{0\} \quad \text{as } t \to \infty. \qquad \square$$

Proposition 5.1 shows in particular that, if N satisfies

$$\left\| \frac{\mathrm{d}}{\mathrm{d}y} N(y, t) |_{y=0} \right\| < r_{\mathbb{C}},$$

then the origin is an asymptotically stable equilibrium point of (20).

Finally let us note that a stable system $\dot{x} = Ax$ can even tolerate *dynamic* structured perturbations as long as the gain of the perturbation operator is smaller than $r_{\mathbb{C}}$. In fact, consider

$$\dot{x}(t) = Ax(t) + B(Dy)(t), \quad x(0) = x_0, \tag{23}$$
$$y(t) = Cx(t),$$

where

$$D: L^2[0, \infty; \mathbb{C}^p] \to L^2[0, \infty; \mathbb{C}^m]$$

is any (nonlinear, time varying) operator with finite gain

$$\| D \| = \inf\{ \gamma \in R;\ \| Dy \|_{L^2} \leq \gamma \| y \|_{L^2} \text{ for all } y \in L^2 \}.$$

We suppose $\sigma(A) \in \mathbb{C}_-$. By Proposition 2.2,

$$\| y(\cdot) \|_{L^2} \leq \| C\, \mathrm{e}^{A\cdot} x_0 \|_{L^2} + \frac{1}{r_{\mathbb{C}}} \| D \| \, \| y(\cdot) \|_{L^2}. \tag{24}$$

Hence if $\| D \| < r_{\mathbb{C}}$ we have $\| y(\cdot) \|_{L^2} < \infty$ for all $x_0 \in \mathbb{C}^n$ and this in turn implies that the origin of (23) is asymptotically stable.

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