Forelæsning 2

Systemer af n differentialligninger af 1. orden.

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{u}(t), \qquad \underline{x} = \underline{x}(t) \qquad t \in I \subset \mathbb{R}$$

$\underline{x}, \underline{u}$ har dimensionen n

$\underline{\underline{A}}$ er en $n \times n$ matrice. $\underline{\underline{A}}$ har reelle koefficienter.

Eksempel

$$y''' + 2y'' - y' + 10y = u(t), \qquad y = y(t)$$

Indfør: $x_1(t) = y(t)$ $\qquad x_1'(t) = y'(t) = x_2(t)$

$x_2'(t) = y''(t) = x_3(t)$

Vi har nu $\quad y''' = -10y + y' - 2y'' + u(t)$

Heraf fås

$$\left\{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -10x_1 + x_2 - 2x_3 + u(t) \end{aligned}\right\} \Rightarrow \left\{\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u(t) \end{bmatrix}\right\}$$

$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$ løsningsantagelse $\underline{x}(t) = \underline{v}\, e^{\lambda t}$

Indsat $\dot{\underline{x}} = \lambda \underline{v} e^{\lambda t} = \underline{\underline{A}}\,\underline{v}\, e^{\lambda t} \Leftrightarrow \boxed{\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}}$

Karakteristiske polynomium $P(\lambda) = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$

Rødder: $P(\lambda) = 0$

Fuldstændige løsning: $\boxed{\underline{x}(t) = \sum_{i=1}^{n} c_i \underline{v}_i e^{\lambda_i t}}$ $\lambda_i \neq \lambda_j,\ i \neq j$

$c_i \in \mathbb{C}$

$\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$ Kompleks konjugeret $\overline{\underline{\underline{A}}\,\underline{v}} = \overline{\lambda \underline{v}} \Rightarrow \underline{\underline{A}}\,\overline{\underline{v}} = \bar{\lambda}\,\overline{\underline{v}}$

egenvektor egenværdi

Løsning på reel form. Sætn. 2.6

a) λ er reel: løsning $\underline{x}(t) = \underline{v}\, e^{\lambda t}$

b) λ er kompleks: $\lambda = a + i\omega$ 2 reelle løsninger

$\underline{x}(t) = \mathrm{Re}[\underline{v} e^{\lambda t}]$ $\underline{x}(t) = \mathrm{Im}[\underline{v} e^{\lambda t}]$

$$\underline{x}(t) = Re[\underline{v}e^{\lambda t}] = Re((\underline{V}_R + i\underline{V}_I)e^{(a+i\omega)t}) =$$

$$Re[(\underline{V}_R + i\underline{V}_I)e^{at}(\cos(\omega t) + i\sin(\omega t))] = \underline{V}_R e^{at}\cos(\omega t) - \underline{V}_I e^{at}\sin(\omega t)$$

Jvf. sætn. 2.6 formlerne (2.11) og (2.12)

Det karakteristiske polynomium $P(\lambda) = \det(\underline{\underline{A}} - \lambda\underline{\underline{I}})$

$$P(\lambda) = a_n(\lambda-\lambda_1)^{p_1}(\lambda-\lambda_2)^{p_2}\cdot\ldots\cdot(\lambda-\lambda_m)^{p_m}$$

p_i er den algebraiske multiplicitet af λ_i

$\exists\, p_i$ lineært uafhængige egenvektorer til λ_i, så er vi færdige.

$\exists\, q_i$ lineært uafhængige egenvektorer til λ_i og $q_i < p_i$,
så har vi et problem!

q_i er den geometriske multiplicitet

q_i er dimensionen for nulrummet til $\underline{\underline{A}} - \lambda_i\underline{\underline{I}}$

$$(\underline{\underline{A}} - \lambda_i\underline{\underline{I}})\underline{v}_i = \underline{0}$$

Sætn. 2.10 λ er en egenværdi for $\underline{\underline{A}}$ med multiplicitet p. Geometriske multiplicitet $q < p$

$\exists\ \underline{b}_{ij}$ således at

$$\underline{x}_1(t) = \underline{b}_{11} e^{\lambda t}$$
$$\underline{x}_2(t) = \underline{b}_{21} e^{\lambda t} + \underline{b}_{22}\, t\, e^{\lambda t}$$
$$\underline{x}_3(t) = \underline{b}_{31} e^{\lambda t} + \underline{b}_{32}\, t\, e^{\lambda t} + \underline{b}_{33}\, t^2 e^{\lambda t}$$
$$\vdots$$
$$\underline{x}_p(t) = \underline{b}_{p1} e^{\lambda t} + \dots + \underline{b}_{pp}\, t^{p-1} e^{\lambda t}$$

Er p lineart uafhængige løsninger til $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$

se eksempel 2.12

Inhomogene ligningssystem $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{u}(t)$

Partikulær løsning $x_p(t)$ $\dot{\underline{x}}_p = \underline{\underline{A}}\,\underline{x} + \underline{u}(t)$

Fuldstændige løsning $\underline{x}(t) = \underline{x}_{hom}(t) + \underline{x}_p(t)$

Fundamentalmatricen

Defn. 2.13

$\underline{x}_1(t), \underline{x}_2(t), \ldots \underline{x}_n(t)$ er løsninger til $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$

Fundamentalmatricen er defineret ved

$$\underline{\underline{\Phi}}(t) = [\,\underline{x}_1(t) \quad \underline{x}_2(t) \quad \cdots \quad \underline{x}_n(t)\,] \qquad t \in I$$

Søjler kan ombyttes frit. $\underline{\underline{\Phi}}(t)$ er ikke entydig bestemt.

Vi har $\quad \dot{\underline{\underline{\Phi}}} = \underline{\underline{A}}\,\underline{\underline{\Phi}}$ (sætn. 2.16)

Fuldstandig løsning $\quad \boxed{\underline{x}(t) = \underline{\underline{\Phi}}(t)\,\underline{c}} \qquad \underline{c} = (c_1, c_2, \ldots, c_n)^T \in \mathbb{C}^n$

Partikulær løsning $\quad \underline{x}(t_0) = \underline{x}_0 \Leftrightarrow \underline{\underline{\Phi}}(t_0)\,\underline{c} = x_0 \Leftrightarrow$

$$\boxed{\underline{c} = \underline{\underline{\Phi}}^{-1}(t_0)\,x_0}$$

Løsning til det inhomogene problem (sætn 2.19)

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{u}(t) \qquad \underline{x}' = \frac{d\underline{x}}{dt}$$

Vi har $\underline{\underline{\Phi}}^{-1}\underline{\underline{\Phi}} = \underline{\underline{I}}$

Differentiation med hensyn til t

$$(\underline{\underline{\Phi}}^{-1})'\underline{\underline{\Phi}} + \underline{\underline{\Phi}}^{-1}\underline{\underline{\Phi}}' = \underline{\underline{0}} \quad \Leftrightarrow \quad \boxed{\underline{\underline{\Phi}}^{-1}\underline{\underline{\Phi}}' = -(\underline{\underline{\Phi}}^{-1})'\underline{\underline{\Phi}}}$$

Vi har $\underline{\underline{\Phi}}' = \underline{\underline{A}}\,\underline{\underline{\Phi}} \Leftrightarrow \underline{\underline{\Phi}}'\underline{\underline{\Phi}}^{-1} = \underline{\underline{A}}\,\underline{\underline{\Phi}}\,\underline{\underline{\Phi}}^{-1} = \underline{\underline{A}}\,\underline{\underline{I}} = \underline{\underline{A}}$

$$\boxed{\underline{\underline{A}} = \underline{\underline{\Phi}}'\underline{\underline{\Phi}}^{-1}}$$

Partikulær løsning $\underline{x}_p(t)$

$$\underline{x}_p' = \underline{\underline{A}}\,\underline{x}_p + \underline{u}(t)$$

$$\underline{\underline{\Phi}}^{-1}\underline{x}_p' = \underline{\underline{\Phi}}^{-1}\underline{\underline{A}}\,\underline{x}_p + \underline{\underline{\Phi}}^{-1}\underline{u} \quad \Leftrightarrow \quad \underline{\underline{\Phi}}^{-1}\underline{u} = \underline{\underline{\Phi}}^{-1}\underline{x}_p' - \underline{\underline{\Phi}}^{-1}\underline{\underline{A}}\,\underline{x}_p$$

Vi får nu

$$\underline{\underline{\Phi}}^{-1}\underline{u} = \underline{\underline{\Phi}}^{-1}\underline{x}_p' - \underline{\underline{\Phi}}^{-1}(\underline{\underline{\Phi}}'\underline{\underline{\Phi}}^{-1})\,\underline{x}_p = \underline{\underline{\Phi}}^{-1}\underline{x}_p' - (\underline{\underline{\Phi}}^{-1}\underline{\underline{\Phi}}')\underline{\underline{\Phi}}^{-1}\underline{x}_p =$$

$$\underline{\underline{\Phi}}^{-1}\underline{x}_p' + (\underline{\underline{\Phi}}^{-1})'\,\underline{\underline{\Phi}}\,\underline{\underline{\Phi}}^{-1}\,\underline{x}_p = \underline{\underline{\Phi}}^{-1}\underline{x}_p' + (\underline{\underline{\Phi}}^{-1})'\,\underline{x}_p = \frac{d}{dt}(\underline{\underline{\Phi}}^{-1}\underline{x}_p)$$

Integration giver $\underline{\underline{\Phi}}^{-1} \underline{x}_p = \int_{t_0}^{t} \underline{\underline{\Phi}}^{-1}(\tau)\, u(\tau)\, d\tau$ $\Leftrightarrow$

$$\boxed{\underline{x}_p(t) = \underline{\underline{\Phi}}(t) \int_{t_0}^{t} \underline{\underline{\Phi}}^{-1}(\tau)\, u(\tau)\, d\tau}$$

Fuldstændig løsning

$$\underline{x}(t) = \underline{\underline{\Phi}}\, \underline{c} + \underline{\underline{\Phi}}(t) \int_{t_0}^{t} \underline{\underline{\Phi}}^{-1}(\tau)\, \underline{u}(\tau)\, d\tau$$

Sætn 2.19