Math 2 exam December 2020
Part B

Problem 1

1. We have $\left|\frac{\sin(nt)}{2^n}\right| \leq \frac{1}{2^n}$, so the convergent series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent majorant, and f is therefore (uniformly) convergent.

2. f is an odd function. A linear combination of odd functions (here $\sin(nx)$) is odd, and this extends to the limit, since the series is convergent.

3. Yes, see part 1.

4. Yes, it's a uniformly convergent series of continuous functions, so the sum function is continuous.

5. We have:

$$\left|f(t) - \sum_{n=1}^{N} \frac{\sin(nt)}{2^n}\right| = \left|\sum_{n=N+1}^{\infty} \frac{\sin(nt)}{2^n}\right| \leq \sum_{n=N+1}^{\infty} \frac{1}{2^n}$$
$$= \frac{1}{2^{N+1}} \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{2^{N+1}} \frac{1}{1-\frac{1}{2}} = \frac{1}{2^{N+1}} 2 = \frac{1}{2^N}\,.$$

   So if $2^{-N} \leq 10^{-3}$ the required inequality is satisfied. Solving this:

   $$2^{-N} \leq 10^{-3} \iff -N \leq \log_2(10^{-3}) = -3\log_2(10) \iff N \geq 3\log_2(10) \approx 9.97$$

   So $N = 10$ works.

   (We could alternatively have used the integral test with $\int_N^{\infty}(1/2^x)\mathrm{d}x$, which leads to $N \geq 10.495$, or with $\int_{N+1}^{\infty}(1/2^x)\mathrm{d}x + \frac{1}{2^{N+1}}$, which leads to $N \geq 10.254$, so in both cases $N = 11$).

6. Parseval's identity gives:

$$\int_0^{2\pi} |f(t)|^2\,\mathrm{d}t = \pi \sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right)^2 = \frac{1}{4}\pi \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{4}\pi \frac{1}{1-\frac{1}{4}} = \frac{\pi}{3}\,.$$

Problem 2

1. Setting $x(t)$ into the equation, the left hand side is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} te^{3t} \\ 2te^{3t}-e^{3t} \end{pmatrix} = \begin{pmatrix} e^{3t}+3te^{3t} \\ 2e^{3t}+6te^{3t}-3e^{3t} \end{pmatrix} = \begin{pmatrix} 3te^{3t}+e^{3t} \\ 6te^{3t}-e^{3t} \end{pmatrix},$$

   and the right hand side is

$$\begin{pmatrix} 5te^{3t}-(2te^{3t}-e^{3t}) \\ 4te^{3t}+(2te^{3t}-e^{3t}) \end{pmatrix} = \begin{pmatrix} 3te^{3t}+e^{3t} \\ 6te^{3t}-e^{3t} \end{pmatrix},$$

   which is equal to the left hand side.

2. The characteristic polynomial for the matrix $\left(\begin{smallmatrix} 5 & -1 \\ 4 & 1 \end{smallmatrix}\right)$ is $P(\lambda) = \lambda^2 - 6\lambda + 9 = (\lambda-3)^2$ which has 3 as a double root, and eigenvector $\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$. Which gives a solution $\mathbf{x}(t) = \left(\begin{smallmatrix} e^{3t} \\ 2e^{3t} \end{smallmatrix}\right)$. This is linearly independent from the previous solution, so the general solution is:

$$\mathbf{x}(t) = c_1\begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} + c_2\begin{pmatrix} te^{3t} \\ 2te^{3t}-e^{3t} \end{pmatrix}, \qquad c_1, c_2 \in \mathbb{R}.$$

3. We need a particular solution, so we look for a solution of the form $e^t\mathbf{v}$. Setting into the equation we get:

$$e^t\mathbf{v} = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix} e^t\mathbf{v} + e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 4 & -1 \\ 4 & 0 \end{pmatrix}\mathbf{v} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

   which has the solution $\mathbf{v} = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. The general solution to the inhomogeneous equation is the sum of the particular solution with the homogeneous general solution:

$$\mathbf{x}(t) = \begin{pmatrix} 0 \\ e^t \end{pmatrix} + c_1\begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} + c_2\begin{pmatrix} te^{3t} \\ 2te^{3t}-e^{3t} \end{pmatrix}, \qquad c_1, c_2 \in \mathbb{R}.$$