Forelæsning 13

Fouriertransformationen

Vi betragter funktioner f i $L^1\{\mathbb{R}\}$ hvor

$$L^1\{\mathbb{R}\} = \{ f : \mathbb{R} \to \mathbb{C} \text{ for hvilke det gælder at } \int_{-\infty}^{\infty} |f(x)|\, dx < \infty \}$$

Definition af den Fouriertransformerede

$$\boxed{\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \gamma}\, dx \quad , \quad \gamma \in \mathbb{R}}$$

Vi skriver også

$$\hat{f}(\gamma) = (Ff)(\gamma)$$

$\hat{f}$ er en kontinuert funktion

$\hat{f}(\gamma) \to 0$ for $\gamma \to \pm\infty$

Den inverse Fouriertransformation er

$$\boxed{f(x) = \int_{-\infty}^{\infty} \hat{f}(\gamma)\, e^{2\pi i x \gamma}\, d\gamma}$$

Regneregler for Fouriertransformeret

1) f er en lige funktion

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)\,(\cos(2\pi x\gamma) - i\sin(2\pi x\gamma))\,dx$$

$$= \int_{-\infty}^{\infty} \cos(2\pi x\gamma)\,dx - i\int_{-\infty}^{\infty} f(x)\sin(2\pi x\gamma)\,dx$$

Da f er lige er $\int_{-\infty}^{\infty} f(x)\sin(2\pi x\gamma)\,dx = 0$

og

$$\boxed{\hat{f}(\gamma) = 2\int_{0}^{\infty} f(x)\cos(2\pi x\gamma)\,dx \quad , \quad f \text{ lige}}$$

2) f er en ulige funktion. I dette tilfælde er

$$\int_{-\infty}^{\infty} f(x)\cos(2\pi x\gamma)\,dx = 0$$

og

$$\boxed{\hat{f}(\gamma) = -2i\int_{0}^{\infty} f(x)\sin(2\pi x\gamma)\,dx \quad , \quad f \text{ ulige}}$$

3) Translations operatoren T_a

$$(T_a f)(x) = f(x-a)$$

$$(F T_a f)(\gamma) = \int_{-\infty}^{\infty} f(x-a)\, e^{-2\pi i \gamma x}\, dx$$

Vi indfører variabeltransformationen

$$y = x - a \quad , \quad dy = dx$$

Vi har nu

$$(F T_a f)(\gamma) = \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i \gamma (y+a)}\, dy$$

$$= e^{-2\pi i \gamma a} \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i \gamma y}\, dy$$

Heraf fås

$$\boxed{(F T_a f)(\gamma) = e^{-2\pi i \gamma a}\, (Ff)(\gamma)}$$

4) Vi betragter $g(x) = f(x)\, e^{2\pi i \theta x}$.

Den Fourier transformerede af $g(x)$ er

$$(Fg)(\gamma) = \hat{g}(\gamma) = \int_{-\infty}^{\infty} g(x)\, e^{-2\pi i \gamma x}\, dx$$

$$(Fg)(\gamma) = \int_{-\infty}^{\infty} f(x)\, e^{2\pi i \theta x}\, e^{-2\pi i \gamma x}\, dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i (\gamma - \theta) x}\, dx = \hat{f}(\gamma - \theta)$$

$$\boxed{(Fg)(\gamma) = (Ff)(\gamma - \theta)}$$

5) Hvis f er differentiabel og $f' \in L^1(\mathbb{R})$ så gælder

$$(Ff')(\gamma) = \int_{-\infty}^{\infty} f'(x)\, e^{-2\pi i \gamma x}\, dx$$

For $f \in L^1(\mathbb{R})$ gælder $f(x) \to 0$ for $x \to \pm\infty$. Det uegentlige integral udregnes således:

$$\int_{-L}^{L} f'(x) e^{-2\pi i \gamma x}\, dx =$$

$$\left[f(x) e^{-2\pi i \gamma x} \right]_{-L}^{L} - \int_{-L}^{L} f(x)(-2\pi i \gamma) e^{-2\pi i \gamma x}\, dx$$

For $b \to \infty$ fås

$$(Ff')(\gamma) = 2\pi i \gamma \int_{-\infty}^{\infty} f(x) e^{-2\pi i \gamma x} \, dx$$

Dvs

$$\boxed{(Ff')(\gamma) = 2\pi i \gamma \; (Ff)(\gamma)}$$

Men betragtes den inverse Fourier transformerede har vi

$$f'(x) = \int_{-\infty}^{\infty} (2\pi i \gamma) \, \hat{f}(\gamma) e^{2\pi i x \gamma} \, d\gamma$$

Dette kan udnyttes til at løse partielle differentialligninger.

Eksempel:

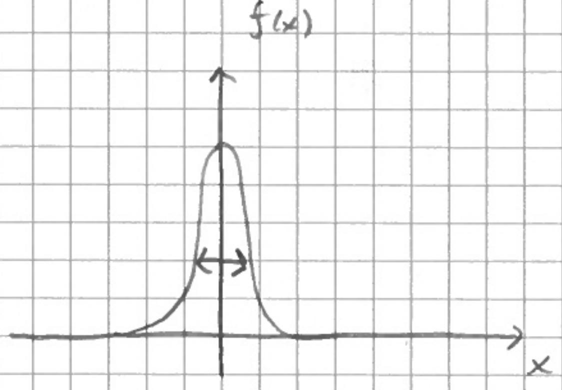
$$f(x) = a \exp\left(-\left(\frac{x}{w}\right)^2\right) \qquad w > 0$$

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \gamma x}\, dx =$$

$$\int_{-\infty}^{\infty} a \exp\left(-\left(\frac{x}{w}\right)^2\right) e^{-2\pi i \gamma x}\, dx =$$

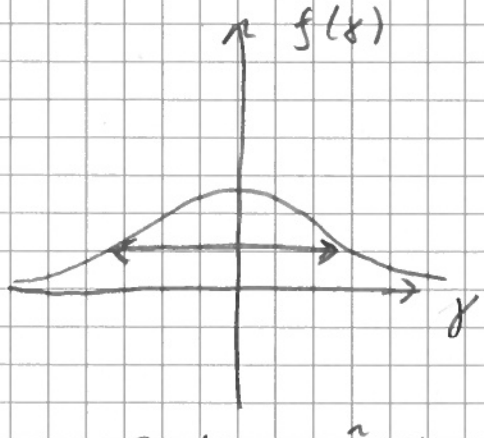
$$a\, w \sqrt{\pi}\, \exp\left(-(w \pi \gamma)^2\right)$$

Result from Maple and Wolfram Alpha



$w \ll 1$

Puls bredde



Bredde af $\hat{f}(\gamma)$

$$\frac{1}{\pi w} \gg 1$$