Mat 2 Exam E20 Part A

Page 1

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Question 1

Consider the two infinite series:

$$A = \sum_{n=1}^{\infty}(-1)^n \frac{2}{1+\sqrt{n}}, \quad \text{and} \quad B = \sum_{n=1}^{\infty} \cos(n^2+1)\frac{2+n^2}{1+n!}.$$

Which statement is true?

- [ ] Both A and B are absolutely convergent.
- [ ] Both A and B are divergent.
- [x] A is conditionally convergent and B is absolutely convergent.
- [ ] A is divergent and B is absolutely convergent.
- [ ] A is convergent and B is divergent.

Question 2

Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(2x)^n}{\sqrt{n^2+1}}$.

The interval of convergence for the series is:

- [ ] $\mathbb{R}$
- [x] $-\frac{1}{2} \le x < \frac{1}{2}$.
- [ ] $-\frac{1}{2} \le x \le \frac{1}{2}$.
- [ ] $-2 \le x < 2$.
- [ ] $-2 \le x \le 2$.
- [ ] $-1 < x < 1$.
- [ ] $-1 \le x \le 1$.
- [ ] $x = 0$.

Question 3

It is given that the characteristic polynomial for a 4th order linear homogeneous differential equation with constant coefficients is: $P(\lambda) = (\lambda - 2)(\lambda + 3)(\lambda^2 + 1)$.

Choose the function below that is a solution to the differential equation:

- [ ] $y(t) = e^{3t} + e^{2t}$.
- [ ] $e^{2t} + e^{-3t} + t\cos(t) + t\sin(t)$.
- [ ] $-e^{-3t} + 7e^{2t} + 2e^{t} + te^{t}$.
- [ ] $e^{3t} + \cos(t) + \sin(t)$.
- [x] $5e^{2t} - 2e^{-3t} + \cos(t)$.
- [ ] $4e^{2t} + 3e^{-3t} + 2e^{-t}$.

Question 4

Given three infinite series, $A = \sum_{n=1}^{\infty} a_n, \quad B = \sum_{n=1}^{\infty} b_n, \quad C = \sum_{n=1}^{\infty} c_n,$

with positive terms $a_n > 0, b_n > 0, c_n > 0,$

suppose that:

$a_n \leq b_n \text{ for all } n,$ and $\lim_{n\to\infty} \frac{c_n}{b_n} = 2.$

Which of the following conclusions follows from this information? (Only one is valid).

- [ ] B and C are divergent, but the convergence status of A is unkown.
- [x] If A is divergent then C is divergent.
- [ ] All three series have the same convergence status.
- [ ] If C is divergent then A is divergent.
- [ ] If $\lim_{n\to\infty} c_n = 0$ then A is convergent.

Question 5

Consider the differential equation

$y''(t) + 2y'(t) + ty(t) = 0.$

By setting:

$y(t) = \sum_{n=0}^{\infty} c_n t^n$

we can rewrite the differential equation as one of the following equations. Choose the correct one:

- [ ] $c_0 + \sum_{n=1}^{\infty} \left((n+2)(n+1)c_{n+2} + 2(n+1)c_{n+1} + c_{n-1}\right) t^n = 0$
- [ ] $c_1 + 2c_2 + c_0 t + \sum_{n=2}^{\infty} \left(n(n-1)c_{n-2} + 2nc_{n-1} + c_{n+1}\right) t^n = 0$
- [ ] $c_0 + \sum_{n=1}^{\infty} \left(n(n-1)c_{n-1} + 2nc_n + c_{n+1}\right) t^n = 0$
- [x] $2(c_1 + c_2) + \sum_{n=1}^{\infty} \left((n+2)(n+1)c_{n+2} + 2(n+1)c_{n+1} + c_{n-1}\right) t^n = 0$
- [ ] $c_1 + 2c_2 + c_0 t + \sum_{n=2}^{\infty} \left(n(n-1)c_{n-2} + 2nc_{n-1} + c_{n-1}\right) t^n = 0$

Page 2

Question 6

Consider the series:

$$\sum_{n=1}^{\infty} \frac{1}{x^n}.$$

The sum of the series is:

- [ ] $\frac{1}{x-1}$, valid for $|x| < 1.$
- [ ] $\frac{x}{1-x}$, valid for $|x| < 1.$
- [ ] $\frac{x}{1-x}$, valid for $|x| > 1.$
- [ ] $\frac{1}{1+x}$, valid for $|x| < 1.$
- [ ] $\frac{1}{1+x}$, valid for $|x| > 1.$
- [x] $\frac{1}{x-1}$, valid for $|x| > 1.$

Question 7

Consider a first order system of differential equations

$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ with

$$\mathbf{A} = \begin{pmatrix} c & 0 & 0 \\ 0 & -1 & a \\ 0 & b & -1 \end{pmatrix},$$

where $a, b, c \in \mathbb{R}.$

The system is asymptotically stable for:

- [ ] $c < 0,\ a \geq 0,\ b \geq 0.$
- [ ] $c < 0$ and $ab > 1.$
- [ ] $c < 0, a < 0, b < 0.$
- [x] $c < 0$ and $ab < 1.$
- [ ] $c > 0$ and $0 < ab < 1.$
- [ ] $c > 0$ and $ab > 0.$

Question 8

Let $f : \mathbb{R} \to \mathbb{R}$ be an *even* and 2π-periodic function, given on the interval $[0, \pi]$ by the formula:

$$f(x) = \begin{cases} x & \text{for } x \in [0, 1[, \\ 1 & \text{for } x \in [1, \pi]. \end{cases}$$

Which of the following statements about the Fourier series of f is valid?

- [ ] The Fourier series converges pointwise, but not uniformly, to f.
- [ ] The Fourier series converges pointwise, but not to f.
- [ ] The Fourier series converges uniformly, but not to f.
- [x] The Fourier series converges uniformly to f.
- [ ] There is at least one point x at which the Fourier series is divergent.

Question 9

Consider the infinite series

$$\sum_{n=1}^{\infty}\left(\frac{\cos(nx)}{n^2}+\frac{\sin(nx)}{n+e^n}\right), \qquad x \in \mathbb{R}.$$

Select the statement that is valid:

- [ ] The series converges at each point, but the sum function is not continuous at every point.
- [ ] The series converges only at the point $x = 0$.
- [x] The series converges at each $x \in \mathbb{R}$, and the sum function is continuous.
- [ ] The series converges only at points $x = (2k+1)\pi,\ k \in \mathbb{Z}$.
- [ ] The series converges only at the points $x = (2k+1)\frac{\pi}{2},\ \ k \in \mathbb{Z}$.
- [ ] The series converges only at points of the form $x = (2k+1)\pi$ or $x = (2k+1)\pi/2$, where k is an integer.

Question 10

A power series $\sum_{n=0}^{\infty} c_n t^n$ satisfies the equation:

$$2(c_1+c_2)+\sum_{n=1}^{\infty}\left((n+2)(n+1)c_{n+2}+2(n+1)c_{n+1}+3c_{n-1}\right)t^n = 0.$$

This leads to the following recurrence relation:

- [ ] $c_2 = -c_1, \quad \text{and } c_n = \frac{2(n-1)c_{n-1}+3c_{n-3}}{n(n-1)},\ \ n \geq 4.$
- [ ] $c_2 = -2c_1, \quad \text{and } c_n = -\frac{2(n+1)c_{n-1}+3c_{n-3}}{n(n+1)},\ \ n \geq 3.$
- [ ] $c_{n+2} = -\frac{2(n+1)c_{n+1}-3c_{n-1}}{(n+2)(n+1)},\ \ n \geq 0.$
- [ ] $c_{n+2} = \frac{2(n+1)c_{n+1}+3c_{n-1}}{(n+2)(n+1)},\ \ n \geq 0.$
- [x] $c_2 = -c_1, \quad \text{and } c_n = -\frac{2(n-1)c_{n-1}+3c_{n-3}}{n(n-1)},\ \ n \geq 3.$