01035 Matematik 2
Løsninger eksamen december 2023

A1 $P(\lambda) = \lambda(4\lambda^2+4\lambda+1) = 4\lambda(\lambda^2+\lambda+\frac{1}{4})$

$= 4\lambda(\lambda+\frac{1}{2})^2$. Rødder $\lambda=0$, $\lambda=-\frac{1}{2}$ (dobbelt)

Fuldstændig løsning

$$y(t) = c_1 e^{0t} + c_2 e^{-\frac{1}{2}t} + c_3 t e^{-\frac{1}{2}t} = \underline{c_1 + c_2 e^{-\frac{1}{2}t} + c_3 t e^{-\frac{1}{2}t}}$$

(c)

A2 $$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = \frac{du}{dt} + 2u$$

$$H(s) = \frac{s+2}{s^2+s-2} = \frac{s+2}{(s+2)(s-1)} = \frac{1}{s-1}$$

$H(s)$ er defineret for $s^2+s-2 \neq 0$, dvs $s \neq -2, 1$

(b)

A3 $$\sum_{n=0}^{\infty} \frac{(-1)^n n}{3^n} x^n. \quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(n+1)}{3^{n+1}} x^{n+1}}{\frac{(-1)^n n}{3^n} x^n}\right|$$

$$= \frac{n+1}{n} \cdot \frac{3^n}{3^{n+1}} |x| = \frac{n+1}{n} \cdot \frac{1}{3}|x| = \left(1+\frac{1}{n}\right)\cdot\frac{1}{3}|x|$$

$\rightarrow \frac{1}{3}|x|$ for $n \to \infty$. Kvotientkriteriet:

$\frac{1}{3}|x| < 1 \Rightarrow |x| < 3 \Rightarrow \rho = 3$ (a)

A4 $\sum_{n=0}^{\infty} \frac{(-1)^n n}{3n^3+8}$.

$$|a_n| = \frac{n}{3n^3+8} < \frac{n}{3n^3} = \frac{1}{3}\frac{1}{n^2} \text{ for } n \geq 1$$

Da $\sum_{n=0}^{\infty} \frac{1}{3}\frac{1}{n^2}$ er konvergent, er $\sum_{n=0}^{\infty} |a_n|$ konvergent i følge sammenligningskriteriet. Rækken er altså absolut konvergent (c)

A5 $\sum_{n=0}^{\infty} \frac{a^{2n}}{e^{-3n}}$.

$$\frac{a^{2n}}{e^{-3n}} = \left(\frac{a^2}{e^{-3}}\right)^n = (e^3 a^2)^n$$

Kvotientrække med kvotient e^3a^2. Konvergent når $|e^3a^2| < 1 \Rightarrow a^2 < e^{-3} \Rightarrow |a| < e^{-3/2}$

(b)

A6 $f \sim \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx$.

$$c_{-1} = \frac{1}{2}(a_1 + ib_1) = \frac{1}{2}\cdot\frac{1}{1^2} = \frac{1}{2} \quad \text{(a)}$$

A7 $\sum_{n=0}^{\infty} \frac{1}{x^{2n}}$. $\frac{1}{x^{2n}} = \left(\frac{1}{x^2}\right)^n$. Kvotientrække med kvotient x^{-2}. Konvergent når $|x^{-2}|<1$, dvs $|x|>1$ med sum $\frac{1}{1-x^{-2}} = \frac{x^2}{x^2-1}$.

$$\sum_{n=0}^{\infty} \frac{1}{x^{2n}} = x^2 \Leftrightarrow \frac{x^2}{x^2-1} = x^2 \Leftrightarrow x^2-1 = 1 \Leftrightarrow x^2 = 2$$

$\Leftrightarrow x = \pm\sqrt{2}$ (som opfylder $|x|>1$) (c)

B1 $\quad \dot{\underline{x}} = \begin{pmatrix} 1 & -1 \\ 6 & -4 \end{pmatrix} \underline{x} + t\begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 7 \end{pmatrix}$

B1.1 $\quad P(\lambda) = \begin{vmatrix} 1-\lambda & -1 \\ 6 & -4-\lambda \end{vmatrix} = (1-\lambda)(-4-\lambda) + 6$

$= \lambda^2 + 3\lambda + 2$. Rødder $\lambda = \dfrac{-3 \pm \sqrt{3^2 - 4\cdot 1 \cdot 2}}{2} = \begin{cases} -1 \\ -2 \end{cases}$

Egenvektorer hørende til $\lambda = -1$: ...

$\left(\begin{array}{cc|c} 1+1 & -1 & 0 \\ 6 & -4+1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 6 & -3 & 0 \end{array}\right) \quad \underline{v} = s \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; s \in \mathbb{R} \setminus \{0\}$

Egenvektorer hørende til $\lambda = -2$:

$\left(\begin{array}{cc|c} 1+2 & -1 & 0 \\ 6 & -4+2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 3 & -1 & 0 \\ 6 & -2 & 0 \end{array}\right) \quad \underline{v} = s \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \; s \in \mathbb{R} \setminus \{0\}$

Fuldstændig løsning til homogen ligning:

$$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t}, \quad c_1, c_2 \in \mathbb{R}$$

B.1.2 Da egenværdierne har negativ realdel er systemet asymptotisk stabilt. Sætning 2.47 + 2.38

B.1.3 Skriv systemet som $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + t\,\underline{a} + \underline{b}$

Antag løsning $\underline{x} = t\underline{v} + \underline{w}$: $\quad \underline{v} = \underline{\underline{A}}(t\underline{v} + \underline{w}) + t\,\underline{a} + \underline{b} \Leftrightarrow$

$\underline{v} = t\,\underline{\underline{A}}\,\underline{v} + \underline{\underline{A}}\,\underline{w} + t\,\underline{a} + \underline{b} \Leftrightarrow t(\underline{\underline{A}}\,\underline{v} + \underline{a}) + \underline{\underline{A}}\,\underline{w} + \underline{b} - \underline{v} = $

Da dette skal gælde for alle $t \in \mathbb{R}$ må

$$\underline{\underline{A}}\,\underline{v} + \underline{a} = \underline{0} \quad \text{og} \quad \underline{\underline{A}}\,\underline{w} + \underline{b} - \underline{v} = 0$$

Først findes $\underline{v}$: $\left(\begin{array}{cc|c} 1 & -1 & -1 \\ 6 & -4 & -4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 2 & 2 \end{array}\right) \sim$

$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right)$ dvs $\underline{\underline{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

Derefter $\underline{w}$: $\underline{b} - \underline{v} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

$\left(\begin{array}{cc|c} 1 & -1 & -1 \\ 6 & -4 & -6 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right)$

dvs $\underline{\underline{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}}$.

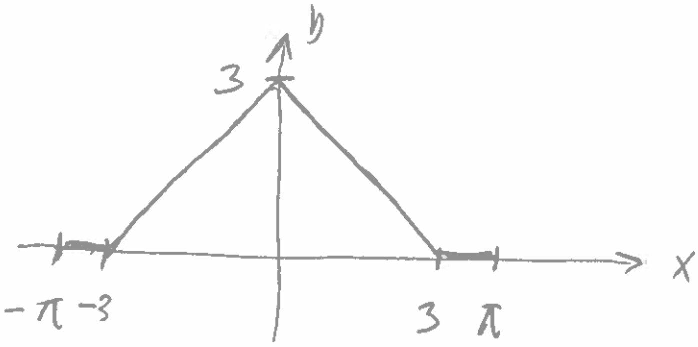
B.1.4.

$$\underline{x}(t) = \underline{x}_{hom}(t) + \underline{x}_{part}(t)$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} + t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

B.2 $f(x) = \begin{cases} 3-x & \text{for } x \in [0,3] \\ 0 & \text{for } x \in]3,\pi] \end{cases}$ 2π-periodisk, lige

B.2.1



B.2.2 Funktionen er stykkevis differentiabel og kontinuert. Korollar 6.17 viser, at Fourier-rækken konvergerer mod f for alle x, og at konvergensen er uniform.

B.2.3 f er lige $\Rightarrow b_n = 0$ for alle n.

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{3}(3-x)\cos nx\,dx =$$

$$\frac{2}{\pi}\,\frac{1-\cos(3n)}{n^2} \text{ for } n \geq 1. \quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx =$$

$$\frac{2}{\pi}\int_0^{3}(3-x)\,dx = \frac{2}{\pi}\left[3x - \tfrac{1}{2}x^2\right]_0^3 = \frac{2}{\pi}\left(9-\frac{9}{2}\right) = \frac{9}{\pi}$$

$$f \sim \frac{9}{2\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi}\,\frac{1-\cos(3n)}{n^2}\cos nx$$

<u>B.2.4.</u> $\left|\frac{2}{\pi}\,\frac{1-\cos(3n)}{n^2}\cos(nx)\right| \leq$

$$\left|\frac{2}{\pi}\,\frac{1-\cos(3n)}{n^2}\right| = \frac{2}{\pi n^2}\left|1-\cos(3n)\right| \leq \frac{2}{\pi n^2}\cdot 2$$

Rækken $\frac{9}{2\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi}\cdot\frac{1}{n^2}$ er altså en majorant-række. Da $\sum_{n=1}^{\infty}\frac{1}{n^2}$ er konvergent, er majorant-rækken det også.