A catalog of 2fi-optimal row-column designs with relatively small runs from the paper "Construction of 2fi-optimal row-column designs"

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We give a catalog of 2fi-optimal row-column designs with relatively small runs for practical use based on the paper "Construction of 2fi-optimal row-column designs". The designs are presented by their array generator matrices $\boldsymbol{\mathcal{G}}$. The catalog contains three parts, corresponding to Theorems 1-3 respectively in the aforementioned paper.

1 2fi-optimal 3^n and 5^n factorial row-column designs for all (p, q) with $n \leq 8$ (corresponding to Theorem 1)

$p = 1, q = 1, n = 2$, $s = 3$ or 5,

$$\boldsymbol{\mathcal{G}} = \begin{pmatrix} \boldsymbol{\mathcal{G}}_c \\ \hline \boldsymbol{\mathcal{G}}_r \end{pmatrix} = \left(\begin{array}{c:c} 1 & 1 \\ \hline 1 & 2 \end{array}\right).$$

$p = 1, q = 2, n = 3$, $s = 3$ or 5,

$$\boldsymbol{\mathcal{G}} = \begin{pmatrix} \boldsymbol{\mathcal{G}}_c \\ \hline \boldsymbol{\mathcal{G}}_r \end{pmatrix} = \left(\begin{array}{c:cc} 1 & 1 & 1 \\ \hline 1 & 2 & 1 \\ 1 & 1 & 2 \end{array}\right).$$

$p = 1, q = 3, n = 4$, $s = 3$ or 5,

$$\boldsymbol{\mathcal{G}} = \begin{pmatrix} \boldsymbol{\mathcal{G}}_c \\ \hline \boldsymbol{\mathcal{G}}_r \end{pmatrix} = \left(\begin{array}{c:ccc} 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right).$$

$p = 1, q = 4, n = 5$, $s = 3$ or 5,

$$\boldsymbol{\mathcal{G}} = \begin{pmatrix} \boldsymbol{\mathcal{G}}_c \\ \hline \boldsymbol{\mathcal{G}}_r \end{pmatrix} = \left(\begin{array}{c:cccc} 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{array}\right).$$

$p = 1, q = 5, n = 6$, $s = 3$ or 5,

$$\boldsymbol{\mathcal{G}} = \begin{pmatrix} \boldsymbol{\mathcal{G}}_c \\ \hline \boldsymbol{\mathcal{G}}_r \end{pmatrix} = \left(\begin{array}{c:ccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 \end{array}\right).$$

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$p=1, q=6, n=7,\ s=3$ or 5,

$$\mathcal{G}=\left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right)=\left(\begin{array}{c|cccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{array}\right).$$

$p=1, q=7, n=8,\ s=3$ or 5,

$$\mathcal{G}=\left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right)=\left(\begin{array}{c|ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{array}\right).$$

$p=2, q=2, n=4, s=3$ or 5,

$$\mathcal{G}=\left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right)=\left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ \hline 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right).$$

$p=2, q=3, n=5, s=3$,

$$\mathcal{G}=\left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right)=\left(\begin{array}{cc|cc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ \hline 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

$p=2, q=3, n=5, s=5$,

$$\mathcal{G}=\left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right)=\left(\begin{array}{cc|cc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 3 \\ \hline 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

$p=2, q=4, n=6, s=3$,

$$\mathcal{G}=\left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right)=\left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \\ \hline 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

$p=2, q=4, n=6, s=5$,

$$\mathcal{G}=\left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right)=\left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 4 \\ \hline 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

$p=2, q=5, n=7, s=3,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r}\right) = \left(\begin{array}{cc:cc:ccc} 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 1 & 1 \\ \hline 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

$p=2, q=5, n=7, s=5,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r}\right) = \left(\begin{array}{cc:cc:ccc} 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 4 & 0 \\ \hline 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

$p=2, q=6, n=8, s=3,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r}\right) = \left(\begin{array}{cc:cc:cccc} 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 & 1 & 1 & 2 \\ \hline 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

$p=2, q=6, n=8, s=5,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r}\right) = \left(\begin{array}{cc:cc:cccc} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 3 & 4 & 0 & 1 \\ \hline 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

$p=3, q=3, n=6, s=3,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r}\right) = \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right).$$

$p=3, q=3, n=6, s=5,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r}\right) = \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ \hline 0 & 0 & 1 & 3 & 1 & 1 \\ 0 & 1 & 0 & 1 & 3 & 1 \\ 1 & 0 & 0 & 1 & 1 & 3 \end{array}\right).$$

$p=3, q=4, n=7, s=3,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r} \right) = \left(\begin{array}{ccc:ccc:c} 1 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

$p=3, q=4, n=7, s=5,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r} \right) = \left(\begin{array}{ccc:ccc:c} 1 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 3 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 3 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

$p=3, q=5, n=8, s=3,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r} \right) = \left(\begin{array}{ccc:ccc:cc} 1 & 0 & 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

$p=3, q=5, n=8, s=5,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r} \right) = \left(\begin{array}{ccc:ccc:cc} 1 & 0 & 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 3 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

$p=4, q=4, n=8, s=3,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r} \right) = \left(\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right).$$

$p=4, q=4, n=8, s=5,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r} \right) = \left(\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 3 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 3 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 3 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 3 \end{array} \right).$$

2 2fi-optimal 2^{n-1} fractional factorial row-column designs for all (p, q) with $n \leq 9$ (corresponding to Theorem 2)

$p = 1, q = 3, n = 5,\ s = 2,$

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right) = \left(\begin{array}{c|ccc|c} 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 \end{array}\right).$$

$p = 1, q = 4, n = 6,\ s = 2,$

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right) = \left(\begin{array}{c|cccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 & 1 \end{array}\right).$$

$p = 1, q = 5, n = 7,\ s = 2,$

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right) = \left(\begin{array}{c|ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 \end{array}\right).$$

$p = 1, q = 6, n = 8,\ s = 2,$

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right) = \left(\begin{array}{c|cccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \end{array}\right).$$

$p = 1, q = 7, n = 9,\ s = 2,$

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right) = \left(\begin{array}{c|ccccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \end{array}\right).$$

$p = 2, q = 2, n = 5,\ s = 2,$

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right) = \left(\begin{array}{cc|cc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{array}\right).$$

$p = 2, q = 3, n = 6, s = 2,$

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right) = \left(\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{array}\right).$$

$p=2, q=4, n=7, s=2,$

$$\mathcal{G}=\left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right)=\left(\begin{array}{cc:ccc:c:c}
1 & 0 & 1 & 0 & 1 & 1 & 1\\
0 & 1 & 1 & 1 & 0 & 1 & 1\\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 0\\
0 & 1 & 1 & 0 & 0 & 1 & 0\\
1 & 1 & 0 & 1 & 0 & 1 & 0\\
\hdashline
0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right).$$

$p=2, q=5, n=8, s=2,$

$$\mathcal{G}=\left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right)=\left(\begin{array}{cc:ccc:c:cc}
1 & 0 & 1 & 0 & 1 & 1 & 1 & 0\\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 1\\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0\\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0\\
\hdashline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{array}\right).$$

$p=2, q=6, n=9, s=2,$

$$\mathcal{G}=\left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right)=\left(\begin{array}{cc:ccc:c:ccc}
1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1\\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1\\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0\\
\hdashline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}\right).$$

$p=3, q=3, n=7, s=2,$

$$\mathcal{G}=\left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right)=\left(\begin{array}{ccc:ccc:c}
1 & 0 & 0 & 0 & 1 & 1 & 1\\
0 & 1 & 0 & 1 & 0 & 1 & 1\\
0 & 0 & 1 & 1 & 1 & 0 & 1\\
\hline
1 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 1 & 1 & 0 & 0 & 1 & 1\\
0 & 0 & 1 & 1 & 1 & 1 & 0
\end{array}\right).$$

$p=3, q=4, n=8, s=2,$

$$\mathcal{G}=\left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right)=\left(\begin{array}{ccc:ccc:c:c}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1\\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0\\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0\\
\hline
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0\\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0\\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right).$$

$p=3, q=5, n=9, s=2,$

$$\mathcal{G}=\left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right)=\left(\begin{array}{ccc:ccc:c:cc}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0\\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1\\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0\\
\hline
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0\\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right).$$

$p=4, q=4, n=9, s=2,$

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right) = \left(\begin{array}{cccc|cccc|c} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right).$$

3 2fi-optimal 3^{n-1} and 5^{n-1} fractional factorial row-column designs for all (p,q) with $n \leq 9$ (corresponding to Theorem 3)

$p=1, q=2, n=4,\ s=3$ or $5,$

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right) = \left(\begin{array}{c|cc|c} 1 & 1 & 1 & 2 \\ \hline 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{array}\right).$$

$p=1, q=3, n=5,\ s=3$ or $5,$

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right) = \left(\begin{array}{c|ccc|c} 1 & 1 & 1 & 1 & 2 \\ \hline 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \end{array}\right).$$

$p=1, q=4, n=6,\ s=3$ or $5,$

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right) = \left(\begin{array}{c|cccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ \hline 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 \end{array}\right).$$

$p=1, q=5, n=7,\ s=3$ or $5,$

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right) = \left(\begin{array}{c|ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ \hline 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 \end{array}\right).$$

$p=1, q=6, n=8,\ s=3$ or $5,$

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r}\right) = \left(\begin{array}{c|cccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ \hline 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \end{array}\right).$$

$p = 1, q = 7, n = 9,\ s = 3$ or 5,

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r} \right) = \left(\begin{array}{c|ccccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ \hline 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \end{array} \right).$$

$p = 2, q = 2, n = 5, s = 3$,

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r} \right) = \left(\begin{array}{cc|cc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ \hline 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 & 1 \end{array} \right).$$

$p = 2, q = 2, n = 5, s = 5$,

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r} \right) = \left(\begin{array}{cc|cc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ \hline 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 & 3 \end{array} \right).$$

$p = 2, q = 3, n = 6, s = 3$,

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r} \right) = \left(\begin{array}{cc|cc|c|c} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 & 0 \\ \hline 1 & 0 & 2 & 1 & 2 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right).$$

$p = 2, q = 3, n = 6, s = 5$,

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r} \right) = \left(\begin{array}{cc|cc|c|c} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 4 & 3 \\ \hline 1 & 0 & 2 & 1 & 2 & 0 \\ 0 & 1 & 2 & 2 & 3 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right).$$

$p = 2, q = 4, n = 7, s = 3$,

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r} \right) = \left(\begin{array}{cc|cc|c|cc} 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 & 1 & 0 \\ \hline 1 & 0 & 2 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right).$$

$p = 2, q = 4, n = 7, s = 5$,

$$\mathcal{G} = \left(\frac{\mathcal{G}_c}{\mathcal{G}_r} \right) = \left(\begin{array}{cc|cc|c|cc} 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 4 & 3 & 2 \\ \hline 1 & 0 & 2 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 2 & 3 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right).$$

$p = 2, q = 5, n = 8, s = 3,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r} \right) = \left(\begin{array}{cc:cc:c:ccc} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 & 1 & 0 & 1 \\ \hline 1 & 0 & 2 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right).$$

$p = 2, q = 5, n = 8, s = 5,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r} \right) = \left(\begin{array}{cc:cc:c:ccc} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 4 & 3 & 0 & 1 \\ \hline 1 & 0 & 2 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 3 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right).$$

$p = 2, q = 6, n = 9, s = 3,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r} \right) = \left(\begin{array}{cc:cc:c:cccc} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 2 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right).$$

$p = 2, q = 6, n = 9, s = 5,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r} \right) = \left(\begin{array}{cc:cc:c:cccc} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 & 3 & 0 & 1 & 1 \\ \hline 1 & 0 & 2 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 3 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right).$$

$p = 3, q = 3, n = 7, s = 3,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r} \right) = \left(\begin{array}{ccc:ccc:c} 1 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right).$$

$p = 3, q = 3, n = 7, s = 5,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r} \right) = \left(\begin{array}{ccc:ccc:c} 1 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 3 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 & 3 & 1 & 2 \\ 1 & 0 & 0 & 1 & 1 & 3 & 1 \end{array} \right).$$

$p=3, q=4, n=8, s=3,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r}\right) = \left(\begin{array}{ccc:ccc:c:c} 1 & 0 & 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

$p=3, q=4, n=8, s=5,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r}\right) = \left(\begin{array}{ccc:ccc:c:c} 1 & 0 & 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 3 & 1 & 1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 3 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 & 3 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

$p=3, q=5, n=9, s=3,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r}\right) = \left(\begin{array}{ccc:ccc:c:cc} 1 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

$p=3, q=5, n=9, s=5,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r}\right) = \left(\begin{array}{ccc:ccc:c:cc} 1 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 3 & 1 & 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 1 & 3 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 3 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

$p=4, q=4, n=9, s=3,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r}\right) = \left(\begin{array}{cccc:cccc:c} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right).$$

$p=4, q=4, n=9, s=5,$

$$\boldsymbol{\mathcal{G}} = \left(\frac{\boldsymbol{\mathcal{G}}_c}{\boldsymbol{\mathcal{G}}_r}\right) = \left(\begin{array}{cccc:cccc:c} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 3 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 & 3 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 1 \end{array}\right).$$