Classification of Darboux Transformations for super KdV (An intrigue of Non Standard Differential Calculus)

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Introduction

- The Darboux Transformation is used for finding solutions of PDE's. We consider the super KdV (Korteweg-de Vries) equation and the Darboux transformation of the KdV.
- We have shown all Darboux Transformations of order one are only of a specific form, reminiscent to the classical case.
- Previously, Liu et al (95,97) found a family of solutions of this form to the super KdV but it was not shown that they were the only possible solutions.
- Our current work is on Darboux Transformations of order two with hopes to generalize for order n.

Problem Statement

The idea is that if we have some equation, $f_{xx} + u(x)f = 0$ we may write it as $L[f] = 0$ where $L = \partial_x^2 + u$, and we transform $L \to L_1$ where $L_1 = \partial_x^2 + u_1$ so that $NL = L_1 M$ for certain operators N and M.

It follows that N and M must be of the same order. What is known is a complete classification of the form of operators M that satisfy the conditions of the transformation in the classical case, using standard calculus. Our goal is to generalize to the supercase, where elements supercommute.

Plan of the talk

- Introduce superalgebra on simple examples
- Own results so far
- What is left to do

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Crash course in supermathematics

- Two types of elements, even elements (behave like standard variables x, y) and odd elements which commute but with a negative sign.
- If μ and α are odd elements, then $\mu\alpha = -\alpha\mu$.
- An even element m commutes with any other element, i.e. $m\beta = \beta m$ regardless of whether β is even or odd.
- $\alpha^2 = 0$ for all odd α, thus odd elements have no inverses.
- We use a specific odd differential operator $D = \partial_\tau + \tau\partial_x$ where τ is an odd variable and x is an even variable. This operator plays an important role in supermathematics.
- Leibniz rule, (or Product Rule), is the same up to a possible sign, i.e. $D(ab) = D[a]b + (-1)^{|a||D|}aD[b]$.

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Supercommutativity

Let $\mathbb{K}$ be a fixed commutative ring (in most applications $\mathbb{K}$ is a field such as $\mathbb{R}$ or $\mathbb{C}$). A **superalgebra** $\mathscr{A} = \mathscr{A}_0 \oplus \mathscr{A}_1$ over $\mathbb{K}$ is a $\mathbb{K}$-module $\mathscr{A}$ with direct sum $\oplus$ decomposition and a bilinear operator $\mathscr{A} \times \mathscr{A} \to A$ such that $\mathscr{A}_i\mathscr{A}_j = \mathscr{A}_{i+j}$ (mod 2). A superring $\mathbb{Z}_2$-graded ring is a superalgebra over the ring $\mathbb{Z}$. For all $a_i \in \mathscr{A}_i$, we say a_i is **homogenous**.

- If $a_0 \in \mathscr{A}_0$, then we define parity of a_0 as $P_{a_0} = |a_0| = 0$, and we say a_0 is even.
- If $a_1 \in \mathscr{A}_1$, then $P_{a_1} = |a_1| = 1$ and we say a_1 is odd.
- If a, b are homogenous, then $|ab| = |a| + |b|$.
- An associative superalgebra satisfies associativity under $(\cdot)$ and a unital superalgebra contains the multiplicative identity element. **Supercommutativity** is, for all homogenous $a, b \in A$, the property $ba = (-1)^{|a|\cdot|b|}ab$

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Lemma

If $D = \partial_\tau + \tau\partial_x$ (τ odd, x even) then $D^2 = \partial_x$.

Proof.

Since τ and ∂_τ are odd, x and ∂_x, we have

$$
\begin{aligned}
D^2 = (\partial_\tau + \tau\partial_x)^2 &= \partial_\tau^2 + \partial_\tau[\tau\partial_x] + \tau\partial_x\partial_\tau + \tau\partial_x\tau\partial_x && (1)\\
&= 0 + \partial_x - \tau\partial_\tau\partial_x + \tau\partial_x\partial_\tau + \tau(\partial_x[\tau] + \tau\partial_x^2) && (2)\\
&= \partial_x - \tau\partial_\tau\partial_x + \tau\partial_\tau\partial_x + \tau^2\partial_x^2 && (3)\\
&= \partial_x && (4)
\end{aligned}
$$

□

Peculiarities of supercalculus

Note D is odd but this differential operator doesn't commute in any sense. So $D^2 \neq 0$. However $\partial_\tau^2 = 0$, and ∂_x commutes as usual. We now provide some examples of simple calculations

- if α is odd then $\alpha^2 = 0$. D is odd but $D^2 = \partial_x$.
- $\underbrace{(\mu D)^2}_{\text{order 2?}} = \mu D \cdot \mu D = \underbrace{\mu D[\mu]}_{\text{function; order 0}}$
- If φ is odd then $\underbrace{(D+\varphi)^2}_{\text{order 2?}} = \underbrace{\partial_x + D[\varphi]}_{\text{order 1?}}$.
- $D[\varphi f] = D[\varphi]f - \varphi D[f]$, where φ is odd.

Our first result: Correct definition of Order

We need to introduce our own notion of order to make up for the fact that $D^2 = \partial_x$. (We rewrite every $\partial_x = D^2$ and consider only operators with even leading coefficient.)

First order Darboux Transformations of Super KdV

- Odd variables will be greek letters, α, β etc. and even variables will be latin, x, y etc. The following variables will all be functions of τ and x.
- Define the super derivative $D = \partial_\tau + \tau\partial_x$ as before.
- Let $N = D + \nu$, $M = D + \mu$,
- Let $L = \partial_x^2 + \alpha D + u$ (the operator form of the KdV, $\varphi_{xx} + \alpha D[\varphi] + u\varphi = 0$), and $L_1 = \partial_x^2 + \alpha_1 D + u_1$.

Our first result,

Lemma

Suppose

$$NL = L_1 M$$

then $N = M$.

After tedious calculation, we show

$$NL = L_1 M$$

$$\vdots$$

$$\mu_{xx} - 2\mu_x D[\mu] + D[\alpha\mu - u] = 0$$

We noticed this is equivalent to a full differential

$$D\left[D[\mu_x] - D[\mu]^2 + \alpha\mu - u\right] = 0$$

and hence

$$D[\mu_x] - D[\mu]^2 + \alpha\mu - u = \lambda$$

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$$D[\mu_x] - D[\mu]^2 + \alpha\mu - u = \lambda \tag{5}$$

The next step was to consider μ in the form of a logarithmic derivative of some even function f.

Lemma

Given a field $\mathbb{K}$ large enough to have coefficients for solutions of $f_\tau + \tau f_x = -\mu f^{-1}$. Every odd $\mu(x,\tau)$ can be represented in the form $\mu = -D[f]f^{-1}$ for a suitable even $f(x,\tau)$.

Note by this substitution we are not putting any restrictions on μ. Substituting this forμ into (5), we see that

$$f_{xx} + \alpha D[f] + uf = \lambda f \text{ i.e. } L[f] = \lambda f \tag{6}$$

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Global Context of Our Results

Current Results

- Previously, Liu (95) showed that, given $ML = L_1 M$ if $L[\varphi] = 0$ then $M = D - D[\varphi]\varphi^{-1}$ satisfies the transformation.
- Here we have shown that M can only be of this form.

Future

- Liu, Nimmo (2009) also found a family of operators M of arbitrary order, which define Darboux transformations of super KdV.
- Analogously, we would like to show that this is the only family of higher order.

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Future and Current Work

Our current goals are revisiting the classical case and seeing how we can generalize these results for order $n \geq 2$ in a way that can be readily adapted into the super case.

Thank you!