SSY281 Model Predictive Control

Assignment 3

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Question 1: Constrained optimization

a

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if $dom\, f$ is convex and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$
$$\forall x, y \in dom\, f \text{ and } 0 \leq \theta \leq 1$$

f is strictly convex if

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta)f(y)$$
$$\forall x, y \in dom\, f \text{ and } 0 < \theta < 1$$

b

A convex set contains line segments between every two points in the set i.e.,

$$x_1, x_2 \in S \implies \theta x_1 + (1-\theta)x_2 \in S, 0 \leq \theta \leq 1$$

c

The optimization problem becomes a convex optimization problem if

- Objective function, $f(x)$ is convex
- Inequality constraints, $g(x)$ is convex
- Equality constraints, $h(x)$ is affine

Question 2: Convexity

a

S_1 is convex.
Proof
We can prove this with the help of the property of convex sets that the intersection

of convex sets preserves the convexity.

$$S_1 = x \in \mathbb{R}^n | \alpha \leq a^T x \leq \beta$$
$$\text{Let } S_1 = S_{11} \cap S_{12}$$
$$\text{where } S_{11} = \{x \in \mathbb{R}^n | a^T x - \beta \leq 0\}$$
$$\text{and } S_{12} = \{x \in \mathbb{R}^n | -a^T x + \alpha \leq 0\}$$

Let

$$a^T x - \beta = f(x)$$
$$-a^T x + \alpha = g(x)$$

Consider two arbitrary points $x_1, x_2 \in f(x)$. Then, from the definition of convex functions,

$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$$
$$\implies a^T(\theta x_1 + (1-\theta)x_2) - \beta \leq \theta f(x_1) + (1-\theta) f(x_2)$$

We know from the definition of S_{11} that

$$f(x) \leq 0$$
$$\implies \theta f(x_1) \leq 0$$
$$\implies (1-\theta) f(x_2) \leq 0$$
$$\therefore a^T(\theta x_1 + (1-\theta)x_2) - \beta \leq 0$$
$$\implies \theta x_1 + (1-\theta)x_2 \in S_{11}$$

Thus all points in the line segment belong to S_{11}. Thus S_{11} is convex.
Now consider two arbitrary points $x_1, x_2 \in g(x)$. Then, from the definition of convex functions,

$$g(\theta x_1 + (1-\theta)x_2) \leq \theta g(x_1) + (1-\theta) g(x_2)$$
$$\implies -a^T(\theta x_1 + (1-\theta)x_2) + \alpha \leq \theta g(x_1) + (1-\theta) g(x_2)$$

We know from the definition of S_{12} that

$$g(x) \leq 0$$
$$\implies \theta g(x_1) \leq 0$$
$$\implies (1-\theta) g(x_2) \leq 0$$
$$\therefore -a^T(\theta x_1 + (1-\theta)x_2) + \alpha \leq 0$$
$$\implies \theta x_1 + (1-\theta)x_2 \in S_{12}$$

Thus all points in the line segment belong to S_{12}. Thus S_{12} is convex.
Since S_1 is the intersection of two convex sets, it is also convex.

b

S_2 is convex.
Proof

$$\begin{gathered} S_2 = \{x| \, \|x-y\| \leq f(y), \forall y \in S\}, S \subseteq \mathbb{R}^n, f(y) \geq 0 \\ \implies S_2 = \{x| \, \|x-y\| - f(y) \leq 0, \forall y \in S\} \end{gathered}$$

Let $\|x-y\| - f(y) = h(x)$. To apply the definition of convexity, we will consider two arbitrary points $x_1, x_2 \in S_2$.

$$h(\theta x_1 + (1-\theta)x_2) = \|\theta x_1 + (1-\theta)x_2)\| - f(y)$$

Applying triangle inequality,

$$\begin{gathered} h(\theta x_1 + (1-\theta)x_2) \leq \theta(\|x_1 - y\| - f(y)) + (1-\theta)(\|x_2 - y\| - f(y)) \\ \implies h(\theta x_1 + (1-\theta)x_2) \leq \theta h(x_1) + (1-\theta)h(x_2) \end{gathered}$$

S_2 satisfies the definition of convexity and is therefore convex.

c

Question 3: Norm problems as linear programs

a

Both optimization problems are minimizing the maximum absolute deviation between vectors Ax and b. In equation (3), we want to solve for the infinity norm, i.e., to find the value of x that minimizes the absolute maximum deviation between i^{th} component of Ax and b. In equation (5), we are minimizing the deviation when the i^{th} component of the deviation $(Ax-b)_i$ is not larger than ϵ. The optimal solution for both the equations is the same. Hence, they are equivalent.

b

Assume x^T is a column vector with length n. Then z^T will have length $n+1$. We want to minimize ϵ such that we can express the optimization problem in the form of equation (2), i.e. for $c^T = [0, 0, 0, \dots, 0, 0, 1]$,

$$\min_{[x^T\epsilon]} \quad c^T[x^T\epsilon]^T = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \epsilon \end{bmatrix} = \epsilon$$

To formulate the constraints,

$$\begin{gathered} Ax_i - b \leq \epsilon \quad \text{and} \\ -\epsilon \leq Ax_i - b \implies -Ax_i - \epsilon \leq -b \\ \implies \begin{bmatrix} A & -1 \\ -A & -1 \end{bmatrix} \begin{bmatrix} x_i \\ \epsilon \end{bmatrix} \leq \begin{bmatrix} b \\ -b \end{bmatrix} \end{gathered}$$

Thus, we can infer that

$$F_i = \begin{bmatrix} A & -1 \\ -A & -1 \end{bmatrix}, \quad z_i = \begin{bmatrix} x_i \\ \epsilon \end{bmatrix}, \quad g_i = \begin{bmatrix} b \\ -b \end{bmatrix}$$

When extended for the entire vector x, the matrices would be

$$F = \begin{bmatrix} A & 0 & 0 & \dots & 0 & -1 \\ -A & 0 & 0 & \dots & 0 & -1 \\ 0 & A & 0 & \dots & 0 & -1 \\ 0 & -A & 0 & \dots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & -1 \\ 0 & 0 & 0 & \dots & A & -1 \\ 0 & 0 & 0 & \dots & -A & -1 \end{bmatrix}, \quad g = \begin{bmatrix} b \\ -b \\ b \\ -b \\ \vdots \\ b \\ -b \end{bmatrix}$$
$$\implies Fz \leq g$$

It is to be noted that the 0s and -1s in the F matrix are vectors that fit the size of A.

c

The task is to minimize ϵ,

$$\min_{x,\epsilon} \quad c^T z \quad \text{st} \quad Fz \leq g$$
$$\text{where} \quad z = \begin{bmatrix} x \\ \epsilon \end{bmatrix}$$

As obtained in the previous problem,

$$c^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad F = \begin{bmatrix} A & -1 \\ -A & -1 \end{bmatrix}, \quad g = \begin{bmatrix} b \\ -b \end{bmatrix}$$

where 0's and 1's are matrices that satisfy the dimensionality of A. On solving using linprog in Matlab,

$$z = \begin{bmatrix} -2.0674 \\ -1.1067 \\ 0.4583 \end{bmatrix}$$

d

Given

$$\min_{x,\epsilon} \quad c^T z \quad \text{st} \quad Fz \leq g$$

The constraint can be rewritten as

$$Fz - g \leq 0$$

Applying the constraint on the objective function,

$$\begin{aligned} \mathcal{L}(z, \mu, \lambda) &= c^T z + \mu^T (Fz - g) + \lambda^T 0 \\ &= c^T z + \mu^T (Fz - g) \end{aligned}$$

$$q(\mu, \lambda) = \inf_z \quad \mathcal{L}(z, \mu, \lambda)$$

We define the dual function with the above Langrangian,

$$\begin{aligned} \max_\mu \quad & -g^T \mu \\ \text{st} \quad & \mu \geq 0 \\ & F^T \mu + c^T = 0 \end{aligned}$$

e

Solved for the above defined dual problem in Matlab and obtained μ as

$$\mu = \begin{bmatrix} 0 \\ 0 \\ 0.4095 \\ 0.4284 \\ 0 \\ 0.1621 \\ 00 \end{bmatrix}$$

f

Question 4: Quadratic programming

a

b

c