# SSY281 Model Predictive Control

## Assignment 2

Shilpa Mary Sudhir (shilpas)

February 13, 2024

# Question 1: Set-point tracking

## a

Here, the number of inputs = number of outputs, ($p = m$), square system.
Hence, we can solve for $(x_s, u_s)$ by substituting for the given system parameters and output set-point in the equation

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_{sp} \end{bmatrix}$$
$$\implies \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ y_{sp} \end{bmatrix}$$

On solving, we get

$$x_s = \begin{bmatrix} 0.1745 \\ 0.0722 \\ -3.1416 \\ 0.0008 \end{bmatrix}$$
$$u_s = \begin{bmatrix} -0.0153 \\ -0.1427 \end{bmatrix}$$

## b

Only the second control input is available i.e., number of inputs $<$ number of outputs, ($p < m$), underdetermined system. We need to find $x_s$ and $u_s$ such that it minimizes the 2-norm of the output error, ($|Cx_s - y_{sp}|^2_{Q_s}$). Solving using quadprog gives,

$$x_s = \begin{bmatrix} 0.0000 \\ 0.0000 \\ -3.1416 \\ 0 \end{bmatrix}$$
$$u_s = -1.2096e-13$$

With this control input, we cannot reach the desired set-point output $[\frac{\pi}{18} \ \ -\pi]^T$, but we can reach $y_s = [0\ \pi]^T$.

### c

Here, only one state is measurable. Number of inputs > number of controlled outputs, $p_z < m$, which means there may be multiple solutions. We want to find $x_s$ and $u_s$ that minimizes $(|u_s - u_{sp}|^2_{R_s} - |Cx_s - y_{sp}|^2_{Q_s})$. Solving using quadprog gives

$$x_s = \begin{bmatrix} 0.1745 \\ 0.0722 \\ 0 \\ 0.0008 \end{bmatrix}$$
$$u_s = \begin{bmatrix} -0.0153 \\ -0.1427 \end{bmatrix}$$

## Question 2: Control of a ball and wheel system

### a

We augment the plant model with the given load disturbance models as,

$$\begin{bmatrix} x(k+1) \\ d(k+1) \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix}$$

The augmented system with the first disturbance model is,

$$A_{e_1} = \begin{bmatrix} 1.0041 & 0.0100 & 0 & 0 & 0 \\ 0.8281 & 1.0041 & 0 & -0.0093 & 0 \\ 0.0002 & 0 & 1.0000 & 0.0098 & 0 \\ 0.0491 & 0.0002 & 0 & 0.9629 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix}, B_{e_1} = \begin{bmatrix} 0.0007 \\ 0.1398 \\ 0.0028 \\ 0.5605 \\ 0 \end{bmatrix}, C_{e_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The augmented system with the second disturbance model is,

$$A_{e_2} = \begin{bmatrix} 1.0041 & 0.0100 & 0 & 0 & 0 & 0 \\ 0.8281 & 1.0041 & 0 & -0.0093 & 0 & 0 \\ 0.0002 & 0 & 1.0000 & 0.0098 & 0 & 0 \\ 0.0491 & 0.0002 & 0 & 0.9629 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix}, B_{e_2} = \begin{bmatrix} 0.0007 \\ 0.1398 \\ 0.0028 \\ 0.5605 \\ 0 \\ 0 \end{bmatrix},$$

$$C_{e_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The augmented system with the third disturbance model is,

$$A_{e_3} = \begin{bmatrix} 1.0041 & 0.0100 & 0 & 0 & 0 & 0.0007 \\ 0.8281 & 1.0041 & 0 & -0.0093 & 0 & 0.1398 \\ 0.0002 & 0 & 1.0000 & 0.0098 & 0 & 0.0028 \\ 0.0491 & 0.0002 & 0 & 0.9629 & 0 & 0.5605 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix}, B_{e_3} = \begin{bmatrix} 0.0007 \\ 0.1398 \\ 0.0028 \\ 0.5605 \\ 0 \\ 0 \end{bmatrix},$$

$$C_{e_3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

For detectability, the following conditions must be satisfied,

$$rank(\mathcal{O}(A, C)) = n$$

$$rank \begin{bmatrix} I - A & -B_d \\ C & C_d \end{bmatrix} = n + n_d$$

where $n$ and $n_d$ are the number of states and number of disturbances respectively. It is observed that the augmented systems for disturbance models 1 and 3 are full rank and hence are detectable. The augmented system with the disturbance model 2 is rank deficit and is hence not detectable.

## b

We design Kalman filters for the detectable systems with disturbance models 1 and 3. The Kalman filter gain, $L$ and the prediction error covariance, $P$ are given by

$$L = PC^T[CPC^T + R]^{-1}$$

$$P = APA^T - APC^T[CPC^T + R]^{-1}CPA^T + Q$$

Given $Q$ and $R$ are identity and substituting the augmented models, the Kalman filter gains are for models 1 and 3 respectively are,

$$L_{e_1} = \begin{bmatrix} 1.1835 & 9.7171 & 0.4284 & 0.4089 & -0.3508 \\ -0.4358 & -3.5104 & 0.2323 & -0.1008 & 0.4763 \end{bmatrix}$$

$$L_{e_3} = \begin{bmatrix} 1.3594 & 11.5489 & 0.0140 & 0.3197 & -0.4815 & -0.0102 \\ 0.0128 & 0.3198 & 0.2478 & 2.0989 & -0.0090 & 0.5109 \end{bmatrix}$$

## c

From lecture notes section 5.2 we have

$$\begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} B_d \hat{d} \\ z_{sp} - HC_d \hat{d} \end{bmatrix}$$

Rearranging and substituting $z_{sp} = 0$ to the form of $M_{ss}$,

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix}^{-1} \begin{bmatrix} B_d \\ -HC_d \end{bmatrix} \hat{d}$$

$$\implies M_{ss} = \begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix}^{-1} \begin{bmatrix} B_d \\ -HC_d \end{bmatrix}$$

Thus, we have $M_{ss}$ for the detectable systems 1 and 3 respectively as,

$$M_{ss_1} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, M_{ss_3} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

## d