1. Suppose that we perform an experiment to calculate the gravitational constant $g$ as follows. We drop a ball from a certain height and measure its distance from the original point at certain time instants. The results of the experiment are shown in the following table.

| Time (s) | 1.00 | 2.00 | 3.00 |
|---|---|---|---|
| Distance (m) | 5.00 | 19.5 | 44.0 |

The equation relating the distance $s$ and the time $t$ at which $s$ is measured is given by

$$s = \frac{1}{2} g t^2.$$

(a) Find a least-squares estimate of $g$ using the experimental results from the table above.

(b) Suppose that we take an additional measurement at time 4.00 and obtain a distance of 78.5. Use the RLS algorithm to calculate an updated least-squares estimate of $g$.

**Solution:**
Recall that Theorem 12.1 states that the unique vector $x^*$ that minimizes $||Ax - b||^2$ is given by the solution to the equation

$$A^T A x = A^T b \implies x^* = (A^T A)^{-1} A^T b$$

To solve this problem, we must first write it in the form of a least squares problem.

$$s = \frac{t^2}{2} g$$

$$A = \begin{bmatrix} 1/2 \\ 2 \\ 9/2 \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ 19.5 \\ 44 \end{bmatrix}$$

So the least squares estimate of g is now given by

$$g^* = (\begin{bmatrix} 1/2 & 2 & 9/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 2 \\ 9/2 \end{bmatrix})^{-1} \begin{bmatrix} 1/2 & 2 & 9/2 \end{bmatrix} \begin{bmatrix} 5 \\ 19.5 \\ 44 \end{bmatrix} = 9.776$$

2. Let $[x_1, y_1]^\top, \dots, [x_p, y_p]^\top$, $p \geq 2$, be points in $\mathbb{R}^2$. We wish to find the straight line of best fit through these points ("best" in the sense that the total squared error is minimized); that is, we wish to find $a^*, b^* \in \mathbb{R}$ to minimize

$$f(a,b) = \sum_{i=1}^{p} (ax_i + b - y_i)^2.$$

Assume that the $x_i$, $i = 1, \dots, p$, are not all equal. Show that there exist unique parameters $a^*$ and $b^*$ for the line of best fit, and find the parameters in terms of the following quantities:

$$\overline{X} = \frac{1}{p}\sum_{i=1}^{p} x_i, \quad \overline{Y} = \frac{1}{p}\sum_{i=1}^{p} y_i, \quad \overline{X^2} = \frac{1}{p}\sum_{i=1}^{p} x_i^2,$$
$$\overline{Y^2} = \frac{1}{p}\sum_{i=1}^{p} y_i^2, \quad \overline{XY} = \frac{1}{p}\sum_{i=1}^{p} x_i y_i.$$

**Solution:**

We can rewrite the problem as:

$$\begin{aligned} f(a,b) &= \sum_{i=1}^{p} (ax_i + b - y_i)^2 \\ &= || \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_p & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} ||^2 = ||Ax - b||^2 \end{aligned}$$

By Theorem 12.1, the matrix A must be invertible. Since $x_i$'s are distinct, the columns of the matrix are linearly independent, making it full rank and thus invertible.
Therefore, there exists a unique pair of parameters $a^*, b^*$ that minimizes the total squared error.
Now, let's apply theorem 12.1 to find the mimimizer $x^*$

$$x^* = (A^T A)^{-1} A^T b$$

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_p & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

$$x^* = (\begin{bmatrix} \overline{X^2} & \overline{X} \\ \overline{X} & 1 \end{bmatrix})^{-1} \begin{bmatrix} \overline{XY} \\ \overline{Y} \end{bmatrix}$$

$$\begin{bmatrix} a^* \\ b^* \end{bmatrix} = \frac{1}{\overline{X^2} - \overline{X}^2} \begin{bmatrix} \overline{XY} - (\overline{X})(\overline{Y}) \\ \overline{X^2}\overline{Y} - (\overline{X})\overline{XY} \end{bmatrix}$$

3. We are given a point $[x_0, y_0]^\top \in \mathbb{R}^2$. Consider the straight line on the $\mathbb{R}^2$ plane given by the equation $y = mx$. Using a least-squares formulation, find the point on the straight line that is closest to the given point $[x_0, y_0]$, where the measure of closeness is in terms of the Euclidean norm on $\mathbb{R}^2$.
   **Hint:** The given line can be expressed as the range of the matrix $A = [1, m]^\top$.

**Solution:**
By the hint, we know that every point on the line can be expressed as

$$v = cA = c\begin{bmatrix}1\\m\end{bmatrix}, c \in \mathbb{R}$$

We can now transform the given problem into the form $||Ax - b||^2$, and apply Theorem 12.1
The measure of closeness is in terms of the Euclidean norm on $\mathbb{R}^2$, which is given by

$$||x - x_0||^2 = || \begin{bmatrix}1\\m\end{bmatrix} c - \begin{bmatrix}x_0\\y_0\end{bmatrix} ||^2$$

$$\begin{aligned}
c^* &= (A^T A)^{-1} A^T b \\
A &= \begin{bmatrix}1\\m\end{bmatrix} \\
b &= \begin{bmatrix}x_0\\y_0\end{bmatrix} \\
c^* &= \frac{x_0 + y_0 m}{1 + m^2} \\
v^* = c^* A &= \begin{bmatrix}\frac{x_0+y_0 m}{1+m^2}\\ \frac{m(x_0+y_0 m)}{1+m^2}\end{bmatrix}
\end{aligned}$$

4. Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $m \geq n$, and $\text{rank}A = n$. Consider the constrained optimization problem

$$\text{minimize } \frac{1}{2}x^\top x - x^\top b \quad \text{subject to } x \in R(A),$$

   where $R(A)$ denotes the range of $A$. Derive an expression for the global minimizer of this problem in terms of $A$ and $b$.

**Solution:**
We can rewrite the problem as a least squares problem, and then apply Theorem 12.1
Observe that

$$\frac{1}{2}x^\top x - x^\top b = \frac{1}{2}||x - b||^2$$

And if $x \in R(A)$, then $\exists y \in \mathbb{R}^n$ such that $x = Ay$
The problem is now to find the unique vector $x^*$ that minimizes: $\frac{1}{2}||Ay - b||^2$

$$y^* = (A^T A)^{-1} A^T b$$
$$x^* = Ay^* = A\left(A^T A\right)^{-1} A^T b$$

Note: another interesting way to solve this problem is to express the objective function in terms of $y$, and minimize it by setting the gradient to 0 and solving for $y^*$.

5. Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $m \leq n$, $\text{rank} A = m$, and $x_0 \in \mathbb{R}^n$. Consider the problem

$$\text{minimize } \|x - x_0\| \quad \text{subject to } Ax = b.$$

Show that this problem has a unique solution given by

$$x^* = A^\top \left(AA^\top\right)^{-1} b + \left(I_n - A^\top \left(AA^\top\right)^{-1} A\right) x_0.$$

**Solution:**
To solve this, we will rewrite the problem as a system of linear equations minimize $||x||$ subject to $Ax = b$. We can then apply Theorem 12.2 Let

$$y = x - x_0$$

$$Ax = A(y + x_0) = b \implies Ay = b + Ax_0$$

Then the problem is now to find the unique vector $y^*$ that minimizes: $||Ay = b - Ax_0||$
By Theorem 12.1, the unique solution $x^*$ to $Ax = b$ that minimizes the norm $||x||$ is given by

$$x^* = A^T(AA^T)^{-1}b$$

Applying this theorem to the system $Ay = b - Ax_0$, we find that

$$\begin{aligned}
y^* &= A^T(AA^T)^{-1}(b + Ax_0) \\
x^* &= y^* + x_0 = A^T(AA^T)^{-1}(b + Ax_0) + x_0 \\
&= A^T(AA^T)^{-1}b + A^T(AA^T)^{-1}Ax_0 + x_0 \\
&= A^T(AA^T)^{-1}b + (I_n - A^T(AA^T)^{-1}A)x_0
\end{aligned}$$

6. Prove the following properties of generalized inverses:
   (a) $(A^\top)^\dagger = (A^\dagger)^\top$.
   (b) $(A^\dagger)^\dagger = A$.

**Solution:**

(a) Recall that the pseudoinverse, $A^\dagger \in \mathbb{R}^{mxn}$ of $A$ can be defined as

$$A^\dagger = (A^T A)^{-1} A^T = (AA^T)^{-1} A^T$$

So, using this definition, we have:

$$\begin{aligned}
(A^T)^\dagger &= (AA^T)^{-1} A^T \\
(A^\dagger)^T &= (AA^T)^{-1} A^T \\
(A^T)^\dagger &= (A^\dagger)^T
\end{aligned}$$

(b)