1. Suppose that we perform an experiment to calculate the gravitational constant g as follows. We drop a ball from a certain height and measure its distance from the original point at certain time instants. The results of the experiment are shown in the following table.

Time (s)	1.00	2.00	3.00
Distance (m)	5.00	19.5	44.0

The equation relating the distance s and the time t at which s is measured is given by

$$s = \frac{1}{2}gt^2.$$

(a) Find a least-squares estimate of g using the experimental results from the table above.

(b) Suppose that we take an additional measurement at time 4.00 and obtain a distance of 78.5. Use the RLS algorithm to calculate an updated least-squares estimate of g.

2. Let $[x_1, y_1]^\top, \ldots, [x_p, y_p]^\top$, $p \geq 2$, be points in $\mathbb{R}^2$. We wish to find the straight line of best fit through these points ("best" in the sense that the total squared error is minimized); that is, we wish to find $a^*, b^* \in \mathbb{R}$ to minimize

$$f(a, b) = \sum_{i=1}^{p} (ax_i + b - y_i)^2.$$

Assume that the x_i, $i = 1, \ldots, p$, are not all equal. Show that there exist unique parameters a^* and b^* for the line of best fit, and find the parameters in terms of the following quantities:

$$\overline{X} = \frac{1}{p}\sum_{i=1}^{p} x_i, \quad \overline{Y} = \frac{1}{p}\sum_{i=1}^{p} y_i, \quad \overline{X^2} = \frac{1}{p}\sum_{i=1}^{p} x_i^2,$$

$$\overline{Y^2} = \frac{1}{p}\sum_{i=1}^{p} y_i^2, \quad \overline{XY} = \frac{1}{p}\sum_{i=1}^{p} x_i y_i.$$

Solution:

Recall that Theorem 12.1 states that the unique vector x^* that minimizes $||Ax - b||^2$ is given by the solution to the equation

$$A^T Ax = A^T b \implies x^* = (A^T A)^{-1} A^T b$$

We can rewrite the problem as:

$$\begin{aligned} f(a, b) &= \sum_{i=1}^{p} (ax_i + b - y_i)^2 \\ &= || \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_p & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} ||^2 = ||Ax - b||^2 \end{aligned}$$

By Theorem 12.1, the matrix A must be invertible. Since x_i's are distinct, the columns of the matrix are linearly independent, making it full rank and thus invertible.
Therefore, there exists a unique pair of parameters a^*, b^* that minimizes the total squared error.
Now, let's apply theorem 12.1 to find the mimimizer x^*

$$x^* = (A^T A)^{-1} A^T b$$

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_p & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

$$x^* = (\begin{bmatrix} \overline{X^2} & \overline{X} \\ \overline{X} & 1 \end{bmatrix})^{-1} \begin{bmatrix} \overline{XY} \\ \overline{Y} \end{bmatrix}$$

$$\begin{bmatrix} a^* \\ b^* \end{bmatrix} = \frac{1}{\overline{X^2} - \overline{X}^2} \begin{bmatrix} \overline{XY} - (\overline{X})(\overline{Y}) \\ \overline{X^2}\overline{Y} - (\overline{X})\overline{XY} \end{bmatrix}$$

3. We are given a point $[x_0, y_0]^\top \in \mathbb{R}^2$. Consider the straight line on the $\mathbb{R}^2$ plane given by the equation $y = mx$. Using a least-squares formulation, find the point on the straight line that is closest to the given point $[x_0, y_0]$, where the measure of closeness is in terms of the Euclidean norm on $\mathbb{R}^2$.

   Hint: The given line can be expressed as the range of the matrix $A = [1, m]^\top$.

4. Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $m \geq n$, and $\text{rank} A = n$. Consider the constrained optimization problem

$$\text{minimize } \frac{1}{2} x^\top x - x^\top b \quad \text{subject to } x \in R(A),$$

   where $R(A)$ denotes the range of A. Derive an expression for the global minimizer of this problem in terms of A and b.

5. Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $m \leq n$, $\text{rank} A = m$, and $x_0 \in \mathbb{R}^n$. Consider the problem

$$\text{minimize } \|x - x_0\| \quad \text{subject to } Ax = b.$$

   Show that this problem has a unique solution given by

$$x^* = A^\top \left(AA^\top\right)^{-1} b + \left(I_n - A^\top \left(AA^\top\right)^{-1} A\right) x_0.$$

6. Prove the following properties of generalized inverses:

   (a) $(A^\top)^\dagger = (A^\dagger)^\top$.

   (b) $(A^\dagger)^\dagger = A$.