Math 164 HW 2

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1. Let $f(x) = x^2 + 4\cos x, x \in \mathbb{R}$. We wish to find the minimizer x^* of f over the interval $[1, 2]$. (Calculator users: Note that in $\cos x$, the argument x is in radians.)

(a) Plot $f(x)$ versus x over the interval $[1, 2]$.

(b) Use the golden section method to locate x^* to within an uncertainty of 0.2. Display all intermediate steps using a table:

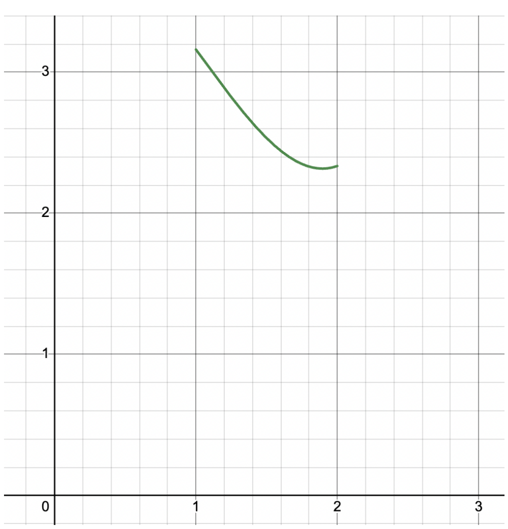
Iteration k	a_k	b_k	$f(a_k)$	$f(b_k)$	New uncertainty interval
1	?	?	?	?	[?, ?]
2	?	?	?	?	[?, ?]
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

(c) Repeat part (b) using the bisection method, with $\epsilon = 0.05$. Display all intermediate steps using a table.

(d) Apply Newton's method, using the same number of iterations as in part (b), with $x^{(0)} = 1$.

Solution:

(a)



(b)

Iteration k	a_k	b_k	$f(a_k)$	$f(b_k)$	New uncertainty interval
1	1.618	1.382	2.429	2.661	[1.382, 2]
2	1.764	1.618	2.343	2.430	[1.618, 2]
3	1.854	1.764	2.320	2.434	[1.764, 2]
4	1.910	1.854	2.317	2.320	[1.854, 2]

We found x* in 4 iterations, it is located in the interval [1.854, 2]

(c) $f(x) = x^2 + 4cos(x) \implies f'(x) = 2x - 4sin(x)$

Iteration k	a_k	b_k	m	$f'(m)$	New uncertainty interval
1	1	2	1.5	-0.989	[1.5, 2]
2	1.5	2	1.75	-0.436	[1.75, 2]
3	1.75	2	1.875	-0.066	[1.875, 2]
4	1.875	2	1.938	0.140	[1.875, 1.938]
5	1.875	1.938	1.906	0.036	[1.875, 1.906]

(d) Newton's Method : $x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$
$f''(x) = 2 - 4cos(x)$

Iteration k	x_n	x_{n+1}
1	1	-7.473
2	-7.473	14.479
3	14.479	6.935
4	6.935	16.636

2. Suppose that $\rho_1, \dots, \rho_N$ are the values used in the Fibonacci search method. Show that for each $k = 1, \dots, N$, $0 \leq \rho_k \leq \frac{1}{2}$, and for each $k = 1, \dots, N-1$,

$$\rho_{k+1} = 1 - \frac{\rho_k}{1 - \rho_k}.$$

Recall that $\rho_k = 1 - \frac{F_{N-k+1}}{F_{N-k+2}}$. Since F_N represents a Fibonacci number, which is always positive by the definition of the Fibonacci sequence, we know that $\frac{F_{N-k+1}}{F_{N-k+2}} > 0$ for all N. Moreover, since by the definition of the Fibonacci sequence, $F_{N-k+1} + F_{N-k} = F_{N-k+2}$, we also know that $F_{N-k+2} > F_{N-k+1}$, and thus $\frac{F_{N-k+1}}{F_{N-k+2}} < 1$, which implies $0 < \rho_k$.
To show that $\rho_k < \frac{1}{2}$, we use the fact that the ratio of consecutive Fibonacci numbers converges to the golden ratio $\phi \approx 1.618$:

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi.$$

Thus, for large N, we have:

$$\rho_k = 1 - \frac{F_{N-k+1}}{F_{N-k+2}} \approx 1 - \frac{1}{\phi}.$$

Since $\phi \approx 1.618$, we have:

$$\frac{1}{\phi} \approx 0.618,$$

and therefore:

$$\rho_k \approx 1 - 0.618 = 0.382 < \frac{1}{2}.$$

Thus, we conclude that $0 < \rho_k < \frac{1}{2}$ for all k.

Now, We want to show that

$$\rho_{k+1} = 1 - \frac{\rho_k}{1-\rho_k}$$

We know that for the Fibonacci search method,

$$\rho_k = 1 - \frac{F_{N-k-1}}{F_{N-k+2}} \implies 1 - \rho_k = \frac{F_{N-k+1}}{F_{N-k+2}}$$

and

$$\rho_{k+1} = 1 - \frac{F_{N-k}}{F_{N-k+1}}(*)$$

Now, to prove the recurrence relation, it will suffice to show that

$$1 - \frac{\rho_k}{1-\rho_k} = 1 - \frac{F_{N-k}}{F_{N-k+1}} \implies \frac{\rho_k}{1-\rho_k} = \frac{F_{N-k}}{F_{N-k+1}}$$

$$\frac{\rho_k}{1-\rho_k} = \frac{F_{N-k}}{F_{N-k+1}} \tag{1}$$

$$(\rho_k)(F_{N-k+1}) = (1-\rho_k)(F_{N-k}) \tag{2}$$

$$(1 - \frac{F_{N-k-1}}{F_{N-k+2}})(F_{N-k+1}) = (\frac{F_{N-k+1}}{F_{N-k+2}})(F_{N-k}) \tag{3}$$

$$F_{N-k+2} - F_{N-k-1} * F_{N-k+1} = F_{N-k+1} * F_{N-k} \tag{4}$$

$$F_{N-k} * F_{N-k+1} = F_{N-k+1} * F_{N-k} \tag{5}$$

We go from (4) to (5) because by the Fibonacci Sequence definition, $F_{N-k+1} + F_{N-k} = F_{N-k+2}$.
Since we know $(*)$, we have shown for each $k = 1, \ldots, N-1$, $\rho_{k+1} = 1 - \frac{\rho_k}{1-\rho_k}$

3. Suppose that we have an efficient way of calculating exponentials. Based on this, use Newton's method to devise a method to approximate $log(2)$ [where "log" is the natural logarithm function]. Use an initial point of $x^{(0)} = 1$, and perform two iterations.

Newton's Method : $x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$ To use Newton's Method to approximate $ln(2)$, we will rewrite this as a "finding the minimum" problem, where our minimum, x^* is at $ln(2)$
Let $f(x) = (e^x - 2)^2$, the minimum is at $f(x) = 0 \implies x^* = ln(2)$
To approximate the minimum, x^*, let's apply Newton's method on $f(x)$

Iteration k	x_n	x_{n+1}
1	1	0.838
2	0.838	0.752

After 2 iterations of Newton's method, we conclude that $log(2) \approx 0.751$

4. Consider the problem of finding the zero of $g(x) = (e^x - 1)/(e^x + 1)$, $x \in \mathbb{R}$, where e^x is the exponential of x. (Note that 0 is the unique zero of g.)

(a) Write down the algorithm for Newton's method of tangents applied to this problem. Simplify using the identity $\sinh x = (e^x - e^{-x})/(2)$.

(b) Find an initial condition $x^{(0)}$ such that the algorithm cycles [i.e. $x^{(0)} = x^{(2)} = x^{(4)} = \cdots$]. You need not explicitly calculate the initial condition; it suffices to provide an equation that the initial condition must satisfy. *Hint:* Draw a graph of g.

(c) For what values of the initial condition does the algorithm converge?

(a.) Newton's method for tangents is $x^{k+1} = x^k - \frac{g(x^k)}{g'(x^k)}$

$$g(x) = \frac{(e^x - 1)}{(e^x + 1)} \tag{6}$$

$$g'(x) = \frac{2e^x}{(e^x + 1)^2} \tag{7}$$

$$\frac{g(x)}{g'(x)} = \frac{(e^x - 1)(e^x + 1)}{2e^x} \tag{8}$$

$$= \frac{1}{2}(e^x - e^{-x}) = sinh(x) \tag{9}$$

$$x^{k+1} = x^k - sinh(x) \tag{10}$$

(b.)

$$x^0 = x^2 \tag{11}$$

$$x^1 = x^0 - sinh(x^0) \tag{12}$$

$$x^2 = x^1 - sinh(x^1) \tag{13}$$

$$x^0 = (x^0 - sinh(x^0)) - sinh((x^0 - sinh(x^0))) \tag{14}$$

$$0 = (-sinh(x^0)) - sinh((x^0 - sinh(x^0))) \tag{15}$$

The initial condition, x^0, must satisfy equation (15)

(c.) For the algorithm to converge, the initial condition must not cycle.
$\forall x > 0, x < sinh(x)$ and $\forall x < 0, x > sinh(x)$
By Newton's method, we have $x^{k+1} = x^k - sinh(x)$
So, for $x > 0$, $|x - sinh(x)| < x \implies -x < x - sinh(x) \implies sinh(x) < 2x$

And conversely, for $x < 0 \implies sinh(x) > 2x$
The algorithm is likely to converge in the interval between the values $\pm(\frac{sinh(x)}{x} = 2)$, this is approximately the interval $[-2.177, 2.177]$

5. Consider using a gradient algorithm to minimize the function

$$f(x) = \frac{1}{2}x^\top \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x$$

with the initial guess $x^{(0)} = \begin{bmatrix} 0.8 \\ -0.25 \end{bmatrix}^\top$.

(a) To initialize the line search, apply the bracketing procedure in Figure 7.11 (in the book) along the line starting at $x^{(0)}$ in the direction of the negative gradient. Use $\epsilon = 0.075$.

(b) Apply the golden section, Fibonacci or bisection method to reduce the width of the uncertainty region to 0.01. Organize the results of your computation in a table format similar to that of Exercise 2.

(a) Let's first calculate the negative gradient. Recall that a quadratic function in the form $f(x) = x^\top Qx$, has a gradient of the form $\nabla f(x) = 2Qx$.

$$\nabla f(x^{(0)}) = 2Qx^{(0)} \tag{16}$$

$$\nabla f(x^{(0)}) = 2\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} 0.8 \\ -0.25 \end{bmatrix} = \begin{bmatrix} 1.35 \\ 0.3 \end{bmatrix} \tag{17}$$

$$x_{k+1} = x_k - 2^k \epsilon \nabla f(x^{(0)}) \tag{18}$$

$$x_{k+1} = \begin{bmatrix} 0.8 \\ -0.25 \end{bmatrix} - 0.075\begin{bmatrix} 1.35 \\ 0.3 \end{bmatrix} \tag{19}$$

Using Python, I computed a few iterations of the bracketing procedure to calculate the desired bracket.

```
(base) Shilpa2@MacBook-Pro-10 Math164 % python3 bracketing.py
Step 0: x = [ 0.8  -0.25], f(x) = 0.5025000000000001
Step 1: x = [ 0.69875 -0.2725 ], f(x) = 0.37209843750000005
Step 2: x = [ 0.49625 -0.3175 ], f(x) = 0.18951093749999998
Step 3: x = [ 0.09125 -0.4075 ], f(x) = 0.1371984375
Step 4: x = [-0.71875 -0.5875 ], f(x) = 1.2840234375000001
```

From these results, we can observe that the desired bracket is between iterations 2 through 4, where $a = x_2 = [0.496, -0.318], c = x_3 = [0.091, -0.408], b = x_4 = [-0.719, -0.588]$.
(b) Let's use the Golden Section method.

Using Python, I obtained these results:

```
Running Golden Section Method
0: a_k = [ 0.49625 -0.3175 ], b_k = [-0.71875 -0.5875 ], f(a_k) = 0.3289, f(b_k) = 0.8456, tolerance = 1.2446
1: a_k = [ 0.49625 -0.3175 ], b_k = [-0.2546613  -0.48436918], f(a_k) = 0.2187, f(b_k) = 0.3289, tolerance = 0.7692
2: a_k = [ 0.49625 -0.3175 ], b_k = [ 0.0321613  -0.42063082], f(a_k) = 0.2306, f(b_k) = 0.2187, tolerance = 0.4754
3: a_k = [ 0.31898389 -0.35689247], b_k = [ 0.0321613  -0.42063082], f(a_k) = 0.2187, f(b_k) = 0.2419, tolerance = 0.2938
4: a_k = [ 0.31898389 -0.35689247], b_k = [ 0.14171778 -0.39628494], f(a_k) = 0.2160, f(b_k) = 0.2187, tolerance = 0.1816
5: a_k = [ 0.31898389 -0.35689247], b_k = [ 0.20942741 -0.38123835], f(a_k) = 0.2188, f(b_k) = 0.2160, tolerance = 0.1122
6: a_k = [ 0.27713704 -0.36619177], b_k = [ 0.20942741 -0.38123835], f(a_k) = 0.2160, f(b_k) = 0.2160, tolerance = 0.0694
7: a_k = [ 0.25127426 -0.37193905], b_k = [ 0.20942741 -0.38123835], f(a_k) = 0.2160, f(b_k) = 0.2166, tolerance = 0.0429
8: a_k = [ 0.25127426 -0.37193905], b_k = [ 0.22541148 -0.37768634], f(a_k) = 0.2159, f(b_k) = 0.2160, tolerance = 0.0265
9: a_k = [ 0.25127426 -0.37193905], b_k = [ 0.23529018 -0.37549107], f(a_k) = 0.2159, f(b_k) = 0.2159, tolerance = 0.0164
10: a_k = [ 0.24516889 -0.3732958 ], b_k = [ 0.23529018 -0.37549107], f(a_k) = 0.2159, f(b_k) = 0.2159, tolerance = 0.0101
11: a_k = [ 0.24516889 -0.3732958 ], b_k = [ 0.23906351 -0.37465255], f(a_k) = 0.2159, f(b_k) = 0.2159, tolerance = 0.0063
12: a_k = [ 0.24516889 -0.3732958 ], b_k = [ 0.24139556 -0.37413432], f(a_k) = 0.2159, f(b_k) = 0.2159, tolerance = 0.0039
13: a_k = [ 0.2437276  -0.37361609], b_k = [ 0.24139556 -0.37413432], f(a_k) = 0.2159, f(b_k) = 0.2159, tolerance = 0.0024
14: a_k = [ 0.2437276  -0.37361609], b_k = [ 0.24228632 -0.37393637], f(a_k) = 0.2159, f(b_k) = 0.2159, tolerance = 0.0015
```