Foundations of data science, summer 2020

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5. Exercise sheet
Hand in solutions until Thursday, 21 May 2020, 12:00

Exercise 5.1 (Eigenvectors with different eigenvalues). (4 points)

Show that if A is a symmetric matrix and α and α' are distinct eigenvalues then their corresponding eigenvectors $|x\rangle$ and $|x'\rangle$ are orthogonal. $\boxed{4}$

Solution. By choice, we have $A\,|x\rangle = \alpha\,|x\rangle$ and $A\,|x'\rangle = \alpha\,|x'\rangle$.

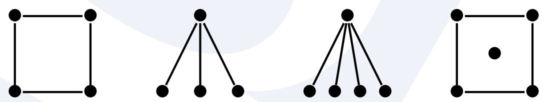
Always $\langle x'|\,A\,|x\rangle = \langle x|\,A^T\,|x'\rangle$ (in the real setting). Since A is symmetric we get

$$\begin{array}{l} \langle x'|\,A\,|x\rangle = \alpha\,\langle x' \,|\, x\rangle \\ \quad \parallel \\ \langle x|\,A\,|x'\rangle = \alpha'\,\langle x \,|\, x'\rangle\,. \end{array}$$

Then $(\alpha - \alpha')\,\langle x' \,|\, x\rangle = 0$. Since $\alpha' \neq \alpha$ we infer $\langle x' \,|\, x\rangle = 0$, ie. $|x\rangle$ and $|x'\rangle$ are orthogonal.

Exercise 5.2 (Eigenvalues of graphs). (10+2 points)

(i) What are the eigenvalues of (the adjacency matrices of) the graphs shown below? What does this say about using eigenvalues to determine whether two graphs are isomorphic? $\boxed{8}$



Solution. The adjacency matrices are not unique, hence there are many possible ways to write them down. One is:

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

with eigenvalues $-2, 0, 2$ for the first graph,

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

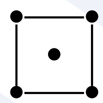
with eigenvalues $-\sqrt{3}, 0, \sqrt{3}$ for the second graph and

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with eigenvalues $-2, 0, 2$ for the third graph.

As we can see, the first and third graph are not isomorphic (they even don't have the same numbers of vertices and edges). But their eigenvalues agree. So eigenvalues can not fully determine if two graphs are isomorphic. Nevertheless, two isomorphic graphs will have the same eigenvalues, so graphs with different eigenvalues will not be isomorphic. $\bigcirc$

The characeristic polynomial of the third example is $T^3(T-2)(T+2)$, ie. 0 is a triple eigenvalue and the eigenvalues -2 and 2 are simple. Looking a litte further we find the 5-vertex graph



which has the same characteristic polynomial. So even worse than before: here the eigenvalues of the two graphs coincide including multiplicities. Still they are not isomorphic: one has a cycle and an isolated vertex, the other is a tree.

2+2

(ii) Let A be the adjacency matrix of an undirected, k-regular graph G, ie. each vertex hase degree k. Prove that the largest eigenvalue α_0 of A is equal to k.

Hint: It is easy to name an eigenvector for eigenvalue k.

Solution. The vector $|v\rangle = |(1, 1, 1, \ldots, 1)\rangle$ is an eigenvector for eigenvalue k, as each row of the adjacency matrix contains k entries that are 1 and all other entries are 0. So by multiplying with $|v\rangle$, we sum up all entries of each row to obtain the new vector, which is then $k \cdot |v\rangle$. Hence k is an eigenvalue of A. It remains to show, that k is the largest eigenvalue.

Assume λ is an eigenvalue of the adjacency matrix and $|x\rangle$ the corresponding eigenvector, ie. $A|x\rangle = \lambda|x\rangle$. Let x_m be the component of $|x\rangle$ such that the absolute value $|x_m|$ is maximal. We have

$$\begin{aligned}
\lambda|x_m| = |(A|x\rangle)_m| &= \left|\sum_{i=1}^{n} A_{m,i}x_i\right| \\
&\leq \sum_{i=1}^{n} A_{m,i}|x_i| \\
&\leq \sum_{i=1}^{n} A_{m,i}|x_m| \\
&= |x_m| \cdot \sum_{i=1}^{n} A_{m,i} = k|x_m|
\end{aligned}$$

by the triangle inequality and the definition of x_m. So we can conclude that $\lambda \leq k$ and hence k is indeed the largest possible eigenvalue. ◯

Exercise 5.3 (Separating balls). (0 points)

Consider a unit ball A centered at the origin and a unit ball B whose center is at distance s from A. Suppose that a random point x is drawn from the mixture distribution: "with probability $\frac{1}{2}$, draw at random from A; with probability $\frac{1}{2}$, draw at random from B". Show that a separation $s \gg \frac{1}{\sqrt{d-1}}$ is sufficient so that $\text{prob}\,(x \in A \cap B) = o\,(1)$, ie. for any $\varepsilon > 0$ there exists c such that if $s \geq \frac{c}{\sqrt{d-1}}$ then $\text{prob}\,(x \in A \cap B) < \varepsilon$. In other words, this extent of separation means that nearly all of the mixture distribution is identifiable.

Hint: Use the theorem about the tropical slice.

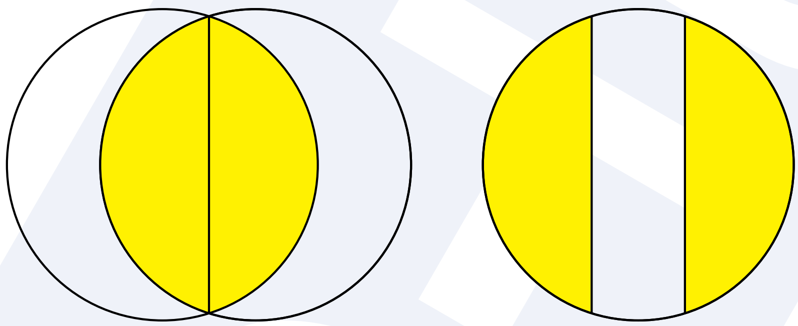
Solution. We have

$$
\begin{aligned}
&\operatorname{prob}\left(x \in A \cap B\right) \\
&= \frac{1}{2}\operatorname{prob}\left(x \in A \cap B \,\middle|\, x \xleftarrow{\text{⚄}} A\right) + \frac{1}{2}\operatorname{prob}\left(x \in A \cap B \,\middle|\, x \xleftarrow{\text{⚄}} B\right) \\
&= \frac{1}{2} \cdot \frac{\operatorname{vol}(A \cap B)}{\operatorname{vol}(A)} + \frac{1}{2} \cdot \frac{\operatorname{vol}(A \cap B)}{\operatorname{vol}(B)},
\end{aligned}
$$

by definition of the uniform distribution. By inserting $\operatorname{vol}(A) = \operatorname{vol}(B)$ we can write

$$\operatorname{prob}\left(x \in A \cap B\right) = \frac{\operatorname{vol}(A \cap B)}{\operatorname{vol}(B)}.$$

To estimate the volume of the intersection, we consider the plane, that perfectly partitions our intersection area in two equally large sphere caps.



Mathematically it is given as the plane orthogonal to the connection of the two ball centers, that goes through the exact midpoint of this connection. Hence it has distance $\frac{s}{2}$ from both midpoints.

The intersection of the balls now consists of two equally large parts. But if we consider only one of these parts and look at how it fits into the corresponding ball, it is exactly one of the remaining caps when we cut out the tropical slice of width $\frac{c}{\sqrt{d-1}}$ (so the slice with distance $\frac{c}{2\sqrt{d-1}}$ from the center of the ball) almost orthogonal to the connection of the ball centers.

So especially, both parts of the intersection together will be twice as large as one part. Similarly, both caps after cutting out the tropical slice will also be twice as large together as only one cap. This is sufficient to conclude, that $\operatorname{vol}(A \cap B) = \operatorname{vol}(B \backslash T)$, where T is a tropic of width $\frac{c}{\sqrt{d-1}}$. When looking at the picture, one can also use a symmetry argument, to conclude that the two parts of $A \cap B$ even have the same shape as $B \backslash T$.

Now we can apply the tropical slice theorem: $1 - \frac{\operatorname{vol} T}{\operatorname{vol} B^d} \leq \frac{2}{\widetilde{c}} e^{-\frac{\widetilde{c}^2}{2}}$. We have to be a little careful, as it is formulated with respect to a tropical slice of width $\frac{2\widetilde{c}}{\sqrt{d-1}}$, so we have to plug in $\widetilde{c} = \frac{c}{2}$. Precisely, we get

$$\operatorname{prob}(x \in A \cap B) \leq \frac{4}{c} e^{-\frac{c^2}{8}}$$

and for $c \to \infty$, we have $\operatorname{prob}(x \in A \cap B) \to 0$. Reformulated, that means that for any ε, we can find a c large enough such that $\operatorname{prob}(x \in A \cap B) < \varepsilon$, which was to show. $\bigcirc$