Foundations of data science, summer 2020

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12. Exercise sheet
Hand in solutions until Thursday, 9 July 2020, 12:00

Exercise 12.1 (Exam exercise). (8 points)

Design an exam exercise. $\boxed{8}$

The estimated time to work out a solution should be about 15 minutes. Award credit points to the parts of your exercise reflecting the estimated working time. Make sure that exercise solutions allow to differentiate between a very good student, a good student and a weak student.

Hint: Asking for mere knowledge is not university level.

Exercise 12.2 (Play with the Perceptron algorithm). (10 points)

(i) Pick a vector $w^* \in \mathbb{R}^2$. Pick $1\,000$ random points x_i uniformly distributed in $[-100, 100]^2 \subset \mathbb{R}^2$ with $|\langle w^* \mid x\rangle| \geq 1$ and compute labels $\ell_i \leftarrow \text{sign}(\langle w^* \mid x_i\rangle) \in \{-1, +1\}$.

  (a) Run the perceptron algorithm (with the kernel function $k(x, y) = x \cdot y$).

  (b) Report about your experiences. $\boxed{3}$

  (c) How does its runtime compare to the lecture's bound on it? $\boxed{2}$

(ii) Pick $1\,000$ random points $\left[\begin{smallmatrix} r\cos\varphi \\ r\sin\varphi \end{smallmatrix}\right]$ with $r \in [0.8, 1.2] \cup [1.6, 2.4]$ uniformly chosen and $\varphi \in [0, 2\pi]$ uniformly chosen and label -1 for $r \leq 1.2$ and label $+1$ for $r \geq 1.6$.

  (a) Run the perceptron algorithm with the Gaussian kernel.

  (b) Report about your experiences. $\boxed{3}$

(iii) Pick $1\,000$ random points $\left[\begin{smallmatrix} 1+r\cos\varphi \\ 1+r\sin\varphi \end{smallmatrix}\right]$ with $r \in [0.8, 1.2] \cup [1.6, 2.4]$ uniformly chosen and $\varphi \in [0, 2\pi]$ uniformly chosen and label -1 for $r \leq 1.2$ and label $+1$ for $r \geq 1.6$.

  (a) Run the perceptron algorithm with the Gaussian kernel.

  (b) Report about your experiences. $\boxed{2}$

Exercise 12.3 (VC dimension). (6+3 points)

A hypercuboid in $\mathbb{R}^d$ is a generalized rectangle, ie. subset of the form

$$[a_0, b_0] \times [a_1, b_1] \times \cdots \times [a_{d-1}, b_{d-1}].$$

In $\mathbb{R}^3$ this is called a cuboid.

$\boxed{3}$ (i) If $\Omega = \mathbb{R}^3$ and $\mathcal{H} = \{\text{cuboid} \subset \mathbb{R}^3\}$, prove that

$$\text{VCdim}(\mathcal{H}) = 6.$$

$\boxed{3}$ (ii) Consider $\Omega = \mathbb{R}^d$ and $\mathcal{H} = \left\{\text{hypercuboid} \subset \mathbb{R}^d\right\}$. With methods similar to those seen in class, prove that

$$\text{VCdim}(\mathcal{H}) < 2d + 1.$$

$\boxed{+3}$ (iii) Can you find a set with $2d$ points in $\mathbb{R}^d$ that shatters? Explain why it does.