Foundations of data science, summer 2020

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6. Exercise sheet
Hand in solutions until Thursday, 28 May 2020, 12:00

Exercise 6.1 (Eigenvalues and -vectors). (10 points)

Take the matrix $A = \begin{bmatrix} 2 & & 4 \\ & -1 & \\ 4 & & 3 \end{bmatrix}$.

(i) Compute eigenvalues and eigenvectors of A. $\boxed{6}$

Solution. Since dimension is very small, one option is to compute $\det(A - \lambda\mathbb{1})$ and find its zeroes. These are the eigenvalues. We find

$$\begin{aligned} \det(A - \lambda\mathbb{1}) &= (-1-\lambda)((2-\lambda)(3-\lambda) - 4^2) \\ &= -(\lambda+1)\left(\left(\lambda - \frac{5}{2}\right)^2 - \frac{65}{4}\right) \end{aligned}$$

and so the eigenvalues are $\lambda_0 = -1$, $\lambda_\pm = \frac{5}{2} \pm \frac{\sqrt{65}}{2}$. Then the eigenvectors are non-trivial solutions of $(A - \lambda)|x\rangle = 0$. One finds

$$\begin{aligned} |v_0\rangle &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} && \text{for } \lambda_0 = -1, \\ |v_\pm\rangle &= \begin{bmatrix} \pm\sqrt{65} - 1 \\ 0 \\ 8 \end{bmatrix} && \text{for } \lambda_\pm = \frac{5}{2} \pm \frac{\sqrt{65}}{2}. \end{aligned}$$

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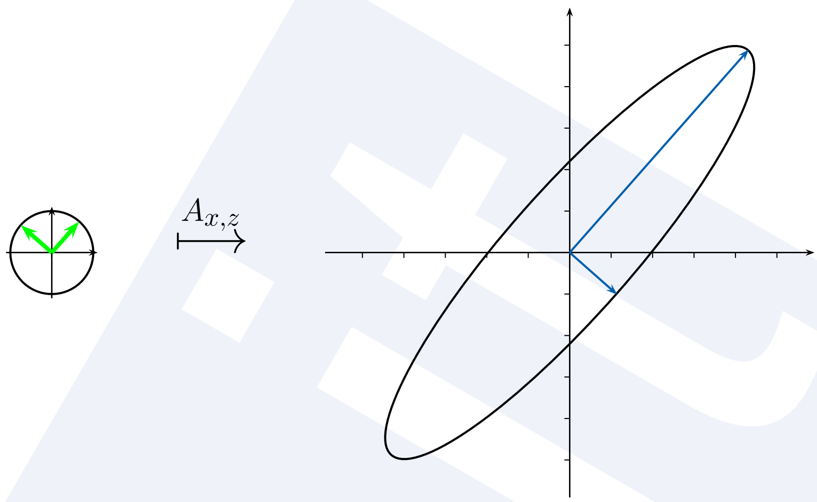
(ii) How does A stretch vectors?

- Describe. $\boxed{2}$

Solution. It flips vectors in direction $|v_0\rangle$ and stretches vectors in direction $|v_\pm\rangle$ by $\lambda_\pm$. In total, a unit vector gets mapped to a certain ellipsoid with radii $-\lambda_0$, λ_+, $-\lambda_-$. (The axes of the ellipsoid are $\frac{\lambda_j}{\| |v_j\rangle \|_2} |v_j\rangle$.) ○

- Plot how A maps the unit circle in the (x, z)-plane and the eigenvectors therein. $\boxed{2}$

Solution. The image of the unit circle under $A_{x,z} = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$ looks like this:



The green arrows are the eigenvectors and the blue ones the scaled versions.

Exercise 6.2 (To norm or not to norm?). (6 points)

For the following expressions, decide (and prove) if the following are norms or not.

$\boxed{2}$ (i) The "0-norm (?)": For a vector $|x\rangle$ we define

$$\left\| |x\rangle \right\|_0 := \# \left\{ i \,\middle|\, x_i \neq 0 \right\}.$$

Solution. This is not a norm. We have for $|v\rangle = |(1,1,1)\rangle \in \mathbb{R}^3$ that

$$\|2 \cdot v\|_0 = 3 \neq 6 = 2 \|v\|_0 .$$

$\boxed{2}$ (ii) The "scaled 1-norm (?)": For a vector $|x\rangle$ we define

$$\left\| |x\rangle \right\|_s = n \left| x_0 \right| + (n-1) \left| x_1 \right| + (n-2) \left| x_2 \right| + \cdots + 1 \left| x_{n-1} \right| .$$

Solution. This is indeed a norm. At first note $|x_i| \geq 0$ with equality if and only if $x_i = 0$. As $\| |x\rangle \|_s$ is the sum of these non-negative entries scaled by positive constants, it follows that $\| |x\rangle \|_s \geq 0$ with equality if and only if all $x_i = 0$, so iff $|x\rangle = 0$.

Further, we have that

$$\begin{aligned} \| |x\rangle + |y\rangle \|_s &= n\,|x_0 + y_0| + (n-1)\,|x_1 + y_1| + \cdots + 1\,|x_{n-1} + y_{n-1}| \\ &\leq n\,|x_0| + (n-1)\,|x_1| + \cdots + 1\,|x_{n-1}| + \\ &\quad n\,|y_0| + (n-1)\,|y_1| + \cdots + 1\,|y_{n-1}| \\ &= \| |x\rangle \|_s + \| |y\rangle \|_s \end{aligned}$$

by the triangle inequality for the absolute value (modulus).

Also by linearity of the absolute value, we have

$$\begin{aligned} \| \lambda\, |x\rangle \|_s &= n\,|\lambda x_0| + (n-1)\,|\lambda x_1| + \cdots + 1\,|\lambda x_{n-1}| \\ &= n\,|\lambda|\,|x_0| + (n-1)\,|\lambda|\,|x_1| + \cdots + 1\,|\lambda|\,|x_{n-1}| \\ &= |\lambda|\,\| |x\rangle \|_s \end{aligned}$$

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(iii) The "last-coordinate-norm (?)": For a vector $|x\rangle$ we define 2

$$\| |x\rangle \|_c = x_{n-1}.$$

Solution. This is not a norm. For one thing, x_{n-1} might be negative, so $\|x\|_c$ might be negative, violating the norm condition. Also $x_{n-1} = 0$ can occur, even if $|x\rangle \neq 0$, for example $|x\rangle = |(3,2,1,0)\rangle \in \mathbb{R}^4$. ○

Exercise 6.3 (Some lemmata). (0 points)

Prove the following lemmata.

(i) **Lemma** (Relations). *If A is symmetric and rank k then*

$$\|A\|_2^2 \leq \|A\|_F^2 \leq k\,\|A\|_2^2\,.$$

Solution. From the lecture we know, that:

- $\|A\|_2^2 = \max_{i<n} \alpha_i^2$.
- $\|A\|_F^2 = \sum_{i<n} \alpha_i^2$.

Also, as A is rank k, we know that A only has k non-zero eigenvalues, hence

$$\|A\|_F^2 = \sum_{i<n} \alpha_i^2 = \sum_{i<k} \alpha_i^2.$$

Additionally, we may assume that the squares of the eigenvalues are ordered, such that α_0^2 takes the maximal value over all α_i^2.

So

$$\|A\|_2^2 = \alpha_0^2 \le \sum_{i<k} \alpha_i^2 = \|A\|_F^2 = \sum_{i<k} \alpha_i^2 \le \sum_{i<k} \alpha_0^2 = k \cdot \alpha_0^2 = k \|A\|_2^2 . \qquad \bigcirc$$

(ii) **Lemma.** *Let A be symmetric. Then* $\|A\|_2 = \max_{\||x\rangle\|_2=1} |\langle x| A |x\rangle|$.

Solution. As A is symmetric, we know that A is diagonalizable. Also, both terms in the given equation are independent under orthonormal changes of the basis, as they maximize over all $|x\rangle$ with $\||x\rangle\|_2 = 1$ and an orthogonal change of the basis preserves the norms of vectors. So we can assume that A is a diagonal matrix without loss of generality.

We know that $\|A\|_2 = |\alpha_0|$, the absolute-largest eigenvalue. This will also be the largest entry on the diagonal of A. Without loss of generality we can assume that this value is the first on the diagonal of A.

That $\max_{\||x\rangle\|_2=1} |\langle x| A |x\rangle| \ge \alpha_0$ is obvious if we consider $|x\rangle = |e_0\rangle$.

That this is indeed maximizing the right hand side can be seen by computing

$$\begin{aligned} |\langle x| A |x\rangle| &= |\alpha_0| x_0^2 + |\alpha_1| x_1^2 + \cdots + |\alpha_{n-1}| x_{n-1}^2 \\ &\le |\alpha_0| (x_0^2 + x_1^2 + \cdots + x_{n-1}^2) \\ &= |\alpha_0| = \|A\|_2 , \end{aligned}$$

as we know that $\||x\rangle\|^2 = \sum_{i<n} x_i^2 = 1$. $\bigcirc$

(iii) **Lemma.** *For each column $|A_{\cdot,j}\rangle$ of A we have* $\||A_{\cdot,j}\rangle\|_2 \le \|A\|_2$.

Solution. We are dealing with an operator norm here, so

$$\|A |x\rangle\|_2 \le \|A\|_2 \cdot \||x\rangle\|_2 .$$

Plug in $|x\rangle = |e_j\rangle$. Then $A |e_j\rangle = |A_{\cdot,j}\rangle$ is column j. Since $\||e_j\rangle\| = 1$ we obtain the claim. $\bigcirc$