Foundations of data science, summer 2020

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2. Exercise sheet
Hand in solutions until Thursday, 30 April 2020, 12:00

Exercise 2.1 (When is Markov sharp?). (4 points)

Show that for any $a > 0$ there exists a probability distribution such that the Markov inequality is sharp, ie. $\operatorname{prob}(X \geq a) = \frac{\mathrm{E}(X)}{a}$. Use the $\operatorname{prob}(\ldots)$-notation to write down the distribution explicitly. $\boxed{4}$

Hint: Recall the proof from the lecture. If it helps you, restrict to the discrete setting.

Exercise 2.2 (Maximum of two dies). (8 points)

Take two independent fair dies $D_i \xleftarrow{⚅} \mathcal{D} := \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\}$ and consider the larger outcome

$$M := \max(D_0, D_1).$$

(i) Compute $\operatorname{prob}(M \leq a)$ for $a \in \mathbb{N}_{\leq 6}$. $\boxed{2}$

(ii) Compute $\operatorname{prob}(M = a)$ for $a \in \mathbb{N}_{\leq 6}$. $\boxed{2}$

(iii) Compute $\mathrm{E}(M)$. $\boxed{2}$

(iv) Compute $\mathrm{E}(M^2)$ and derive $\operatorname{var}(M)$. $\boxed{2}$

Exercise 2.3 (Random variables). (0+8 points)

Consider the following experiment:

1. Throw a coin $C \xleftarrow{⚅} \{0, 1\}$.
2. Choose $X \xleftarrow{⚅} [0, 1]$ uniformly.
3. Roll a die $D \xleftarrow{⚅} \{1, 2, 3, 4, 5, 6\}$.
4. If $C = 0$ then
5.   Let $Z \leftarrow D + X$.
6. Else

7. Let $Z \leftarrow D - X$.
8. Return Z

The output Z of this algorithm is a random variable.

$\boxed{+3}$ (i) Compute its expection $\mathrm{E}(Z)$.

$\boxed{+4}$ (ii) Compute its second moment $\mathrm{E}(Z^2)$.

$\boxed{+1}$ (iii) Compute its variance $\mathrm{var}(Z)$.