Foundations of data science, summer 2020

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3. Exercise sheet
Hand in solutions until Thursday, 7 May 2020, 12:00

Exercise 3.1 (Cylinder). (8 points)

Given a d-dimensional circular cylinder of radius r and height h

(i) What is the surface in terms of vol($B^{d'}$) and surface($B^{d'}$) for appropriate d'? $\boxed{6}$

Solution. We define the d-dimensional cylinder of radius r and height h as

$$C^d_{r,h} := r \cdot B^{d-1} \times [0, h].$$

The surface of a cylinder can be calculated as the sum of the surfaces of both caps and the body. We find that surface(cap) $=$ vol(rB^{d-1}) and surface(body) $=$ vol($[0, h]$)$\cdot$surface(rB^{d-1}). Putting this together yields

$$\begin{aligned} \text{surface}(C^d_{r,h}) &= 2 \cdot \text{vol}(rB^{d-1}) + \text{vol}([0, h]) \cdot \text{surface}(rB^{d-1}) \\ &= 2 \cdot r^{d-1} \cdot \text{vol}(B^{d-1}) + h \cdot r^{d-2} \cdot \text{surface}(B^{d-1}). \end{aligned}$$

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(ii) What is the volume? $\boxed{2}$

Solution. To calculate the volume, we follow a similar approach. We observe that the volume of the cylinder is the height of the body multiplied by the volume of a cap. We thus have

$$\begin{aligned} \text{vol}(C^d_{r,h}) &= \text{vol}([0, h]) \cdot \text{vol}(rB^{d-1}) \\ &= h \cdot r^{d-1} \cdot \text{vol}(B^{d-1}). \end{aligned}$$

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Exercise 3.2 (Annuli). (7 points)

(i) Compute and estimate the volume of the $\frac{1}{100}$-annulus compared to the volume of the d-dimensional ball B^d. $\boxed{1}$

Solution. By the formula from the lecture we know that the volume of the annulus of width ε of the d-dimensional ball, here denoted A^d_ε, is

$$\operatorname{vol}\left(A^d_\varepsilon\right) = \operatorname{vol}(B^d) - (1-\varepsilon)^d \operatorname{vol}(B^d).$$

So for $\varepsilon = \frac{1}{100}$ we obtain

$$\operatorname{vol}\left(A^d_{\frac{1}{100}}\right) = \operatorname{vol}(B^d) - \left(1 - \frac{1}{100}\right)^d \operatorname{vol}(B^d),$$

so compared to the d-dimensional ball we have

$$\frac{\operatorname{vol}\left(A^d_{\frac{1}{100}}\right)}{\operatorname{vol}(B^d)} = 1 - \left(1 - \frac{1}{100}\right)^d \geq 1 - e^{-\frac{1}{100}d}.$$

So especially we have

$$\frac{\operatorname{vol}\left(A^d_{\frac{1}{100}}\right)}{\operatorname{vol}(B^d)} \xrightarrow{d\to\infty} 1.$$

$\boxed{1}$ (ii) Compute and estimate the volume of the $\frac{1}{\sqrt{d}}$-annulus compared to the volume of the d-dimensional ball B^d.

Solution. With the same formula as above, for $\varepsilon = \frac{1}{\sqrt{d}}$ we obtain

$$\operatorname{vol}\left(A^d_{\frac{1}{\sqrt{d}}}\right) = \operatorname{vol}(B^d) - \left(1 - \frac{1}{\sqrt{d}}\right)^d \operatorname{vol}(B^d),$$

so compared to the d-dimensional ball we have

$$\frac{\operatorname{vol}\left(A^d_{\frac{1}{\sqrt{d}}}\right)}{\operatorname{vol}(B^d)} = 1 - \left(1 - \frac{1}{\sqrt{d}}\right)^d \geq 1 - e^{-\frac{1}{\sqrt{d}}d} = 1 - e^{-\sqrt{d}}.$$

So especially, we have

$$\frac{\operatorname{vol}\left(A^d_{\frac{1}{\sqrt{d}}}\right)}{\operatorname{vol}(B^d)} \xrightarrow{d\to\infty} 1.$$

$\boxed{1}$

(iii) Compute and estimate the volume of the $\frac{1}{d^2}$-annulus compared to the volume of the d-dimensional ball B^d.

Solution. With the same formula as above, for $\varepsilon = \frac{1}{d^2}$ we obtain

$$\operatorname{vol}\left(A^d_{\frac{1}{d^2}}\right) = \operatorname{vol}(B^d) - \left(1 - \frac{1}{d^2}\right)^d \operatorname{vol}(B^d),$$

so compared to the d-dimensional ball we have

$$\frac{\operatorname{vol}\left(A^d_{\frac{1}{d^2}}\right)}{\operatorname{vol}(B^d)} = 1 - \left(1 - \frac{1}{d^2}\right)^d \geq 1 - e^{-\frac{1}{d^2}d} = 1 - e^{-\frac{1}{d}}.$$

Actually applying an appropriate upper bound we find

$$\frac{\operatorname{vol}\left(A^d_{\frac{1}{d^2}}\right)}{\operatorname{vol}(B^d)} \xrightarrow{d\to\infty} 0.$$

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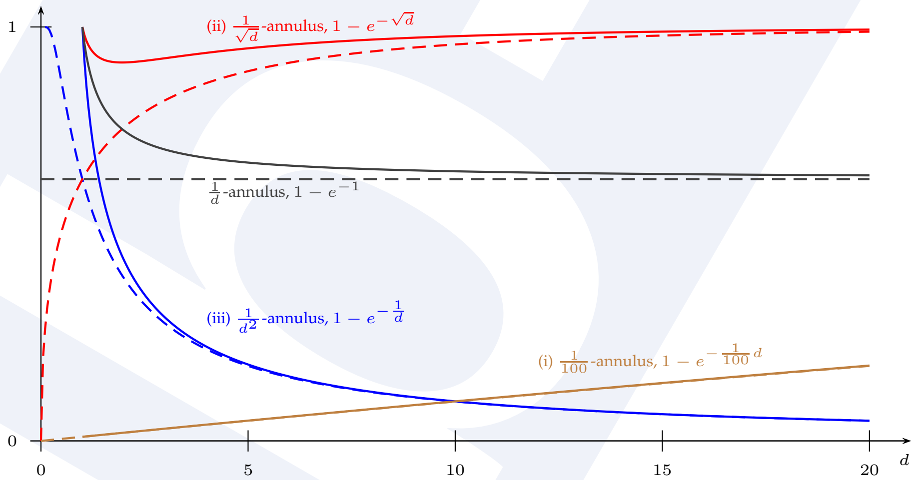
(iv) Plot the three functions and the relative volume of the $\frac{1}{d}$-annulus for $d = 1..20$. $\boxed{2}$

Solution.



Here, the true functions are drawn as solid lines, the estimates as dashed lines.

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(v) For which ε does the ε-annulus have at least 99% of the ball volume? $\boxed{2}$

Solution. If we consider the previous picture, the graph in which we have control of both lower and higher dimensions, is the graph of $\varepsilon = \frac{1}{d}$. The idea now is to consider $\varepsilon = \frac{c}{d}$ to get a different constant. We compute that the volume of the annulus compared to the ball itself is

$$(1 - \left(1 - \frac{c}{d}\right)^d) \geq 1 - e^{-c}.$$

To obtain $1 - e^{-c} \geq 0.99$, we need to solve $e^{-c} \leq 0.01$. As the exponential function is monotone increasing, this is equivalent to

$$-c \leq \ln(0.01) \Leftrightarrow c \geq \ln(100).$$

So especially for $c = \ln(100) = 4.605⫠$ we have that an $\varepsilon = \frac{c}{d}$-annulus contains more than 99% of the volume of any d-dimensional ball.[1] ○

Exercise 3.3 (Gamma). (0 points)

+0

(i) Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Hint: Change the variable and use $\int_{-\infty}^{\infty} e^{-x^2}\, \mathrm{d}x = \sqrt{\pi}$.

Solution. We are combining a few elementary transformations on integrals to obtain that. It needs one 'standard' substitution and one trick.

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty x^{\frac{1}{2}-1} e^{-x}\, \mathrm{d}x \\
&\xlongequal[\mathrm{d}x = 2y\,\mathrm{d}y]{x=y^2,} \int_0^\infty y^{-1} e^{-y^2} 2y\, \mathrm{d}y \\
&= \int_{-\infty}^{\infty} e^{-y^2}\, \mathrm{d}y = \sqrt{\pi}.
\end{aligned}$$

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[1]Side remark: to indicate how a real number was rounded we append a special symbol. Examples: $\pi = 3.14$⫠ $= 3.142$Y $= 3.1416$⊤ $= 3.14159$⫠. The height of the platform shows the size of the left-out part and the direction of the antenna indicates whether actual value is larger or smaller than displayed. We write, say, $e = 2.72$⊤ $= 2.71$⊓ as if the shorthand were exact.

For the last integral (from the hint) the simplest way I know is to square it:

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\begin{aligned}
\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y \cdot \int_{-\infty}^{\infty} e^{-z^2}\,\mathrm{d}z &= \int_{\mathbb{R}^2} e^{-(y^2+z^2)}\,\mathrm{d}y\ \mathrm{d}z \\
&\overset{\substack{(y,z)=(r\cos\varphi, r\sin\varphi)\\ \mathrm{d}y\ \mathrm{d}z = r\ \mathrm{d}r\ \mathrm{d}\varphi}}{=\!=\!=\!=} \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\ \mathrm{d}r\ \mathrm{d}\varphi \\
&= \int_0^{\infty} 2re^{-r^2}\,\mathrm{d}r \cdot \frac{1}{2}\int_0^{2\pi}\mathrm{d}\varphi \\
&= \underbrace{\left[-e^{-r^2}\right]_0^{\infty}}_{=1} \cdot \pi \\
&= \pi
\end{aligned}
$$

and thus $\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y = \sqrt{\pi}$.

(ii) Recall the formula for integration by parts (look it up if need be. . .), $\int_a^b f(x)g'(x)\,\mathrm{d}x = \dots$, where f and g are any suitably nice functions. Use this formula to show that indeed $\Gamma(z+1) = z\Gamma(z)$. $\boxed{+0}$

Solution. We split the integrand into the parts x^z with derivate zx^{z-1} and e^{-x} with anti-derivative $-e^{-x}$:

$$
\begin{aligned}
\Gamma(z+1) &= \int_0^{\infty} \underbrace{x^{(z+1)-1}}_{=:f\downarrow}\underbrace{e^{-x}}_{=:g\uparrow}\,\mathrm{d}x \\
&= \underbrace{\left[-x^z e^{-x}\right]_0^{\infty}}_{=0} + z\int_0^{\infty} x^{z-1}e^{-x}\,\mathrm{d}x = z\Gamma(z).
\end{aligned}
$$

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