Lecture on
Fuzzy and Crisp Relations(Unit-3-Lecture 4)

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Outline of Presentation

- Introduction.
- The crisp set v.s. the fuzzy set.
- Crisp Relation.
- Fuzzy relation.
- References.

Introduction

- A Crisp relation represents presence or absence of association, interaction or interconnection between elements of ≥ 2 sets. This concept can be generalized to various degrees or strengths of association or interaction between elements. A fuzzy relation generalizes these degrees to membership grades.
- Fuzzy set theory formally speaking is one of these theories, which was initially intended to be an extension of dual logic and/or classical set theory.

The crisp set v.s. the fuzzy set

- The crisp set is defined in such a way as to partition the individuals in some given universe of discourse into two groups: members and nonmembers.
  However, many classification concepts do not exhibit this characteristic.
  For example, the set of tall people, expensive cars, or sunny days.
- A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set.
  For example: a fuzzy set representing our concept of sunny might assign a degree of membership of 1 to a cloud cover of 0 percent, 0.8 to a cloud cover of 20 percent, 0.4 to a cloud cover of 30 percent, and 0 to a cloud cover of 75 percent.

Crisp Relation

- A crisp relation is used to represents the presence or absence of interaction, association, or interconnectedness between the elements of more than a set. This crisp relational concept can be generalized to allow for various degrees or strengths of relation or interaction between elements.

Fuzzy relation

- Degrees of association can be represented by grades of the membership in a fuzzy relation in the same way as degrees of set membership are represented in the fuzzy set. In fact, just as the crisp set can be viewed as a restricted case of the more general fuzzy set concept, the crisp relation can be considered to be a restricted case of the fuzzy relations.

Fuzzy Relations

Cartesian Product

- *Let $A_1, A_2, \ldots, A_n$ be fuzzy sets in $U_1, U_2, \ldots U_n$, respectively.*

  The Cartesian product of $A_1, A_2, \ldots, A_n$ is a fuzzy set in the space $U_1 \times U_2 \times \ldots \times U_n$ with the membership function as:

  $$\mu_{A1 \times A2 \times \ldots \times An}(x_1, x_2, \ldots, x_n) = \min[\mu_{A1}(x_1), \mu_{A2}(x_2), \ldots \mu_{An}(x_n)]$$

- *So, the Cartesian product of $A_1, A_2, \ldots, A_n$ are donated by $A_1 \times A_2 \times \ldots \times A_n$*

Fuzzy Relations

Cartesian Product: Example

- *Let A =* {(3, 0.5), (5, 1), (7, 0.6)}
- *Let B =* {(3, 1), (5, 0.6)}
- Find the product
- *The product is all set of pairs from A and B with the minimum associated memberships*
- *Ax B =* {[(3, 3), min (0.5, 1)], [(5, 3), min(1, 1)], [(7, 3), min(0.6, 1)], [(3, 5), min(0.5, 0.6)], [(5, 5), min(1, 0.6)], [(7, 5), min(0.6, 0.6)]}

  = {[(3, 3), 0.5], [(5, 3), 1], [(7, 3), 0.6], [(3, 5), 0.5], [(5, 5), 0.6], [(7, 5), 0.6]}

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Fuzzy Relations

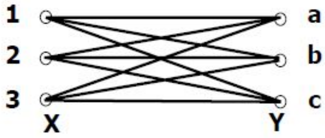
Crisp Relations

- *The relation between any two sets is the Cartesian product of the elements of $A_1 \times A_2 \times \ldots\ldots \times A_n$*
- *For X and Y universes* $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$
- $$\mu_{x \times y}(x, y) = \begin{cases} 1, & (x, y) \in X \times Y \\ 0, & (x, y) \notin X \times Y \end{cases}$$
- *This relation can be represented in a matrix format*

Fuzzy Relations

Crisp Relations: Example

- *Universe X* = {1, 2, 3}
- *Universe Y* = {a, b, c}



$$R = \begin{array}{c} 1 \\ 2 \\ 3 \\ \\ \end{array} \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ \begin{array}{ccc} a & b & c \end{array} \end{array}$$

Fuzzy Relations

Fuzzy Relations

- *Fuzzy relations are mapping elements of one universe, to those of another universe, Y, through the Cartesian product of two universes. X, Universe X = {1, 2, 3}*
- $R(X, Y) = \{[(x, y), \mu_R(x, y)] \mid (x, y) \in (X \times Y)\}$
- *Where the fuzzy relation R has membership function*
- $\mu_R(x, y) = \mu_{A \times B}(x, y) = \min(\mu_A(x), \mu_B(y))$

Fuzzy Relations

Fuzzy Relations

- *It represents the strength of association between elements of the two sets*
- *Ex: R = "x is considerably larger than y"*
- *R (X, Y)= Relation between sets X and Y*
- *R (x, y) = memebership function for the relation R (X, Y)*
- $R(X, Y) = \{R(x, y) / (x, y) \mid (x, y) \in (X \times Y)\}$

Fuzzy Relations

Operations on Fuzzy Relations

- *Since the fuzzy relation from X to Y is a fuzzy set in X × Y, then the operations on fuzzy sets can be extended to fuzzy relations. Let R and S be fuzzy relations on the Cartesian space* X × Y *then*:
- *Union:* $\mu_{R \cup S}(x, y) = \max[\mu_R(x, y), \mu_S(x, y)]$
- *Intersection:* $\mu_{R \cap S}(x, y) = \min[\mu_R(x, y), \mu_S(x, y)]$
- *Complement:* $\mu_{\bar{R}}(x, y) = 1 - \mu_R(x, y)$

Fuzzy Relations

Fuzzy Relations: Example

- *Assume two Universes*: A = {3, 4, 5} and B = {3, 4, 5, 6, 7}

- $$\mu_R(x, y) = \begin{cases} (y-x)/(y+x+2) & \text{if } y > x \\ 0, & \text{if } y \leq x \end{cases}$$

- *This can be expressed as follow:*

$$R = \begin{array}{c} 3 \\ 4 \\ 5 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 0 & 0.11 & 0.2 & 0.27 & 0.33 \\ 0 & 0 & 0.09 & 0.17 & 0.23 \\ 0 & 0 & 0 & 0.08 & 0.14 \end{bmatrix} \\ \begin{matrix} 3 & \;\;4 & \;\;5 & \;\;6 & \;\;7 \end{matrix} \end{array}$$

Fuzzy Relations

Fuzzy Relations: Example

$$
\begin{array}{cc}
 & \\
R = & \begin{array}{c} 3 \\ 4 \\ 5 \end{array} \begin{pmatrix} 0 & 0.11 & 0.2 & 0.27 & 0.33 \\ 0 & 0 & 0.09 & 0.17 & 0.23 \\ 0 & 0 & 0 & 0.08 & 0.14 \end{pmatrix} \\
 & \begin{array}{ccccc} \;\;\;3 & \;\;\;4 & \;\;\;5 & \;\;\;6 & \;\;\;7 \end{array}
\end{array}
$$

- *This matrix represents the membership grades between elements in X and Y*
- $\mu_R(x, y) = \{[0/(3, 3)], [0.11/(3, 4)], [0.2/(3, 5)], \ldots\ldots\ldots\ldots\ldots\ldots, [0.14/(5, 7)]\}$

Fuzzy Relations

Fuzzy Relations: Example

- Assume two fuzzy sets: $A = \{0.2/x_1 + 0.5/x_2 + 1/x_3\}$

  $B = \{0.3/y_1 + 0.9/y_2\}$

- Find the fuzzy relation (the Cartesian product)

$$A \times B = R = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \\ \end{array} \begin{array}{c} \begin{pmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{pmatrix} \\ \begin{array}{cc} y_1 & \quad y_2 \end{array} \end{array}$$

Fuzzy Relations

Composition of Fuzzy Relations

- *Composition of fuzzy relations used to combine fuzzy relations on different product spaces*
- *Having a fuzzy relation; R (X ×Y) and S (Y ×Z), then Composition is used to determine a relation T (X × Z),*

Fuzzy Relations

Composition of Fuzzy Relations

- *The max-min composition can be interpreted as indicating the strength of the existence of relation between the elements of X and Z*
- *Calculations of* (R o S) is almost similar to matrix multiplication
- *Fuzzy relations composition have the same properties of:*

Distributivity: R o (S U T) = (R o S) U (R o T)

Associativity: R o (S o T) = (R o S) o T

Fuzzy Relations

Composition of Fuzzy Relations: Example

- *Assume the following universes: $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, and $Z = \{z_1, z_2, z_3\}$, with the following fuzzy relations.*

$$R = \begin{array}{c} x_1 \\ x_2 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \\ \begin{array}{cc} y_1 & y_2 \end{array} \end{array} \quad \text{and} \quad S = \begin{array}{c} y_1 \\ y_2 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \\ \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \end{array}$$

- *Find the fuzzy relation between X and Z using the max-min and max-product composition*

Fuzzy Relations

Composition of Fuzzy Relations: Example

- *By max-min composition*

$$\mu_T(x_1, z_1) = \max [\min (0.7, 0.9), \min (0.5, 0.1)] = 0.7$$

$$T = \begin{array}{c} x_1 \\ x_2 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \\ \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \end{array}$$

- *By max-product composition*

$$\mu_T(x_2, z_2) = \max [(0.8, 0.6), (0.4, 0.7)] = 0.48$$

$$T = \begin{array}{c} x_1 \\ x_2 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 0.63 & 0.42 & 0.25 \\ 0.72 & 0.48 & 0.20 \end{bmatrix} \\ \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \end{array}$$

Refrences

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Queries

Thanking You