Lecture on
Fuzzy and Crisp Relation Properties(Unit 3 Lecture 5)

Presented By

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Outline of Presentation

- Introduction.
- The Relation Properties.
- References.

Introduction

- In real world, there exist much fuzzy knowledge (i.e. vague, uncertain inexact etc).
- Human thinking and reasoning (analysis, logic, interpretation) frequently involved fuzzy information.
- Human can give satisfactory answers, which are probably true.
- Our systems are unable to answer many question because the systems are designed based upon classical set theory (Unreliable and incomplete).
- We want, our system should be able to cope with unreliable and incomplete information.
- Fuzzy system have been provide solution.

Difference

Classical set theory	Fuzzy set theory
Classes of objects with sharp boundaries.	Classes of objects with un-sharp boundaries.
A classical set is defined by crisp(exact) boundaries, i.e., there is no uncertainty about the location of the set boundaries.	A fuzzy set is defined by its ambiguous boundaries, i.e., there exists uncertainty about the location of the set boundaries.
Widely used in digital system design	Used in fuzzy controllers.

Crisp (classic) sets, fuzzy sets

- Given an *universe of discourse* (*crisp*) X
- For a classic (crisp) set $A \subset X$, for each element $x \in X$, either $x \in A$ or $x \notin A$.
- For the set A it can be defined a characteristic function $\nu_A : X \rightarrow \{0, 1\}$, with $\nu_A(x) = 1$ iff (if and only if) $x \in A$ and $\nu_A(x) = 0$ iff $x \notin A$
- For a fuzzy set $\tilde{A}$, an element $x \in X$ belongs to the fuzzy set $\tilde{A} \subset X$ **in a certain degree**
- The characteristic function of a crisp set will be extended to the *membership function* of a fuzzy set, which can take values in the real numbers interval $[0, 1]$

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Definition

"If X is a collection of objects" (named the *universe of discourse*) "denoted generically by x, then a fuzzy set $\tilde{A} \subset X$ is a set of ordered pairs

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$$

where $\mu_{\tilde{A}}(x) : X \to [0, 1]$ is called *membership function* or *degree of membership* (also, degree of compatibility or degree of truth). of x in A" (Zimmermann [Zim91])

If the interval of real numbers [0, 1] is replaced with the discrete set $\{0, 1\}$, then the fuzzy set $\tilde{A}$ becomes a classic (crisp) set.

- Fuzzy sets can be discrete or continuous
- The interval $[0, 1]$ can be extended to $[0, k]$, where $k > 0$
- It is possible to define fuzzy sets on more complex structures than intervals or real numbers, e.g. $\mathbb{L}$-*fuzzy sets*, where $\mathbb{L}$ is a partially ordered set (see chapter 3, Extensions of fuzzy sets)
- Example of discrete fuzzy set (Zimmermann [Zim91]):
  - MF: comfortable house for a 4 person family as a function of the number of bedrooms:
  - The universe discourse: $X = \{1, 2, \ldots, 10\}$
  - $\tilde{A} \subset X$ will be
    $\tilde{A} = \{(1, 0.1), (2, 0.5), (3, 0.8), (4, 1.0), (5, 0.7), (6, 0.2)\}$

1. Determine the union and intersection of the fuzzy sets $\tilde{A} =$ "comfortable house for a 4 persons - family" and $\tilde{B} =$ "small house", where
$\tilde{A} = \{(1, 0.1), (2, 0.5), (3, 0.8), (4, 1.0), (5, 0.7), (6, 0.2)\}$ and
$\tilde{B} = \{(1, 1), (2, 0.8), (3, 0.4), (4, 0.1)\}$:
$\tilde{A} \cup \tilde{B} = \{(1, \max(0.1, 1)), (2, \max(0.5, 0.8)), (3, \max(0.8, 0.4)),$
$(4, \max(1, 0.1)), (5, \max(0.7, 0)), (6, \max(0.2, 0)\} =$
$\{(1, 1), (2, 0.8), (3, 0.8), (4, 1), (5, 0.7), (6, 0.2)\}$
$\tilde{A} \cap \tilde{B} = \{(1, \min(0.1, 1)), (2, \min(0.5, 0.8)), (3, \min(0.8, 0.4)),$
$(4, \min(1, 0.1)), (5, \min(0.7, 0)), (6, \min(0.2, 0)\} =$
$\{(1, 0.1), (2, 0.5), (3, 0.4), (4, 0.1), (5, 0), (6, 0)\}$
$\tilde{A} \cup \tilde{B}$ can be read as "comfortable house for a 4 persons - family or small", and $\tilde{A} \cap \tilde{B}$ as "comfortable house for a 4 persons - family and small"

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3.4. Composition of Fuzzy Relations:

- A fuzzy relation *R* is defined on sets *A, B and* another fuzzy relations *S* is defined on sets *B,C.*

That is, $R \subseteq A \times B, \quad S \subseteq B \times C.$

The composition $S \bullet R = SR$ *of the two relations R and S expresses the* relation from *A to C.*

- *This composition is defined by* an *inner product*. The inner product is similar to an ordinary matrix (dot) product, except *Multiplication* is replaced by **Minimum** and *Summation* by **Maximum**. Thus this composition is defined by the following

$$\mu_{S \bullet R}(a, c) = \max [\min (\mu_R (a, b), \mu_S (b, c))]$$

fuzzy sets

example:

Consider the fuzzy *sets A, B* and *C to represent sets of events.*

The relation $R \subseteq AxB$, *gives the* possibility of occurrence of B after A, and the relation $S \subseteq BxC$ *gives the* possibility of occurrence of C after B.

R	b_1	b_2	b_3	b_4
a_1	0.1	0.2	0.0	1.0
a_2	0.3	0.3	0.0	0.2
a_3	0.8	0.9	1.0	0.4

S	c_1	c_2	c_3
b_1	0.9	0.0	0.3
b_2	0.2	1.0	0.8
b_3	0.8	0.0	0.7
b_4	0.4	0.2	0.3

For example, by the relation R, the possibility of b_1 *to occur after* a_1 *is 0.1. And by the relation S, the* possibility of occurrence of c_1 after b_1 is 0.9.

3.5. Properties of Fuzzy Relations:

the properties of commutativity, associativity, distributivity, involution, idempotency and De Morgan's principles all hold for fuzzy relations. But since there is overlap between a relation and its complement:

$$\mathrm{R} \cup \overline{\mathrm{R}} \neq \mathbf{E}$$

$$\mathrm{R} \cap \overline{\mathrm{R}} \neq \mathbf{O}$$

Where **O=** the null relation, and **E**=the complete relation e.g.

$$\mathbf{O} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

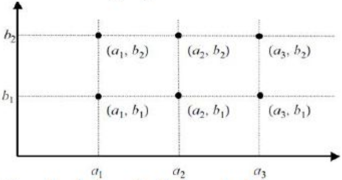
3.2. Crisp Relation

- A *Crisp Relation* **R** from a set A to a set B assigns to each ordered pair exactly one of the following statements:
  (i)"a is related to b" or (ii) "a is not related to b"

- The **Cartesian Product** AxB is the set of all possible combinations of the items of A and B. For example when:

$\mathbf{A = \{a_1,a_2,a_3\}}$ and $\mathbf{B = \{b_1,b_2\}}$

The Cartesian product yields the shown figure.



Which means:

$\mathbf{AxB = \{(a_1,b_1),(a_1,b_2),(a_2,b_1),(a_2,b_2),(a_3,b_1),(a_3,b_2)\}}$

Crisp Relation

1.2.1 Example

Let $X = \{1,4,5\}$ and $Y = \{3,6,7\}$

Classical matrix for the crisp relation when $R = x < y$ is

$$R = \begin{array}{c} \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

1.2.2 Example

Let $A = \{2,4,6,8\}$ and $B = \{2,4,6,8\}$

Classical matrix for the crisp relation $R = x = y$

$$R = \begin{array}{c} \\ 2 \\ 4 \\ 6 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 4 & 6 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

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3.5. Fuzzy Tolerance and Equivalence Relations:

Let R be the fuzzy relation defined on the set of cities and representing the concept very near. We may assume that a city is certainly (i.e., to a degree of 1) very near to itself. The relation is therefore **reflexive**. Furthermore, if city A is very near to city B, then B is certainly very near to A. Therefore, the relation is also **symmetric**. Finally, if city A is very near to city B to some degree, say .7, and city B is very near to city C to some degree, say .8, it is possible (although not necessary) that city A is very near to city C to a smaller degree, say 0.5. Therefore, the relation is **nontransitive**.

A fuzzy relation is a fuzzy equivalence relation if all three of the following properties for matrix relations define it:

Reflexivity $\mu R(x_i, y_i) = 1$

Symmetry $\mu R(x_i, y_j) = \mu R(x_j, y_i)$

Transitivity $\mu R(x_i, y_j) = \lambda 1$ and $\mu R(x_j, y_k) = \lambda 2$

$\rightarrow$ $\mu R(x_i, y_k) = \lambda$ where $\lambda \geq \min[\lambda 1, \lambda 2]$.

2.3 Reflexive Relation

Let R be a fuzzy relation in $X \times X$ then R is called reflexive, if

$$\mu_R(x,x) = 1 \qquad \forall x \in X$$

2.3.1 Example

Let $X = \{1, 2, 3, 4\}$

$$R = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0.9 & 0.6 & 0.2 \\ 0.9 & 1 & 0.7 & 0.3 \\ 0.6 & 0.7 & 1 & 0.9 \\ 0.2 & 0.3 & 0.9 & 1 \end{bmatrix} \end{array}$$

is reflexive relation

2.4 Antireflexive relations

Fuzzy relation $R \subset X \times X$ is antireflexive if

$$\mu_R(x, x) = 0, x \in X$$

2.4.1 Example

$$R_1 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0.6 \\ 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \end{array} \text{ is antireflexive relation}$$

Fuzzy properties

2.5 Symmetric Relation

A fuzzy relation R is called symmetric if,

$$\mu_R(x, y) = \mu_R(y, x) \quad \forall x, y \in X$$

2.5.1 Example

Let $X = \{x_1, x_2, x_3\}$

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.7 \\ 0.1 & 1 & 0.6 \\ 0.7 & 0.6 & 0.5 \end{bmatrix} \end{array}$$ is a symmetric relation.

2.6 Antisymmetric Relation

Fuzzy relation $R \subset X \times X$ is antisymmetric iff

$$\text{if } \mu_R(x,y) > 0 \text{ then } \mu_R(y,x) = 0 x, y \in X, x \neq y$$

2.6.1 Example

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0.7 \\ 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix} \end{array}$$ is antisymmetric relation.

2.7 Transitive Relation

Fuzzy relation $R \subset X \times X$ is transitive in the sense of max-min iff

$$\mu_R(x,z) \geq \max_{y \in X}\left(\min\left(\mu_R(x,y), \mu_R(y,z)\right)\right) x, z \in X$$

since $R^2 = R \circ R$ if

$$\mu_{R^2}(x,z) = \max_{y \in X}\left(\mu_R(x,y), \mu_R(y,z)\right)$$

then R is transitive if $R \circ R = R\left(R \circ R \subseteq R\right)$

and $R^2 \subset R$ means that $\mu_{R^2}(x,y) \leq \mu_R(y,x)$

2.7.1 Example

Let $X = \{x_1, x_2, x_3\}$

is $R = \begin{bmatrix} 0.7 & 0.9 & 0.4 \\ 0.1 & 0.3 & 0.5 \\ 0.2 & 0.1 & 0 \end{bmatrix}$ a transitive relation?

Solution

$$R \circ R = \begin{bmatrix} 0.7 & 0.9 & 0.4 \\ 0.1 & 0.3 & 0.5 \\ 0.2 & 0.1 & 0 \end{bmatrix} \circ \begin{bmatrix} 0.7 & 0.9 & 0.4 \\ 0.1 & 0.3 & 0.5 \\ 0.2 & 0.1 & 0 \end{bmatrix}$$

$$R^2 = \begin{bmatrix} 0.7 & 0.7 & 0.5 \\ 0.2 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.2 \end{bmatrix}$$

Fuzzy properties

Since $\mu_{R^2}(x_i, x_j)$ is not always less than or equal to $\mu_R(x_i, x_j)$, hence R is not transitive.

2.7.2 Example

Let $X = \{x_1, x_2\}$

is $R = \begin{bmatrix} 0.4 & 0.2 \\ 0.7 & 0.3 \end{bmatrix}$ a transitive relation?

Solution

$$R \circ R = \begin{bmatrix} 0.4 & 0.2 \\ 0.7 & 0.3 \end{bmatrix} \circ \begin{bmatrix} 0.4 & 0.2 \\ 0.7 & 0.3 \end{bmatrix}$$

using max-min composition

$$R^2 = \begin{bmatrix} \max(\min(0.4, 0.4), \min(0.2, 0.7)) & \max(\min(0.4, 0.2), \min(0.2, 0.3)) \\ \max(\min(0.7, 0.4), \min(0.3, 0.7)) & \max(\min(0.7, 0.2), \min(0.3, 0.3)) \end{bmatrix}$$

$$R^2 = \begin{bmatrix} \max(0.4, 0.2) & \max(0.2, 0.2) \\ \max(0.4, 0.3) & \max(0.2, 0.3) \end{bmatrix}$$

$$R^2 = \begin{bmatrix} 0.4 & 0.2 \\ 0.4 & 0.3 \end{bmatrix}$$

$\mu_{R^2}(x_i, x_j)$ is less than or equal to $\mu_R(x_i, x_j)$, so R is transitive.

Refrences

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Queries

Thanking You