Financial time series and modeling volatility (Section 10.3.5)

"Stylized Facts" of Financial Returns

Define $X_t = 100*(\ln(P_t) - \ln(P_{t-1}))$ (log returns)

- heavy tailed

$$P(|X_1| > x) \sim C\, x^{-\alpha}, \quad 0 < \alpha < 4.$$

- uncorrelated

$\hat{\rho}_X(h)$ near 0 for all lags h > 0 (MGD sequence?)

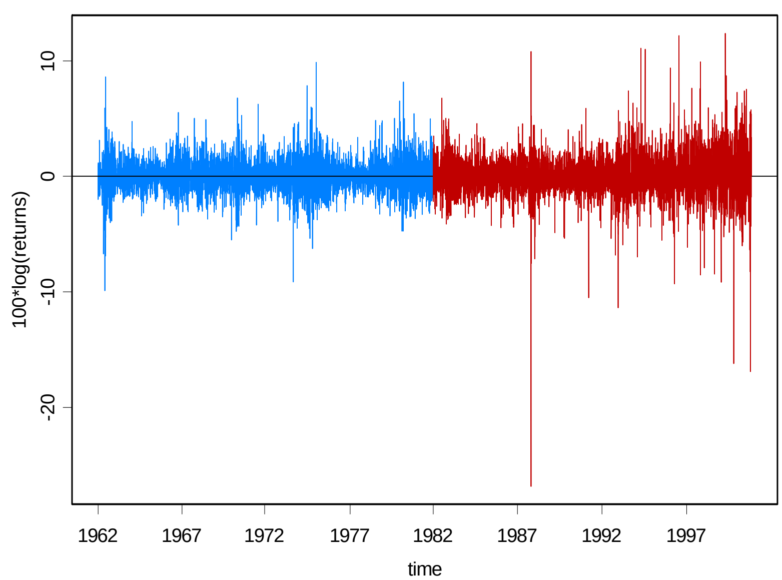
- $|X_t|$ and X_t^2 have slowly decaying autocorrelations

$\hat{\rho}_{|X|}(h)$ and $\hat{\rho}_{X^2}(h)$ converge to 0 *slowly* as h increases.

- process exhibits 'stochastic volatility'.

1

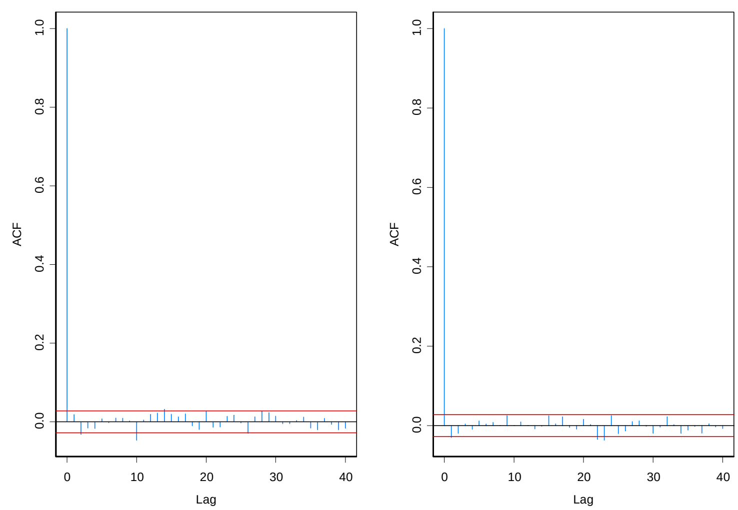
Log returns for IBM 1/3/62-11/3/00 (blue=1961-1981)



2

Sample ACF IBM (a) 1962-1981, (b) 1982-2000

(a) ACF of IBM (1st half) (b) ACF of IBM (2nd half)

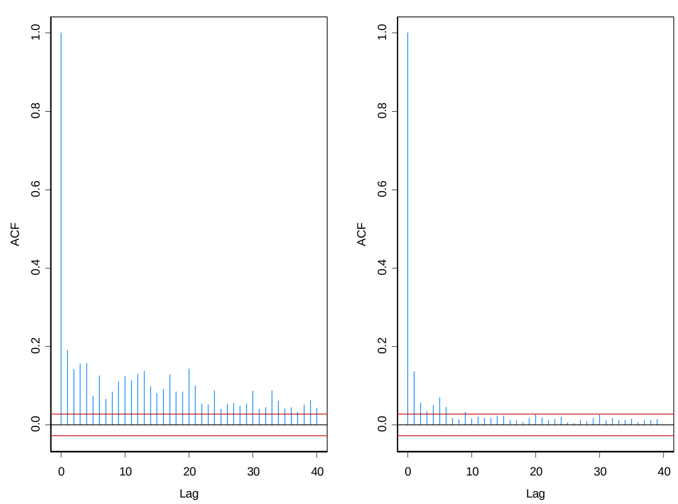


Remark: Both halves look like white noise.

3

ACF of squares for IBM (a) 1961-1981, (b) 1982-2000

(a) ACF, Squares of IBM (1st half) (b) ACF, Squares of IBM (2nd half)



Remark: Series are not independent white noise?

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ARCH and GARCH Models

ARCH(p) (Engle(1982))

$\{Z_t\}$ is a causal strictly and weakly stationary solution of

$$Z_t = \sqrt{h_t} e_t, \quad \{e_t\} \sim \mathrm{IID}(0,1),$$

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-i}^2,$$

where $\alpha_0 > 0$ and $\alpha_i \geq 0, \; i = 1, \ldots, p$.

ARCH = AutoRegressive Conditional Heteroscedasticity.

(See later for existence conditions.)

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ARCH and GARCH Models (cont)

Properties:

If $\{Z_t\}$ is a causal stationary solution, then

1. $E(Z_t | Z_s, s < t) = E[E(Z_t | e_s, s < t) | Z_s, s < t] = 0.$
2. $EZ_t = 0.$
3. $E(Z_s Z_t) = 0, s \neq t.$
4. $E(Z_t^2 | Z_s, s < t) = E[E(h_t e_t^2 | e_s, s < t) | Z_s, s < t]$

$$= h_t = \alpha_0 + \textstyle\sum_{i=1}^{p} \alpha_i Z_{t-i}^2,$$

so h_t= conditional variance of Z_t given $Z_s, s < t$.

5. $E(Z_t^2) = Eh_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i E Z_t^2,$

so that $\quad EZ_t^2 = \alpha_0 / (1 - \sum_{i=1}^{p} \alpha_i).$

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ARCH and GARCH Models (cont)

Theorem 1 (Proof later)

(i) A NS condition for the existence of a causal SS and WS solution $\{Z_t\}$ of the ARCH(p) equations is

$$\sum_{i=1}^{p} \alpha_i < 1 \qquad (1)$$

and $\{Z_t\}$ is the unique such solution.

(ii) If (1) is satisfied and

$$(Ee_t^4)(\sum_{i=1}^{p} \alpha_i)^2 < 1, \qquad (2)$$

then $EZ_t^4 < \infty$.

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ARCH and GARCH Models (cont)

Further Properties:

If conditions (1) and (2) are both satisfied then

6. $\{Z_t^2\}$ is an AR(p) process and all of its correlations are non-negative (generating persistence of volatility). To see this, note that $U_t = Z_t^2 - h_t$ is a MGD sequence and hence WN. It follows that

$$\begin{aligned} Z_t^2 &= Z_t^2 - h_t + h_t \\ &= U_t + \alpha_0 + \alpha_1 Z_{t-1}^2 + \cdots + \alpha_p Z_{t-p}^2. \end{aligned}$$

7. Z_t has heavier tails than e_t in the sense that its kurtosis $(EZ_t^4/(EZ_t^2)^2)$ is greater than or equal to that of e_t.

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ARCH and GARCH Models (cont)

Illustration: The following graphs show the time-series plot, sample ACF and qq plot of 10000 simulated values of an ARCH(2) process $\{Z_t\}$ with

$$\alpha_0 = 1, \quad \alpha_1 = 0.4, \quad \alpha_2 = 0.2$$

and Gaussian noise $\{e_t\}$. The sample ACF's of $\{|Z_t|\}$ and $\{Z_t^2\}$ are also shown.

The sample ACF of $\{Z_t\}$, unlike that of $\{Z_t^2\}$, shows no evidence of the dependence in the series. For this particular ARCH process the model ACF of $\{Z_t^2\}$ has the values 0.5 at lag 1 and 0.4 at lag 2. These are in good agreement with the sample ACF shown.

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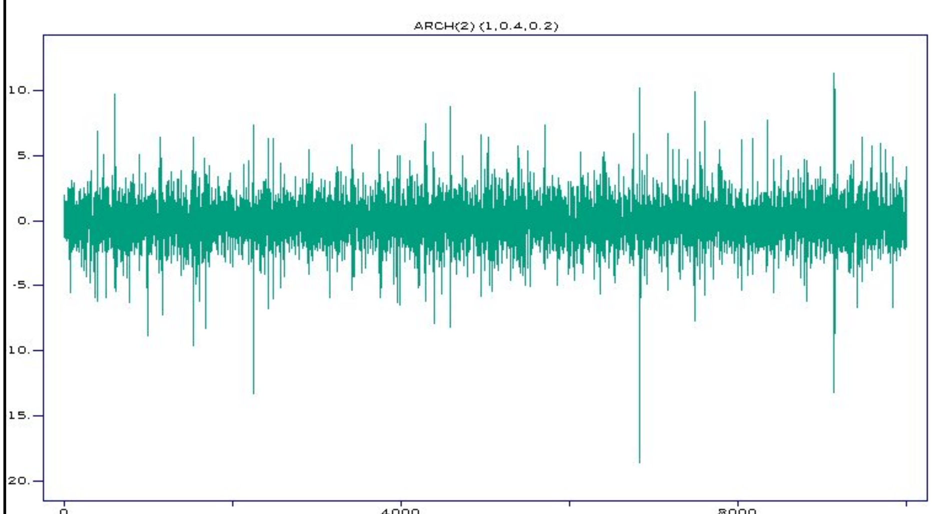
ARCH and GARCH Models (cont)



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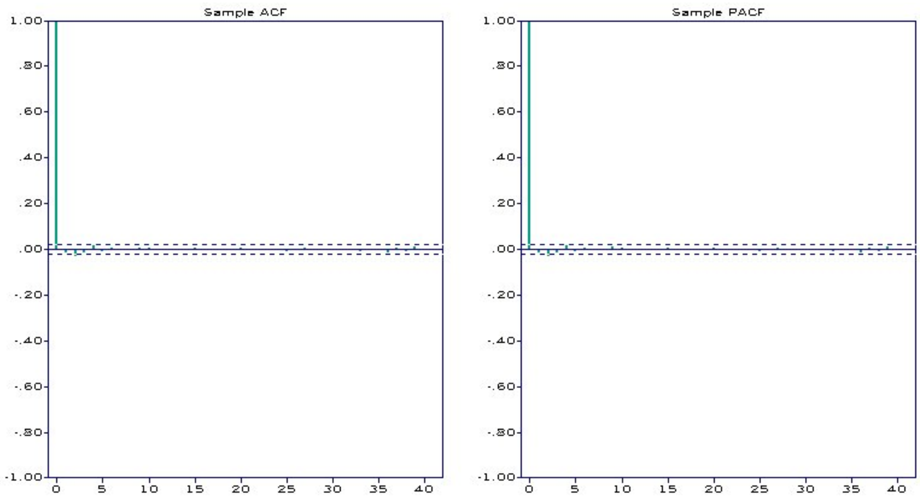
ARCH and GARCH Models (cont)



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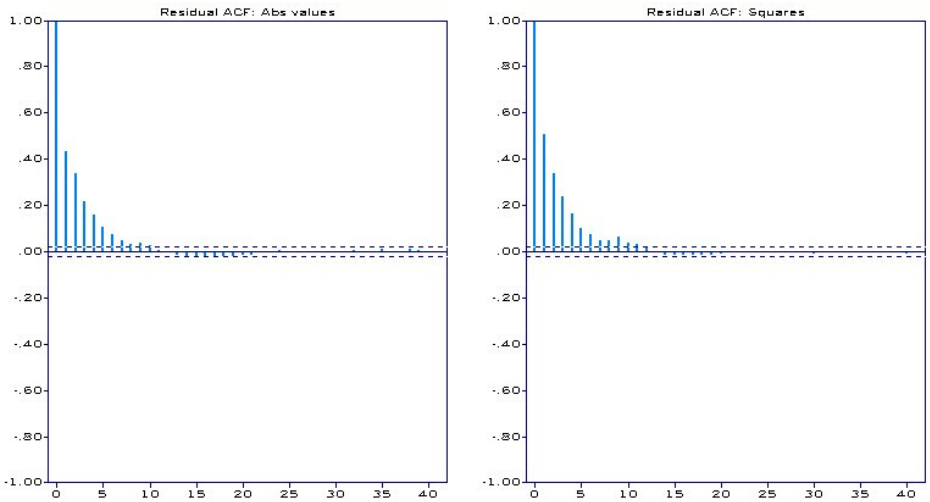
ARCH and GARCH Models (cont)



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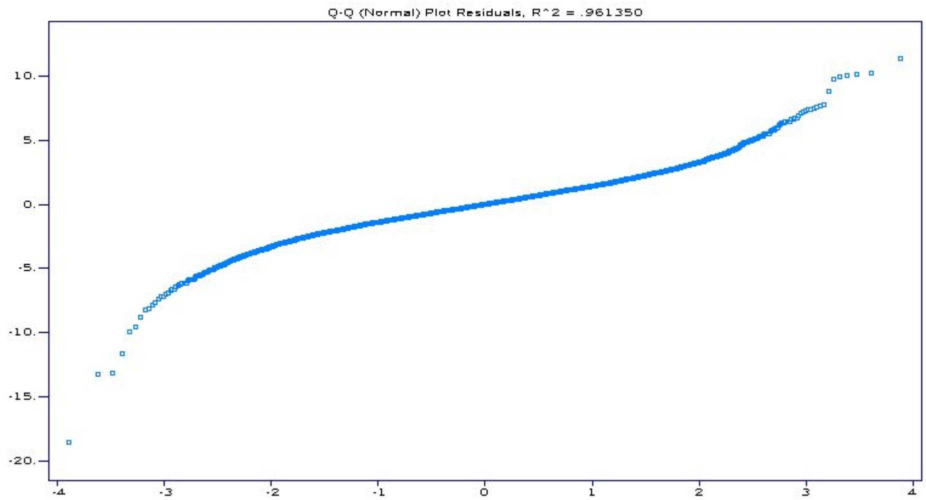
ARCH and GARCH Models (cont)



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ARCH and GARCH Models (cont)

GARCH(p,q) (Bollerslev(1986))

$\{Z_t\}$ is a causal strictly and weakly stationary solution of

$$Z_t = \sqrt{h_t} e_t, \quad \{e_t\} \sim \mathrm{IID}(0,1),$$

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-i}^2 + \sum_{i=1}^{q} \beta_i h_{t-i},$$

where $\alpha_0 > 0$, $\alpha_i \geq 0$ and $\beta_i \geq 0$ for each i.

GARCH = Generalized ARCH.

(See later for existence conditions.)

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ARCH and GARCH Models (cont)

6. $\{Z_t^2\}$ follows an ARMA(m,q)process, where $m = \max(p,q)$.

$$\begin{aligned} Z_t^2 &= Z_t^2 - h_t + h_t \\ &= U_t + \alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-i}^2 + \sum_{i=1}^{q} \beta_i h_{t-i} \\ &= U_t + \alpha_0 + \sum_{i=1}^{m} (\alpha_i + \beta_i) Z_{t-i}^2 - \sum_{i=1}^{q} \beta_i (Z_{t-i}^2 - h_{t-i}) \\ &= \alpha_0 + \sum_{i=1}^{m} (\alpha_i + \beta_i) Z_{t-i}^2 + U_t - \sum_{i=1}^{q} \beta_i U_{t-i} \end{aligned}$$

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ARCH and GARCH Models (cont)

Theorem 3 (GARCH(p,q)) (see Bollerslev (1986)). The equations,

$$Z_t = \sqrt{h_t} e_t, \quad \{e_t\} \sim \mathrm{IID}(0,1),$$

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-i}^2 + \sum_{i=1}^{q} \beta_i h_{t-i},$$

where $\alpha_0 > 0$, $\alpha_i \geq 0$ and $\beta_i \geq 0$ for each i,

have a causal weakly stationary solution if and only if

$$\sum_{i=1}^{p} \alpha_i + \sum_{i=1}^{q} \beta_i < 1.$$

There is exactly one such solution.

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ARCH and GARCH Models (cont)

Parameter Estimation for Finite-Variance GARCH Models

Our model is

$$Z_t = \sqrt{h_t}\, e_t, \quad \{e_t\} \sim \text{IID}(0,1),$$

with

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-i}^2 + \sum_{i=1}^{q} \beta_i h_{t-i},$$

$\alpha_0 > 0$, $\alpha_j, \beta_j \geq 0$ for $j \geq 1$, and

$$\sum_{i=1}^{p} \alpha_i + \sum_{i=1}^{q} \beta_i < 1.$$

For modeling purposes it is usually assumed in addition that either

$$e_t \sim N(0,1),$$

or that

$$\sqrt{\frac{\nu}{\nu-2}} e_t \sim t_\nu, \quad \nu > 2,$$

where t_ν denotes Student's t-distribution with ν degrees of freedom. Other distributions for e_t can however be used.

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ARCH and GARCH Models (cont)

Note:

In the analysis of empirical financial data such as percentage daily stock returns (defined as $100 \ln(P_t/P_{t-1})$, where P_t is the closing price on trading day t), it is often found that better fits to the data are obtained by using the heavier-tailed Student's t-distribution for the distribution of Z_t given $\{Z_s, s < t\}$.

The "persistence of volatility" (large (small) fluctuations in the data tend to be followed by fluctuations of comparable magnitude) is reflected by GARCH models through the correlation in the sequence $\{h_t\}$ of conditional variances.

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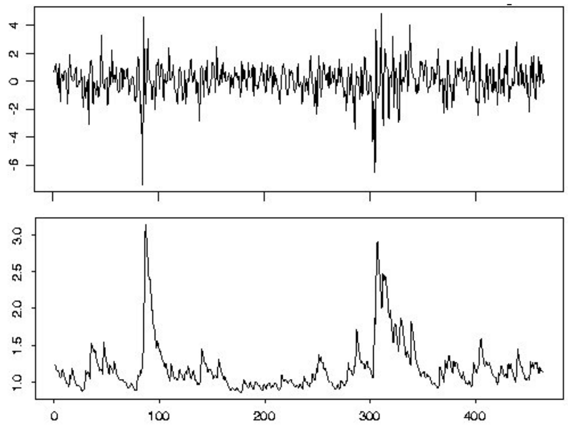
ARCH and GARCH Models (cont)

Example (Fitting a GARCH model to stock returns.)

The top graph shows the percentage daily returns of the Dow-Jones Industrial Index for the period July 1st, 1997, through April 9th, 1999, contained in the file E1032.TSM. The graph suggests that there are sustained periods of both high volatility (in October, 1997, and August, 1998) and of low volatility in between.



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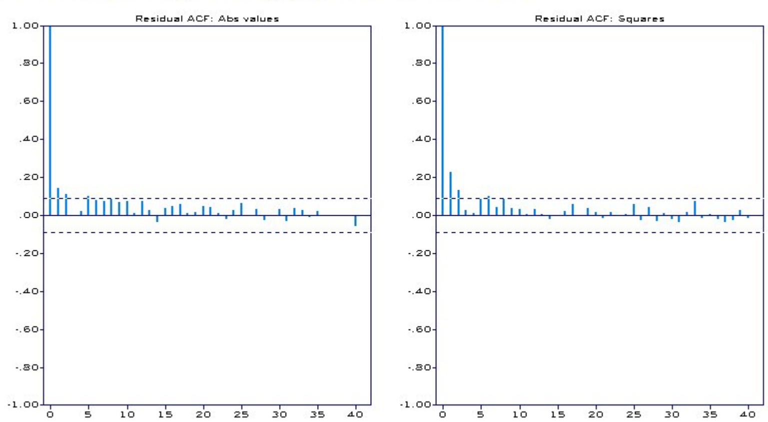
ARCH and GARCH Models (cont)

The sample autocorrelation function of this series, has very small values, however the sample autocorrelations of the absolute values and squares of the data are significantly different from zero, indicating dependence in spite of the lack of autocorrelation. These properties suggest that an ARCH or GARCH model might be appropriate for this series.



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ARCH and GARCH Models (cont)

The model,

$$Y_t = a + Z_t,$$

where $\{Z_t\}$ is Gaussian-driven GARCH(p, q) can be fitted using ITSM as follows.

Open the project E1032.TSM and click on the red button labeled GAR at the top of the ITSM screen. In the resulting dialog box enter the desired values of p and q, e.g. 1 and 1 for GARCH(1,1).

With `Use normal noise` selected, click on `OK` and then click on the red MLE button. Subract the sample mean, which will be used as the estimate of a (unless you wish to assume that the parameter a is zero).

The `GARCH Maximum Likelihood Estimation` box will then open. When you click on `OK` the conditional likelihood maximization will proceed.

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ARCH and GARCH Models (cont)

Denoting by $\{\hat{Z}_t\}$ the (possibly) mean-corrected observations, the program ITSM maximizes the likelihood of $\hat{Z}_{p+1}, \ldots, \hat{Z}_n$ conditional on the known values $\hat{Z}_1, \ldots, \hat{Z}_p$, and with assumed values 0 for each $\hat{Z}_t$, $t \leq 0$, and $\hat{\sigma}^2$ for each h_t, $t \leq 0$, where $\hat{\sigma}^2$ is the sample variance of $\{\hat{Z}_1, \ldots, \hat{Z}_n\}$. In other words the program maximizes

$$L(\alpha_0, \ldots, \alpha_p, \beta_1, \ldots, \beta_q) = \prod_{t=p+1}^{n} \frac{1}{\sigma_t} \phi\left(\frac{\hat{Z}_t}{\sigma_t}\right),$$

with respect to the coefficients $\alpha_0, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$, where ϕ denotes the standard normal density, and the standard deviations $\sigma_t = \sqrt{h_t}$, $t \geq 1$, are computed from the GARCH recursions with Z_t replaced by $\hat{Z}_t$, and with $\hat{Z}_t = 0$ and $h_t = \hat{\sigma}_t^2$ for $t \leq 0$.

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ARCH and GARCH Models (cont)

Comparison of models with different orders p and q can be made with the aid of the AICC, which is defined in terms of the conditional likelihood L as

$$\text{AICC} := -2\frac{n}{n-p}\ln L + 2(p+q+2)n/(n-p-q-3).$$

The factor $\frac{n}{n-p}$ multiplying the first term on the right has been introduced to correct for the fact that the number of factors in he conditional likelihood is only $n-p$. Notice also that the GARCH(p,q) model has $p+q+1$ coefficients.

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ARCH and GARCH Models (cont)

Estimated mean:

$$\hat{a} = 0.0608$$

Minimum-AICC Gaussian GARCH model for $\hat{Z}_t = Y_t - \hat{a}$: GARCH(1,1) with

$$\hat{\alpha}_0 = 0.1300, \hat{\alpha}_1 = 0.1266, \hat{\beta}_1 = 0.7922,$$

$$\text{AICC value} = 1469.02.$$

The bottom graph shown earlier shows the corresponding estimated conditional standard deviations, $\hat{\sigma}_t = \sqrt{\hat{h}_t}$, which clearly reflect the changing volatility of the series $\{Y_t\}$. This graph is obtained from ITSM by clicking on the red SV (stochastic volatility) button.

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ARCH and GARCH Models (cont)

Model-checking:

Under the fitted model, the GARCH residuals, $\{\hat{Z}_t/\hat{\sigma}_t\}$, should be approximately IID N(0,1).

Check independence: Sample ACF's of the absolute values and squares of the residuals (fifth red button at the top of the ITSM window) look OK.

Check normality: `Garch>Garch residuals> QQ-Plot(normal)` should give approximately a straight line through the origin with slope 1. But deviations are large for large values of $|\hat{Z}_t|$, suggestng a heavier-tailed model, e.g. one with conditional t-distribution. Jarque-Bera test for normality has p-value=.00000 to 5 decimal places - reject normality!

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ARCH and GARCH Models (cont)

Fitting a t-GARCH Model:

To fit a t-GARCH model the conditional likelihood is replaced by

$$L(\alpha_0,\ldots,\alpha_p,\beta_1,\ldots,\beta_q,\nu) = \prod_{t=p+1}^{n} \frac{\sigma_t^{-1}\sqrt{\nu}}{\sqrt{\nu-2}}\ t_\nu\left(Z_t\frac{\sigma_t^{-1}\sqrt{\nu}}{\sqrt{\nu-2}}\right).$$

Maximization is now carried out with respect to the coefficients $\alpha_0,\ldots,\alpha_p$, $\beta_1,\ldots,\beta_q$ and the degrees of freedom ν of the t-density, t_ν.

Proceed as before but select `t-distribution for noise` in each of the dialog boxes where it appears.

Good idea to initialize the coefficients by first fitting a Gaussian GARCH model and then optimizing with t-distributed noise.

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ARCH and GARCH Models (cont)

Estimated mean:

$\hat{a} = 0.0608$

Minimum-AICC t-GARCH model for $\hat{Z}_t = Y_t - \hat{a}$:

t-GARCH(1,1) with

$\hat{\alpha}_0 = 0.1324, \hat{\alpha}_1 = 0.0672, \hat{\beta}_1 = 0.8400, \hat{\nu} = 5.714$

AICC value = 1437.89.

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ARCH and GARCH Models (cont)

Model-checking:

Under the fitted model, the GARCH residuals, $\{\hat{Z}_t/\hat{\sigma}_t\}$, should be approximately IID and t-distributed with 5.714 degrees of freedom.

Check independence: Sample ACF's of the absolute values and squares of the residuals (fifth red button at the top of the ITSM window) look OK.

Check t-distribution: Selecting the 6th red button at the top of the ITSM window will give a qq plot using quantiles of the t-distribution with the fitted degrees of freedom (5.714 in this case). The graph is closer to linear than for the Gaussian model.

The improvement in AICC strongly suggests the superiority of the t-driven model.

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ARCH and GARCH Models (cont)

The estimated mean is $\hat{a} = 0.0608$ as before, and the minimum-AICC GARCH model for the residuals, $\hat{Z}_t = Y_t - \hat{a}$, is the GARCH(1,1) with estimated parameter values

$$\hat{\alpha}_0 = 0.1324, \hat{\alpha}_1 = 0.0672, \hat{\beta}_1 = 0.8400, \hat{\nu} = 5.714,$$

and an AICC value (as in (10.3.17) with q replaced by $q+1$) of 1437.89. Thus from the point of view of AICC, the model with conditional t-distribution is substantially better than the conditional Gaussian model. The sample ACF of the absolute values and squares of the GARCH residuals are much the same as those found using Gaussian noise, but the qq plot (obtained by clicking on the red QQ button) is closer to the expected line than was the case for the model with Gaussian noise. □

ARCH and GARCH Models (cont)

ARMA and regression models with GARCH errors

ITSM can be used to fit an ARMA or regression model with GARCH errors by using the procedure described in the last lecture to fit a GARCH model to the residuals $\{\hat{Z}_t\}$ from the ARMA (or regression) fit.

Example Open the file SUNSPOTS.TSM, subtract the mean and use the option `Model>Estimation>Autofit` with the default ranges for p and q.

This gives an ARMA(3,4) model for the mean-corrected data.

Clicking on the second green button at the top of the ITSM window, we see that the sample ACF of the ARMA residuals is compatible with iid noise.

However the sample ACF's of the absolute and squared residuals suggest dependence.

To fit a GARCH(1,1) model to the ARMA residuals:

(i) Click on the red GAR button, enter the value 1 for both p and q and click `OK`.

(ii) Click on the red MLE button, click `OK` in the dialog box, and the **`GARCH ML Estimates`** window will open, showing the estimated parameter values.

(iii) Repeat the steps in the previous sentence two more times and the window will display the following ARMA(3,4) model for the mean-corrected sunspot data and the fitted GARCH model for the ARMA noise process $\{Z_t\}$.

$$X_t = 2.463Z_{t-1} - 2.248Z_{t-2} + .757Z_{t-3} + Z_t - .948Z_{t-1}$$

$$-.296Z_{t-2} + .313Z_{t-3} + .136Z_{t-4},$$

where

$$Z_t = \sqrt{h_t}e_t$$

and

$$h_t = 31.152 + .223Z_{t-1}^2 + .596h_{t-1}.$$

ARCH and GARCH Models (cont)

The AICC value for the GARCH fit (805.12) should be used for comparing alternative GARCH models for the ARMA residuals. The AICC value adjusted for the ARMA fit (821.70) should be used for comparison with alternative ARMA models (with or without GARCH noise). Standard errors of the estimated coefficients are also displayed.

Simulation using the fitted ARMA(3,4) model driven by GARCH(1,1) noise can be carried out by selecting the option n `Model>Simulate`. If you retain the default settings in the **`ARMA Simulation`** dialog box and click `OK` you will see a simulated realization of the model for the *original* data in SUNSPOTS.TSM.

ARCH and GARCH Models (cont)

The AICC value for the GARCH fit (805.12) should be used for comparing alternative GARCH models for the ARMA residuals. The AICC value adjusted for the ARMA fit (821.70) should be used for comparison with alternative ARMA models (with or without GARCH noise). Standard errors of the estimated coefficients are also displayed.

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Forecasting with GARCH

Since the GARCH process $\{Z_t\}$ is a martingale difference sequence,

$$E(Z_{t+m}|Z_s, s \leq t) = 0 \quad \forall m. \tag{1}$$

The past is therefore of no help in predicting Z_{t+h} and the best (in terms of MSE) predictor is the same as the best linear predictor.

However

$$E(Z_{t+1}^2|Z_s, s \leq t) = h_{t+1} = \alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t+1-i}^2 + \sum_{i=1}^{q} \beta_i h_{t+1-i},$$

so that the past is valuable for forecasting the future variance of $\{Z_t\}$. (This is in contrast with the case $\{Z_t\} \sim \mathrm{IID}(0, \sigma^2)$, when $E(Z_{t+1}^2|Z_s, s \leq t) = E(Z_{t+1}^2) = \sigma^2$.)

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Forecasting with GARCH (cont)

Calculation of $E(Z_{t+j}^2|\mathcal{F}_t)$ for an ARCH(1) process:

$$\begin{aligned}
E(Z_{t+1}^2|\mathcal{F}_t) &= h_{t+1} = \alpha_0 + \alpha_1 Z_t^2 \\
E(Z_{t+2}^2|\mathcal{F}_t) &= E[h_{t+2}e_{t+2}^2|\mathcal{F}_t] \\
&= E[E(h_{t+2}e_{t+2}^2|e_s, s \le t+1)|\mathcal{F}_t] \\
&= E[h_{t+2}|\mathcal{F}_t] \\
&= E[\alpha_0 + \alpha_1 Z_{t+1}^2|\mathcal{F}_t] \\
&= \alpha_0(1+\alpha_1) + \alpha_1^2 Z_t^2.
\end{aligned}$$

Repeating this argument gives

$$E(Z_{t+k}^2|\mathcal{F}_t) = \alpha_0(1 + \alpha_1 + \cdots + \alpha_1^{k-1}) + \alpha_1^k Z_t^2.$$

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