Chapter 4: Spectral Analysis

Spectral density: If $\{X_t\}$ is stationary with $\sum_{h=-\infty}^{\infty}|\gamma(h)|<\infty$ then

(0) $$f(\lambda) := \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h)$$
$$= \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}(\cos(h\lambda) - i\sin(h\lambda))\gamma(h)$$
$$= \frac{1}{2\pi}\left[\gamma(0) + 2\sum_{h=1}^{\infty}\cos(h\lambda)\gamma(h)\right]$$

Properties of the spectral density:

(1) $f(\lambda+2\pi) = f(\lambda)$ (periodic with period $2\pi \Rightarrow$ only need to specify f on $(-\pi, \pi]$)

(2) $f(\lambda) = f(-\lambda)$ (even function $\Rightarrow$ only need to specify f on $[0, \pi]$)

(3) $f(\lambda) \geq 0$.

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Proof: $$0 \leq \frac{1}{n}E\left|\sum_{t=1}^{n} X_t e^{-it\lambda}\right|^2 = \sum_{h=-n}^{n}(1-|h|/n)\gamma(h)e^{-ih\lambda}$$
$$\to \sum_{h=-\infty}^{\infty}\gamma(h)e^{-ih\lambda} = 2\pi f(\lambda).$$

(4) $$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)d\lambda$$

Proof: Multiply both sides of the defining equation for $f(\lambda)$ by $e^{ik\lambda}$ and integrate. Use the fact that

$$\gamma(h) = \int_{-\pi}^{\pi} e^{-ik\lambda}e^{ih\lambda}d\lambda = \begin{cases} 2\pi, & \text{if } h = k \\ 0, & \text{if } h \neq k \end{cases}$$

Note: Eqns (0) and (4) show that $\gamma(.)$ determines $f(.)$ and vice versa.

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Proposition 4.1.1 A real valued f on [-π, π] is the spectral density of a stationary time series if and only if

(1) $f(\lambda) = f(-\lambda)$

(2) $f(\lambda) \geq 0.$

(3) $\int_{-\pi}^{\pi} f(\lambda) d\lambda < \infty$

Proof: If $\Sigma_h |\gamma(h)| < \infty$, f has these properties as argued above. If $\Sigma_h |\gamma(h)| = \infty$, see TSTM. Conversely if f has these properties, then define

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda.$$

It follows that $\gamma(h)$ is even and non-negative definite and hence (see Theorem 2.1.1) the ACVF of a stationary time series.

Remark: f is the spectral density of a stationary TS with ACVF $\gamma(h)$ iff $\gamma(h) = \int e^{ih\lambda} f(\lambda) d\lambda$

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Corollary 4.1.1 An absolutely summable function γ() is an ACVF of a stationary time series if and only if it is even and

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) e^{-ih\lambda} \geq 0 \text{ for all } \lambda \in (-\pi, \pi].$$

Proof: We have already seen that this condition is necessary. Now suppose the condition holds. Then

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda.$$

It follows that $\gamma(h)$ is even and non-negative definite and hence (see Theorem 2.1.1) the ACVF of a stationary time series.

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Spectral distribution function.

The spectral density f is like a pdf on [-π, π] except

$$\int_{-\pi}^{\pi} f(\lambda)d\lambda = \gamma(0) \quad \text{(instead of 1)}$$

The spectral distribution function is defined as

$$F(\lambda) = \int_{-\pi}^{\lambda} f(\omega)d\omega$$

and behaves like a probability distribution function except that

$$F(\pi) = \int_{-\pi}^{\pi} f(\omega)d\omega = \gamma(0).$$

Remark: Not every stationary TS has a spectral density, but every stationary TS has a spectral distribution function.

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)d\lambda \quad \text{if spectral density exists}$$

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} dF(\lambda) \quad \text{for all stationary TS!}$$

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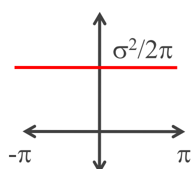
Example 1. White noise satisfies $\Sigma_h |\gamma(h)| < \infty \Rightarrow$ it has spectral density given by

$$f(\lambda) := \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h) = \frac{\gamma(0)}{2\pi} = \frac{\sigma^2}{2\pi}$$



In other words, the spectral density is the same for all $\lambda \epsilon (-\pi, \pi]$ and hence the name *white noise*.

Example 2. Suppose the TS is the random sinusoid given by $X_t = A \cos \omega t + B \sin \omega t$, where A, B uncorrelated $(0, \sigma^2)$. This process does not have a spectral density, but the spectral df is

$$F(\lambda) = \begin{cases} 0, & \lambda < -\omega, \\ \sigma^2/2, & -\omega \le \lambda < \omega, \\ \sigma^2, & \omega \le \lambda. \end{cases}$$

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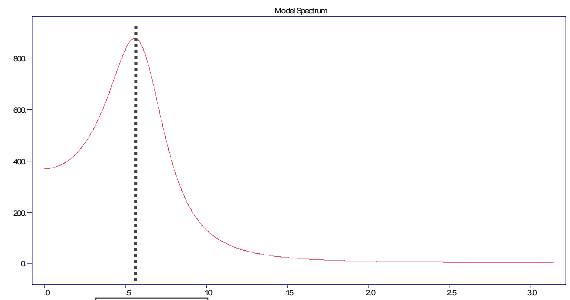
Example 3. The model for the mean corrected sunspot data found in Problem 4.7 is an AR(2) whose spectral density, computed by ITSM, is



$\lambda = .56$

The maximum occurs around $\lambda = .56$ rad/year showing that the most significant frequencies in the decomposition of $\gamma(h)$ (and X_t) into sine waves are those around .56 rad/year. But this corresponds to a period of $T = 2\pi/.56 = 11.22$ years. That is the time series is roughly periodic with period 11.22 years.

Example 4. AR(1) process. The spectral density of the AR(1) process

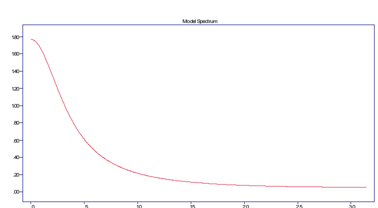
$$X_t = \phi X_{t-1} + Z_t \qquad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$$

is

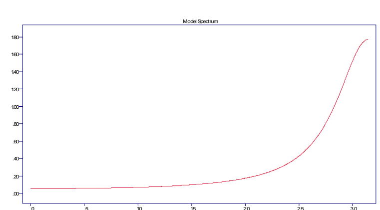
$$f(\lambda) = \frac{\sigma^2}{2\pi\left(1-\phi^2\right)}\left(1+\sum_{h=1}^{\infty}\phi^h\left(e^{-ih\lambda}+e^{ih\lambda}\right)\right)$$

$$= \frac{\sigma^2}{2\pi\left(1-\phi^2\right)}\left(1+\frac{\phi e^{i\lambda}}{1-\phi e^{i\lambda}}+\frac{\phi e^{-i\lambda}}{1-\phi e^{-i\lambda}}\right)$$

$$= \frac{\sigma^2}{2\pi}\left(1-2\phi\cos\lambda+\phi^2\right)^{-1}.$$



$\phi = .7$



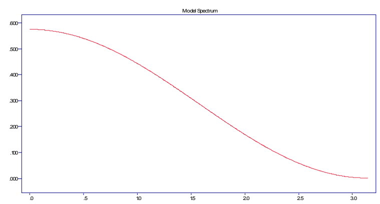
$\phi = -.7$

Example 5. MA(1) process. The spectral density of the MA(1) process

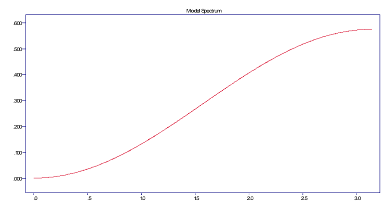
$$X_t = Z_t + \theta Z_{t-1} \qquad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$$

is

$$f(\lambda) = \frac{\sigma^2}{2\pi}\left(1+\theta^2+\theta\left(e^{-i\lambda}+e^{i\lambda}\right)\right) = \frac{\sigma^2}{2\pi}\left(1+2\theta\cos\lambda+\theta^2\right).$$



$\theta = .9$



$\theta = -.9$

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Testing for non-negative definiteness.

Suppose we have an even function $\gamma(.)$ on the integers and we want to know if it is a possible ACVF of a stationary TS. By Theorem 2.1.1, it needs to be nnd which, in practice, can be difficult to check directly. If $\Sigma_h |\gamma(h)| < \infty$, then $\gamma(.)$ is nnd iff

$$f(\lambda) := \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h) \geq 0 \quad \text{for all } \lambda.$$

This can then be used as a test for nnd. To illustrate this, recall Ex 2.1.1.; for which values of ρ is the following function an ACVF?

$$\gamma(h) = \begin{cases} 1, & h = 0 \\ \rho, & h = \pm 1 \\ 0, & |h| > 1 \end{cases}$$

$$f(\lambda) := \frac{1}{2\pi}(1+2\rho\cos\lambda) \geq 0 \quad \text{for all } \lambda$$

if and only if $|\rho| \leq .5$.

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Omit Section 4.2.

Note that the periodogram (or smoothed periodogram) is an estimator of the spectral density. The appropriateness of a fitted model can be checked by comparing the periodogram computed from the data with the spectral density computed from the model. (This is analogous to comparing the sample ACF with the model ACF.)

Section 4.3. Time invariant linear filters.

$$\{X_t\} \xrightarrow{\qquad} \boxed{\{\psi_k\}} \xrightarrow{\qquad} \{Y_t = \Sigma_k\, \psi_k\, X_{t-k} = \psi(B)X_t\}$$

Time invariant: coefficients ψ_k are independent of time

Linear filter: $\psi(B)(aU_t + bV_t) = a\psi(B)U_t + b\,\psi(B)V_t$

Proposition 4.3.1 Suppose $\{X_t\}$ is stationary, mean 0, ACVF $\gamma_X(.)$, spectral density $f_X(.)$, and linear filter $\psi(B) = \Sigma_k\, \psi_k B^k$ with $\Sigma_k\, |\psi_k| < \infty$. If

$$Y_t = \psi(B)X_t = \sum_{k=-\infty}^{\infty} \psi_k X_{t-k},$$

then $\{Y_t\}$ is stationary, mean 0, with ACVF and spectral density given by

$$\gamma_Y(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_X(h+k-j) \qquad \text{(Prop 2.2.1)}$$

$$f_Y(\lambda) = |\psi(e^{-i\lambda})|^2 \, f_X(\lambda),$$

respectively.

Proof: From Prop 2.2.1, we already have the formula for the ACVF.

$$\gamma_Y(h) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\psi_j\psi_k\gamma_X(h+k-j) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\psi_j\psi_k\int_{-\pi}^{\pi}e^{ih\lambda+ik\lambda-ij\lambda}f_X(\lambda)d\lambda$$

$$= \int_{-\pi}^{\pi}e^{ih\lambda}\left(\sum_{k=-\infty}^{\infty}e^{ik\lambda}\psi_k\right)\left(\sum_{j=-\infty}^{\infty}e^{-ij\lambda}\psi_j\right)f_X(\lambda)d\lambda$$

$$= \int_{-\pi}^{\pi}e^{ih\lambda}\left|\sum_{j=-\infty}^{\infty}e^{-ij\lambda}\psi_j\right|^2 f_X(\lambda)d\lambda = \int_{-\pi}^{\pi}e^{ih\lambda}f_Y(\lambda)d\lambda$$

Transfer function: $\psi(e^{-i\lambda}) = \sum_{k=-\infty}^{\infty}e^{-ik\lambda}\psi_k, \qquad -\pi \le \lambda \le \pi$

Power transfer function: $\left|\psi(e^{-i\lambda})\right|^2 = \psi(e^{-i\lambda})\psi(e^{i\lambda}), \qquad -\pi \le \lambda \le \pi$

The power transfer function of the filter relates the spectral density of the output to the spectral density of the input via the relation

$$f_Y(\lambda) = |\psi(e^{-i\lambda})|^2 f_X(\lambda),$$

(In particular if $\left|\psi(e^{-i\lambda_0})\right|^2 = 0$, then the filter eliminates the component of the input with frequency λ_0.)

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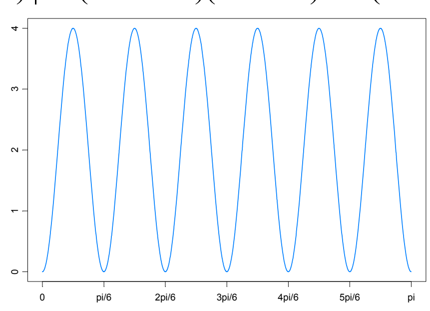
Example 1. $\nabla_{12} = (1-B^{12})$ is the differencing filter used to eliminate seasonal components of period 12. Its power transfer function is

$$|\psi(e^{-i\lambda})|^2 = (1-e^{-12i\lambda})(1-e^{12i\lambda}) = 2(1-\cos(12\lambda))$$



This graph of the power transfer function shows that it eliminates sine waves of frequency 0 (constant), $\pi/6$ (period 12), $2\pi/6$ (period 6), $3\pi/6$ (period 4), etc. Since every period 12 function is a linear combination of a period 12 sine wave and its harmonics, the filter eliminates constants and all period 12 functions.

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Section 4.4 Spectral density of an ARMA process.

From the ARMA equations,

$$\phi(B)X_t = \theta(B)Z_t \qquad \{Z_t\} \sim WN(0, \sigma^2),$$

$$X_t = \psi(B)Z_t\,,$$

where $\psi(B)=\theta(B)/\phi(B)$, we have

$$f_X(\lambda) = |\psi(e^{-i\lambda})|^2 f_Z(\lambda) = |\theta(e^{-i\lambda})|^2 / |\phi(e^{-i\lambda})|^2\, \sigma^2/(2\pi)$$

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}$$

Example 2. AR(2) process, $(1-\phi_1 B - \phi_2 B^2)\, X_t = Z_t$

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$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{(1-\phi_1 e^{-i\lambda} - \phi_2 e^{-i2\lambda})(1-\phi_1 e^{i\lambda} - \phi_2 e^{i2\lambda})}$$

$$= \frac{\sigma^2}{2\pi} \frac{1}{1+\phi_1^2+\phi_2^2-2\phi_1\cos\lambda - 2\phi_2\cos 2\lambda + 2\phi_1\phi_2\cos\lambda}$$

$$= \frac{\sigma^2}{2\pi} \frac{1}{1+\phi_1^2+\phi_2^2+2\phi_2+2(\phi_1\phi_2-\phi_1)\cos\lambda - 4\phi_2\cos^2\lambda}$$

where we have used the identity, $\cos 2\lambda = 2\cos^2\lambda - 1$

For $\phi_2 < 0$, the denominator is a minimum at $\cos\lambda = (\phi_1\phi_2 - \phi_1)/(4\phi_2)$.

For the SUNSPOTS data, the Yule-Walker estimates of the AR(2) model fit are $\phi_1 = 1.318$, $\phi_2 = -.6341$, which produces a peak in the spectral density at

$$\cos\lambda = .849 \text{ i.e., } \lambda = .557$$

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Spectral representation of a zero-mean stationary TS.

Example 4.1.2. Linear combination of sinusoids.

$$X_t = \sum_{j=1}^{k} (A_j \cos(\omega_j t) + B_j \sin(\omega_j t)),$$

where $A_1, \ldots, A_k, B_1, \ldots, B_k$ are uncorrelated with mean 0 and variance $\sigma^2_j = \mathrm{Var}(A_j) = \mathrm{Var}(B_j)$. Then the ACVF and the spectral df for $\{X_t\}$ is given by

$$\gamma(h) = \sum_{j=1}^{k} \sigma_j^2 \cos(\omega_j h), \quad F(\lambda) = \sum_{j=1}^{k} \sigma_j^2 F_j(\lambda),$$

where

$$F_j(\lambda) = \begin{cases} 0, & \text{if } \lambda < -\omega_j \\ .5, & \text{if } -\omega_j \le \lambda < \omega_j \\ 1.0, & \text{if } \lambda \ge \omega_j. \end{cases}$$

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Spectral representation of a zero-mean stationary TS (cont)

It turns out that every stationary TS has this type of representation,

$$X_t = \int_{(-\pi,\pi]} e^{ih\lambda} dZ(\lambda), \qquad \text{spectral representation of ts.}$$

$$\gamma(h) = \int_{(-\pi,\pi]} e^{ih\lambda} dF(\lambda), \qquad \text{spectral representation of ACVF.}$$

where $Z(\lambda)$ is a process of orthogonal increments with

$$E(dZ(\lambda)\overline{dZ(\lambda)}) = \begin{cases} F(\lambda) - F(\lambda-) & \text{discrete case} \\ f(\lambda) d\lambda & \text{continuous case} \end{cases}$$

In our example,

$$E(dZ(\lambda)\overline{dZ(\lambda)}) = \begin{cases} \sigma_j^2 / 2, & \text{if } \lambda = \pm\omega_j \\ 0, & \text{otherwise.} \end{cases}$$

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