ARMA Modeling and Forecasting (Chap 5)

5.1 Preliminary Estimation

Useful for

- order identification (requires the fitting of a number of competing models).
- initial parameter estimates for likelihood optimization.

ARMA(p,q) Model: Based on observations $x_1,\ldots, x_n$,

from the model

$$\phi(B)\, X_t = \theta(B)\, Z_t\,, \qquad \{Z_t\} \sim \mathrm{WN}(0,\sigma^2),$$

want to estimate $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_q)$,

where the orders p and q are assumed known (for the moment).

AR(p) Processes:

Yule-Walker Estimation (moment estimates)

Burg Estimation

ARMA(p,q) Processes:

Innovations Algorithm

Hannan-Rissanen Estimates

Yule-Walker Estimates (for AR(p) processes):

Recall from Chapter 3, that

$$\gamma(0)-\phi_1\gamma(1) - \cdots - \phi_p \gamma(p) = \sigma^2$$

$$\gamma(k)-\phi_1\gamma(k-1) - \cdots - \phi_p \gamma(k-p) = 0, \quad k=1,\ldots,p$$

or in matrix form,

$$\Gamma_p \phi = \gamma_p$$

where Γ_p is the covariance matrix of $X_1, \ldots, X_p$, and $\phi = (\phi_1, \ldots, \phi_p)'$, $\gamma_p = (\gamma(1), \ldots, \gamma(p))'$. The Y-W estimates are then found by replacing the ACVF $\gamma(\ )$, by its estimated value $\hat{\gamma}(\ )$.

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Sample Yule-Walker Equations:

$$\hat{\phi} = \hat{\Gamma}_p^{-1}\hat{\gamma}_p,$$

$$\hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\gamma}_p'\hat{\Gamma}_p^{-1}\hat{\gamma}_p.$$

(These estimates are computed in ITSM .)

Remarks:

- The fitted model is causal.
- The sample ACVF and fitted model ACVF agree at lags h=0,1,. . . ,p.
- Estimates are asymptotically efficient (i.e. have the same limit behavior as MLE).

$$\hat{\phi} \approx N(\phi, n^{-1}\sigma^2\Gamma_p^{-1}).$$

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Order Selection for AR Models

Usually there is no true AR model. Goal is to find an AR model which represents the data in some sense.

Two Techniques.

- If $\{X_t\}$ is an AR(p) process with $\{Z_t\} \sim \text{IID}(0,\sigma^2)$, then $\hat{\phi}_{mm} = \hat{\alpha}(m) \sim N(0, n^{-1})$ for all $m > p$. ($\hat{\alpha}(m)$ is the sample PACF). Estimate p as the smallest value m such that

$$|\hat{\alpha}(k)| < 1.96\, n^{-.5} \text{ for } k > m.$$

- Estimate p by minimizing the AICC statistic

$$\text{AICC} = -2\ln L(\phi_p) + 2(p+1)n/(n-p-2)$$

where L( ) denotes the Gaussian likelihood.

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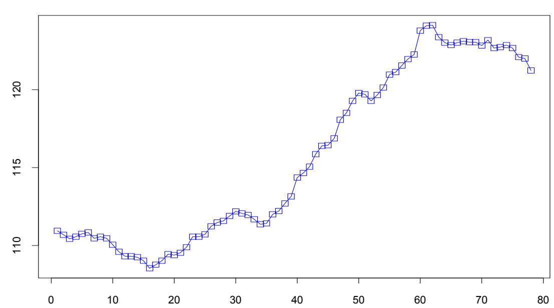
Ex 5.1.1 (Dow-Jones Utilities Index; DOWJ.DAT).
Plot of $D_1, \ldots, D_{78}$



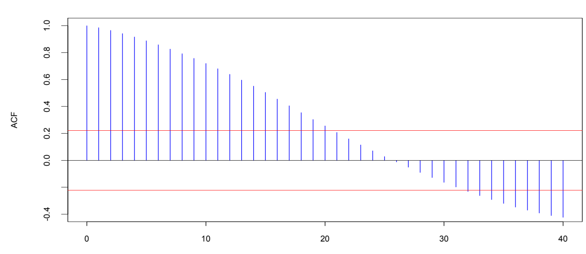


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Fig 5.1 ACF of differenced series $Y_t = D_t - D_{t-1}$

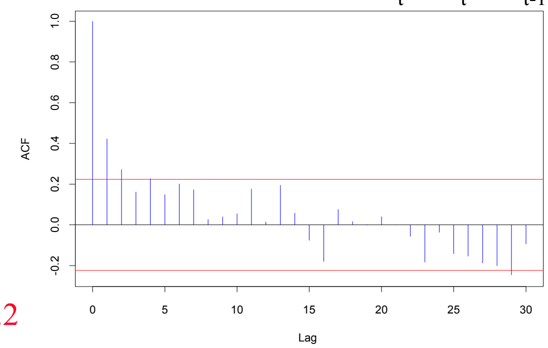
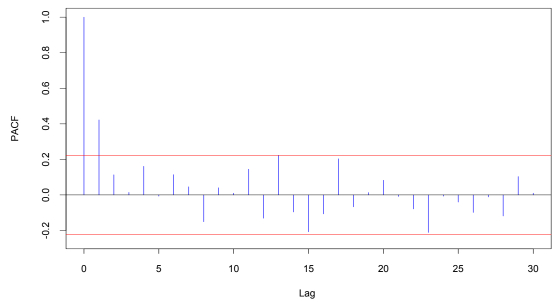


Fig 5.2



Preliminary Model for Dow-Jones:

Using the Preliminary Estimation option of ITSM, the fitted model is

$$X_t = \underset{(.094)}{.4219}\, X_{t-1} + Z_t\,, \quad \{Z_t\} \sim \mathrm{WN}(0, .1479)$$

where

$$X_t = Y_t - .1336, \quad Y_t = D_t - D_{t-1}\,,$$

Remark: We can also arrive at this model using the automatic AICC minimization option in Preliminary Estimation.

5.1.2 Burg's Algorithm

The Yule-Walker estimates $(\hat{\phi}_1, \ldots, \hat{\phi}_p)$ satisfy

$$\tilde{P}_p X_{p+1} = \hat{\phi}_1 X_p + \cdots + \hat{\phi}_p X_1$$

where $\tilde{P}_p$ is the prediction operator relative to the fitted Y-W model. As can be seen from the Durbin-Levinson algorithm, these coefficients are determined from the sample PACF ,

$$\hat{\alpha}(h) = \hat{\phi}_{hh}\ , \ h = 0, \ldots, p.$$

Burg's algorithm uses a different estimate of the PACF based on minimizing the forward and backward one-step prediction errors.

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Properties of Burg's estimates:

- The fitted model is causal.
- Estimates are asymptotically efficient (i.e. have the same limit behavior as MLE).
- Estimates tend to perform better than the Y-W estimates especially if a zero of the AR polynomial is close to the unit circle.

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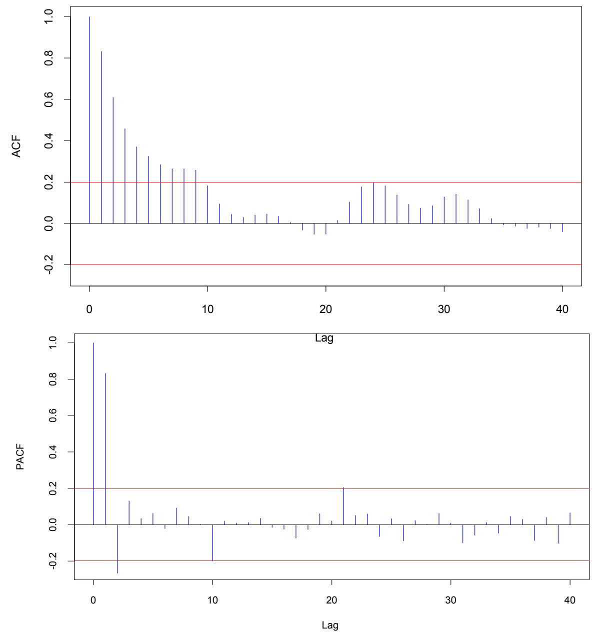
Ex 5.1.4 (Lake Data).



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Model for the Lake data.

The sample ACF and PACF's suggest an AR(2) model for the mean corrected data

$$X_t = Y_t - 9.0041.$$

Burg's model:

$$X_t = 1.0449\, X_{t-1} - .2456\, X_{t-2} + Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, .4705)$$

Y-W fitted model:

$$X_t = 1.0583\, X_{t-1} - .2668\, X_{t-2} + Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, .4920)$$

Minimum AICC model using Burg or Y-W is p=2.

Burg's gives smaller WN variance and larger likelihood

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5.1.3 The Innovations Algorithm

The Yule-Walker estimates were found by applying the Durbin-Levinson Algorithm with the ACF replaced by the sample ACF. The Innovations algorithm estimates are obtained in the same fashion--the ACF is replaced by the sample ACF.

The fitted innovations MA(m) model:

$$X_t = Z_t + \hat{\theta}_{m1} Z_{t-1} + \cdots + \hat{\theta}_{mm} Z_{t-m}, \ \{Z_t\} \sim \mathrm{WN}(0, \hat{v}_m)$$

where $\hat{\theta} = (\hat{\theta}_{m1}, \ldots, \hat{\theta}_{mm})'$ and $\hat{v}_m$ are found from the innovations algorithm with the ACVF replaced by the sample ACVF.

Innovations estimates for MA(q) model.

$$X_t = Z_t + \hat{\theta}_{m1} Z_{t-1} + \cdots + \hat{\theta}_{mq} Z_{t-q}, \ \{Z_t\} \sim \mathrm{WN}(0, \hat{v}_m),$$

where m_n is a sequence of integers,

$$m_n \longrightarrow \infty, \quad m_n = o(n^{1/3}).$$

(These estimates are computed in ITSM.)

Order Selection for MA Models

Three Techniques.

- If $\{X_t\}$ is an MA(q) process with $\{Z_t\} \sim \text{IID}(0,\sigma^2)$, then $\hat{\rho}(m) \sim N(0, n^{-1}(1+2\rho^2(1) + \cdots + 2\rho^2(q))$ for all $m > q$. Estimate q as the smallest value m such that $\hat{\rho}(k)$ is not significantly different from 0, for all $k > m$.
- Examine the coefficient vector in the fitted innovations MA(m) model to see which are not significantly different from 0.
- Estimate q by minimizing the AICC statistic

$$\text{AICC} = -2 \ln L(\theta_q) + 2(q+1)n/(n-q-2)$$

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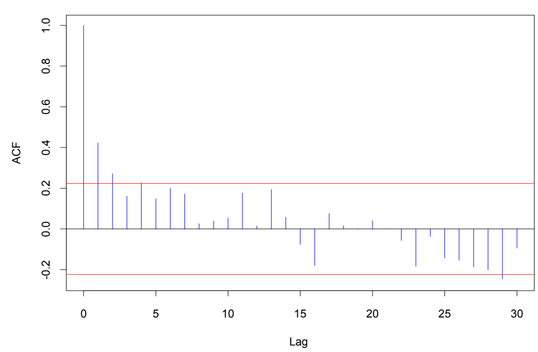
Ex 5.1.5 (Dow-Jones Data). $X_t = D_t - D_{t-1} - .1336$



ACF suggests MA(2).

MA(2) model using innovations estimates: (m=17)

$$X_t = Z_t + \underset{(.114)}{.4269}\, Z_{t-1} + \underset{(.124)}{.2704}\, Z_{t-2}\,,\ \{Z_t\} \sim \text{WN}(0,.1470)$$

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MA(17) model:

MA Coefficient							
.427	.270	.118	.159	.135	.157	.128	-.006
.015	-.003	.197	-.046	.202	.129	-.021	-.258
.076							

COEFFICIENT/(1.96*STANDARD ERROR)							
1.911	1.113	.473	.631	.533	.613	.497	-.023
.057	-.006	.759	-.176	.767	.480	-.079	-.956
.276							

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Innovations Algorithm Estimates When $p > 0$

Step 1. Use Innovations algorithm to fit MA(m) model

$$X_t = Z_t + \hat{\theta}_{m1} Z_{t-1} + \cdots + \hat{\theta}_{mm} Z_{t-m}, \ \{Z_t\} \sim \mathrm{WN}(0, \hat{v}_m),$$

where $m > p + q$.

Step 2. Replace ψ_j by $\hat{\theta}_{mj}$ in the following equations:

$$\psi_j = \theta_j + \sum_{i=1}^{\min(j,p)} \phi_i \psi_{j-i}, \ j = 1, \ldots, p+q.$$

Step 3. Solve the equations in Step 2 for ϕ and θ.

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5.1.4 Hannan-Rissanen Algorithm

Step 1. Fit a high order AR(m) (m > max(p,q)) using Yule -Walker estimates and compute estimated residuals

$$\hat{Z}_t = X_t - \hat{\phi}_{m1} X_{t-1} - \cdots - \hat{\phi}_{mm} X_{t-m}, \quad t = m+1, \ldots, n.$$

Step 2. Estimate ϕ and θ, by regressing X_t onto $(X_{t-1}, \ldots, X_{t-p}, \hat{Z}_{t-1}, \ldots, \hat{Z}_{t-q})$, i.e. by minimizing the sum of squares

$$S(\phi, \theta) = \sum_{t=m+1}^{n} (X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} - \theta_1 \hat{Z}_{t-1} - \cdots - \theta_q \hat{Z}_{t-q})^2$$

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Ex 5.1.7 (Lake Data). We fit an ARMA(1,1) model using the Innovations and Hannan-Rissanen estimates to the mean corrected data $X_t = Y_t - 9.0041$.

Innovations fitted model :

$$X_t - \underset{(.115)}{.7234} X_{t-1} = Z_t + \underset{(.099)}{.3596} Z_{t-1}, \quad \{Z_t\} \sim \mathrm{WN}(0,.4757)$$

Hannan-Rissanen fitted model :

$$X_t - \underset{(.078)}{.6961} X_{t-1} = Z_t + \underset{(.147)}{.3788} Z_{t-1}, \quad \{Z_t\} \sim \mathrm{WN}(0,.4757)$$

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5.2 Maximum Likelihood Estimation

Suppose $\{X_t\}$ is a causal ARMA(p,q) process.

The Gaussian Likelihood :

$$L(\phi, \theta, \sigma^2) = (2\pi\sigma^2)^{-n/2}(r_0 \cdots r_{n-1})^{-.5} \exp\{-\frac{1}{2\sigma^2} \sum_{t=1}^{n} \frac{(X_t - \hat{X}_t)^2}{r_{t-1}}\}$$

where $\hat{X}_t = P_{t-1} X_t$, $\sigma^2 r_{t-1} = E(X_t - \hat{X}_t)^2$.

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Maximum Likelihood Estimators:

$$\hat{\sigma}^2 = n^{-1} S(\hat{\phi}, \hat{\theta}),$$

where

$$S(\hat{\phi}, \hat{\theta}) = \sum_{t=1}^{n} (X_t - \hat{X}_t)^2 / r_{t-1},$$

and $\hat{\phi}, \hat{\theta}$ are the values of ϕ, θ which minimize the **reduced likelihood**

$$l(\phi, \theta) = \ln(n^{-1} S(\phi, \theta)) + n^{-1} \sum_{t=1}^{n} \ln(r_{t-1}).$$

Remark: Minimization of $l(\phi, \theta)$ must be done numerically which is carried out in ITSM.

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Order Selection

AICC Criterion:

Choose p, q, ϕ_p and θ_q to minimize

$$\text{AICC} = -2 \ln L(\phi_p , \theta_q) + 2(p+q+1)n/(n-p-q-2).$$

For fixed p, q, AICC is minimized when ϕ_p , θ_q are the maximum likelihood estimates.

Large Sample Behavior of MLE :

For a large sample,

$(\hat{\phi} , \hat{\theta})'$ is approx $N((\phi , \theta)' , n^{-1}V(\phi , \theta))$ (see p.162)

Ex. 5.2.5 (The Lake Data).

Preliminary ARMA(1,1) model:

$$X_t - .7234\, X_{t-1} = Z_t + .3596\, Z_{t-1},\ \{Z_t\} \sim \text{WN}(0,.4757)$$

(Model fitted to the mean-corrected data, $X_t = Y_t - 9.004$ using the innovations algorithm found in the Preliminary Estimation option of ITSM.)

Maximum likelihood ARMA(1,1) model:

$$X_t - \underset{(.0771)}{.7453}\, X_{t-1} = Z_t + \underset{(.1116)}{.3208}\, Z_{t-1},\ \{Z_t\} \sim \text{WN}(0,.4750)$$

Estimates are found using option ARMA Parameter Estimation in ITSM.

Maximum Likelihood AR(2) Model:

$X_t - 1.0415\, X_{t-1} + .2494\, X_{t-2} = Z_t$, $\{Z_t\} \sim \mathrm{WN}(0,.4790)$

AICC = 213.54 (AR(2) Model)

AICC = 212.77 (ARMA(1,1) model)

Based on AICC, ARMA(1,1) model is slightly better.

5.3 Diagnostic Checking

Residuals:

$$\hat{W}_t = (X_t - \hat{X}_t)/\, r_{t-1}^{1/2}, \quad t = 1, \ldots, n.$$

Properties of $\{\hat{W}_t\}$ should reflect those of the WN sequence $\{Z_t\}$, namely $\{\hat{W}_t\}$ should be approximately

- uncorrelated if $\{Z_t\} \sim \mathrm{WN}(0,\sigma^2)$
- independent if $\{Z_t\} \sim \mathrm{IID}(0,\sigma^2)$
- normally distributed if $Z_t \sim \mathrm{N}(0,\sigma^2)$

Rescaled Residuals :

$$\hat{R}_t = \hat{W}_t \;/\hat{\sigma}$$

Diagnostic Checks :

Graph of $\{\hat{R}_t\}$.

- Inspect for departures from WN (eg. trend, cyclical component, non-constant variance, ...)

Sample ACF of $\{\hat{R}_t\}$.

- Compare with bounds $\pm$ 1.96 $n^{-.5}$.
- Ljung-Box Portmanteau Test
- McLeod-Li Portmanteau Test

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Tests for Randomness

- Turning points
- Difference Sign Test
- Rank Test
- Minimum AICC YW Model for Residuals

All these tests can be carried out in ITSM after model fitting by selecting Statistics>Residual Analysis>Tests of Randomness.

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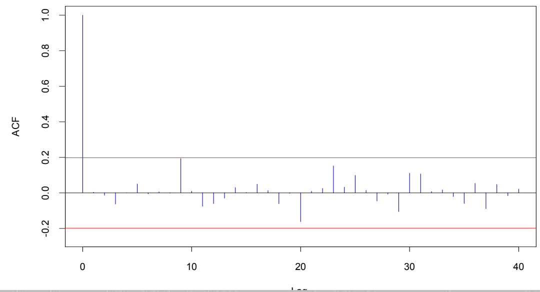
Applied to residuals of Lake Data from ARMA(1,1) fit:

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Ljung - Box statistic = 10.234 Chi-Square ( 20 ), p-value = .96372

McLeod - Li statistic = 16.548 Chi-Square ( 22 ), p-value = .78777

# Turning points = 69.000~AN(64.000,sd = 4.1352), p-value = .22661

# Diff sign points = 50.000~AN(48.500,sd = 2.8723), p-value = .60151

Rank test statistic = .20830E+04~AN(.23765E+04,sd = .16290E+03), p-value = .07160

Jarque-Bera test statistic (for normality) = .28493 Chi-Square (2), p-value = .86722

Order of Min AICC YW Model for Residuals = 0
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5.4 Forecasting

Ex 5.4.1. (Overshorts).

Maximum likelihood MA(1) Model :

$$X_t + 4.035 = Z_t - .818\, Z_{t-1},\ \{Z_t\} \sim \text{WN}(0,2040.75)$$

To predict the next 5 days of overshorts, we have

$$P_{57} X_{57+h} = \begin{cases} -4.035 + \theta_{57,1}(X_{57} - \hat{X}_{57}), & \text{if h=1,} \\ -4.035, & \text{if h>1.} \end{cases}$$

#	XHAT	SQRT(MSE)	XHAT+MEAN
58	1.0097	45.1753	-3.0254
59	0.0000	58.3602	-4.0351
59	0.0000	58.3602	-4.0351
60	0.0000	58.3602	-4.0351
61	0.0000	58.3602	-4.0351

95% Prediction Interval for X_{61} :

$$-4.0351 \mp 1.96(58.3602) = (-118.42, 110.35)$$

This interval assumes that the noise is Gaussian.

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Effect of parameter estimation on forecast error :

Data : $X_1, \ldots, X_n$

Model : $X_t = \phi X_{t-1} + Z_t$, $\{Z_t\} \sim \mathrm{IID}(0,\sigma^2)$

MLE estimate : $\hat{\phi}$

One-step forecast : $\hat{X}_{n+1} = \hat{\phi} X_n$

Mean-square error of forecast :

$$E(X_{n+1} - \hat{\phi} X_n)^2 = E((\phi - \hat{\phi})X_n + Z_{n+1})^2,$$

$$= E((\phi - \hat{\phi})X_n)^2 + \sigma^2.$$

Now, $E((\phi - \hat{\phi})^2|X_n) \sim E(\phi - \hat{\phi})^2 X_n^2 \sim ((1-\phi^2)/n)X_n^2$,

so, $E(X_{n+1} - \hat{\phi} X_n)^2 \sim n^{-1}(1-\phi^2)(1-\phi^2)^{-1}\sigma^2 + \sigma^2$

$$= \sigma^2(1 + 1/n)$$

Error in parameter estimation contributes σ^2/n to MSE.

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5.5 Order Selection

AICC Statistic :

$$-2 \ln L_X(\phi,\theta, S_X(\phi,\theta)/n) + 2(p+q+1)n/(n-p-q-2)$$

AIC Statistic :

$$-2 \ln L_X(\phi,\theta, S_X(\phi,\theta)/n) + 2(p+q+1)$$

BIC Statistic :

$$(n-p-q) \ln (n\,\sigma^2/(n-p-q)) + n(1 + .5 \ln (2\pi)) + (p+q) \ln \Big(\Big(\sum_{t=1}^{n} X_t^2 - n\,\sigma^2\Big)/(p+q)\Big).$$

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Properties of AICC, AIC, BIC :

- BIC is consistent; estimated orders converge to true orders.
- AICC and AIC are not consistent. Designed to be 'unbiased' estimates of Kullback Leibler index.
- AICC and AIC are efficient; fitted model achieves optimal rate of convergence for the mean square (one-step) prediction error (see Section 9.3 of TSTM).

BIC AICC AIC

fewer parameters ⟵⟶ more parameters

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