تمرین درس یادگیری عمیق

سری اول

سؤال ۱)

$$X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix} \qquad W^{[1]} = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix} = \begin{bmatrix} 0.15 & 0.20 \\ 0.25 & 0.30 \end{bmatrix} \qquad b^{[1]} = \begin{bmatrix} 0.35 \\ 0.35 \end{bmatrix}$$

$$Z^{(1)} = W^{[1]}X + b^{[1]} = \begin{bmatrix} 0.15 & 0.2 \\ 0.25 & 0.3 \end{bmatrix}\begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix} + \begin{bmatrix} 0.35 \\ 0.35 \end{bmatrix} = \begin{bmatrix} 0.3775 \\ 0.3925 \end{bmatrix}$$

$$A^{[1]} = \sigma(Z^{(1)}) = \begin{bmatrix} 0.5932 \\ 0.5968 \end{bmatrix}$$

$$W^{[2]} = \begin{bmatrix} w_5 & w_6 \\ w_7 & w_8 \end{bmatrix} \qquad b^{[2]} = \begin{bmatrix} 0.60 \\ 0.60 \end{bmatrix} \qquad Z^{(2)} = W^{[2]}A^{[1]} + b^{[2]} = \begin{bmatrix} 1.105 \\ 1.224 \end{bmatrix}$$

$$A^{[2]} = \sigma(Z^{[2]}) = \begin{bmatrix} 0.751 \\ 0.772 \end{bmatrix}$$

$$Loss = \frac{1}{2}\left[(Y_1 - \hat{Y}_1)^2 + (Y_2 - \hat{Y}_2)^2\right]$$

$$\hat{Y}_1 = 0.751 \qquad \hat{Y}_2 = 0.772 \qquad Loss = \frac{1}{2}\left[(0.01 - 0.751)^2 + (0.02 - 0.772)^2\right] = 0.557$$

$$\frac{\partial L}{\partial w_8} = \frac{\partial L}{\underbrace{\partial A_2^2}_{Y^2}} \times \frac{\partial A_2^2}{\partial z_2^{(2)}} \times \frac{\partial z_2^{(2)}}{\partial w_8} = \left(-(Y_2 - \hat{Y}_2)\right) \times \left(A_2^{(2)}(1 - A_2^{(2)})\right) \times h_2 = \left(-(0.02 - 0.772)\right) \times \left(0.772(1 - 0.772)\right)$$

$$\times\ 0.5968 = 0.078$$

$$w_8 = w_8 - \alpha \frac{\partial L}{\partial w_8} = 0.55 - 0.5 \times 0.078 = 0.5105$$

$$\frac{\partial L}{\partial w_7} = \frac{\partial L}{\partial y_2} \times \frac{\partial y_2}{\partial z_2^2} \times \frac{\partial z_2^2}{\partial w_7} = \left(-(0.02 - 0.772)\right) \times \left(0.772(1 - 0.772)\right) \times \overset{0.5932}{h_1} = 0.0785$$

$$w_7 = w_7 - \alpha \frac{\partial L}{\partial w_7} = 0.50 - 0.5 \times 0.785 = 0.4607$$

s.a.m

$$\frac{\partial L}{\partial w_6} = \frac{\partial L}{\partial y_1} \times \frac{\partial y_1}{\partial z_1^{(2)}} \times \frac{\partial z_1^{(2)}}{\partial w_6} = \left(-(y_1 - \hat{y}_1)\right) \times \left(\overbrace{A_1^{(2)}}^{A_1^2} \times (1 - A_1^{(2)})\right) \times h_2 = 0.082$$

$$w_6 = w_6 - \alpha \frac{\partial L}{\partial w_6} = 0.45 - 0.5 \times 0.082 = 0.408$$

$$\frac{\partial L}{\partial w_5} = \frac{\partial L}{\partial y_1} \times \frac{\partial y_1}{\partial z_1^{(2)}} \times \frac{\partial z_1^{(2)}}{\partial w_5} = \left(-(y_1 - \hat{y}_1)\right) \times \left(A_1^{(2)} \times (1 - A_1^{(2)})\right) \times h_1 = 0.0821$$

$$w_5 = w_5 - \alpha \frac{\partial L}{\partial w_5} = 0.4 - 0.5 \times 0.0821 = 0.358$$

$$\frac{\partial L}{\partial b_1^2} = \frac{\partial L}{\partial y_1} \times \frac{\partial y_1}{\partial z_1^{(2)}} \times \frac{\partial z_1^{(2)}}{\partial b_1^2} = 0.138 \qquad b_1^2 = b_1^2 - 0.5 \frac{\partial L}{\partial b_1^2} = 0.6 - 0.5 \times 0.138 = 0.5307$$

$$\frac{\partial L}{\partial b_2^2} = \frac{\partial L}{\partial y_2} \times \frac{\partial y_2}{\partial z_2^{(2)}} \times \frac{\partial z_2^2}{\partial b_2^2} = 0.132 \qquad b_2^2 = b_2^2 - 0.5 \times 0.132 = 0.5338$$

$$\frac{\partial L}{\partial w_1} = \left(\frac{\partial L}{\partial y_1} \times \frac{\partial y_1}{\partial z_1^2} \times \frac{\partial z_1^2}{\partial A_1^{(1)}} + \frac{\partial L}{\partial y_2} \times \frac{\partial y_2}{\partial z_2^2} \times \frac{\partial z_2^2}{\partial A_1^{(1)}}\right) \times \frac{\partial A_1}{\partial z_1^1} \times \frac{\partial z_1}{\partial w_1}$$

$$= \left(-(y_1 - \hat{y}_1) \times (A_1^2 (1 - A_1^2)) \times w_5 + (-(y_2 - \hat{y}_2)) \times (A_2^2 (1 - A_2^2) \times w_7)\right) \times (A_1^1 (1 - A_1^1)) \times x_1$$

$$= \left((-(0.01 - 0.751) \times (0.751(1 - 0.751)) \times 0.4) + (-(0.02 - 0.772) \times (0.772(1 - 0.772)) \times 0.5)\right) \times (0.5932(1 - 0.5932) \times 0.05$$

$$(0.0554 + 0.0661) \times 0.012 = 0.0014 \qquad w_1 = w_1 - \alpha \frac{\partial L}{\partial w_1} = 0.149$$

$$\frac{\partial L}{\partial w_2} = \left(\frac{\partial L}{\partial y_1} \times \frac{\partial y_1}{\partial z_2^2} \times \underbrace{\frac{\partial z_2^2}{\partial A_2^1}}_{w_6} + \frac{\partial L}{\partial y_2} \times \frac{\partial y_2}{\partial z_2^2} \times \underbrace{\frac{\partial z_2^2}{A_2^1}}_{w_8}\right) \times \frac{\partial A_2^1}{\partial z_2^1} \times \frac{\partial z_2^1}{\partial w_2}$$

$$= \left(-(y_1 - \hat{y}_1) \times (A_1^2 (1 - A_1^2) \times w_6 + (-(y_2 - \hat{y}_2)) \times (A_2^2 (1 - A_2^2)) \times w_8\right) \times (A_2^1 (1 - A_2^1)) \times x_1$$

$$= (0.0623 + 0.0728) \times (0.5968(1 - 0.5968) \times 0.05 = 0.0008 \qquad w_2 = 0.1995$$

s.a.m

$$\frac{\partial L}{\partial w_3} = \left( \frac{\partial L}{\partial y_1} \times \frac{\partial y_1}{\partial z_1^2} \times \frac{\partial z_1^2}{\partial A_1^1} + \frac{\partial L}{\partial y_2} \times \frac{\partial y_2}{\partial z_2^2} \times \frac{\partial z_2^2}{\partial A_1^1} \right) \times \frac{\partial A^1}{\partial z_1^1} \times \frac{\partial z_1^1}{\partial w_3}$$

$$= (0.0554 + 0.0667) \times 0.0241 = 0.0029 \qquad w_3 = w_3 - \alpha \frac{\partial L}{\partial w_3} = 0.2485$$

$$\frac{\partial L}{\partial w_4} = \left( \frac{\partial L}{\partial y_1} \times \frac{\partial y_1}{\partial z_2^2} \times \frac{\partial z_2^2}{\partial A_2^1} + \frac{\partial L}{\partial y_2} \times \frac{\partial y_2}{\partial z_2^2} \times \frac{\partial z_2^2}{\partial A_2^1} \right) \times \frac{\partial A_2^1}{\partial z_2^1} \times \frac{\partial z_2^1}{\partial w_4}$$

$$= (0.0623 + 0.0728) \times (0.5968(1 - 0.5968)) \times 0.1 = 0.0032 \qquad w_4 = 0.2983$$

$\frac{\partial L}{\partial b_1^1}$ → محاسبه می‌شود $\frac{\partial L}{\partial w_3}$ دقیقاً مشابه → محاسبه می‌شود $\frac{\partial z_1^1}{\partial b_1^1}$ فقط در آخر $= 0.0294$ $\qquad b_1^1 = b_1^1 - \alpha \frac{\partial L}{\partial b_1^1} = 0.335$

$\frac{\partial L}{\partial b_2^2}$ → محاسبه می‌شود $\frac{\partial L}{\partial w_4}$ دقیقاً مشابه → محاسبه می‌شود $\frac{\partial z_1^1}{\partial b_2^1}$ فقط در آخر $= 0.0325$ $\qquad b_2^2 = 0.3337$

سوال دوم :

$$f = w_1^{[3]} a_1^{[2]} + w_2^{[3]} a_2^{[2]}$$

activation function در همه‌ی sigmoid است.

الف) $\delta_1 = \dfrac{\partial f}{\partial z_1^{[2]}}$

$$\frac{\partial f}{\partial z_1^{[2]}} = \frac{\partial f}{\partial a_1^{[2]}} \times \frac{\partial a_1^{[2]}}{\partial z_1^{[2]}}$$

$$= w_1^{[3]} \times \left(a_1^{[2]}\left(1 - a_1^{[2]}\right)\right)$$

ب) $\delta_2 = \dfrac{\partial f}{\partial z^{[2]}} =$

$$\frac{\partial f}{\partial z^{[2]}} = \begin{bmatrix} \frac{\partial f}{\partial z_1^{[2]}} \\ \frac{\partial f}{\partial z_2^{[2]}} \end{bmatrix}$$

$$\frac{\partial f}{\partial z_1^{[2]}} = w_1^{[3]} \times \left(a_1^{[2]}\left(1 - a_1^{[2]}\right)\right)$$

$$\frac{\partial f}{\partial z_2^{[2]}} = \frac{\partial f}{\partial a_2^{[2]}} \times \frac{\partial a_2^{[2]}}{\partial z_2^{[2]}} = w_2^{[3]} \times \left(a_2^{[2]}\left(1 - a_2^{[2]}\right)\right)$$

ج) $\delta_3 = \dfrac{\partial f}{\partial z^{[1]}}$

$$\frac{\partial f}{\partial z_1^{[1]}} = \left(\frac{\partial f}{\partial z_1^{[2]}} \times \frac{\partial z_1^{[2]}}{\partial a_1^{[1]}} + \frac{\partial f}{\partial z_2^{[2]}} \times \frac{\partial z_2^{[2]}}{\partial a_1^{[1]}}\right) \times \frac{\partial a_1^{[1]}}{\partial z_1^{[1]}}$$

$$\frac{\partial z_1^{[2]}}{\partial a_1^{[1]}} = w_{11}^{[2]} \qquad \frac{\partial z_2^{[2]}}{\partial a_1^{[1]}} = w_{21}^{[2]}$$

* اینها را در قسمت الف و ب بدست آوردیم.

$$\frac{\partial f}{\partial z_1^{[1]}} = \Big(w_1^{[3]} \times \underbrace{\left(a_1^{[2]}\left(1 - a_1^{[2]}\right)\right)}_{\times w_{11}^{[2]}} + w_2^{[3]} \times \underbrace{\left(a_2^{[2]}\left(1 - a_2^{[2]}\right)\right)}_{\times w_{21}^{[2]}}\Big) \times \left(z_1^{[1]}\left(1 - z_1^{[1]}\right)\right)$$

$$\frac{\partial f}{\partial z_2^{[1]}} = \left(\frac{\partial f}{\partial z_1^{[2]}} \times \frac{\partial z_1^{[2]}}{\partial a_2^{[1]}} + \frac{\partial f}{\partial z_2^{[2]}} \times \frac{\partial z_2^{[2]}}{\partial a_2^{[1]}}\right) \times \frac{\partial a_2^{[1]}}{\partial z_2^{[1]}} \qquad \frac{\partial z_1^{[2]}}{\partial a_2^{[1]}} = w_{12}^{[2]} \qquad \frac{\partial z_2^{[2]}}{\partial a_2^{[1]}} = w_{22}^{[2]}$$

$$= \left(w_1^{[3]} \times \left(a_1^{[2]}\left(1 - a_1^{[2]}\right)\right) \times w_{12}^{[2]} + w_2^{[3]}\left(a_2^{[2]}\left(1 - a_2^{[2]}\right)\right) w_{22}^{[2]}\right) \times \left(z_2^{[1]}\left(1 - z_2^{[1]}\right)\right)$$

s.a.m

$$\delta_4 = \frac{\partial f(x)}{\partial w_{11}^{[1]}} \qquad \frac{\partial f}{\partial w_{11}^{[1]}} = \frac{\partial f}{\partial z_1^{[1]}} \times \frac{\partial z_1^{[1]}}{\partial w_{11}^{[1]}}$$

$$\frac{\partial z_1^{[1]}}{\partial w_{11}^{[1]}} = x_1$$

* در قسمت قبل محاسبه کردیم.

$$\frac{\partial f}{\partial w_{11}^{[1]}} = \left(w_1^{[3]}\left(a_1^{[2]}\left(1-a_1^{[2]}\right)\right) \times w_{11}^{[2]} + w_2^{[3]}\left(a_2^{[2]}\left(1-a_2^{[2]}\right)\right) w_{21}^{[2]}\right) \times \left(z_1^{[1]}\left(1-z_1^{[1]}\right)\right) \times x_1$$

سؤال ۳)

رویکرد یک می‌تواند دو کلاس را از هم تفکیک کند. با قرار دادن یک مقدار به عنوان قرارداد به سادگی می‌توان اعضای دو کلاس را از هم جدا کرد و خروجی ما $0$ و $1$ خواهد بود. در رویکرد دو نیز ما می‌توانیم مسائل دو کلاسه را از هم جدا کنیم. با توجه به این مسئله خروجی Softmax دو احتمال را به ما می‌دهد که مشخص می‌کند نمونه مربوط به کدام کلاس است. در نتیجه می‌توان گفت این دو رویکرد می‌توانند مانند هم عمل کنند. حال باید نشان دهیم که $(W_s, b_s)$ را می‌توان برحسب $(W_l, b_l)$ نوشت.

رویکرد 1: تابع تصمیم $= \hat{y} = \sigma(W_l X + b_l) = \dfrac{1}{1+e^{-(W_l X + b_l)}}$

مرز تصمیم $= W_l X + b_l = 0$

رویکرد 2: تابع تصمیم $= \hat{y} = \text{softmax}(W_s X + b_s) = \text{softmax}\left(\begin{bmatrix} w_{11} & \cdots & w_{1d} \\ w_{21} & \cdots & w_{2d} \end{bmatrix} X + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right)$

مرز تصمیم $= \hat{y}_1 \geq \hat{y}_2 \Rightarrow \dfrac{e^{W_1 X + b_1}}{e^{W_1 X + b_1} + e^{W_2 X + b_2}} \geq \dfrac{e^{W_2 X + b_2}}{e^{W_1 X + b_1} + e^{W_2 X + b_2}}$ مخرج برابر $e^{W_1 X + b_1} \geq e^{W_2 X + b_2}$

$\xrightarrow[\text{از طرفین}]{\text{لگاریتم}} W_1 X + b_1 \geq W_2 X + b_2 \rightarrow (W_1 - W_2) X + (b_1 - b_2) \geq 0$

اگر در نظر بگیریم $W_l = W_1 - W_2$ , $b_l = b_1 - b_2$ $\Rightarrow$ آنگاه مرز تصمیم‌ها یکی می‌شوند.

در نتیجه رویکرد 2 نمی‌تواند مدل‌های پیچیده‌تری نسبت به رویکرد 1 را مدل کند.

s.a.m

پ.ن: در رویکرد 2، من وزن‌ها را برای در نظر گرفتن برای $d$ بعد.