

Definition of matrix calculus

Suppose X and Y is matrix, with arbitrary dimensions. As $X \in \mathcal{R}^{n_1 \times \dots \times n_k}$ and $Y \in \mathcal{R}^{m_1 \times \dots \times m_l}$. Define the matrix calculus as follows

$$\frac{\partial X}{\partial Y} \in \mathcal{R}^{n_1 \times \dots \times n_k \times m_1 \times \dots \times m_l}, \left(\frac{\partial X}{\partial Y} \right)_{i_1 \dots i_k j_1 \dots j_l} = \frac{\partial X_{i_1 \dots i_k}}{\partial Y_{j_1 \dots j_l}}$$

So, the following criterions for matrix calculus hold

- (1) When the column of matrix is seen as **first dimension**, matrix calculus follows Jacobian or numerator formulation
- (2) When the row of matrix is seen as first dimension, matrix calculus follows Hessian or denominator formulation

The two tips are illustrated in the table below

Result of differentiating various kinds of aggregates with other kinds of aggregates							
Column is first dimension		Scalar y		Column vector y (size $m \times 1$)		Matrix Y (size $m \times n$)	
		Notation	Type	Notation	Type	Notation	Type
Scalar x	Numerator	$\frac{\partial y}{\partial x}$	Scalar	$\frac{\partial y}{\partial x}$	Size- m column vector	$\frac{\partial Y}{\partial x}$	$m \times n$ matrix
	Denominator	$\frac{\partial y}{\partial x}$		$\frac{\partial y}{\partial x}$	Size- m row vector	$\frac{\partial Y}{\partial x}$	
Column vector x (size $n \times 1$)	Numerator	$\frac{\partial y}{\partial x}$	Size- n row vector	$\frac{\partial y}{\partial x}$	$m \times n$ matrix	$\frac{\partial Y}{\partial x}$	
	Denominator	$\frac{\partial y}{\partial x}$	Size- n column vector	$\frac{\partial y}{\partial x}$	$n \times m$ matrix	$\frac{\partial Y}{\partial x}$	
Matrix X (size $p \times q$)	Numerator	$\frac{\partial y}{\partial X}$	$q \times p$ matrix	$\frac{\partial y}{\partial X}$		$\frac{\partial Y}{\partial X}$	$p \times q \times m \times n$ matrix
	Denominator	$\frac{\partial y}{\partial X}$	$p \times q$ matrix	$\frac{\partial y}{\partial X}$		$\frac{\partial Y}{\partial X}$	$q \times p \times n \times m$ matrix

Figure 1 Illustration of matrix calculus.