Pricing Options with Mathematical Models

17. Black-Scholes-Merton pricing

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

The model

- The risk-free asset satisfies the ODE

$$dB(t) = rB(t)dt, \quad B(0) = 1$$

implying $B(t) = e^{rt}$.

- The stock satisfies the SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

Using Ito's rule we can check that

$$S(u) = S(t)e^{(\mu - \frac{1}{2}\sigma^2)(u-t) + \sigma[W(u) - W(t)]}$$

Black-Scholes-Merton pricing: PDE approach

- We want to find the price of a European path-independent claim with payoff $C(T) = g(S(T))$.

- It is reasonable to guess that the price will be a function $C(t, S(t))$ of the current time and price of the underlying. If so, from Ito's rule,

$$dC = \left[C_t + \frac{1}{2}\sigma^2 S^2 C_{ss} + \mu S C_s \right] dt + \sigma S C_s dW$$

- On the other hand, with $\pi(t)$ = amount invested in stock at time t, a self-financing wealth process satisfies:

$$dX(t) = \frac{\pi(t)}{S(t)} dS(t) + \frac{X(t) - \pi(t)}{B(t)} dB(t)$$

$$dX = [rX + (\mu - r)\pi]dt + \sigma\pi dW$$

Replication produces a PDE

- If we want replication, $C(t) = X(t)$, we need the dt terms to be equal, and the dW terms to be equal.
- Comparing the dW terms we get that the number of shares needs to be equal to the so-called **delta of the option**

$$\frac{\pi(t)}{S(t)} = C_s(t, S(t))$$

- Using this and comparing the dt terms we get the Black-Scholes PDE:

$$C_t + \frac{1}{2}\sigma^2 s^2 C_{ss} + r(sC_s - C) = 0$$

- subject to the boundary condition,

$$C(T, s) = g(s)$$

The bottom line

- If the PDE has a unique solution $C(t, s)$, it means we can replicate the option by holding delta shares at each time. The option price at time t when the stock price is equal to s is given by $C(t, s)$, and the option delta is the derivative of the option price with respect to the underlying.
- The PDE and the option price do not depend on the mean return rate μ of the underlying!

Black-Scholes formula

- For an European call option $g(s) = (s-K)^+$, the solution of the PDE is given by the Black-Scholes formula:

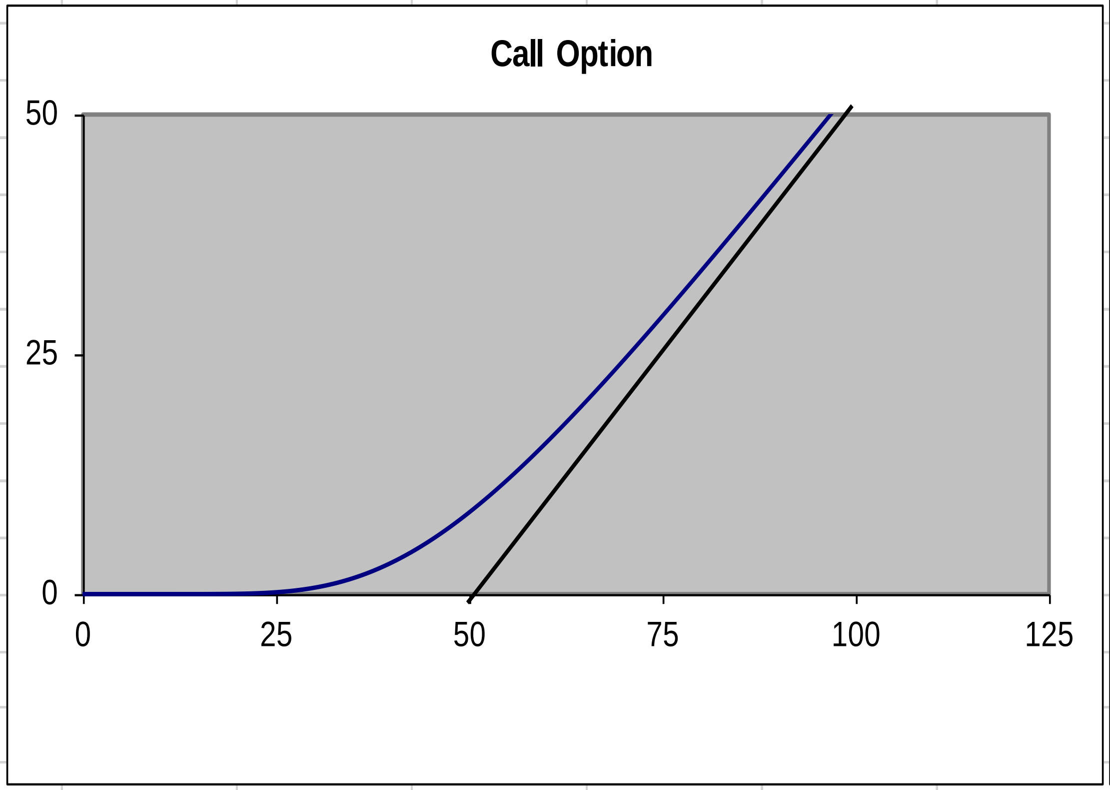
$$C(t, S(t)) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$$

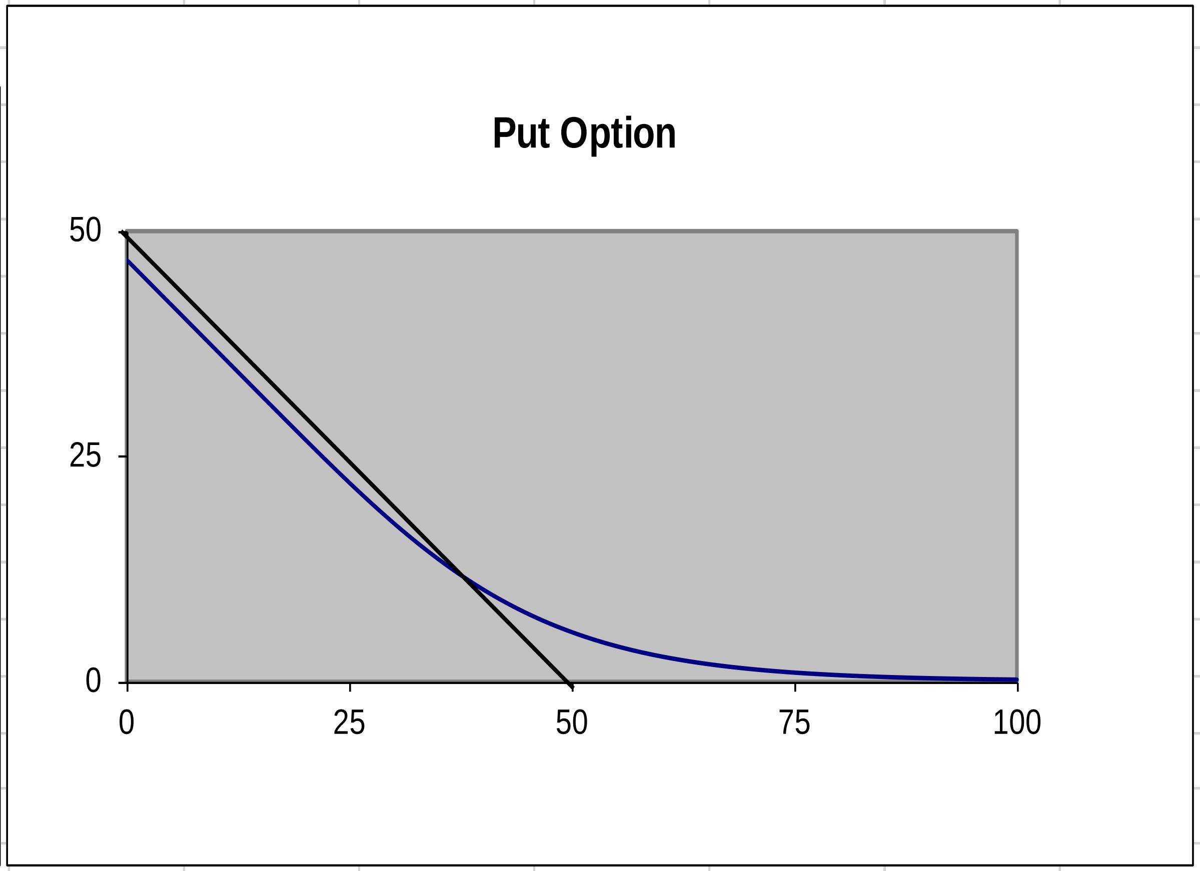
where

$$N(x) := P[Z \le x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}[\log(S(t)/K) + (r+\sigma^2/2)(T-t)] \\ d_2 &= \frac{1}{\sigma\sqrt{T-t}}[\log(S(t)/K) + (r-\sigma^2/2)(T-t)] \\ &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

Graphs of the PDE solutions





Pricing Options with Mathematical Models

18. Risk-neutral pricing: Black-Scholes-Merton model

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Risk-neutral probability in B-S-M model

- Let us find the dynamics of S under the risk-neutral probability Q.
- Denote by W^Q the Brownian motion under Q.
- We claim that if we replace μ by r, that is, if the stock satisfies

$$\frac{dS(t)}{S(t)} = rdt + \sigma dW^Q(t)$$

then the discounted stock price is a Q-martingale.

- Indeed, this is because Ito's rule then gives

$$d(e^{-rt}S(t)) = e^{-rt}dS(t) + S(t)d(e^{-rt})$$

$$= e^{-rt}[rS(t)dt + \sigma S(t)dW^Q(t)] - S(t)re^{-rt}dt = 0 \times dt + \sigma \bar{S}(t)dW^Q(t)$$

Girsanov theorem

- In order to have the above dynamics and also

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t),$$

we need to have

$$W^Q(t) = W(t) + \frac{\mu - r}{\sigma} t$$

- The famous **Girsanov theorem** tells us that this is possible: there exists a unique probability Q under which so-defined W^Q is a Brownian motion.

- The discounted wealth process is also a Q-martingale:

$$\begin{aligned} d(e^{-rt}X(t)) &= e^{-rt}(\mu - r)\pi(t)dt + e^{-rt}\sigma\pi(t)dW(t) \\ &= e^{-rt}\sigma\pi(t)dW^Q(t) \end{aligned}$$

Black-Scholes formula as an expected value

- Option prices can then be computed taking expected values under Q.
- To do that, we note that we can write

$$S(T) = S(0)e^{\sigma W^Q(T)+(r-\frac{1}{2}\sigma^2)T}$$

- We have to compute

$$E^Q[e^{-rT}(S(T)-K)^+]$$

$$= E^Q[e^{-rT}S(T)\mathbf{1}_{\{S(T)>K\}}] - Ke^{-rT}E^Q[\mathbf{1}_{\{S(T)>K\}}]$$

Computing the expected values

- For the second term we compute the price of a Digital (Binary) option:

$$E^Q e^{-rT}\mathbf{1}_{\{S(T)>K\}} = e^{-rT} Q(S(T) > K)$$

$$\begin{aligned} Q(S(T) > K) &= Q\left(S(0)e^{(r-\sigma^2/2)T+\sigma W^Q(T)} > K\right) \\ &= Q\left(\frac{W^Q(T)}{\sqrt{T}} > -d_2\right) \\ &= N(d_2) \end{aligned}$$

where the middle equality follows by taking logs and re-arranging.

- The first term is computed using the formula

$$E\left[g\left(\frac{W^Q(T)}{\sqrt{T}}\right)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-x^2/2}dx$$

Reminder: Black-Scholes formula

$$C(t, S(t)) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$N(x) := P[Z \leq x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}[\log(S(t)/K) + (r + \sigma^2/2)(T-t)] \\ d_2 &= \frac{1}{\sigma\sqrt{T-t}}[\log(S(t)/K) + (r - \sigma^2/2)(T-t)] \\ &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

Another way to get the PDE

- Under risk-neutral probability Q, by Ito's rule

$$dC(t, S(t)) = [C_t + rS(t)C_s + \frac{1}{2}\sigma^2 S^2(t)C_{ss}]dt + \sigma C_s S(t) dW^Q(t)$$

Then, discounting,

$$d(e^{-rt}C(t, S(t)))$$

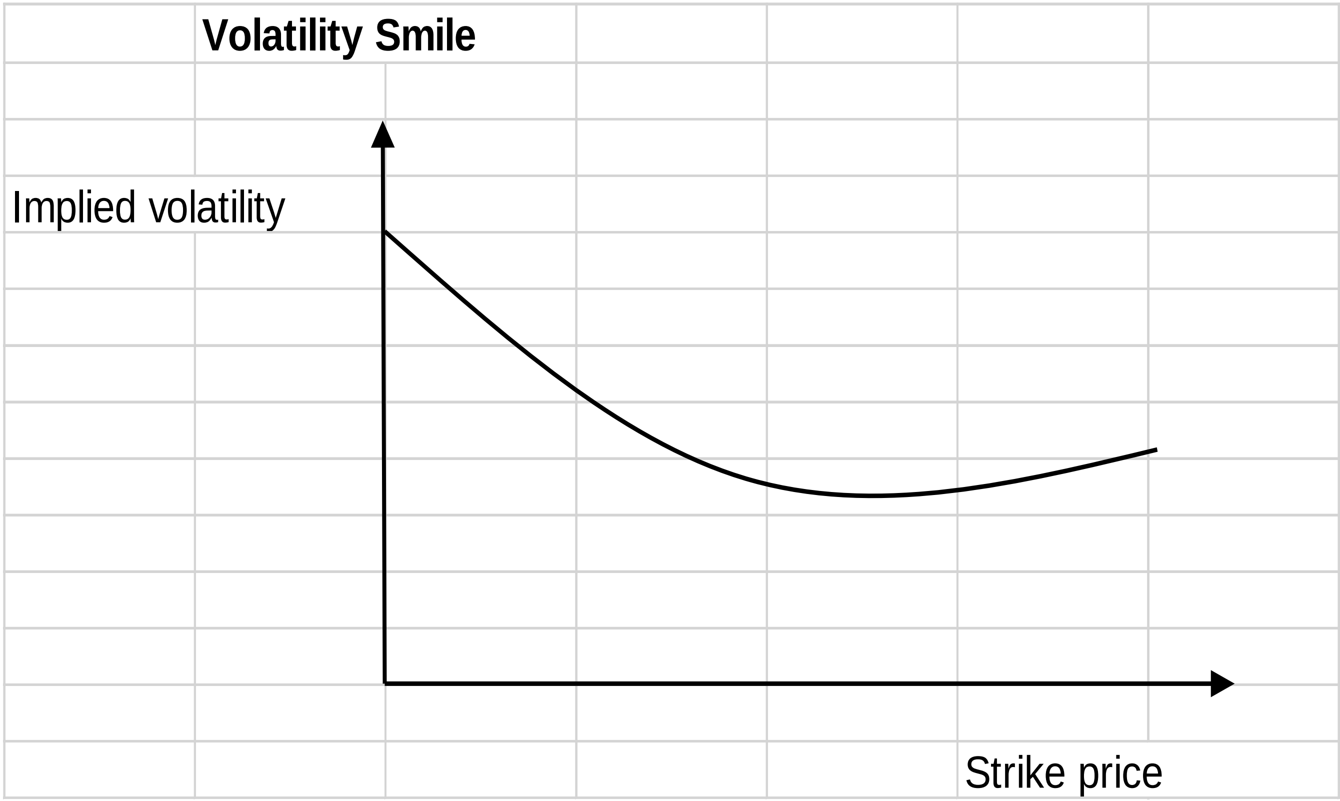
$$= e^{-rt}[(C_t + rS(t)C_s + \frac{1}{2}\sigma^2 S^2(t)C_{ss} - rC)]dt$$

$$+e^{-rt}\sigma C_s S(t) dW^Q$$

- This has to be a Q martingale, which means that the dt term has to be zero, resulting in the Black-Scholes PDE.

Implied volatility

- It is the value of σ that matches the theoretical Black-Scholes price of the option with the observed market price of the option



- In the Black - Scholes model, volatility is the same for all options on the same underlying

- However, this is not the case for implied volatilities: **volatility smile**