Pricing Options with Mathematical Models

14. Brownian motion process

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

History

- Brown, 1800's
- Bachelier, 1900
- Einstein 1905, 1906
- Wiener, Levy, 1920's, 30's
- Ito, 1940's
- Samuelson, 1960's
- Merton, Black, Scholes, 1970's

A short introduction to the Merton-Black-Scholes model

- Risk-free asset

$$B(t) = e^{rt}$$

- Stock has a **lognormal distribution**:

$$\log S(t) = log\, S(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\mathrm{z}(\mathrm{t})$$

where $\mathrm{z}(\mathrm{t})$ is a standard normal random variable. Thus,

$$S(t) = S(0)\ \ e^{\ (\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\mathrm{z}(\mathrm{t})}$$

and it can be shown that

$$ES(t) = S(0)\ e^{\mu t},\ \ \frac{1}{t}Var\left[\log\frac{S(t)}{S(0)}\right] = \sigma^2$$

Discretized Brownian motion

- W(0) = 0
- $\mathrm{W}(t_{k+1}) = \mathrm{W}(t_k) + \sqrt{\Delta t}\, \mathrm{z}(t_k)$

where $\mathrm{z}(t_k)$ are independent standard normal random variables.

Thus,

- $\mathrm{W}(t_l) - \mathrm{W}(t_k) = \sqrt{\Delta t} \ \sum_{i=k}^{l-1} \mathrm{z}(t_i)$

is normally distributed, with zero mean and variance $(l-k)\Delta t = t_l$-t_k

Brownian motion definition

- **(i)** $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$, for $s < t$.

- **(ii)** The process W has independent increments: for any set of times $0 \leq t_1 < t_2 < \cdots < t_n$, the random variables

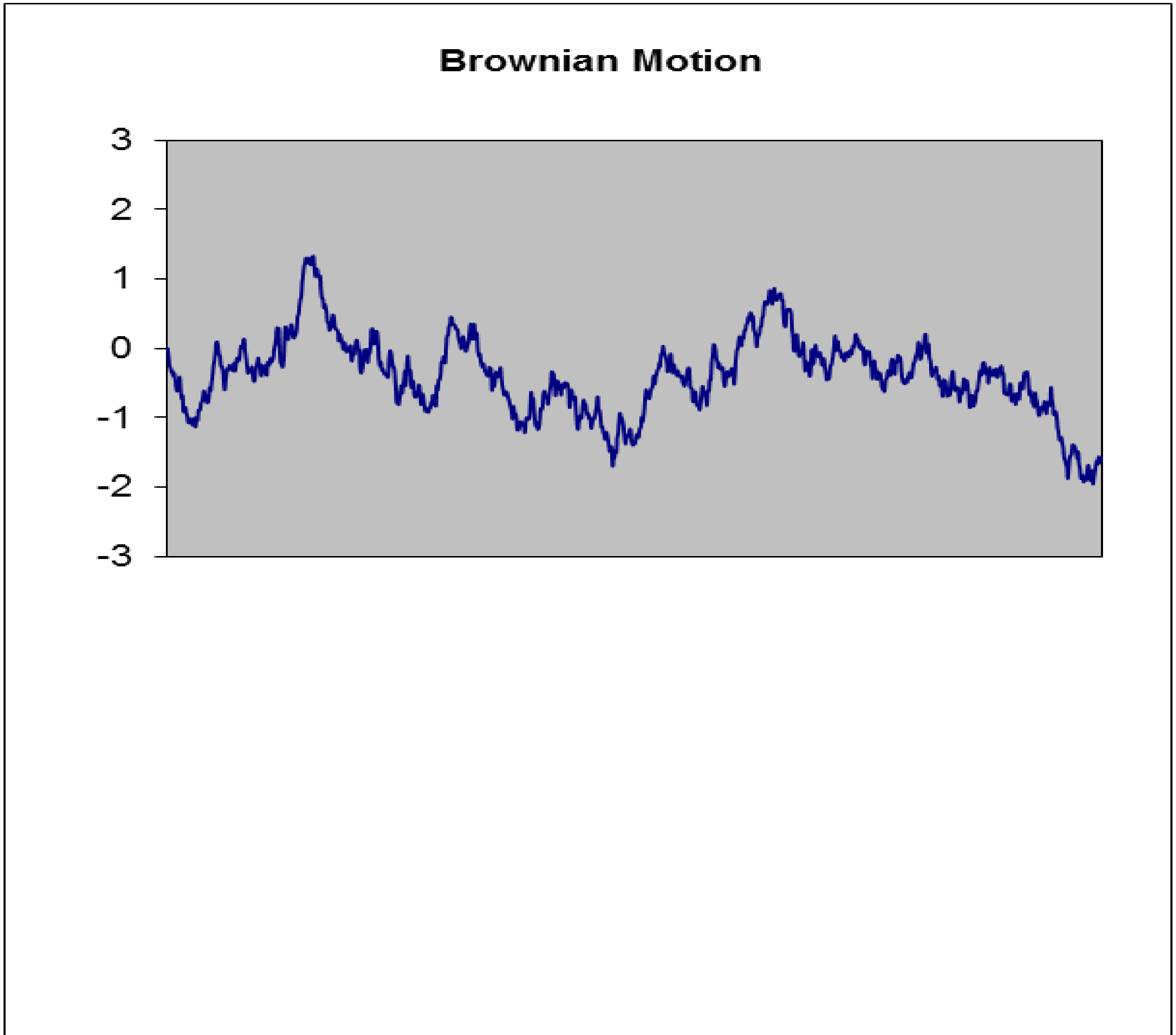
$$W(t_2) - W(t_1), W(t_3) - W(t_2), \ldots, W(t_n) - W(t_{n-1})$$

  are independent.

- **(iii)** $W(0) = 0$.

- **(iv)** The sample paths $\{W(t); t \geq 0\}$ are continuous functions of t.

A simulated path of Brownian motion



Brownian motion properties

- Not differentiable: $E\frac{[W(t)-W(s)]^2}{(t-s)^2} = \frac{1}{t-s} \to \infty$ as $(t-s) \to 0$.
- A Markovian process: the distribution of the future value $W(t)$ given information up to time $s < t$ depends only on $W(s)$ and not on the past values.
- Martingale property:

$$E_s W(t) = W(s),\ t > s$$

because

$$\begin{aligned} E_s W(t) &= E[W(t)|W(s)] = E[W(t) - W(s)|W(s)] + E[W(s)|W(s)] \\ &= E[W(t) - W(s)] + W(s) \\ &= W(s) \end{aligned}$$

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15. Stochastic integral

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Stochastic Differential Equations

- Modeling a process in time with an Ordinary Differential Equation:

$$\frac{dX(t)}{dt} = \mu(t, X(t))$$

which may be informally written as

$$dX(t) = \mu(t, X(t))dt$$

- We would like to have a Stochastic Differential Equation (SDE):

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t)$$

- We will define it in the integral form:

$$X(t) = X(0) + \int_0^t \mu(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s)$$

Stochastic integral, Ito integral

- Fix a process Y adapted to the information given by W, such that

$$E\left[\int_0^t Y^2(u)du\right] < \infty$$

- **Construction:** split interval $[0,t]$ into n subintervals $[t_i, t_{i+1}]$ and consider

$$I_n(t) := \sum_i Y(t_i)[W(t_{i+1}) - W(t_i)]$$

Ito showed that there is a limit $I(t)$,

$$E[(I_n(t) - I(t))^2] \to 0$$

and called the limit the **stochastic integral**:

$$I(t) = \int_0^t Y(s)dW(s)$$

Stochastic integral properties

- Process $I(t) = \int_0^t Y(u)dW(u)$ is a martingale with mean zero, or

$$
\begin{aligned}
E\left[\int_0^t Y(u)dW(u)\right] &= 0 \\
E_s\left[\int_0^t Y(u)dW(u)\right] &= \int_0^s Y(u)dW(u)
\end{aligned}
$$

and the variance is

$$
E\left[\left(\int_0^t Y(u)dW(u)\right)^2\right] = E\left[\int_0^t Y^2(u)du\right]
$$

Reasons why the martingale property

- We have

$$E_s \int_0^t Y(u)dW(u) = E_s \int_0^s Y(u)dW(u) + E_s \int_s^t Y(u)dW(u)$$

$$= \int_0^s Y(u)dW(u) + E_s \int_s^t Y(u)dW(u)$$

- We claim that

$$E_s \int_s^t Y(u)dW(u) = 0$$

For example, for $t_{j+1}, t_j > s$,

$$\begin{aligned}
E_s[Y(t_j)(W(t_{j+1}) - W(t_j))] &= E_s E_{t_j}[Y(t_j)(W(t_{j+1}) - W(t_j))] \\
&= E_s \left\{ Y(t_j) E_{t_j}[(W(t_{j+1}) - W(t_j))] \right\} \\
&= 0
\end{aligned}$$

Reasons why the variance

- We have, for example

$$
\begin{aligned}
E\left[Y^2(t_1)(W(t_2)-W(t_1))^2\right] &= E\left[E_{t_1}\left\{Y^2(t_1)(W(t_2)-W(t_1))^2\right\}\right] \\
&= E\left[Y^2(t_1)E\{(W(t_2)-W(t_1))^2\}\right] \\
&= E[Y^2(t_1)(t_2-t_1)]
\end{aligned}
$$

Here, we used the fact that

$$E_{t_1}\{(W(t_2)-W(t_1))^2\} = E\{(W(t_2)-W(t_1))^2\}$$

because $(W(t_2)-W(t_1))$ is independent of the information available up to time t_1.

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16. Ito’s rule, Ito’s lemma

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Ito’s rule

- Standard calculus:

$$\frac{d}{dt}f(t,x(t)) = \frac{\partial}{\partial t}f(t,x(t)) + \frac{\partial}{\partial x}f(t,x(t))\frac{d}{dt}x(t)$$

- or, denoting partial derivatives with subscripts,

$$df(t,x(t)) = f_t(t,x(t))dt + f_x(t,x(t))dx(t)$$

- In stochastic calculus, for

$$dX(t) = \mu(t)dt + \sigma(t)dW(t)$$

$$df(t,X(t)) = f_t(t,X(t))dt + f_x(t,X(t))dX(t) + \frac{1}{2}\sigma^2 f_{xx}(t,X(t))dt$$

or

$$df = \left[f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx}\right]dt + \sigma f_x dW$$

Reason why – quadratic variation

- Split interval $[0, t]$ into pieces of length Δt.
- Consider the sum of absolute increments to the p-th power

$$Q_p(t, W) := \sum_i |W(t_{i+1}) - W(t_i)|^p$$

- For $p = 2$, its limit is called **quadratic variation** and for Brownian motion we have

$$Q_2(t, W) \to t \ , \quad \text{as } \Delta t \to 0$$

while $Q_1(t, W) \to \infty$.

- For a differentiable function f,

$$Q_1(t, f) \to \int_0^t |f'(s)| ds \ , \quad \text{and } Q_2(t, f) \to 0$$

“Proof” of Ito’s rule

- Taylor expansion:

$$f(t+\Delta t, X(t+\Delta t)) - f(t, X(t)) = f_t \Delta t + f_x \Delta X$$

$$+ \frac{1}{2} f_{xx} (\Delta X)^2 \quad + \text{ higher order terms}$$

- The second order term does not disappear:

$$(\Delta X)^2 = (\mu \Delta t + \sigma \Delta W)^2 = \mu^2 (\Delta t)^2 + 2\mu\sigma \Delta W \Delta t + \sigma^2 (\Delta W)^2$$

In the limit when $\Delta t \to 0$ this gives

$$(dX)^2 = \sigma^2 dt$$

- We get Itô’s rule: $\quad df = f_t dt + f_x dX + \frac{1}{2} f_{xx} \sigma^2 dt$

More on Ito's rule

- We can write

$$df = f_t dt + f_x dX + \frac{1}{2} f_{xx} dX \cdot dX$$

using the following informal rules:

$$dt \cdot dt = 0, \quad dt \cdot dW = 0, \quad dW \cdot dW = dt$$

- This gives

$$dX \cdot dX = (\mu dt + \sigma dW) \cdot (\mu dt + \sigma dW) = \sigma^2 dt$$

Example: $W^2(t)$

$$\int_0^t W(s)dW(s) = ?$$

Consider function $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$. Since

$$dW = 0 \times dt + 1 \times dW$$

we have, by Ito's rule,

$$dW^2(t) = 2W(t)dW(t) + \frac{1}{2} \times 2dt$$

which can be written as

$$W^2(t) - W^2(0) = \int_0^t 2W(s)dW(s) + \int_0^t ds$$

which gives

$$2\int_0^t W(s)dW(s) = W^2(t) - t$$

Exponential of Brownian motion

- The process
$$Y(t) = e^{aW(t)+bt}$$
is a function $Y(t) = f(t, W(t))$ with
$$f(t,x) = e^{ax+bt},\ \ f_t(t,x) = bf(t,x),\ \ f_x(t,x) = af(t,x),\ \ f_{xx}(t,x) = a^2 f(t,x)$$

- Applying Itô's rule we get
$$dY = \left[b + \frac{1}{2}a^2\right] Y dt + aY dW$$

- If $b = -\frac{1}{2}a^2$, so that $Y(t) = e^{aW(t)-\frac{1}{2}a^2 t}$ we have a martingale:
$dY = aY dW$, and from $E_s[Y(t)] = Y(s)$ we get
$$E_s[e^{aW(t)}] = e^{aW(s)+\frac{1}{2}a^2(t-s)}$$

Two-dimensional Ito's rule

- Correlated Brownian motions:

$$E[W_X(t)W_Y(t)] = \rho t, \quad dW_X dW_Y = \rho dt$$

- Consider a model with two processes

$$dX = \mu_X dt + \sigma_X dW_X(t), \quad dY = \mu_Y dt + \sigma_Y dW_Y(t)$$

- Ito's rule:

$$df(X(t), Y(t)) = f_x dX + f_y dY + \frac{1}{2} f_{xx} (dX)^2 + \frac{1}{2} f_{yy} (dY)^2 + f_{xy} dX dY$$

$$= f_x dX + f_y dY + [\frac{1}{2} f_{xx} \sigma_X^2 + \frac{1}{2} f_{yy} \sigma_Y^2 + f_{xy} \rho \sigma_X \sigma_Y] dt$$

- **Product Rule:** $d(XY) = XdY + YdX + \rho\sigma_X\sigma_Y dt$