Pricing Options with Mathematical Models

19. Variations on Black-Scholes-Merton

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Dividends paid continuously

- Assume the stock pays a dividend at a continuous rate q. Total value of holding one share of stock is

$$G(t) := S(t) + \int_0^t qS(u)du$$

- Therefore, the wealth process of investing in this stock and the bank account is

$$dX = (X - \pi)dB/B + \pi dG/S$$

$$dX(t) = [rX(t) + \pi(t)(\mu + q - r)]dt + \pi(t)\sigma dW(t)$$

- To get the discounted wealth process to be a martingale, that is,

$$dX(t) = rX(t)dt + \pi(t)\sigma dW^Q(t)$$

we need to have

$$W^Q(t) = W(t) + t(\mu + q - r)/\sigma$$

- This makes the stock dynamics

$$dS(t) = S(t)[(r-q)dt + \sigma dW^Q(t)]$$

and the pricing PDE is

$$C_t + \frac{1}{2}\sigma^2 s^2 C_{ss} + (r-q)sC_s - rC = 0$$

- The solution, for the European call option, is obtained by replacing the underlying price s with $se^{-q(T-t)}$:

$$C(t,s) = se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}[\log(s/K) + (r-q+\sigma^2/2)(T-t)] \\ d_2 &= \frac{1}{\sigma\sqrt{T-t}}[\log(s/K) + (r-q-\sigma^2/2)(T-t)] \end{aligned}$$

Dividends paid discretely

- Assume the stock pays deterministic dividends, and denote the process of discounted dividends by $\bar{D}(t)$.
- Assume that the process

$$S_G(t) = S(t) - \bar{D}(t)$$

satisfies

$$dS_G = S_G[\mu dt + \sigma dW(t)]$$

Then, the option price is obtained by replacing $s = S(t)$ by $S(t) - \bar{D}(t)$.

Options on futures

- Since $F(t) = e^{r(T-t)}S(t)$,

$$dF = F(\mu - r)dt + F\sigma dW$$

- With $W^Q(t) = W(t) + t(\mu - r)/\sigma$, we get

$$dF = F\sigma dW^Q$$

- Thus, the PDE for path independent options is

$$C_t + \frac{1}{2}\sigma^2 f^2 C_{ff} - rC = 0$$

- The solution for the call option is

$$
\begin{aligned}
C(t,f) &= e^{-r(T-t)}[fN(d_1) - KN(d_2)] \\
d_1 &= \frac{1}{\sigma_F\sqrt{T-t}}[\log(f/K) + (\sigma_F^2/2)(T-t)] \\
d_2 &= \frac{1}{\sigma_F\sqrt{T-t}}[\log(f/K) - (\sigma_F^2/2)(T-t)]
\end{aligned}
$$

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20. Currency options

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Currency options in the B-S-M model

- Consider the payoff, evaluated in the domestic currency, equal to

$$(R(T) - K)^+$$

  where $R(T)$ is the exchange rate, the time T domestic value of one unit of foreign currency.

- Assume that the exchange rate process is given by

$$dR(t) = R(t)[\mu_R dt + \sigma_R dW(t)]$$

- The pricing formula is the same as in the case of a dividend-paying underlying, but with q replaced by r_f, the foreign risk-free rate.

Reasons why

- We trade in the domestic and foreign risk-free accounts.
- The dollar value of one unit of the foreign account is

$$R^*(t) := R(t)e^{r_f \cdot t}$$

$$dR^* = R^* \left[(\mu_R + r_f)dt + \sigma_R dW\right]$$

- The wealth dynamics (in domestic currency) of a portfolio of π dollars in the foreign account and the rest in the domestic account are

$$dX = \frac{X - \pi}{B} dB + \frac{\pi}{R^*} dR^* = [rX + \pi(\mu_R + r_f - r)]dt + \pi \sigma_R dW$$

- This is exactly the same as for dividends with q replaced by r_f.

$$W^Q(t) = W(t) + t(\mu_R - (r - r_f))/\sigma_R$$

Call option formula

- The price of the call option is

$$c(t,R) = RN(d_1) - Ke^{-r(T-t)}[N(d_2)]$$

where

$$d_1 = \frac{1}{\sigma_R\sqrt{T-t}}[\log(R/K) + (r - r_f + \sigma_R^2/2)(T-t)]$$

$$d_2 = \frac{1}{\sigma_R\sqrt{T-t}}[\log(R/K) + (r - r_f - \sigma_R^2/2)(T-t)] = d_1 - \sigma_R\sqrt{T-t} \quad .$$

Example: Quanto options

- - $S(t)$: a domestic equity index

  - Payoff: $S(T) - F$ units of **foreign currency**; quanto forward

- As in the previous slide, we have

$$W^Q(t) = W(t) + t(\mu_R - (r - r_f))/\sigma_R$$

  and thus

$$dR(t) = R(t)[(r - r_f)dt + \sigma_R dW^Q(t)]$$

- Assume

$$dS(t) = S(t)[rdt + \sigma_S dZ^Q(t)]$$

  where BMP Z^Q has instantaneous correlation ρ with W^Q. We have

$$d(S(t)R(t)) = S(t)R(t)[(2r - r_f + \rho\sigma_R\sigma_S)dt + \sigma_R dW^Q(t) + \sigma_S dZ^Q(t)]$$

- $S(T) - F$ units of foreign currency is the same as $(S(T) - F)R(T)$ units of domestic currency. The domestic value is

$$e^{-rT}(E^Q[S(T)R(T)] - FE^Q[R(T)])$$

- To make it equal to zero

$$F = \frac{E^Q[S(T)R(T)]}{E^Q[R(T)]}$$

- We have

$$E^Q[S(T)R(T)] = S(0)R(0)e^{(2r-r_f+\rho\sigma_S\sigma_R)T}$$
$$E^Q[R(T)] = R(0)e^{(r-r_f)T}$$

- We get

$$F = S(0)e^{(r+\rho\sigma_S\sigma_R)T}$$

- If

$$dX = aX dt + bX dW + cX dZ$$

then

$$EX(t) = X(0)e^{at}$$

- This is because

$$d(EX(t)) = a \times (EX(t))dt$$

and the solution to this ODE, that has initial value $X(0)$, is the one above.

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21. Exotic options

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Most popular exotic options

- Barrier options: they pay a call/put payoff only if the underlying price reaches a given level (barrier) before maturity. Thus, they depend on the maximum or the minimum price of the underlying during the life of the option.
- Asian options: a call/put written on the average stock price until maturity. Useful when the price of the underlying may be very volatile.
- Compound options: the underlying is another option. Call on a call:

$$E_0^Q e^{-rT_1}\left[\mathrm{BS}(T_1) - K_1\right]^+$$

Example: a forward start option

- A call with the strike price $S(t_1)$, $t_1 > 0$. Note that

$$S(0)\frac{S(T)}{S(t_1)} = S(0)\exp\{\sigma(W^Q(T) - W^Q(t_1)) + (r - \sigma^2/2)(T - t_1)\}$$

- We first compute the value at t_1:

$$\begin{aligned} E_{t_1}^Q\left[e^{-r(T-t_1)}(S(T) - S(t_1))^+\right] &= E_{t_1}^Q\left[e^{-r(T-t_1)}\frac{S(t_1)}{S(0)}\left(\frac{S(0)S(T)}{S(t_1)} - S(0)\right)^+\right] \\ &= \frac{S(t_1)}{S(0)}\mathrm{BS}(T - t_1, S(0)) \quad . \end{aligned}$$

- Today's value

$$E_0^Q\left[e^{-rt_1}\frac{S(t_1)}{S(0)}\mathrm{BS}(T - t_1, S(0))\right] = \mathrm{BS}(T-t_1, S(0))E_0^Q\left[e^{-rt_1}\frac{S(t_1)}{S(0)}\right] = \mathrm{BS}(T-t_1, S(0))$$

Example: a chooser option

- The holder can decide at time t_1 whether the payoff will be a call or a put, with the same strike price and maturity. Thus, the value at time t_1 is, using put-call parity,

$$\max(c(t_1), p(t_1)) = \max(c(t_1), c(t_1) + Ke^{-r(T-t_1)} - S(t_1))$$

$$= c(t_1) + \max(0, Ke^{-r(T-t_1)} - S(t_1))$$

- It is a package of a call option with maturity T and strike price K, and a put option with maturity t_1 and strike price $Ke^{-r(T-t_1)}$.

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22. Pricing options on more underlyings

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Two risky assets

$$dS_1 = S_1[\mu_1 dt + \sigma_1 dW_1]$$
$$dS_2 = S_2[\mu_2 dt + \sigma_2 dW_2]$$

Equivalently,

$$\begin{aligned} dS_1 &= S_1[\mu_1 dt + \sigma_1 dB_1] \\ dS_2 &= S_2[\mu_2 dt + \sigma_2 \rho dB_1 + \sigma_2 \sqrt{1-\rho^2} dB_2] \end{aligned}$$

This is because, given two independent Brownian Motions B_1 and B_2, we can set

$$W_1 = B_1, \quad W_2 = \rho B_1 + \sqrt{1-\rho^2} B_2$$

Wealth process

- $$dX = \frac{\pi_1}{S_1} dS_1 + \frac{\pi_2}{S_2} dS_2 + \frac{X - (\pi_1 + \pi_2)}{B} dB \quad .$$

This gives

$$dX = [rX + \pi_1(\mu_1 - r) + \pi_2(\mu_2 - r)]dt + \pi_1 \sigma_1 dW_1 + \pi_2 \sigma_2 dW_2 \quad .$$

For the discounted wealth process to be a martingale under the risk-neutral probability Q, we need to have

$$dX = rX dt + \pi_1 \sigma_1 dW_1^Q + \pi_2 \sigma_2 dW_2^Q$$

for some Q-Brownian Motions W_i^Q with correlation ρ. For that to be the case, we must have

$$W_i^Q(t) = W_i(t) + t(\mu_i - r)/\sigma_i$$

The pricing PDE with two factors

- Suppose $C(T) = g(S_1(T), S_2(T)$. Using the two-dimensional Ito's rule

$$
\begin{aligned}
dC &= \left[C_t + rS_1C_{s_1} + rS_2C_{s_2} + \frac{1}{2}\sigma_1^2S_1^2C_{s_1s_1} + \frac{1}{2}\sigma_2^2S_2^2C_{s_2s_2} + \rho\sigma_1\sigma_2S_1S_2C_{s_1s_2}\right]dt \\
&\quad + \sigma_1S_1C_{s_1}dW_1^Q + \sigma_2S_2C_{s_2}dW_2^Q \quad .
\end{aligned}
$$

Comparing the dt term with the wealth equation, or making the drift of the discounted C equal to zero,

$$
C_t+\frac{1}{2}\sigma_1^2s_1^2C_{s_1s_1}+\frac{1}{2}\sigma_2^2s_2^2C_{s_2s_2}+\rho\sigma_1\sigma_2s_1s_2C_{s_1s_2}+r(s_1C_{s_1}+s_2C_{s_2}-C) = 0 \ .
$$

$$
C(T, s_1, s_2) = g(s_1, s_2)
$$

$$
\frac{\pi_1}{S_1} = C_{s_1}, \quad \frac{\pi_2}{S_2} = C_{s_2}
$$

Example: exchange option

The payoff is

$$g(S_1(T), S_2(T)) = (S_2(T) - S_1(T))^+ = \max(S_2(T) - S_1(T), 0)$$

Since we have

$$(s_2 - s_1)^+ = s_1 \left(\frac{s_2}{s_1} - 1 \right)^+$$

it is reasonable to expect that the option price will be of the form

$$C(t, s_1, s_2) = s_1 D(t, z)$$

for some function D and a new variable $z = s_2/s_1$. After some computations, we can show that D has to satisfy

$$D_t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) z^2 D_{zz} = 0, \quad D(T, z) = (z - 1)^+$$

Example: exchange option (continued)

This is the Black-Scholes PDE corresponding to the European call option with strike price $K = 1$, interest rate $r = 0$, and volatility

$$\sigma_E = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \quad .$$

Using the Black-Scholes formula for D, and the fact that $C = s_1 D$, we get

$$C(t, s_1, s_2) = s_2 N(d_1) - s_1 N(d_2) \quad ,$$

$$d_1 = \frac{1}{\sigma_E\sqrt{T-t}}[\log(s_2/s_1) + (\sigma_E^2/2)(T-t)]$$

$$d_2 = \frac{1}{\sigma_E\sqrt{T-t}}[\log(s_2/s_1) - (\sigma_E^2/2)(T-t)] = d_1 - \sigma_E\sqrt{T-t} \quad ,$$