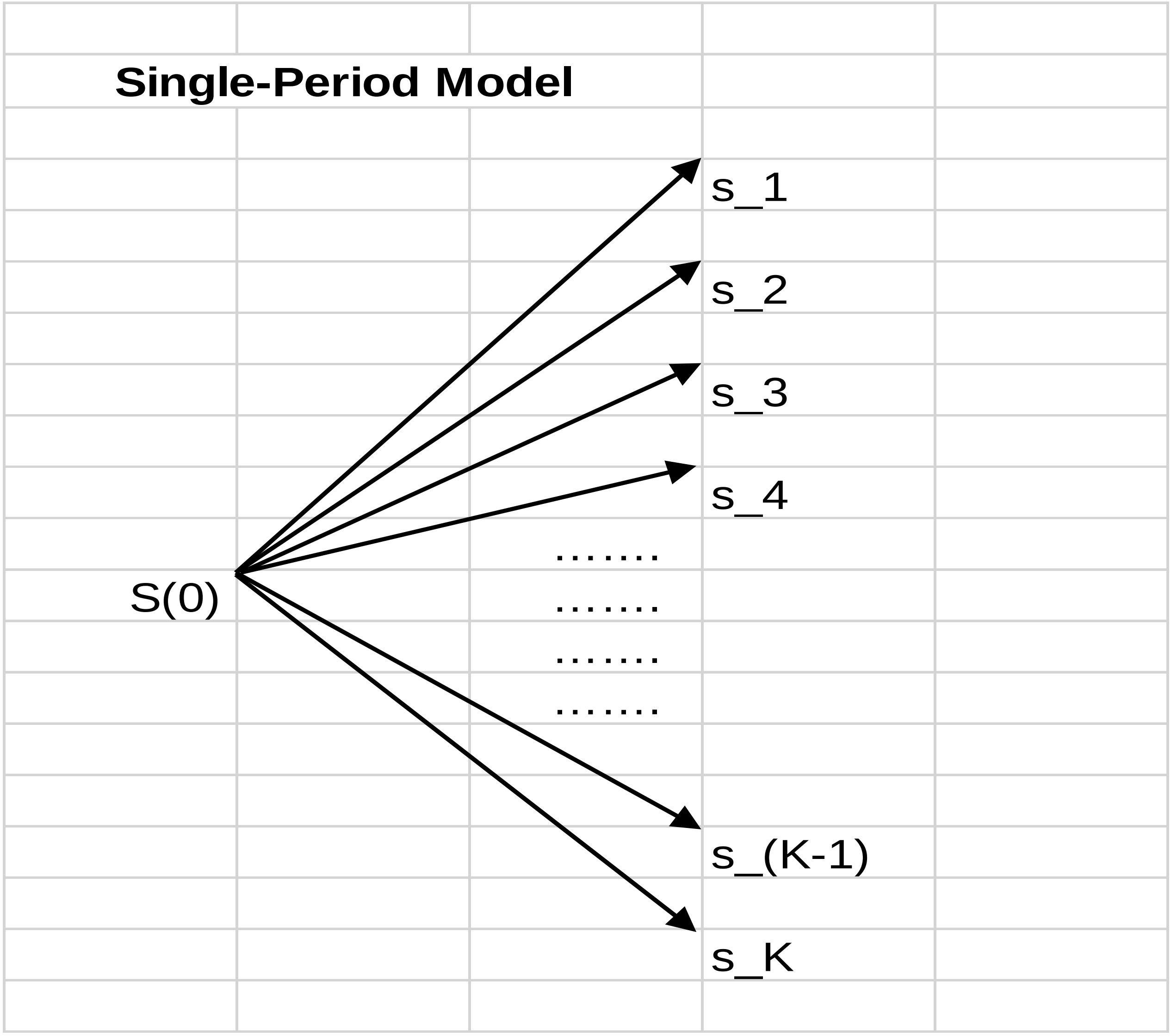
Pricing Options with Mathematical Models

10. Discrete-time models

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.



$$P(S(T) = s_i) = p_i$$

- Risk-free asset, bank account:

$$B(0) = 1, B(1) = 1 + r$$

- Initial wealth:

$$X(0) = x$$

- Number of shares in asset i: δ_i

- End-of-period wealth:

$$X(1) = \delta_0 B(1) + \delta_1 S_1(1) + \cdots + \delta_N S_N(1)$$

- Budget constraint, self-financing condition:

$$X(0) = \delta_0 B(0) + \delta_1 S_1(0) + \cdots + \delta_N S_N(0)$$

- **Profit/loss, P&L**, or the **gains** of a portfolio strategy:

  G(1) = X(1) - X(0)

- Discounted version of process Y: $\overline{Y}(t) = Y(t)/B(t)$
- Change in price: $\Delta S_i(1) = S_i(1) - S_i(0)$
- We have

$$G(1) = \delta_0 r + \delta_1 \Delta S_1(1) + \cdots + \delta_N \Delta S_N(1)$$

$$X(1) = X(0) + G(1)$$

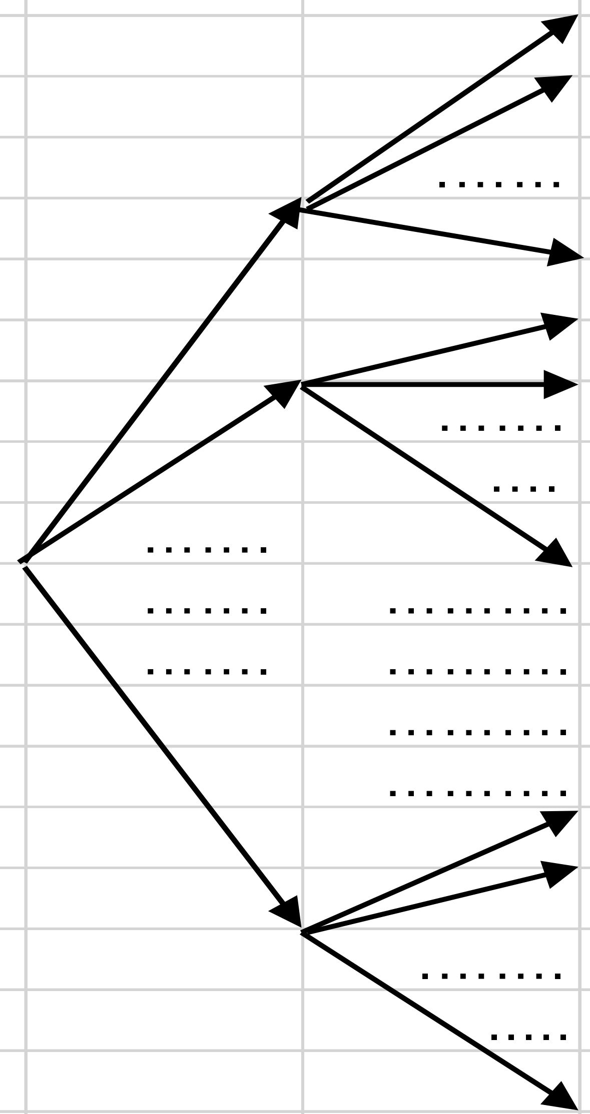
Denoting

$$\Delta \bar{S}_i(1) = \bar{S}_i(1) - S_i(0), \qquad \overline{G}(1) = \delta_1 \Delta \bar{S}_1(1) + \cdots + \delta_N \Delta \bar{S}_N(1)$$

one can verify that

$$\bar{X}(1) = X(0) + \overline{G}(1)$$

Multi-Period Model



- Risk-free asset, bank account:
$$B(0) = 1, B(t) = \big(1 + r(t)\big)B(t-1)$$
- Number of shares in asset i during the period $[t-1, t)$: δ_i(t)

- Wealth process:

$$X(t) = \delta_0(t)B(t) + \delta_1(t)S_1(t) + \cdots + \delta_N(t)S_N(t)$$
- Self-financing condition:

$$X(t) \\ = \delta_0(t+1)B(t) + \delta_1(t+1)S_1(t) + \cdots + \delta_N(t+1)S_N(t)$$

- Change in price: $\Delta S_i(t) = S_i(\mathrm{t}) - S_i(t-1)$

- $\mathrm{G(t)} = \sum_{s=1}^{t} \delta_0(s)\, \Delta B(s) + \sum_{s=1}^{t} \delta_1(s)\, \Delta S_1(s) + \cdots + \sum_{s=1}^{t} \delta_N(s)\, \Delta S_N(s)$

- It can be checked that $X(t) = X(0) + G(t)$

Denoting

$$\Delta \bar{S}_i(t) = \bar{S}_i(\mathrm{t}) - \bar{S}_i(\mathrm{t}-1),$$

$$\overline{G}(t) = \sum_{s=1}^{t} \delta_1(s)\, \Delta \bar{S}_1(s) + \cdots + \sum_{s=1}^{t} \delta_N(s)\, \Delta \bar{S}_N(s)$$

one can verify that

$$\bar{X}(t) = X(0) + \bar{G}(t)$$

For example, with one risky asset and two periods:

- Change in price: $\Delta S_i(t) = S_i(\mathrm{t}) - S_i(t-1)$
- $$\mathrm{G}(2) = \delta_0(1)\big(B(1) - B(0)\big) + \delta_0(2)\big(B(2) - B(1)\big)$$
$$+\delta_1(1)\big(S(1) - S(0)\big) + \delta_1(2)\big(S(2) - S(1)\big)$$
- Using self-financing

$$\delta_0(1)B(1) + \delta_1(1)S(1) = \delta_0(2)B(1) + \delta_1(2)S(1)$$

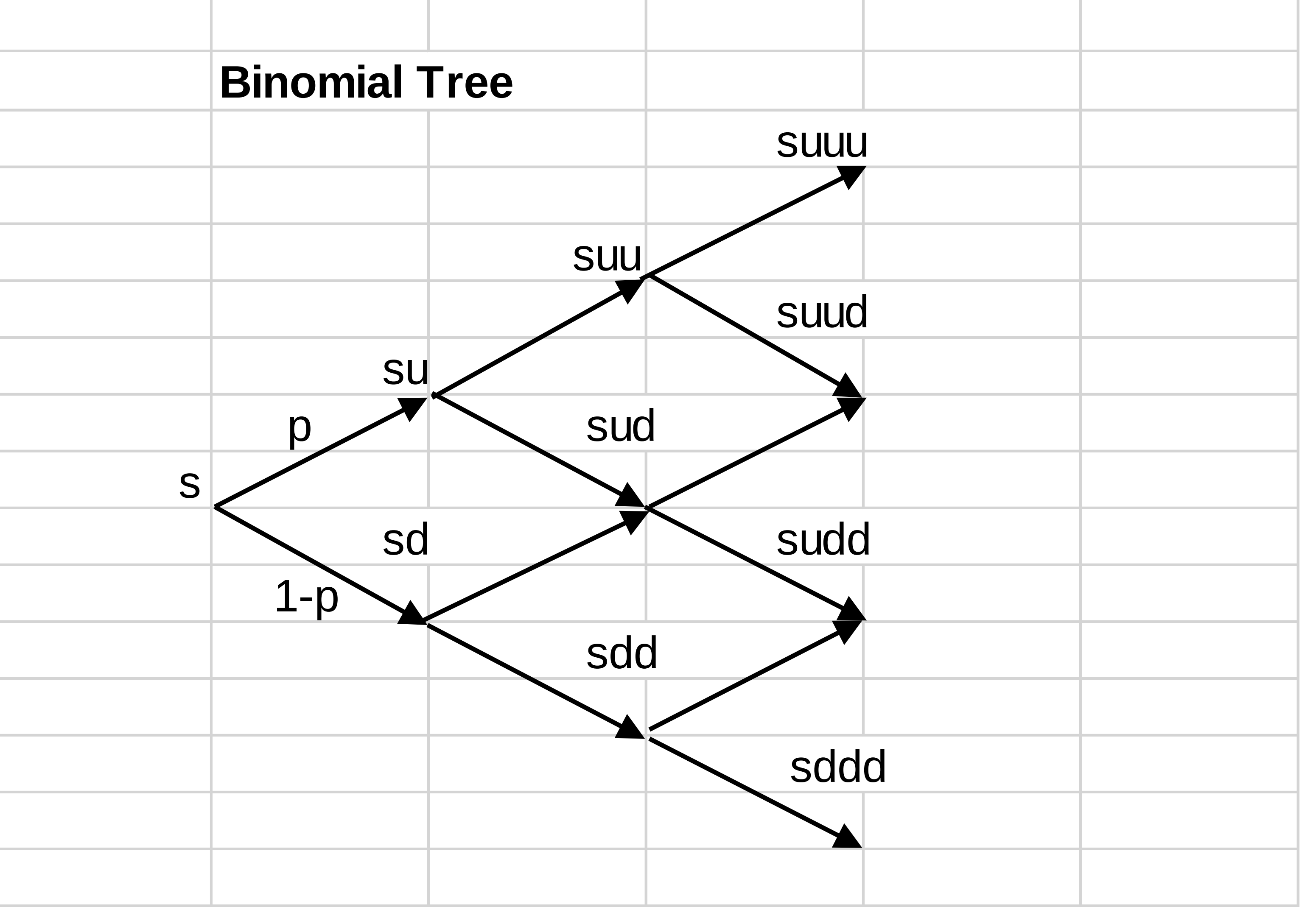
we get

- $\mathrm{G}(2) = \delta_0(2)B(2) + \delta_1(2)S(2) - \delta_0(1)B(0) - \delta_1(1)S(0)$
- This is the same as

$$G(2) = X(2)-X(0)$$

Binomial Tree (Cox-Ross-Rubinstein) model

- $p = P\big(S(t+1) = u\,S(t)\big)$, $1 - p = P\big(S(t+1) = d\,S(t)\big)$
- u > 1+r > d



Pricing Options with Mathematical Models

11. Risk-neutral pricing

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Martingale property

- Insurance pricing:

$$C(t) = E_t[\, e^{-r(T-t)} C(T) \,]$$

where E_t is expectation given the information up to time t.

- For a stock, this would mean:

$$e^{-rt} S(t) = E_t[\, e^{-rT} S(T) \,]$$

- If so, we say that $M(t) = e^{-rt} S(t)$ is a **martingale process**:

$$M(t) = E_t[\, M(T) \,]$$

Martingale probabilities (measures)

- Typically, the stock price process will not be a martingale under the actual (physical) probabilities, but it may be a martingale under some other probabilities.
- Those are called **martingale**, or **risk-neutral**, or **pricing probabilities**.
- Such probabilities are typically denoted Q, q_i, sometimes $P^*,\ p_i^*$.
- We write:

$$e^{-rt}S(t) = E_t^Q\ [\, e^{-rT}S(T)\,]\,,$$

or $$e^{-rt}S(t) = E_t^*\ [\, e^{-rT}S(T)\,]$$

Risk-neutral pricing formula

- Thus, we expect to have, for some risk-neutral probability Q

$$\boldsymbol{Price\ of\ claim\ today}$$
$$\boldsymbol{= expected\ value, under\ Q, of\ the\ claim's\ discounted\ future\ payoff}$$

or

$$\mathrm{C}(t) = E_t^Q\ [\ e^{-r(T-t)}C(T)\ ]$$

if $C(T)$ is paid at T, and the continuously compounded risk-free rate r is constant.

- How to justify this formula?
- Which Q? Are there any? How many?

Example: A Single Period Binomial model

- r=0.005, S(0)=100, $s^u = 101, s^d = 99$, that is, u=1.01, d=0.99.
- The payoff is an European Call Option, with payoff

$$\max\{S(1) - 100, 0\}$$

- It will be \$1 if the stock goes up and \$0 if the stock goes down. Looking for the **replicating portfolio**, we solve

$$\delta_0(1 + 0.005) + \delta_1 101 = 1\ ;$$
$$\delta_0(1 + 0.005) + \delta_1 99 = 0.$$

We get

$$\delta_0 = -49.254,\ \delta_1 = 0.5$$

Example continued

- $\delta_0 = -49.254$, $\delta_1 = 0.5$
- This means borrow 49.254, and buy one share of the stock. This costs

$$C(0) = 0.5 \times 100 - 49.254 = 0.746$$

This is the no arbitrage price:

1) Suppose the price is higher, say 1.00. Sell the option for 1.00, invest 1-0.746 at the risk-free rate; use 0.746 to set up the replicating strategy; have 1 if stock goes up, and 0 if it goes down, exactly what you need. Arbitrage.
2) Suppose the price is lower, say 0.50. Buy the option for 0.50, sell short half a share for 50, invest 49.254 at the risk-free rate; This leaves you with extra 0.246 today. If stock goes up you make 1.00 from the option; together with 49.254×1.005, this covers 101/2 to close your short position. If stock goes down, use 49.254×1.005 to cover 99/2 when closing your short position. Arbitrage.

Martingale pricing

- Suppose the discounted wealth process $\bar{X}$ is a martingale under Q, and suppose it replicates C(T), so that X(T)=C(T). By the martingale property,

$$\bar{X}(t) = E_t^Q \bar{X}(T) = E_t^Q \bar{C}(T)$$

For example, if discounting is continuous at a constant rate r, this gives

$$X(t) = E_t^Q [\, e^{-r(T-t)} C(T) \,]$$

This is the cost of replication at time t, therefore, for any such probability Q,

the price/value of the claim at time t is equal to the expectation, under Q, of the discounted future payoff of the claim.

Single Period Binomial model

- The future wealth value is

$$X(1) = \delta_0(1+r) + \ \delta_1 S(1)$$

thus, when discounted,

$$\bar{X}(1) = \delta_0 + \ \delta_1 \bar{S}(1)$$

Therefore, if the discounted (non-dividend paying) stock is a martingale, so is the discounted wealth. For the stock to be a Q-martingale, we need to have

$$S(0) = E^Q \frac{S(1)}{1+r} = \frac{1}{1+r}(q \times s^u + (1-q) \times s^d)$$

Solving for q, we get, with $s^u = S(0)u, s^d = S(0)d$,

$$q = \frac{(1+r)-d}{u-d}, 1-\mathrm{q} = \frac{u-(1+r)}{u-d}$$

Example (the same as above)

$$s^u = 100 \times 1.01, s^d = 100 \times 0.99,$$

$$q = \frac{(1+r) - d}{u - d} = \frac{1.005 - 0.99}{1.01 - 0.99} = 0.75$$

Thus, the price of the call option is

$$\mathrm{C}(0) = E^Q \frac{C(1)}{1+r} = \frac{1}{1+r}\left(q \times C^u + (1-q) \times C^d\right)$$

$$= \frac{1}{1 + 0.005}(0.75 \times (101 - 100) + (1 - 0.75) \times 0)$$

=0.746 (the same as above)

Forwards

- Let D denote the process used for discounting, for example
$$\mathrm{D}(t) = e^{-rt}$$
- .We want the forward price $F(t)$ to be such that the value of the forward contract zero at the initial time t:
$$0 = E_t^Q[\ \{S(T) - F(t)\ \}\frac{D(T)}{D(t)}\ ]$$

Since DS is a Q-martingale, we have $E_t^Q[D(T)S(T)] = D(t)S(t)$, and we get
$$F(t) = S(t)\frac{D(t)}{E_t^Q[D(T)]}$$
which, for the above $\mathrm{D}(t)$, is the same as
$$F(t) = S(t)e^{r(T-t)} = S(t)B(t,T)$$

Pricing Options with Mathematical Models

12. Fundamental theorems of asset pricing

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Risk-Neutral Pricing

FINANCIAL MARKETS

No Arbitrage = Risk-Neutral Measures

Arbitrage = No Risk-Neutral Measures

Complete Markets =
Unique Risk-Neutral Measure =
One Price, the Cost of Replication

Incomplete Markets =
Many Risk-Neutral Measures =
Many possible no-arbitrage prices

Expected Value

Solution to a PDE

Equivalent martingale measures (EMM's)

Recall

$$q = \frac{(1+r)-d}{u-d}, 1-\mathrm{q} = \frac{u-(1+r)}{u-d}$$

Thus, q and $1-q$ are strictly between zero and one if and only if

$$d < 1+r < u$$

Then, the events of non-zero P probability also have non-zero Q probability, and vice-versa. We say that P and Q are **equivalent probability measures**, and Q is called an **equivalent martingale measure** (EMM). Note also that Q is the only EMM.

First fundamental theorem of asset pricing

$$\boldsymbol{No\ arbitrage = existence\ of\ at\ least\ one\ \ EMM}$$

Definition of arbitrage: there exists a strategy such that, for some T,
$X(0) = 0, X(T) \geq 0$ with probability one, and

$$P(X(T) > 0) > 0$$

One direction: suppose there exists an EMM Q, and a strategy with $X(T)$ as above. Then,

$$X(0) = E^{Q}\,\bar{X}(T) > 0, \text{ so, no arbitrage.}$$

Second fundamental theorem of asset pricing

- **Definition of completeness**: a market (model) is complete if every claim can be replicated by trading in the market.

$$\boldsymbol{Completeness\ and\ no\ arbitrage}$$
$$\boldsymbol{= existence\ of\ exactly\ one\ \ EMM}$$

- In a complete market, every claim has a unique price, equal to the cost of replication, also equal to the expectation under the unique EMM.
- Even in an incomplete market, one assumes that there is one EMM Q (among many), that the market chooses to price all the claims.
- How to compute it?

Example: Binomial tree model is arbitrage free and complete if $d < 1 + r < u$

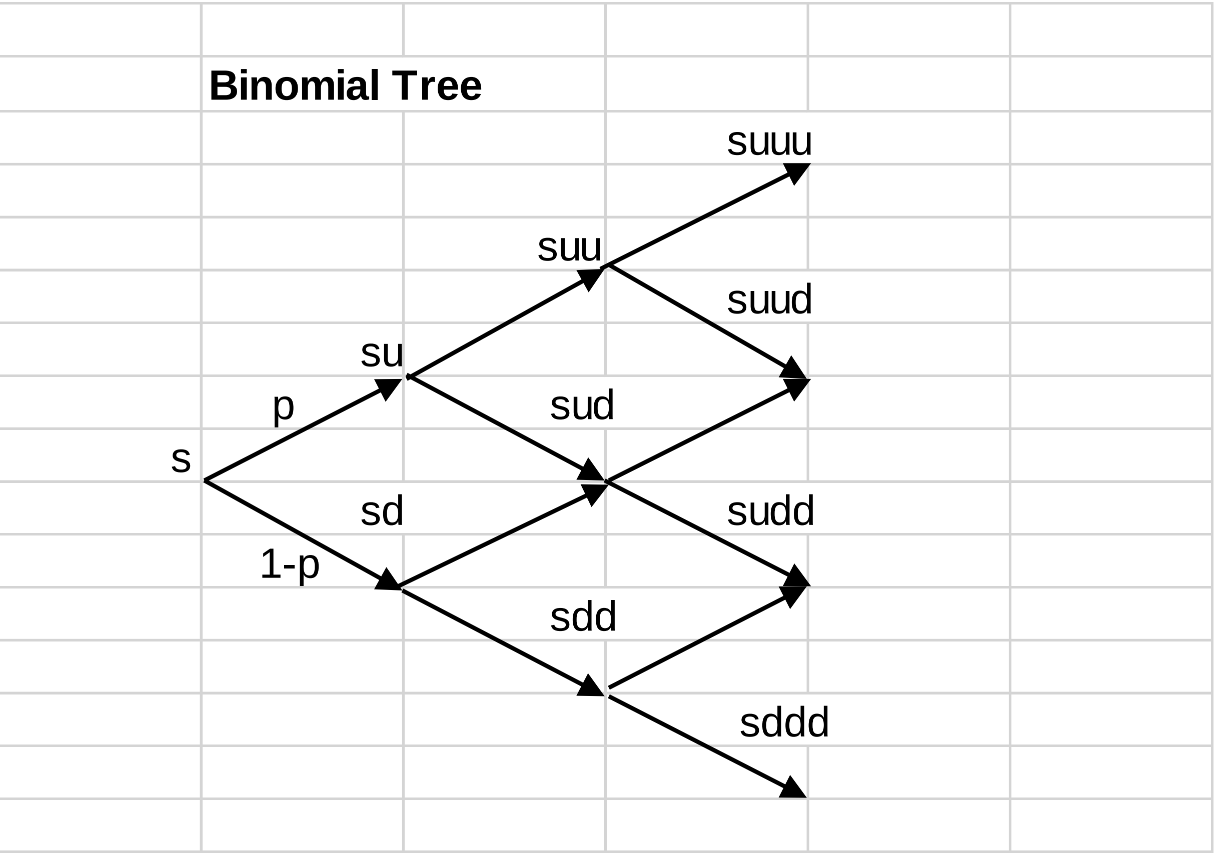
Pricing Options with Mathematical Models

13. Binomial tree pricing

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Binomial Tree (Cox-Ross-Rubinstein) model

- $p = P\big(\, S(t+1) = u\, S(t)\big)$,
- $1 - p = P\big(\, S(t+1) = d\, S(t)\big)$
- u > $e^{r\Delta t}$ > d



Expectation formula

- CLAIM: Given a random variable X whose value will be known at time T, process $\mathrm{M(t)} = E_t[X]$ is a martingale. Indeed, for $s < t$,

$$E_s[M(t)] = E_s E_t\,[\mathrm{X}] = E_s[X] = M(s)$$

where the middle equality is the so- called ***law of iterated expectations***.

- Since

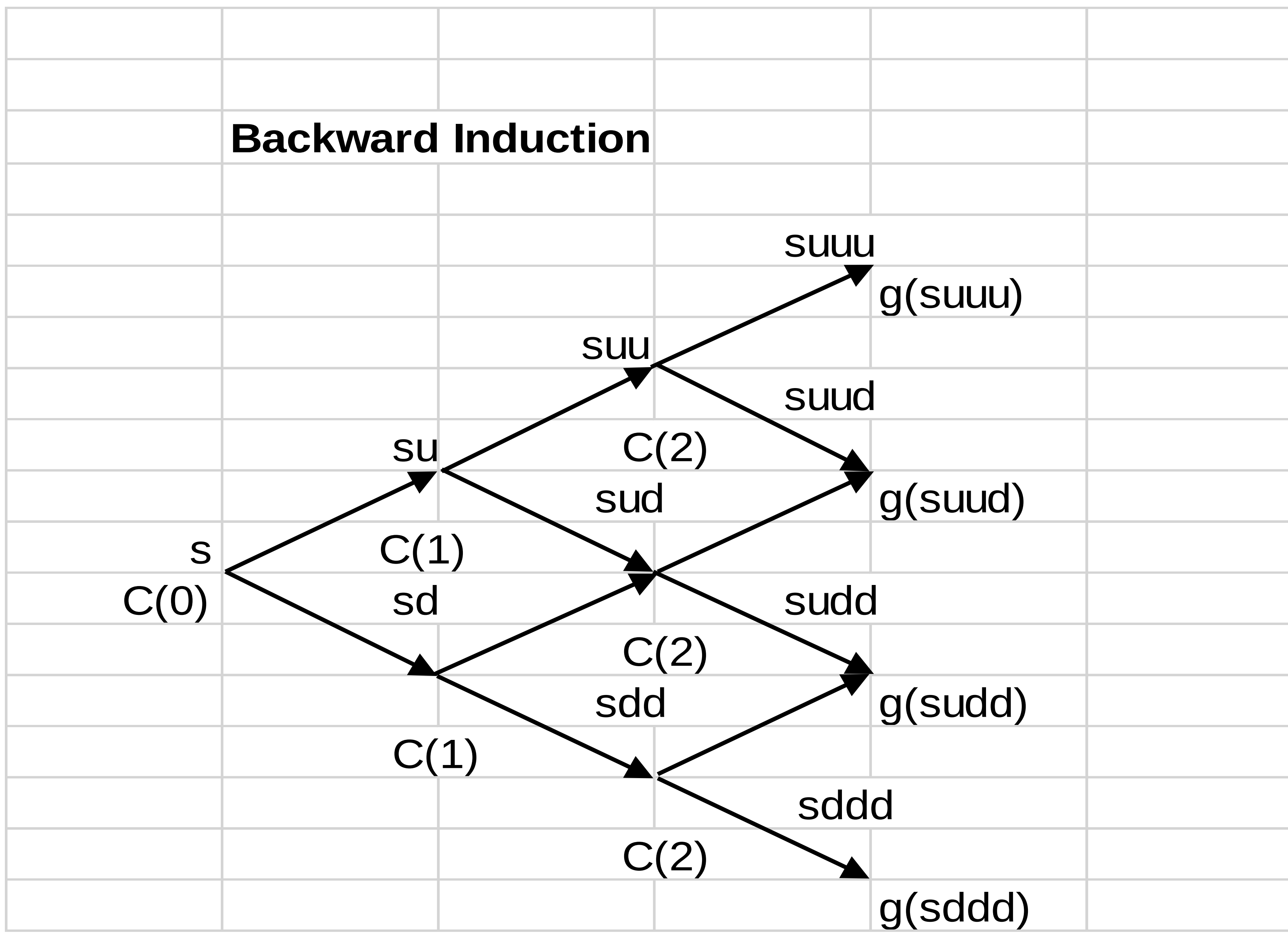
$e^{-rt}\mathrm{C}(t) = E_t^Q\,[\,e^{-rT}C(T)\,]$

we conclude that $e^{-rt}\mathrm{C}(t)$ is a Q-martingale. Therefore,

$e^{-rt}\mathrm{C}(t) = E_t^Q\,[\,e^{-r(t+\Delta t)}C(t+\Delta t)\,]$, and we get the **expectation formula**

$$\mathrm{C}(t) = E_t^Q\left[\,e^{-r\Delta t}C(t+\Delta t)\right]$$
$$= e^{-r\Delta t}[\mathrm{q}\times C^u(t+\Delta t) + (1-q)\times C^d(t+\Delta t)]$$

Pricing path-independent payoff $g(S(T))$



Example: a call option

			European Option Price			
					121.0000	
					21.0000	
				110.0000		
				15.1088		
S(0)	100.0000	S(0)	100.0000		99.0000	
u	1.1000	C(0)	10.8703		0.0000	
d	0.9000					
K	100.0000					
r	0.0500					
p*	0.7564			90.0000		
Delta t	1.0000			0.0000		
					81.0000	
					0.0000	

$$q = \frac{e^{r\Delta t} - d}{u - d} = 0.7564$$

$$15.1088 = e^{-r\Delta t}[\, q \times 21 + (1 - q) \times 0 \,]$$

$$10.87 = e^{-r\Delta t}[\, q \times 15.1088 + (1 - q) \times 0 \,]$$

American options

- Consider an American option that pays $g(\tau)$dollars if exercised at time $\tau \leq T$.
- It can be shown that, in a complete market with discrete time and time intervals of length Δt, its no-arbitrage price $A(t)$is given by

$$\mathrm{A}(t) = \max_{t \leq \tau \leq T} E_t^Q[\, e^{-r(\tau - t)} g(\tau) \,]$$

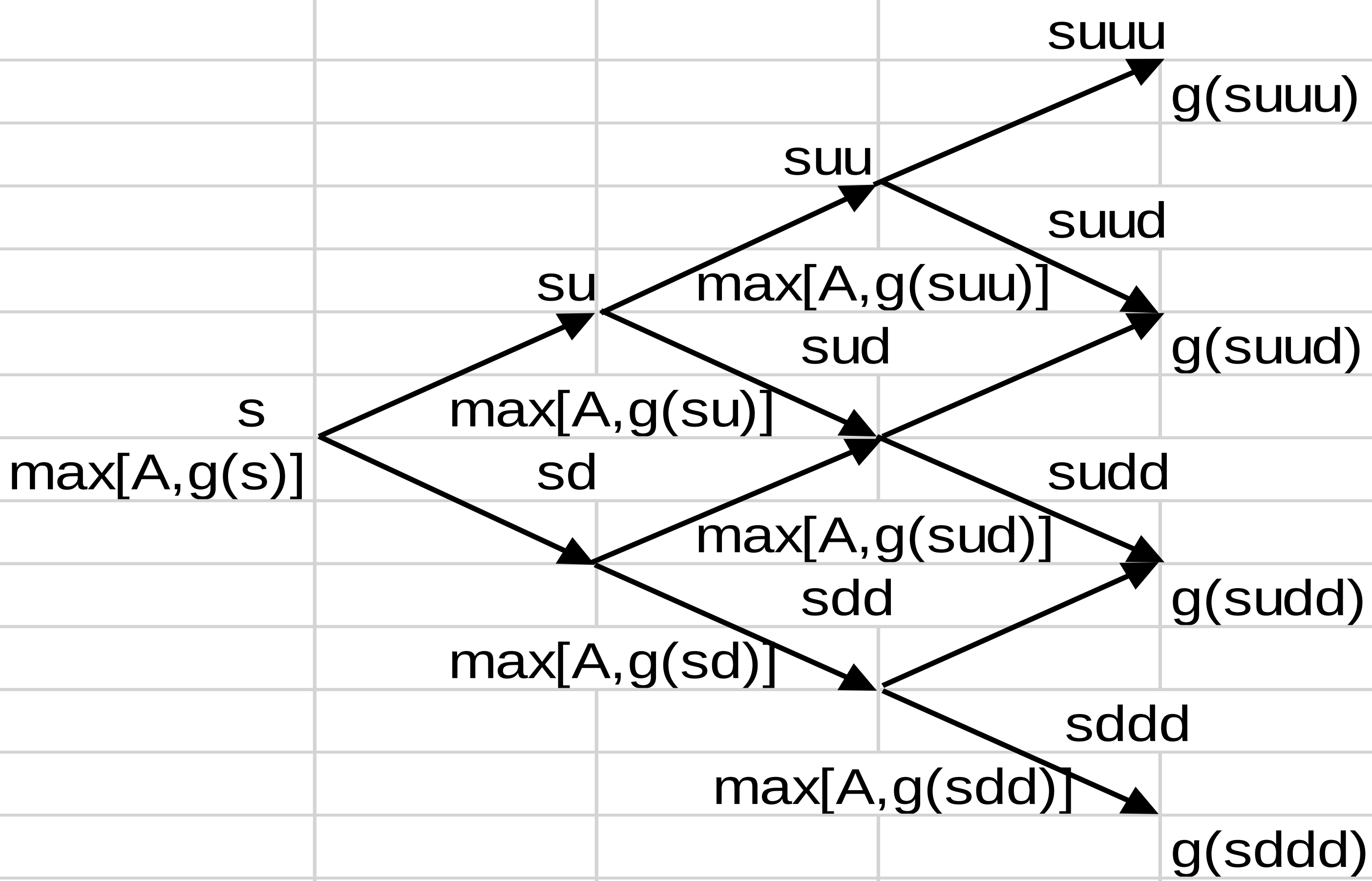
- The expectation formula is given by the dynamic programming principle:

$$\mathrm{A}(t) = \max\ \{g(t)\,, E_t^Q[e^{-r\Delta t} g(t + \Delta t)\,]\,\}$$

or, in the binomial model,

$$\mathrm{A}(t) = \max[g(t),\ e^{-r\Delta t}\{\mathrm{q} \times A^u(t + \Delta t) + (1 - q) \times A^d(t + \Delta t)\}]$$

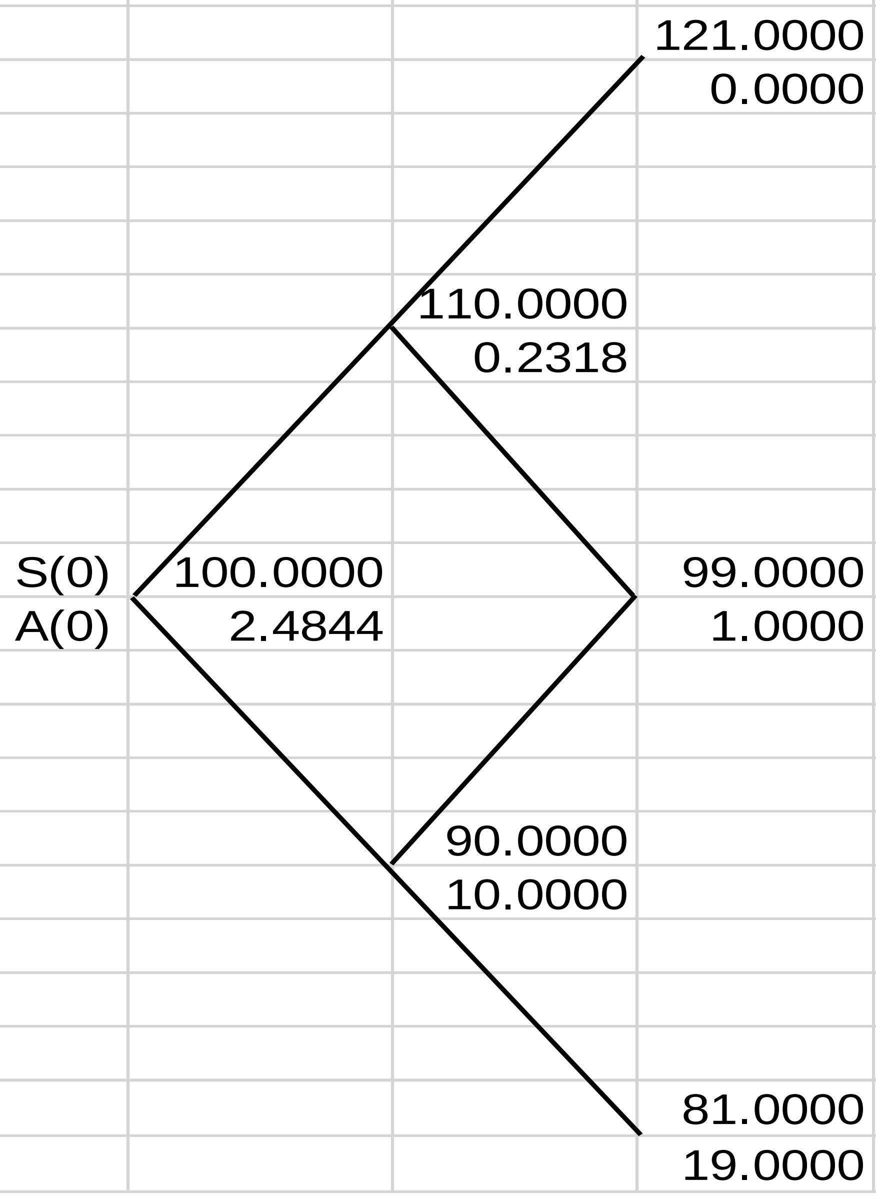
Backward Induction for American Options



Example: a put option

American Option Price

S(0)	100.0000
u	1.1000
d	0.9000
K	100.0000
r	0.0500
p*	0.7564
Delta t	1.0000



$$q = \frac{e^{r\Delta t} - d}{u - d} = 0.7564$$

$$10 = \max\{10, e^{-r\Delta t}[\, q \times 1 + (1 - q) \times 19] \,\}$$
$$= \max\{10, 5.1229\}$$

$$0.2318 = \max\{0, e^{-r\Delta t}[\, q \times 0 + (1 - q) \times 1 \,] \,\}$$

$$2.4844 = \max\{0, e^{-r\Delta t}[\, q \times 0.2318 + (1 - q) \times 10 \,]\}$$