Pricing Options with Mathematical Models

23. Stochastic volatility

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Complete markets

$$dS(t) = S(t)[r(t, S(t))dt + \sigma(t, S(t))dW^Q(t)]$$

- We have

$$S(T) = S(t)e^{\int_t^T [r(u,S(u)) - \frac{1}{2}\sigma^2(u,S(u))]du + \int_t^T \sigma(u,S(u))dW^Q(u)}$$

The PDE for the value of the payoff $g(S(T))$ is

$$C_t + \frac{1}{2}\sigma^2(t, s)C_{ss} + r(t, s)[sC_S - C] = 0$$

Constant Elasticity of Variance, CEV model, with $0 < \beta < 1$:

$$\sigma(t, s) = \frac{\sigma}{s^\beta}$$

.

Complete markets (continued)

- If $r(t)$ and $\sigma(t)$ are deterministic functions of time, random variable $\int_t^T \sigma(u)dW(u)$ has normal distribution with zero mean and variance $\int_t^T \sigma^2(u)du$.
- For a payoff $g(S(T))$, the value at time t is obtained by replacing $\sigma^2 \times (T-t)$ with $\int_t^T \sigma^2(u)du$ and replacing $r \times (T-t)$ with $\int_t^T r(u)du$.

Incomplete markets

- Consider two independent BMP's W_1 and W_2, and

$$\begin{aligned} dS(t) &= S(t)[\mu(t)dt + \sigma_1(t, V(t))dW_1(t) + \sigma_2(t, V(t))dW_2(t)] \\ dV(t) &= \alpha(t)dt + \gamma(t)dW_2(t) \end{aligned}$$

- Denote by $\kappa(t)$ *any* (adapted) stochastic process. For each, there is a risk-neutral measure Q_κ. In particular, for any such process κ we can set

$$\begin{aligned} dW_1^{Q_\kappa}(t) &= dW_1(t) + \frac{1}{\sigma_1(t)}[\mu(t) - r(t) - \sigma_2(t)\kappa(t)]dt \\ dW_2^{Q_\kappa}(t) &= dW_2(t) + \kappa(t)dt \end{aligned}$$

- It can be checked that discounted S is then a Q_κ-martingale and

$$dV(t) = [\alpha(t) - \kappa(t)\gamma(t)]dt + \gamma(t)dW_2^{Q_\kappa}(t)$$

-

Incomplete markets (continued)

- For constant κ and constant patrameters, the PDE becomes

$$
\begin{aligned}
& C_t + \frac{1}{2} C_{ss} s^2 (\sigma_1^2 + \sigma_2^2) + \frac{1}{2} C_{vv} \gamma^2 \\
+ \quad & s C_{sv} \gamma \sigma_2 + r(s C_S - C) + C_v(\alpha - \kappa\gamma) = 0
\end{aligned}
$$

- The parameters are **calibrated** to the market, that is, chosen so that the market prices of liquidly traded options are matched to the model prices as well as possible.

Examples

- Heston's model:

$$dS(t) = S(t)[rdt + \sqrt{V(t)}dW^Q(t)]$$

$$dV(t) = A(B - V(t))dt + \gamma\sqrt{V(t)}dZ^Q(t)$$

for some other risk-neutral Brownian motion Z^Q having correlation ρ with W^Q. Price is a function $C(t, s, v)$ satisfying

$$0 = C_t + \frac{1}{2}v[s^2 C_{ss} + \gamma^2 C_{vv}] + r(sC_s - C) + A(B - v)C_v + \rho\gamma v s C_{sv}$$

- SABR model:

$$dS(t) = S(t)[rdt + \sigma(t)\frac{1}{S^\beta(t)}dW^Q(t)]$$

$$d\sigma(t) = \alpha\sigma(t)dZ^Q(t)$$

Pricing Options with Mathematical Models

24. Jump-diffusion models

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Merton's jump-diffusion model

- Suppose the jumps arrive according to a Poisson process, that is, at independent exponentially distributed intervals.
- The number $N(t)$ of jumps up to time t is given by Poisson distribution:

$$P[N(t) = k] = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

- The stock price satisfies the following dynamics:

$$dS(t) = S(t)[r - \lambda m]dt + S(t)\sigma dW^Q(t) + dJ(t) \quad ,$$

where m is such that the discounted stock price is a Q-martingale, and dJ is the actual jump size.

Merton's jump-diffusion model (continued)

- More precisely, $dJ(t) = 0$ if there is no jump at time t, and $dJ(t) = S(t)X_i - S(t)$ if the i-th jump occurs at time t, where X_i are iid random variables. Therefore,

$$S(t) = S(0) \cdot X_1 \cdot X_2 \cdot \ldots \cdot X_{N(t)} \cdot e^{(r-\sigma^2/2-\lambda m)t+\sigma W^Q(t)}$$

- The price of payoff $g(S(T))$ is

$$C(0) = \sum_{k=0}^{\infty} E^Q \left[e^{-rT} g(S(T)) \,\middle|\, N(T) = k \right] Q(N(T) = k)$$

which is equal to

$$\textstyle\sum_{k=0}^{\infty} \quad E^Q \left[e^{-rT} g\left(S(0) X_1 \cdot \ldots X_k \cdot e^{(r-\sigma^2/2-\lambda m)T+\sigma W^Q(T)} \right) \right] \times e^{-\lambda T} \frac{(\lambda T)^k}{k!}$$

Merton's jump-diffusion model (continued)

- If X_i's are lognormally distributed, the price of the option can be represented as

$$C(0) = \sum_{k=0}^{\infty} e^{-\tilde{\lambda}T} \frac{(\tilde{\lambda}T)^k}{k!} BS_k$$

for $\tilde{\lambda} = \lambda(1+m)$ and BS_k is the Black and Scholes formula with appropriately chosen $r = r_k$ and $\sigma = \sigma_k$.