

Pricing Options with Mathematical Models

1. OVERVIEW

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

- What we want to accomplish:

Learn the basics of option pricing so you can:

- (i) continue learning on your own, or in more advanced courses;
- (ii) prepare for graduate studies on this topic, or for work in industry, or your own business.

- The prerequisites we need to know:
 - (i) Calculus based probability and statistics, for example computing probabilities and expected values related to normal distribution.
 - (ii) Basic knowledge of differential equations, for example solving a linear ordinary differential equation.
 - (iii) Basic programming or intermediate knowledge of Excel

- A rough outline:
 - Basic securities: stocks, bonds
 - Derivative securities, options
 - Deterministic world: pricing fixed cash flows, spot interest rates, forward rates

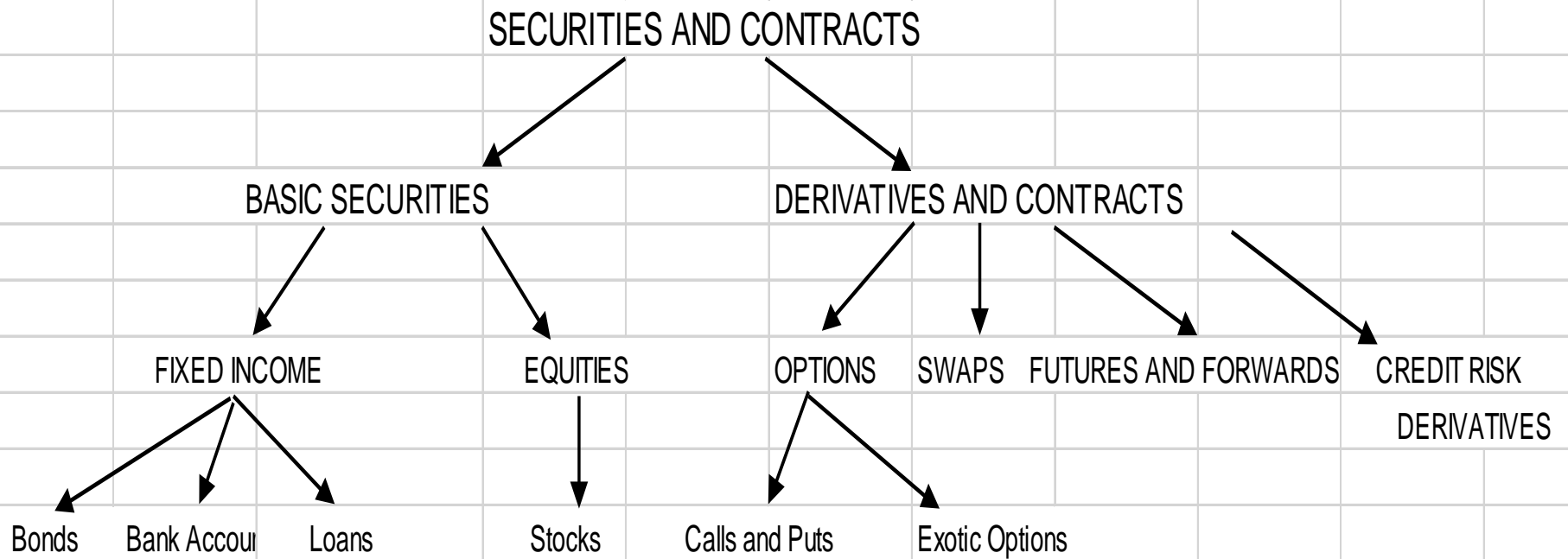
- A rough outline (continued):
 - Stochastic world, pricing options:
 - Pricing by no-arbitrage
 - Binomial trees
 - Stochastic Calculus, Ito's rule, Brownian motion
 - Black-Scholes formula and variations
 - Hedging
 - Fixed income derivatives

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2. Stocks, Bonds, Forwards

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A Classification of Financial Instruments



Stocks

- Issued by firms to finance operations
- Represent ownership of the firm
- Price known today, but not in the future
- May or may not pay dividends

Bonds

- Price known today
- Future payoffs known at fixed dates
- Otherwise, the price movement is random
- Final payoff at maturity: face value/nominal value/principal
- Intermediate payoffs: coupons
- Exposed to default/credit risk

Derivatives

- Sell for a **price/value/premium** today.
- Future value **derived** from the value of the underlying securities (as a function of those).
- Traded at exchanges – standardized contracts, no credit risk;
- or, over-the-counter (OTC) – a network of dealers and institutions, can be non-standard, some credit risk.

Why derivatives?

- To hedge risk
- To speculate
- To attain “arbitrage” profit
- To exchange one type of payoff for another
- To circumvent regulations

Forward Contract

- An agreement to buy (**long**) or sell (**short**) a given **underlying** asset S :
 - At a predetermined future date T (**maturity**).
 - At a predetermined price F (**forward price**).
- F is chosen so that the contract has zero value today.
- Delivery takes place at maturity T :
 - Payoff at maturity: $S(T) - F$ or $F - S(T)$
 - Price F set when the contract is established.
 - $S(T) = \text{spot (market) price}$ at maturity.

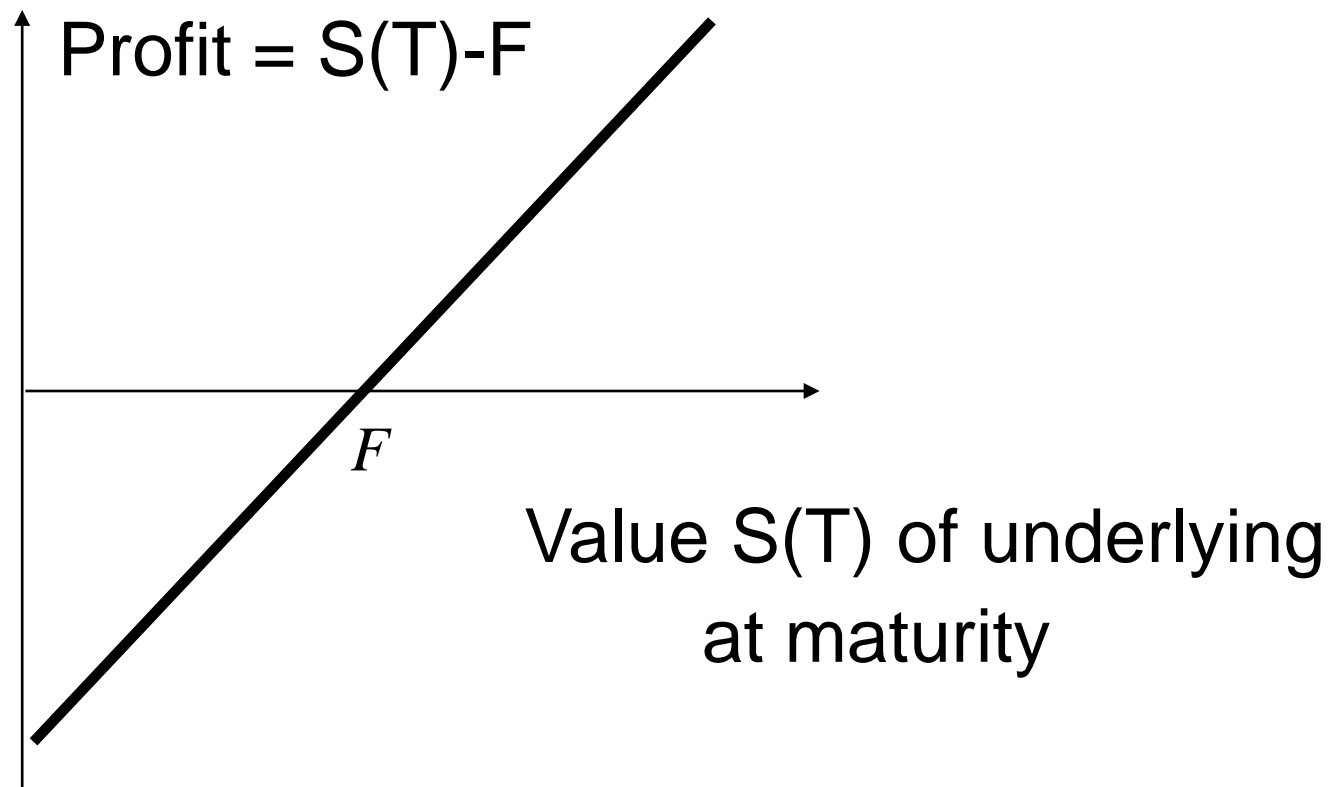
Forward Contract (continued)

- Long position: obligation to buy
- Short position: obligation to sell
- Differences with options:
 - Delivery has to take place.
 - Zero value today.

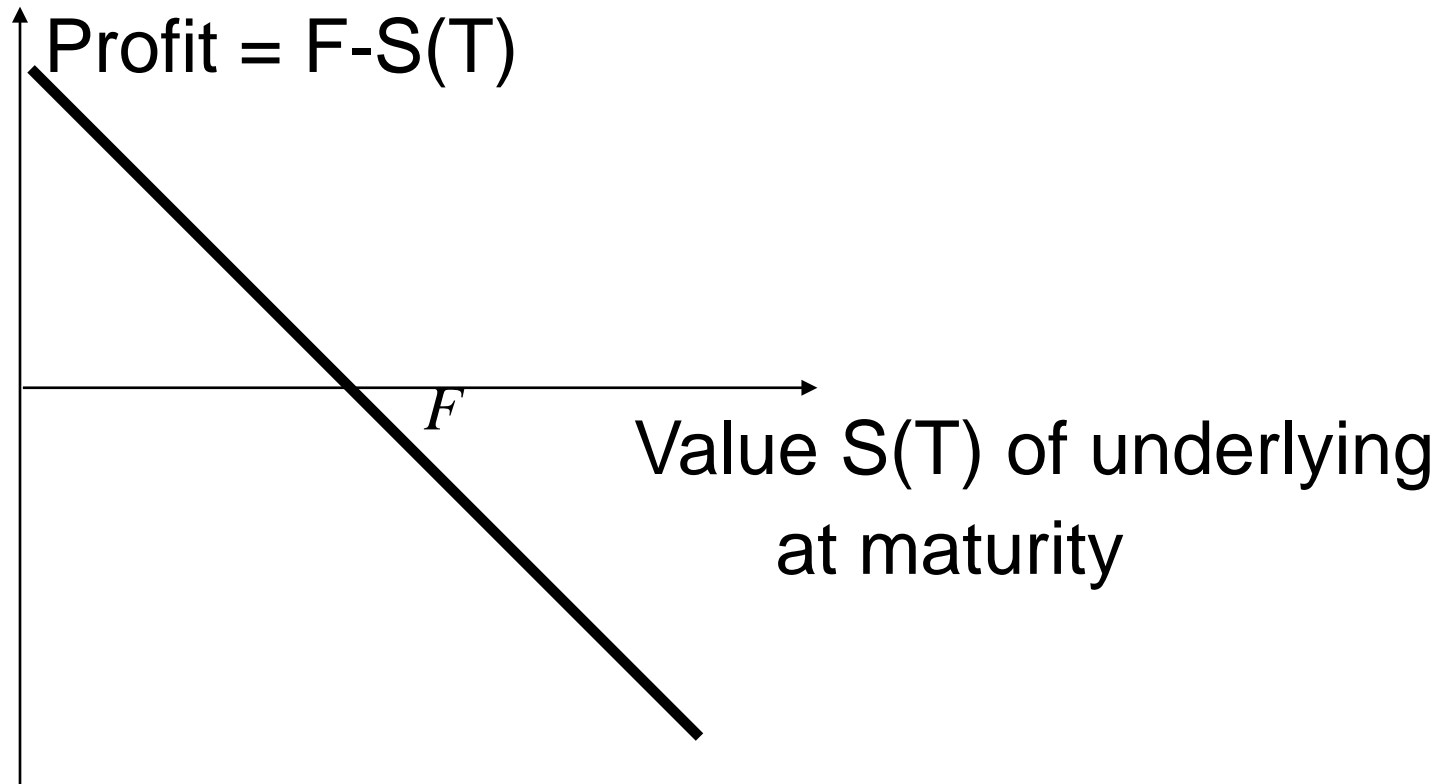
Example

- On May 13, a firm enters into a long forward contract to buy one million euros in six months at an exchange rate of 1.3
- On November 13, the firm pays $F = \$1,300,000$ and receives $S(T)$ = one million euros.
- How does the payoff look like at time T as a function of the dollar value of $S(T)$ spot exchange rate?

Profit from a long forward position



Profit from a short forward position



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3. Swaps

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Swaps

- Agreement between two parties to exchange two series of payments.
- Classic interest rate swap:
 - One party pays **fixed** interest rate payments on a notional amount.
 - Counterparty pays **floating** (random) interest rate payments on the same notional amount.
- Floating rate is often linked to LIBOR (London Interbank Offer Rate), reset at every payment date.

Motivation

- The two parties may be exposed to different interest rates in different markets, or to different institutional restrictions, or to different regulations.

A Swap Example

- New pension regulations require higher investment in fixed income securities by pension funds, creating a problem: liabilities are long-term while new holdings of fixed income securities may be short-term.
- Instead of selling assets such as stocks, a pension fund can enter a swap, exchanging returns from stocks for fixed income returns.
- Or, if it wants to have an option not to exchange, it can buy **swaptions** instead.

Swap Comparative Advantage

- US firm B wants to borrow AUD, Australian firm A wants to borrow USD
- Firm B can borrow at 5% in USD, 12.6% AUD
- Firm A can borrow at 7% USD, 13% AUD
- Expected gain = $(7-5) - (13-12.6) = 1.6\%$
- Swap:



- Bank gains 1.3% on USD, loses 1.1% on AUD, gain = 0.2%
- Firm B gains $(12.6-11.9) = 0.7\%$
- Firm A gains $(7-6.3) = 0.7\%$
- Part of the reason for the gain is credit risk involved

A Swap Example: Diversifying

- Charitable foundation CF receives 50mil in stock X from a privately owned firm.
- CF does not want to sell the stock, to keep the firm owners happy
- Equity swap: pays returns on 50mil in stock X, receives return on 50mil worth of S&P500 index.
- A bad scenario: S&P goes down, X goes up; a potential cash flow problem.

Swap Example: Diversifying II

- An executive receives 500mil of stock of her company as compensation.
- She is not allowed to sell.
- Swap (if allowed): pays returns on a certain amount of the stock, receives returns on a certain amounts of a stock index.
- Potential problems: less favorable tax treatment; shareholders might not like it.

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4. Call and Put Options

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Vanilla Options

- **Call** option: a right to buy the underlying
- **Put** option: a right to sell the underlying
- **European** option: the right can be **exercised** only at **maturity**
- **American** option: can be exercised at any time before maturity

Various underlying variables

- Stock options
- Index options
- Futures options
- Foreign currency options
- Interest rate options
- Credit risk derivatives
- Energy derivatives
- Mortgage based securities
- Natural events derivatives ...

Exotic options

- **Asian options:** the payoff depends on the average underlying asset price
- **Lookback options:** the payoff depends on the maximum or minimum of the underlying asset price
- **Barrier options:** the payoff depends on whether the underlying crossed a barrier or not
- **Basket options:** the payoff depends on the value of several underlying assets.

Terminology

- *Writing an option*: selling the option
- *Premium*: price or value of an option
- Option **in/at/out of the money**:
 - *At*: strike price equal to underlying price
 - *In*: immediate exercise would be profitable
 - *Out*: immediate exercise would not be profitable

Long Call

Outcome at maturity

	$S(T) \leq K$	$S(T) > K$
Payoff:	0	$S(T) - K$
Profit:	$-C(t, K, T)$	$S(T) - K - C(t, K, T)$

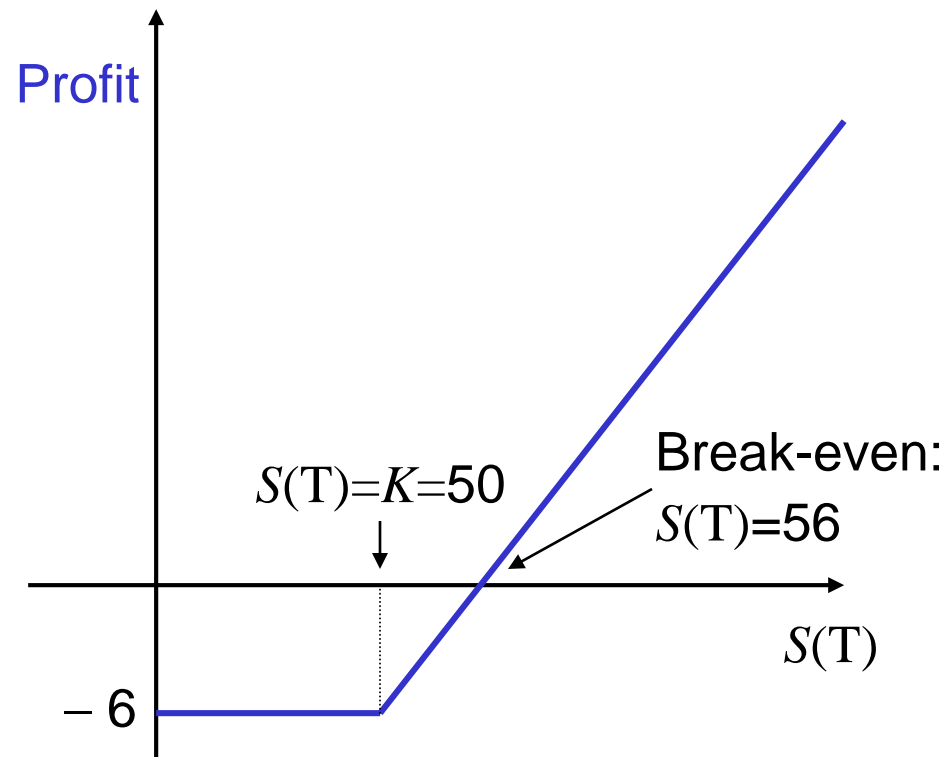
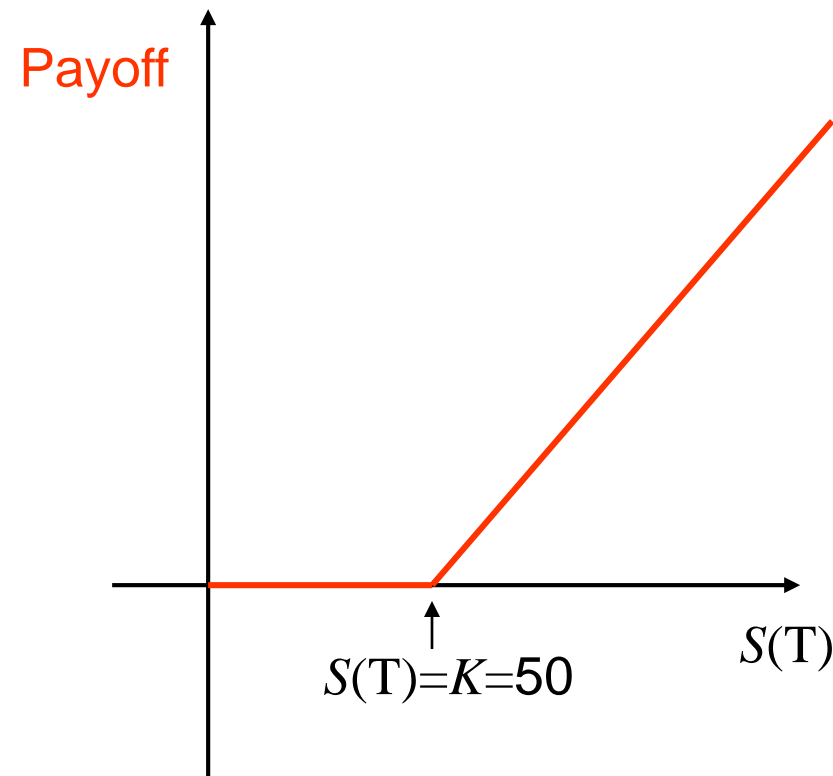
A more compact notation:

Payoff: $\max [S(T) - K, 0] = (S(T) - K)_+$

Profit: $\max [S(T) - K, 0] - C(t, K, T)$

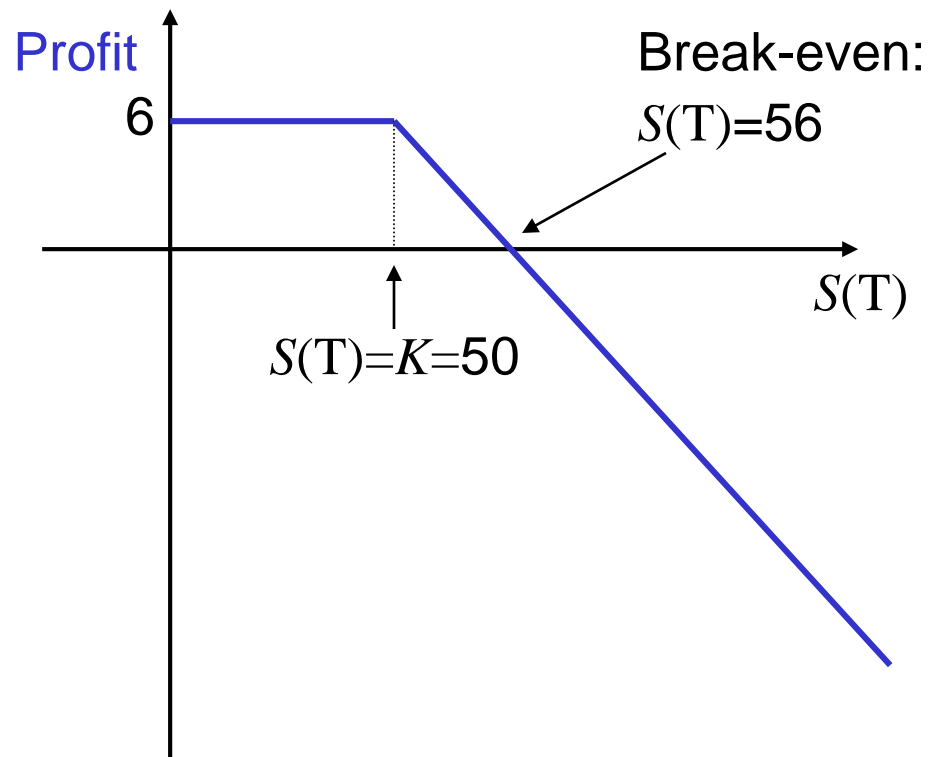
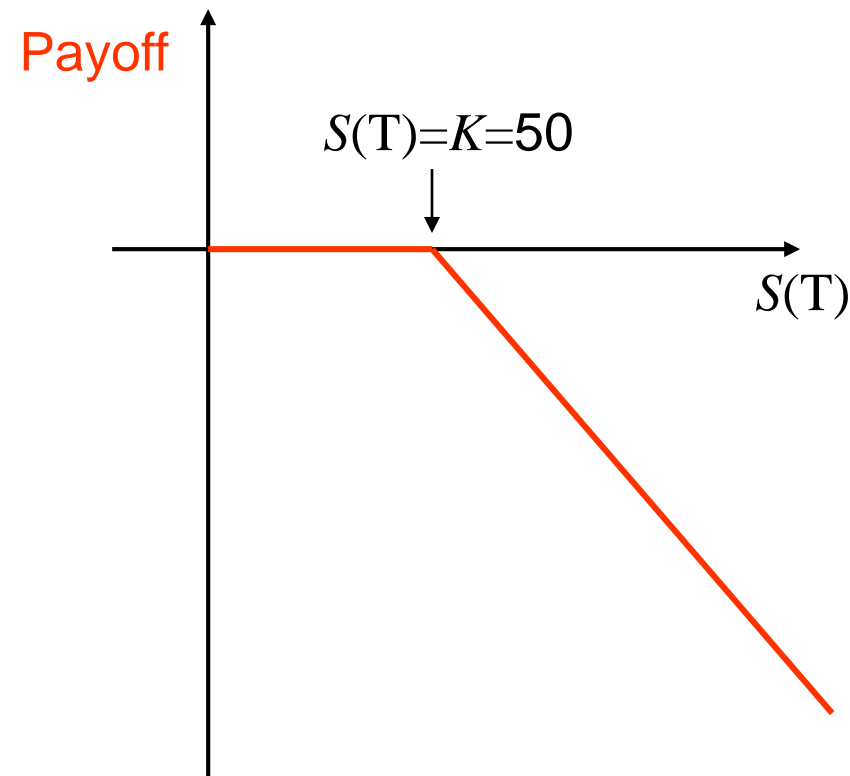
Long Call Position

- Assume $K = \$50$, $C(t, K, T) = \$6$
- Payoff: $\max [S(T) - 50, 0]$
- Profit: $\max [S(T) - 50, 0] - 6$



Short Call Position

- $K = \$50$, $C(t, K, T) = \$6$
- Payoff: $-\max [S(T) - 50, 0]$
- Profit: $6 - \max [S(T) - 50, 0]$



Long Put

Outcome at maturity

$$S(T) \leq K$$

$$S(T) > K$$

Payoff:

$$K - S(T)$$

$$0$$

Profit:

$$K - S(T) - P(t, K, T) \quad - P(t, K, T)$$

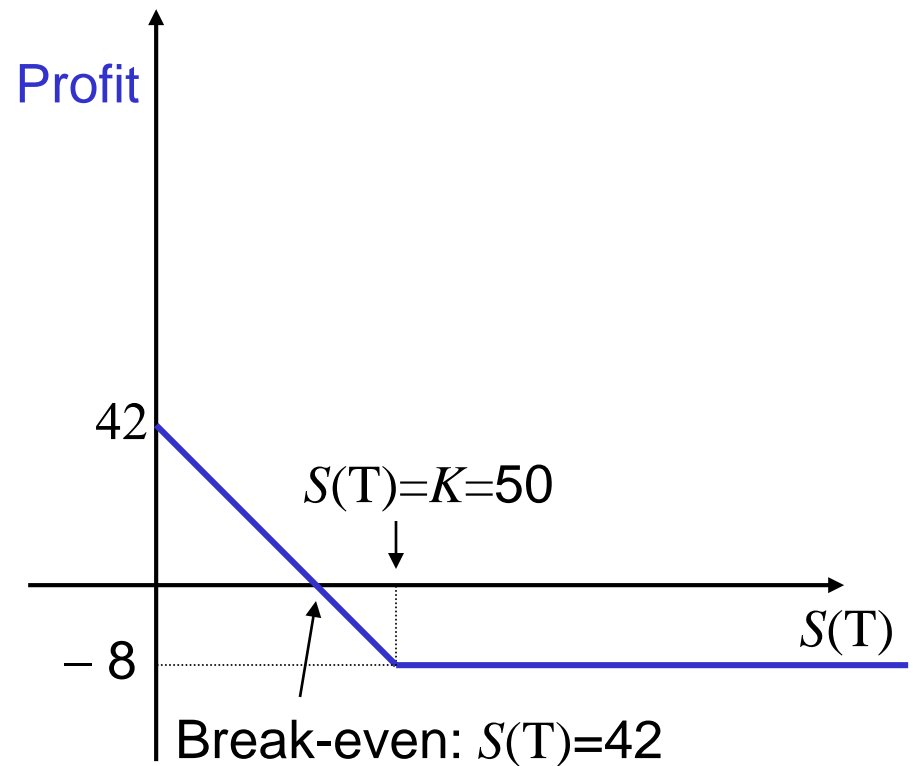
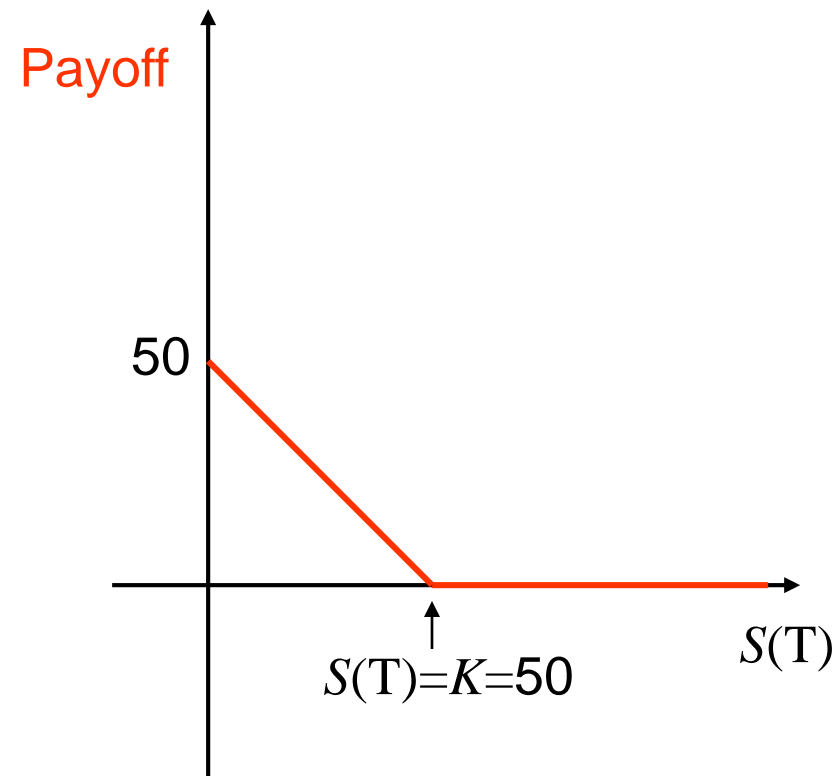
A more compact notation:

$$\text{Payoff:} \quad \max [K - S(T), 0] = (K - S(T))_+$$

$$\text{Profit:} \quad \max [K - S(T), 0] - P(t, K, T)$$

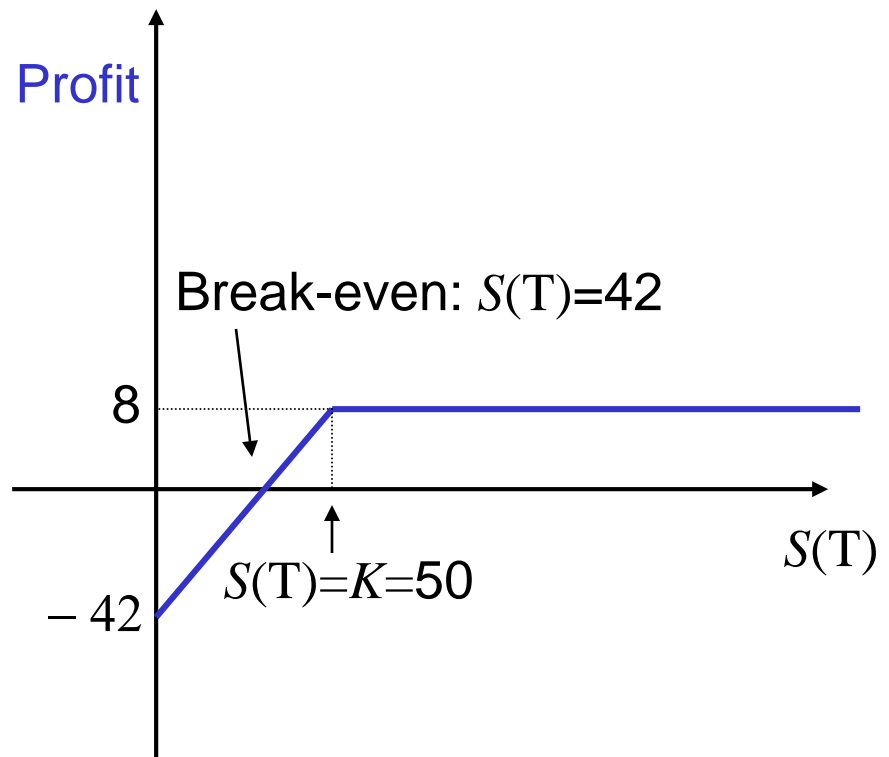
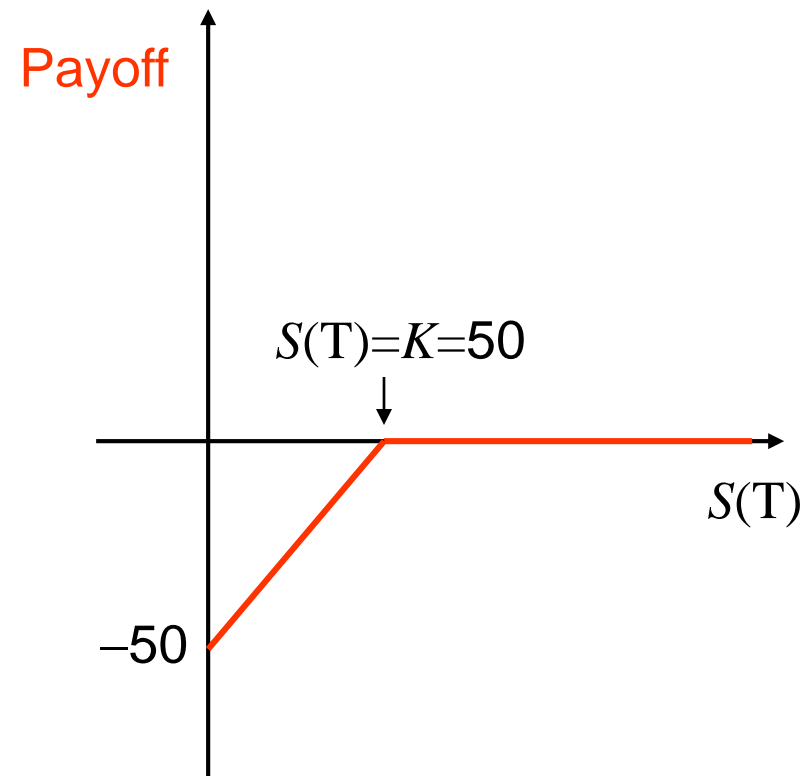
Long Put Position

- Assume $K = \$50$, $P(t, K, T) = \$8$
- Payoff: $\max [50 - S(T), 0]$
- Profit: $\max [50 - S(T), 0] - 8$



Short Put Position

- $K = \$50$, $P(t, K, T) = \$8$
- Payoff: $-\max [50 - S(T), 0]$
- Profit: $8 - \max [50 - S(T), 0]$



Implicit Leverage: Example

- Consider two securities
 - Stock with price $S(0) = \$100$
 - Call option with price $C(0) = \$2.5$ ($K = \$100$)
- Consider three possible outcomes at $t=T$:
 - Good: $S(T) = \$105$
 - Intermediate: $S(T) = \$101$
 - Bad: $S(T) = \$98$

Implicit Leverage: Example (continued)

Suppose we plan to invest \$100

Invest in:	Stocks	Options
Units	1	40
Return in:		
Good State	5%	100%
Mid State	1%	-60%
Bad State	-2%	-100%

EQUITY LINKED BANK DEPOSIT

- Investment = 10,000
- Return = 10,000 if an index below the current value of 1,300 after 5.5 years
- Return = $10,000 \times (1 + 70\% \text{ of the percentage return on index})$
- Example: Index=1,500. Return =
 $= 10,000 \cdot (1 + (1,500/1,300 - 1) \cdot 70\%) = 11,077$
- Payoff = Bond + call option on index

HEDGING EXAMPLE

Your bonus compensation: 100 shares of the company, each worth \$150.

Your hedging strategy: buy 50 put options with strike $K = 150$

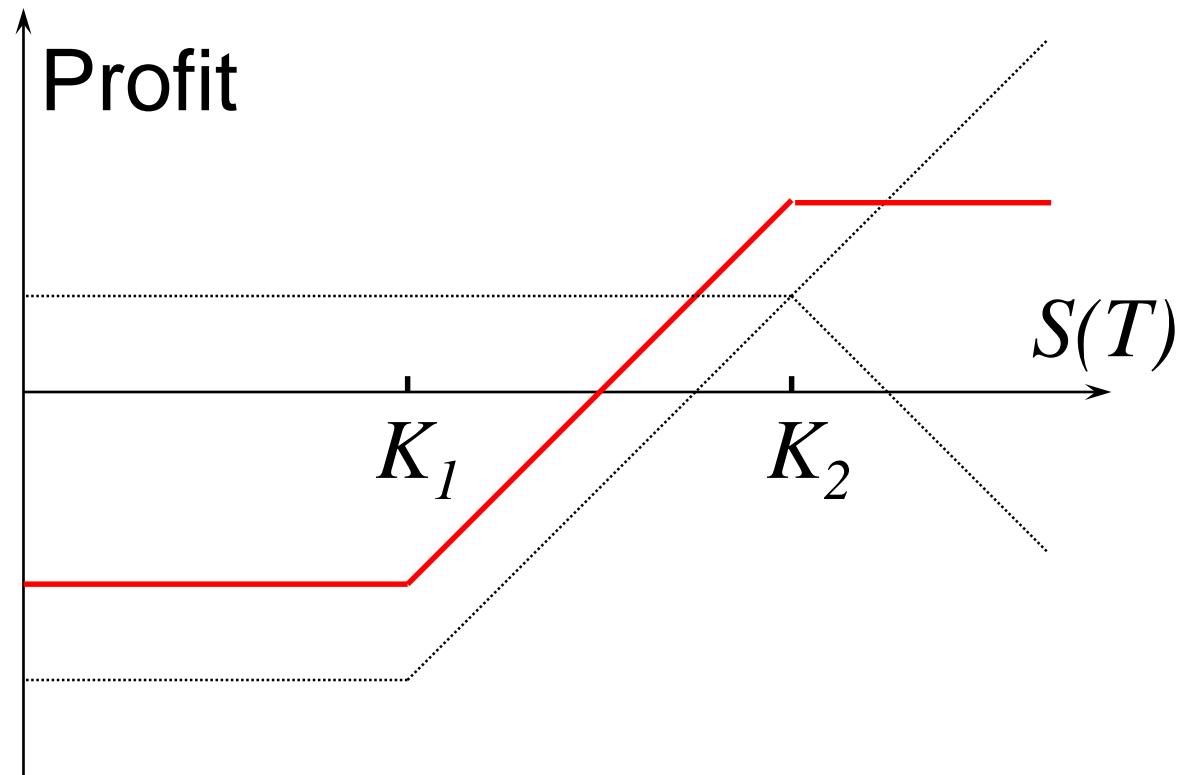
If share value falls to \$100: you lose \$5,000 in stock, win \$2,500 minus premium in options

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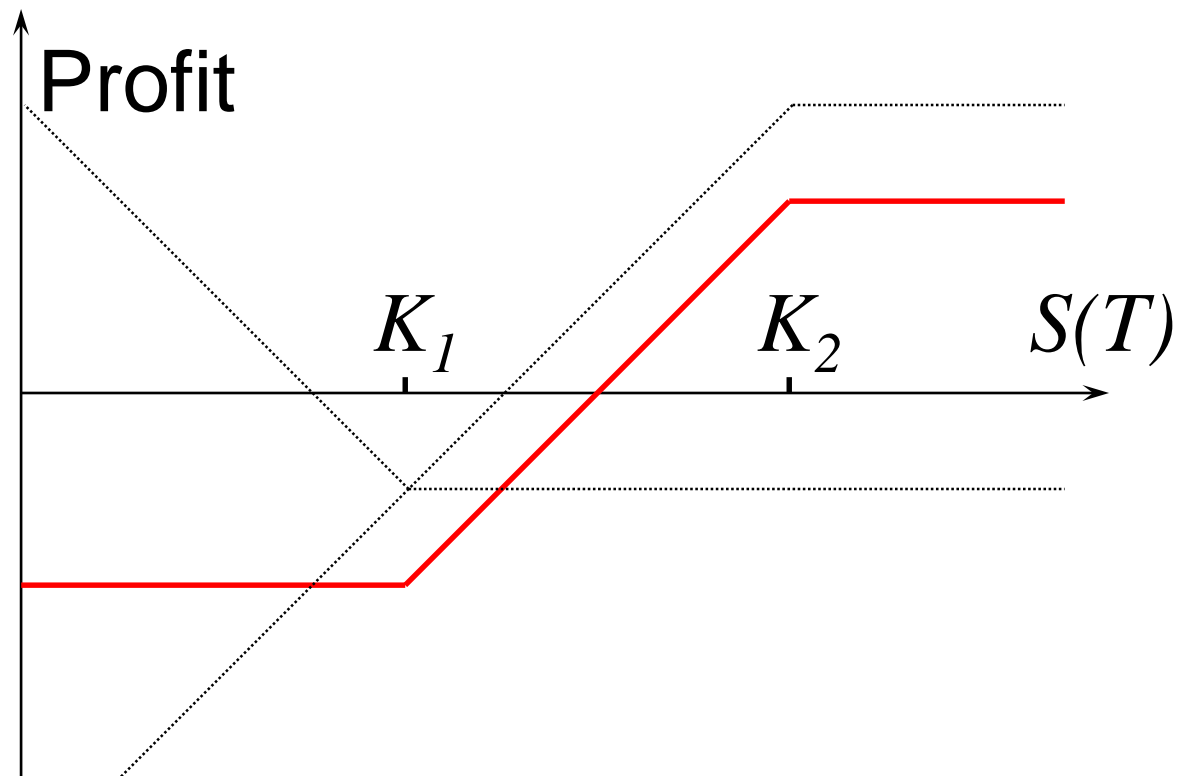
5. Options Combinations

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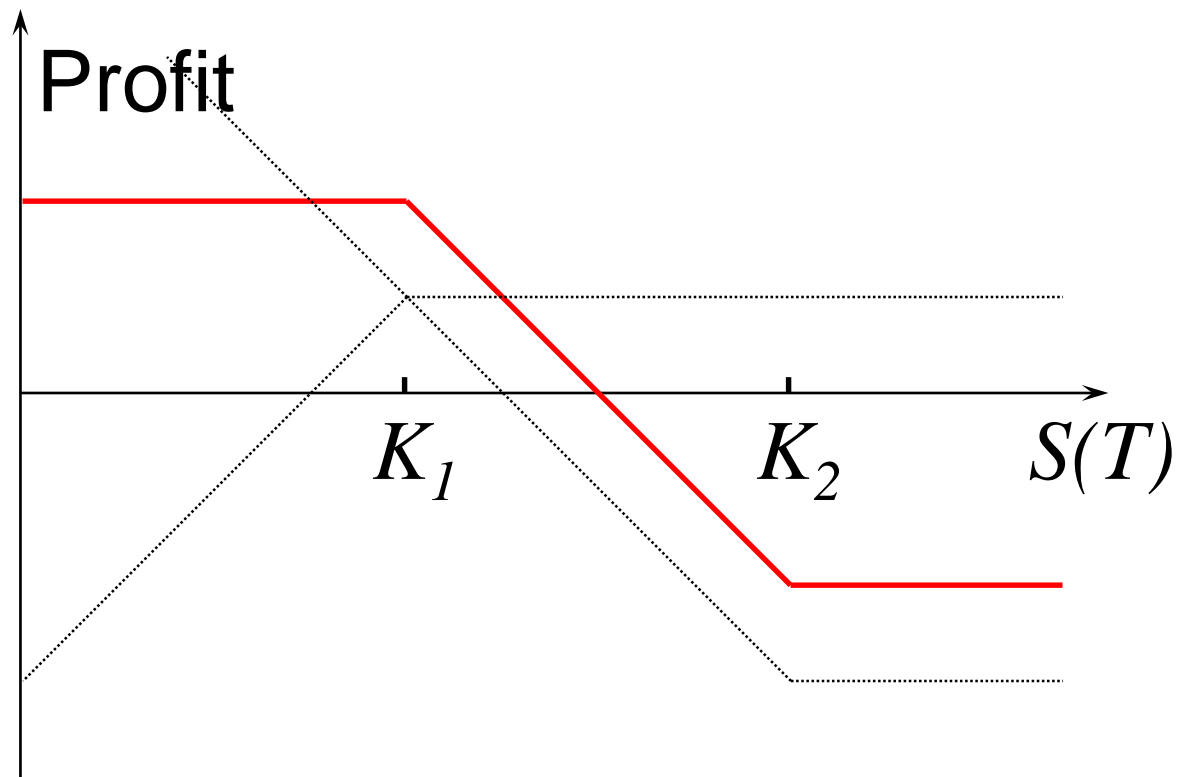
Bull Spread Using Calls



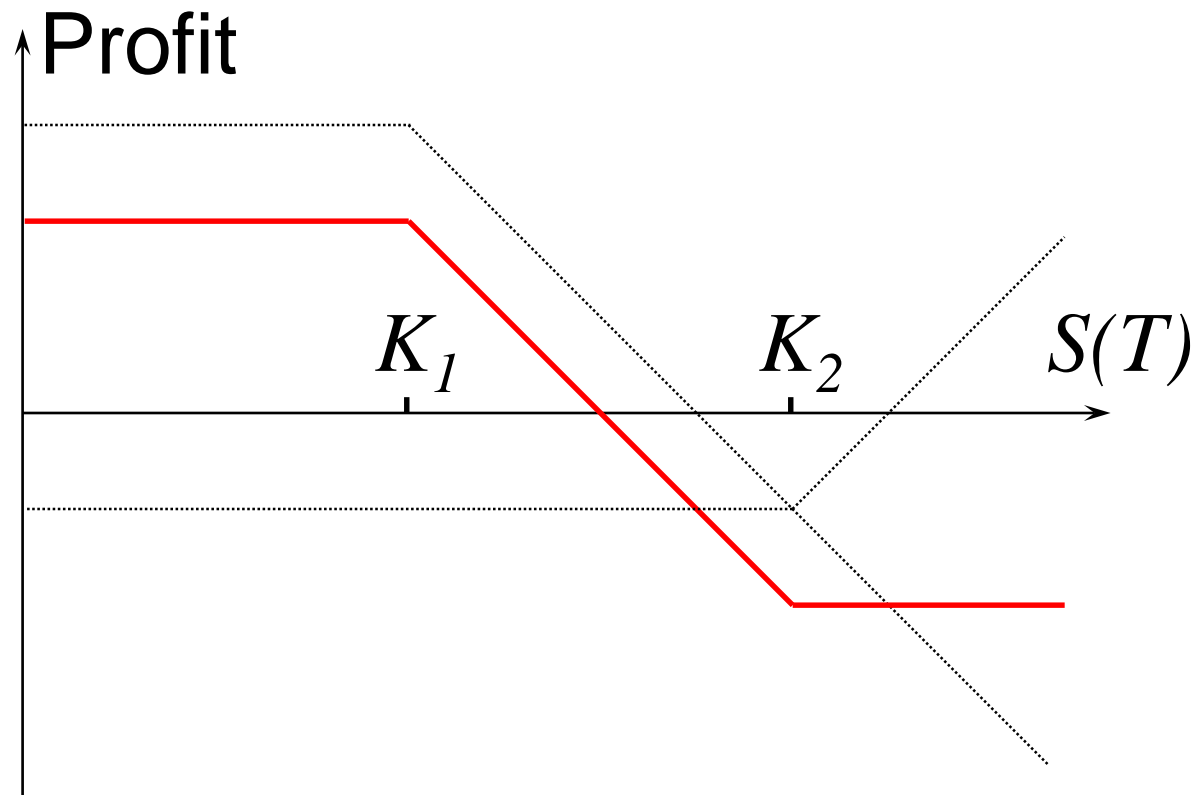
Bull Spread Using Puts



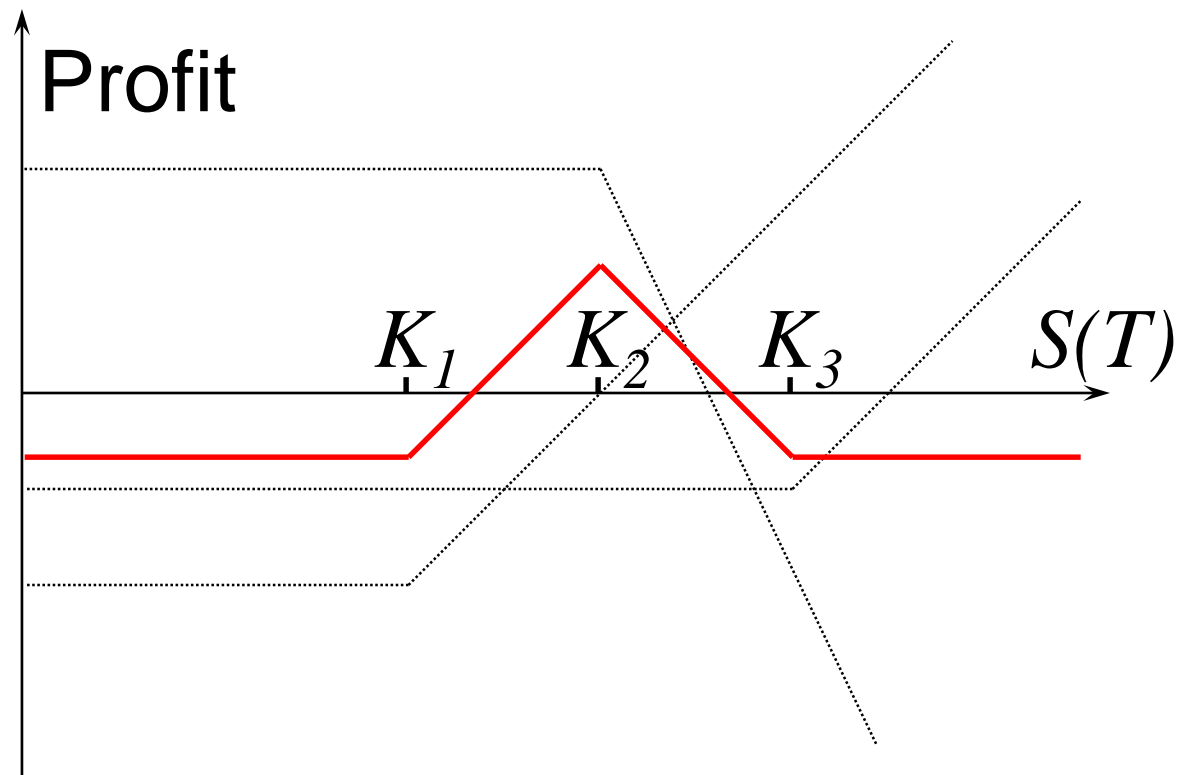
Bear Spread Using Puts



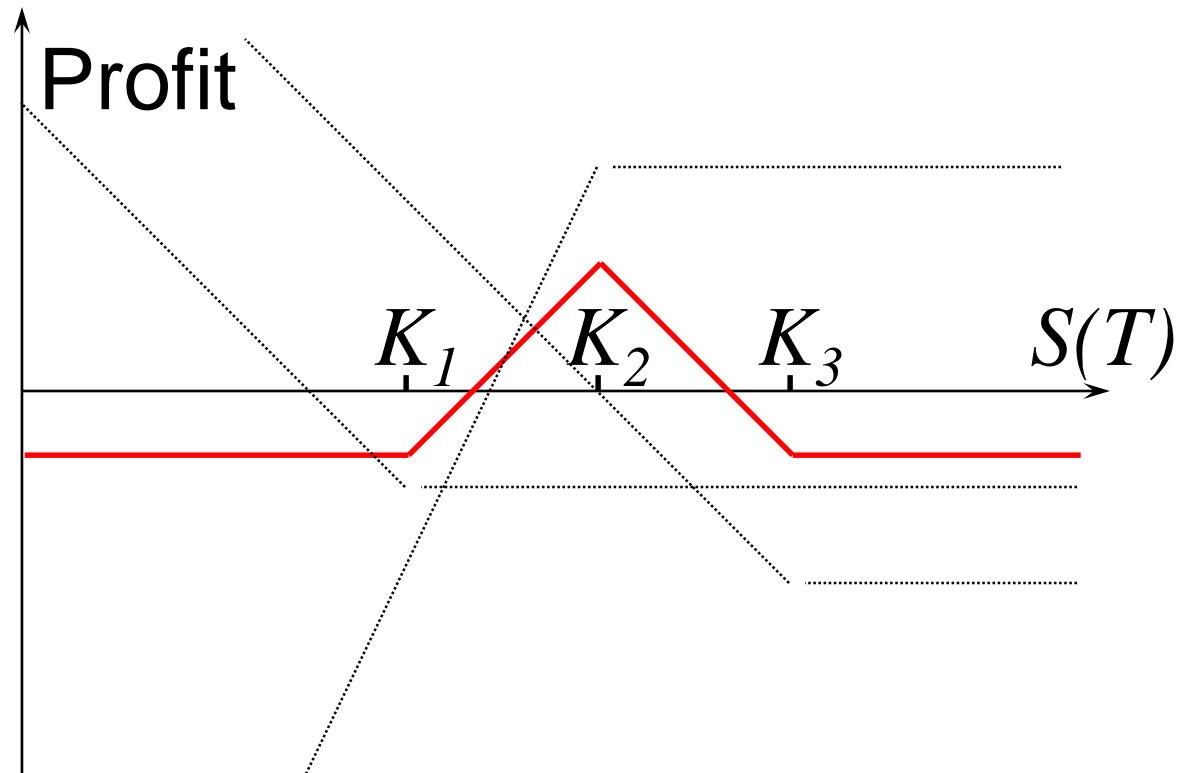
Bear Spread Using Calls



Butterfly Spread Using Calls



Butterfly Spread Using Puts



Bull Spread (Calls)

- Two strike prices: K_1, K_2 with $K_1 < K_2$
- Short-hand notation: $C(K_1), C(K_2)$

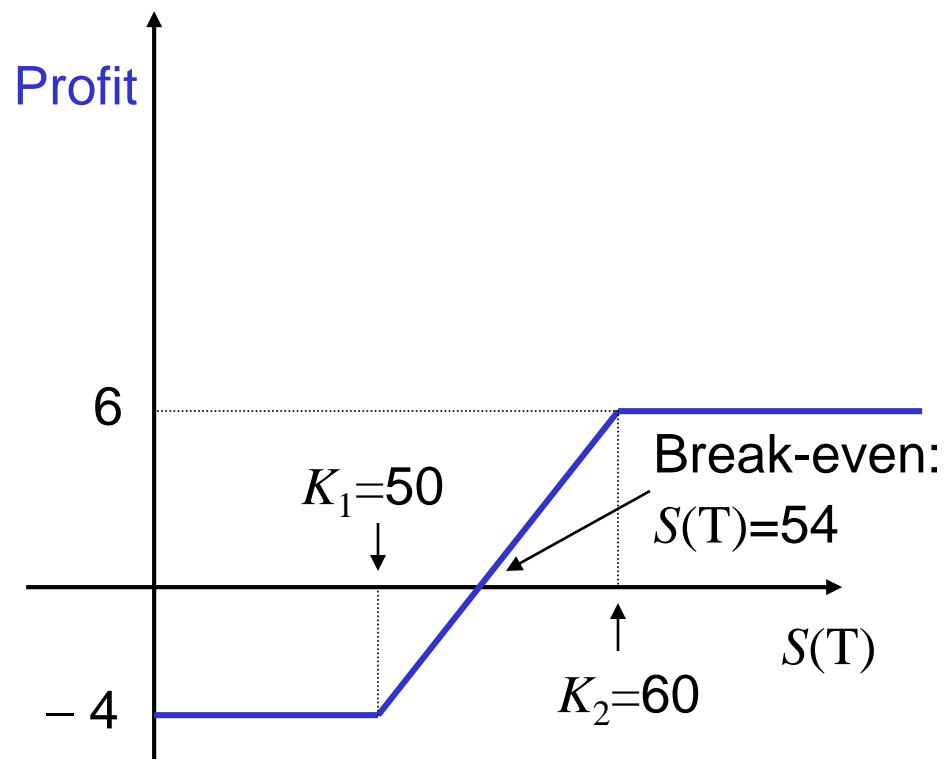
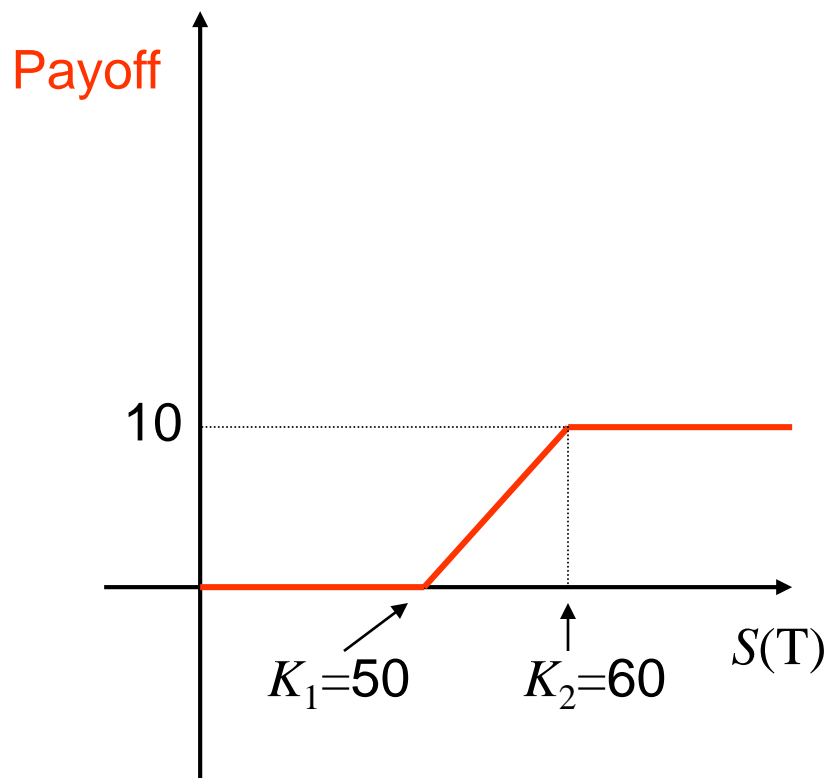
Outcome at Expiration

$S(T) \leq K_1$ $K_1 < S(T) \leq K_2$ $S(T) > K_2$

Payoff:	0	$S(T) - K_1$	$S(T) - K_1 - (S(T) - K_2) =$ $= K_2 - K_1$
Profit:	$C(K_2) - C(K_1)$	$C(K_2) - C(K_1)$ $+ S(T) - K_1$	$C(K_2) - C(K_1) + K_2 - K_1$

Bull Spread (Calls)

- Assume $K_1 = \$50$, $K_2 = \$60$, $C(K_1) = \$10$, $C(K_2) = \$6$
- Payoff: $\max [S(T) - 50, 0] - \max [S(T) - 60, 0]$
- Profit: $(6-10) + \max [S(T)-50,0] - \max [S(T)-60,0]$



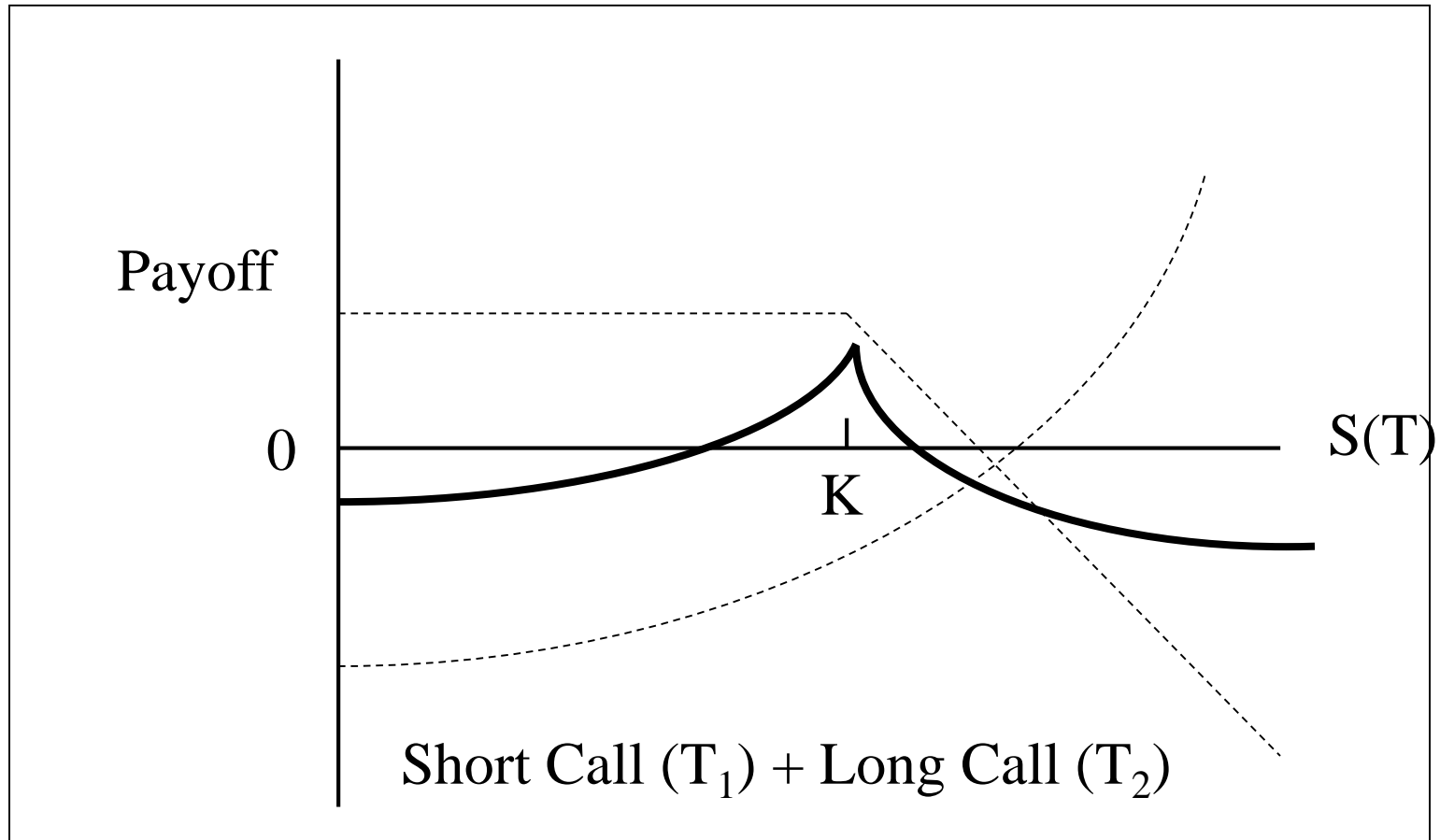
Bear Spread (Puts)

- Again two strikes: K_1, K_2 with $K_1 < K_2$
- Short-hand notation: $P(K_1), P(K_2)$

Outcome at Expiration

	$S(T) \leq K_1$	$K_1 < S(T) \leq K_2$	$S(T) > K_2$
Payoff:	$K_2 - S(T) - (K_1 - S(T)) =$ $= K_2 - K_1$	$K_2 - S(T)$	0
Profit:	$P(K_1) - P(K_2) + K_2 - K_1$	$P(K_1) - P(K_2) +$ $+ K_2 - S(T)$	$P(K_1) - P(K_2)$

Calendar Spread



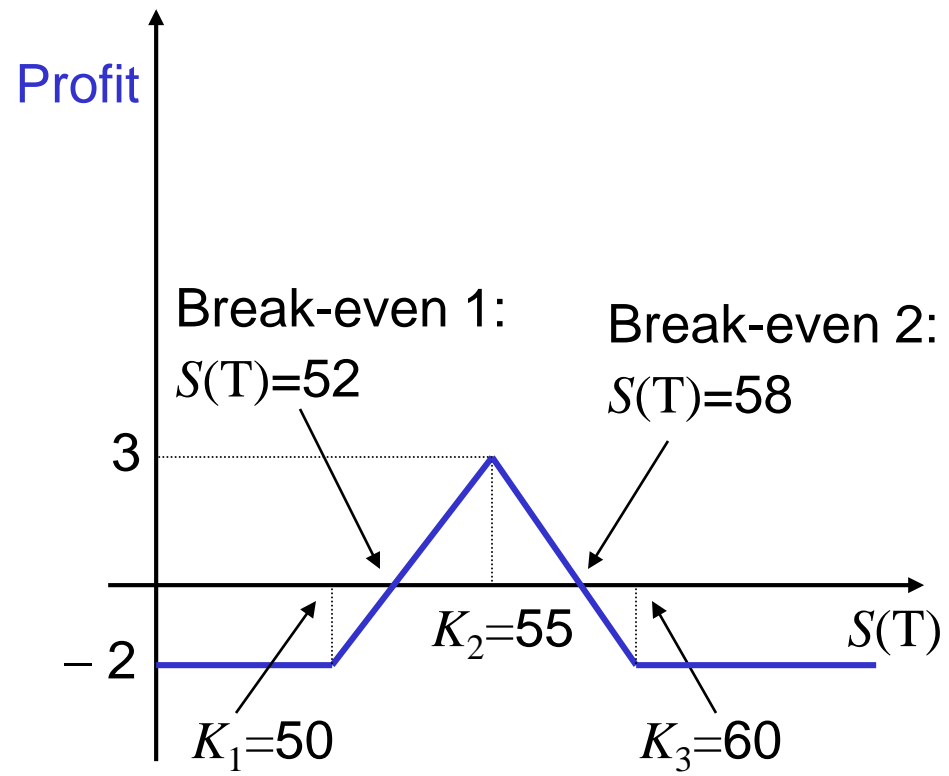
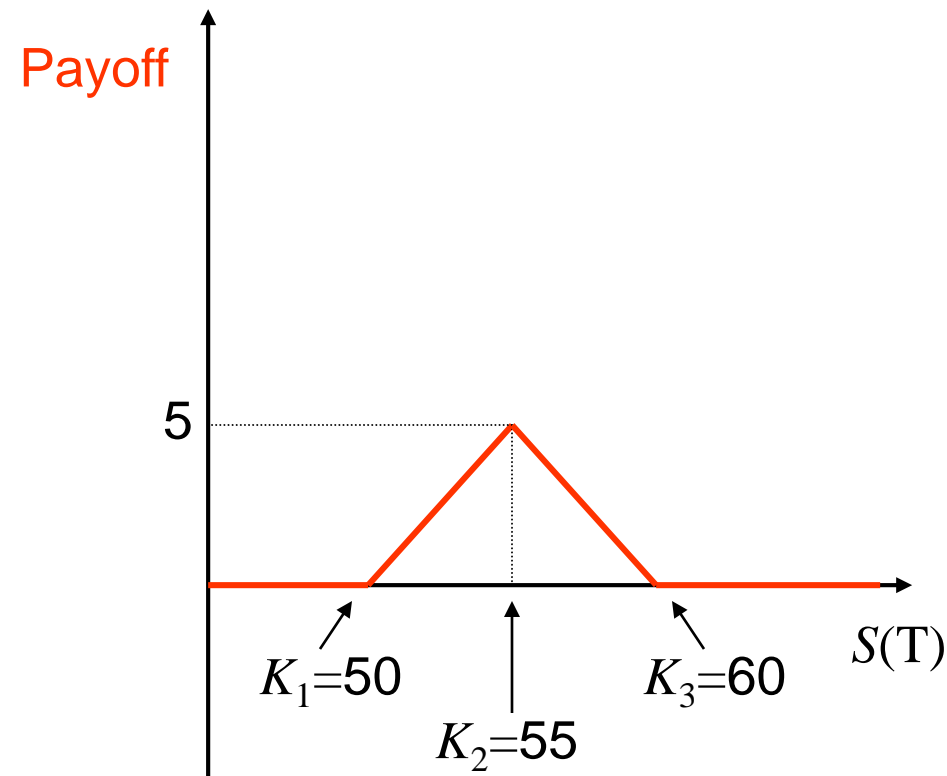
Butterfly Spread

- Positions in **three** options of the same class, with same maturities but different strikes K_1, K_2, K_3
 - Long butterfly spreads: buy one option each with strikes K_1, K_3 , sell two with strike K_2
- $K_2 = (K_1 + K_3) / 2$



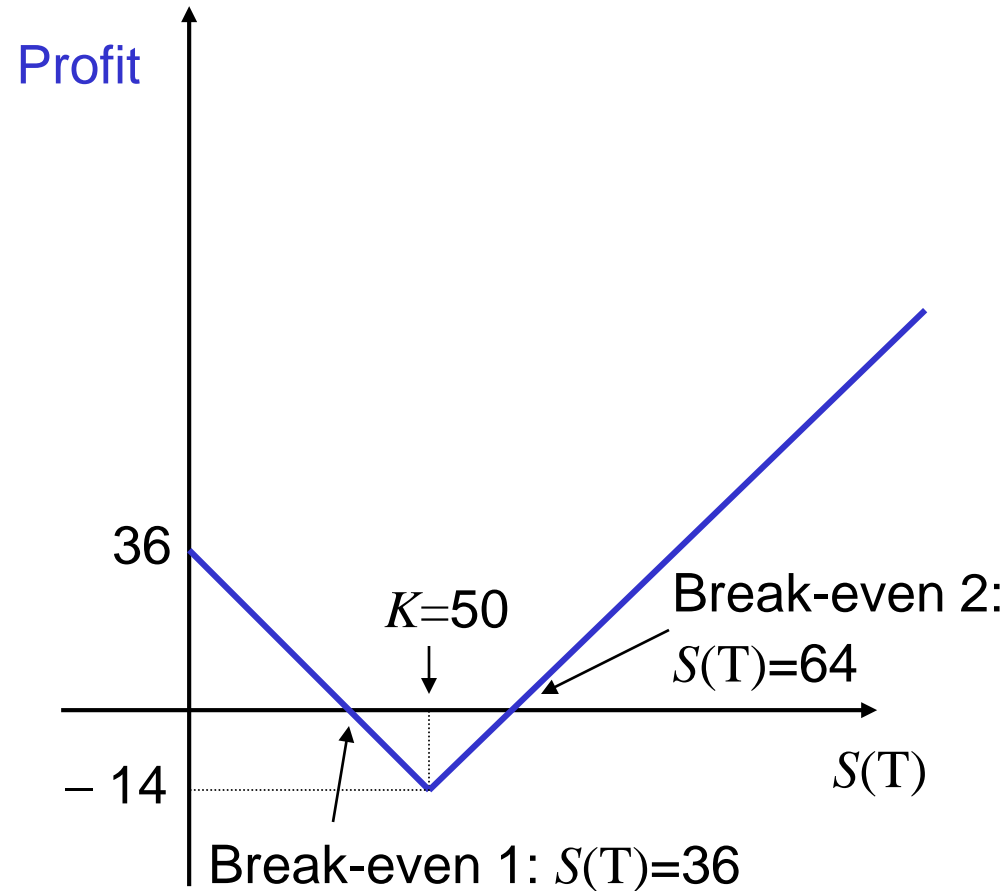
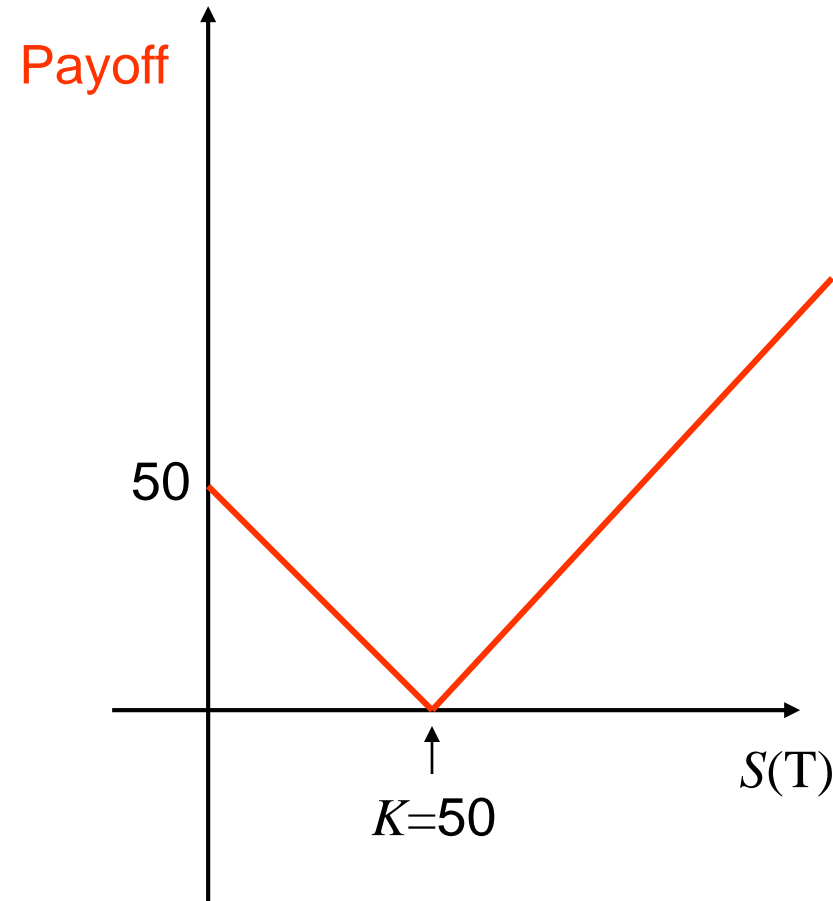
Long Butterfly Spread (Puts)

- $K_1 = \$50$, $K_2 = \$55$, $K_3 = \$60$
- $P(K_1) = \$4$, $P(K_2) = \$6$, $P(K_3) = \$10$



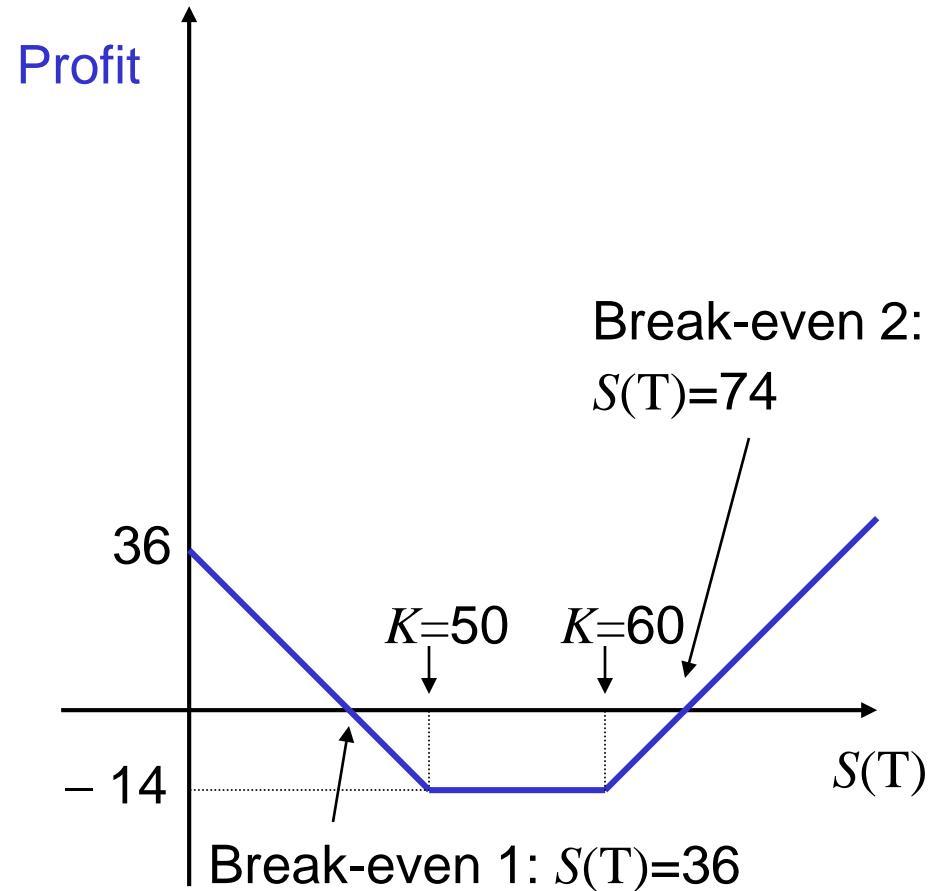
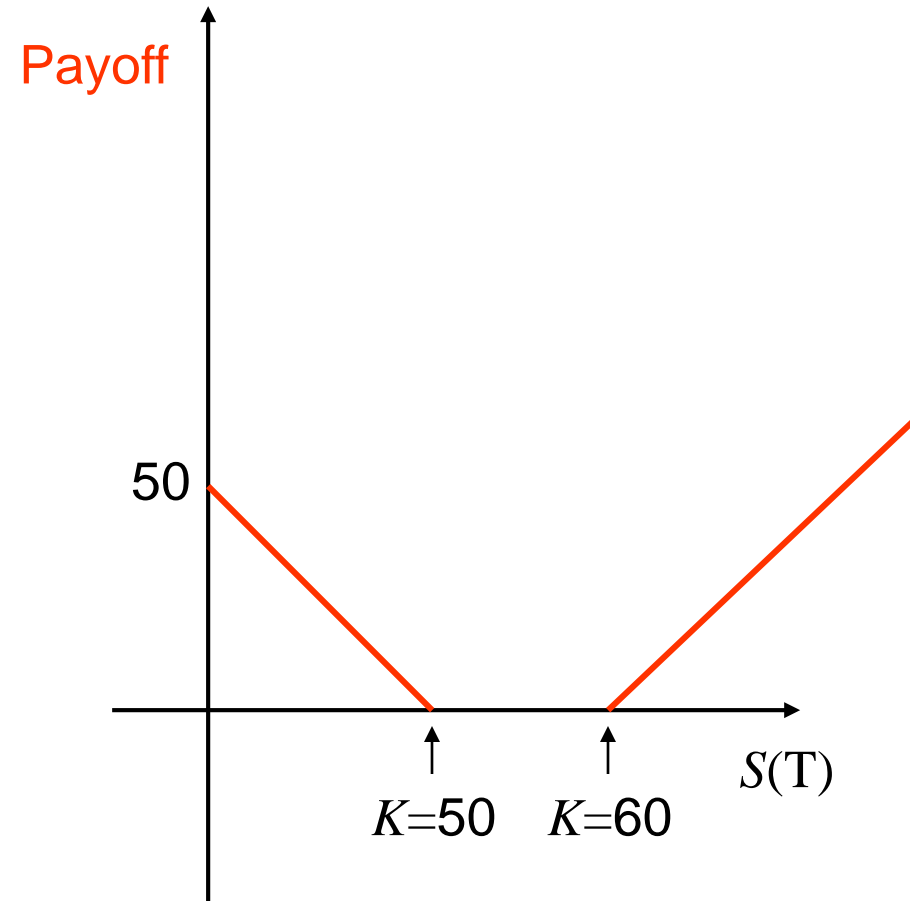
Bottom Straddle

Assume $K = \$50$, $P(K) = \$8$, $C(K) = \$6$



Bottom Strangle

Assume $K_1 = \$50$, $K_2 = \$60$, $P(K_1) = \$8$, $C(K_2) = \$6$



Arbitrary payoff shape

- Suppose we want to have a payoff of the form $f(S(T))$ for some function $f(\cdot)$. Assume that call options written on $S(T)$ are traded for all possible strike values K .
- CLAIM: If $f(\cdot)$ is smooth and $f'(\infty) \cdot 0 = 0$, then
- $$f(s) = f(0) + f'(0)s + \int_0^\infty f''(K) \max(S - K, 0) dK$$

Proof sketch

$$\int_0^{\infty} f''(K) \max(s - K, 0) dK$$

$$= (\text{integration by parts}) =$$

$$= f'(\infty) \cdot 0 - f'(0) \cdot s - \int_0^{\infty} f'(K) d[\max(s - K, 0)]$$

$$= -f'(0) \cdot s + \int_0^s f'(K) dK$$

$$= -f'(0) \cdot s + f(s) - f(0) .$$

