Pricing Options with Mathematical Models

25. Static hedging with futures

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Perfect hedge with futures

- There is a futures contract that trades:
  - exactly the asset we want to hedge.
  - with the exact maturity we want to hedge.
- Otherwise, “asset mismatch” or “maturity mismatch”. Then, the solution is “crosshedging”.

Crosshedging

- Hedging payoff $S_1(T)$ with futures F_2 maturing at $U \geq T$: the unknown quantity X, called **basis** is, with δ the number of futures conttracts,

$$X = S_1(T) - \delta F_2(T,U)$$

- The risk is measured as

$$Var_t[X] = Var_t[S_1(T)] + \delta^2 Var_t[F_2(T,U)] - 2\delta Cov_t[S_1(T), F_2(T,U)]$$

- If we want to minimize the variance, taking the derivative with respect to δ and setting it equal to zero gives us the optimal δ:

$$\delta = \frac{Cov_t[S_1(T), F_2(T,U)]}{Var_t[F_2(T,U)]} = \rho \frac{\sigma_S}{\sigma_F} \quad ,$$

where ρ is the correlation between $S_1(T)$ and $F_2(T,U)$, and σ_S^2, σ_F^2 are their variances.

Crosshedging (continued)

- The minimal variance is

$$Var_t[X] = Var_t[S_1(T)] - \frac{Cov_t^2[S_1(T), F_2(T,U)]}{Var_t[F_2(T,U)]} \quad .$$

- In the case of perfect hedge, $S_1 = S_2$, and $U = T$, we have $S_1(T) = F_1(T) = F_2(T,U)$. Then, $\delta = 1$, and, using the fact that $Cov[S,S] = Var[S]$, we see that $Var_t[X] = 0$ and there is no basis risk involved.

Example

- Consider a U.S. firm that will receive 1 million of currency A, six months from now. The firm will hedge by shorting δ units of six-month futures contracts on a highly correlated currency B.

- Suppose that the exchange rates are $Q_A = 0.1$ dollar$/A$ and $Q_2 = 0.2$ dollar$/B$. The exchange rate A/B has to be $Q = Q_A/Q_B = 0.5$. However, this does not mean that the company should short 0.5 million of B futures.

- Suppose historical data gives us $\sigma_{Q_A} = 0.03$, and $\sigma_{Q_B} = 0.02$, and that the correlation between the two is 0.9. Thus, the covariance is equal to $0.9 \cdot 0.02 \cdot 0.03 = 0.00054$. We get

$$\delta = 0.00054/0.0004 = 1.35$$

- Therefore, for each unit of A the U.S. company should short an amount of B equivalent to 1.35 of A, thus $1,000,000 \cdot 1.35/2 = 675,000$ of B.

- The minimal variance (per unit of currency) is 0.000171, quite a bit smaller than the variance of the dollar$/A$ exchange rate, equal to $\sigma^2_{Q_A} = 0.0009$.

Rolling the hedge forward: the story of Metallgesellschaft

- In early 1990s, MG sold huge volume of long-term fixed price forward-type contracts to deliver oil.
- Hedging: rolling over short-term future contracts to receive oil.
- Oil price went down: good for fixed price contracts.
- Bad for futures: large margin calls.
- All contracts closed out at a huge loss.

Pricing Options with Mathematical Models

26. Static hedging with bonds

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Duration

- Suppose the bond price is given by

$$P = \sum_{i=1}^{T} \frac{C_i}{(1+y)^i}$$

- The sensitivity of the price with respect to yield is

$$\frac{\partial P}{\partial y} = -\sum_{i=1}^{T} i \frac{C_i}{(1+y)^{i+1}} = -\frac{P}{1+y} \sum_{i=1}^{T} i \frac{1}{P} \frac{C_i}{(1+y)^i}$$

- (Macaulay) **duration** is defined as

$$D = \sum_{i=1}^{T} i \frac{1}{P} \frac{C_i}{(1+y)^i}$$

- It is an average of the coupon payment times, weighted by the relative size of the coupons; it is equal to maturity T for the zero-coupon bond.

Bond immunization

- A second order measure of yield risk, the **convexity**, is defined as

$$C = \frac{1}{P}\frac{\partial^2 P}{\partial y^2}$$

- Duration is a static version of the delta of an option, and convexity is a static version of the gamma of an option.

- **Hedging future cash payments** that have to be delivered at specified times: use a portfolio of bonds with the same duration and the same convexity as the cash payments. This is called **immunization**. It is based on then following approximation:

$$\Delta P \approx -\frac{D}{1+y} P \Delta y + \frac{1}{2} C P (\Delta y)^2$$

Pricing Options with Mathematical Models

27. Perfect hedging - replication

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Replication in binomial trees

- Consider a call with $S(0) = 55.5625, \sigma = 0.32, r = 0, K = 55, T = 0.08,$

$$\begin{aligned} S^u &= S(0)e^{\sigma\sqrt{T-t}} = 55.5625e^{0.32\sqrt{0.08}} = 60.826 \\ S^d &= S(0)e^{-\sigma\sqrt{T-t}} = 55.5625e^{-0.32\sqrt{0.08}} = 50.7544 \end{aligned}$$

- Option payoff is either $60.826 - 55 = 5.826$ or zero. To replicate it, we solve

$$\begin{aligned} \delta_0 &+ 60.826\delta_1 = 5.826 \\ \delta_0 &+ 50.7544\delta_1 = 0 \end{aligned}$$

- Suppose $S(T) = 68.8125$. Then, the final profit/loss is $\delta_1(68.8125 - 55.5625) + 55 - 68.8125 = -6.1479$

- In general, the number δ_1 of shares of the underlying is given by

$$\delta_1 = \frac{C^u - C^d}{S^u - S^d}$$

Replication in the B-S-M model

- In the B-S-M model the delta of payoff $C(T) = g(S(T))$ is the derivative of its price with respect to the underlying,

$$\Delta_C := \frac{\partial C(t,s)}{\partial s}$$

- For the European call it is

$$\Delta_C = N(d_1)$$

- For the perfect (theoretical) hedge, re-balancing must take place continuously. This requires that the model and its parameters are exactly correct.

A real data example

- A call option on Microsoft stock 20 consecutive days in year 2000, $T = 20/252 = 0.08$ years.
- Daily data: $Y(i) = \log S(i+1) - \log S(i)$
- Sample standard deviation $= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y}_i)^2} = 0.019978$
- Annual st dev $\sigma = 0.019978 \times \sqrt{252} = 0.317139$
- $K = 55, S(0) = 55.5625$. We set $r = 0$.

A real data example (continued)

- Using B-S-M, $C(0) = 2.2715, \Delta(0) = 0.5629$
- Borrow $\Delta(0)S(0) - C(0)$
- Next day: $S(1) = 55.3750, T - 1 = 19/252$
- Portfolio value: $X(0) = C(0)$ and

$$X(1) = X(0) + \Delta(0)[S(1) - S(0)] = 2.166$$

- $\Delta(1) = 0.5483$, sell $\Delta(0) - \Delta(1)$ shares
- $X(2) = X(1) + \Delta(1)[S(2) - S(1)] = 2.2003$

Replication Experiment

Time	Stock Price	Call Price	Delta	Wealth
0	55.5625	2.2715	0.5629	2.2715
1	55.3750	2.1176	0.5483	2.1660
2	55.4375	2.1009	0.5539	2.2003
3	56.5625	2.7256	0.6481	2.8235
4	59.1250	4.5831	0.8268	4.4841
5	60.3125	5.5702	0.8899	5.4659
6	61.3125	6.4562	0.9313	6.3558
7	60.6250	5.7970	0.9166	5.7155
8	62.6875	7.7346	0.9724	7.6060
9	61.2500	6.3360	0.9507	6.2082
10	63.2500	8.2681	0.9873	8.1095
11	64.1875	9.1933	0.9953	9.0351
12	64.2500	9.2531	0.9972	9.0973
13	65.0000	10.0007	0.9992	9.8452
14	63.0000	8.0025	0.9974	7.8467
15	64.1875	9.1877	0.9997	9.0310
16	65.8125	10.8125	1.0000	10.6556
17	68.2500	13.2500	1.0000	13.0931
18	68.1250	13.1250	1.0000	12.9681
19	68.8125	13.8125	1.0000	13.6556

A real data example (continued)

- Loss in option:
$$S(T) - K = S(T) - 55 = 13.8125$$
- Portfolio value: $X(T) = 13.6556$
- Loss with hedging: $13.8125 - 13.6556 = 0.1569$
- Loss without hedging: $13.8125 - C(0) = 11.5410$

Pricing Options with Mathematical Models

28. Hedging portfolio sensitivities

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Option sensitivities

- In general, a portfolio value X is sensitive to changes in values of all parameters.

- Partial derivatives of the portfolio value with respect to parameters are called greeks

· **Delta**: $\Delta = \frac{\partial}{\partial s} X$

· **Theta**: $\Theta = \frac{\partial}{\partial t} X$

· **Gamma**: $\Gamma = \frac{\partial^2}{\partial s^2} X$

· **Vega**: $\mathcal{V} = \frac{\partial}{\partial \sigma} X$

· **rho**: $\rho = \frac{\partial}{\partial r} X$

Approximation by Taylor expansion

- From Taylor's expansion,

$$X(t+\Delta t, s+\Delta s) = X(t,s) + \frac{\partial X(t,s)}{\partial s}\Delta S + \frac{\partial X(t,s)}{\partial t}\Delta t$$
$$+\frac{1}{2}\frac{\partial^2 X(t,s)}{\partial s^2}\Delta S^2 + \frac{1}{2}\frac{\partial^2 X(t,s)}{\partial t^2}\Delta t^2 + \frac{\partial^2 X(t,s)}{\partial s \partial t}\Delta S \Delta t + \cdots$$

where

$$\Delta S = S(t+\Delta t) - S(t)$$

- Approximation:

$$X(t+\Delta t, s+\Delta s) \approx X(t,s) + \Delta \cdot \Delta S + \Theta \cdot \Delta t + \frac{1}{2}\Gamma \cdot \Delta S^2$$

Approximation by Taylor expansion (continued)

- If the portfolio is delta-neutral, that is, its Δ is zero, then

$$\Delta X \approx \Theta \Delta t + \frac{1}{2}\Gamma \Delta S^2$$

- If Γ is strictly positive, any change in the value of the underlying tends to increase the value of the portfolio.

- If σ and r are stochastic:

$$\begin{aligned} & X(t+\Delta t, s+\Delta s, \sigma+\Delta\sigma, r+\Delta r)) \\ = \; & X(t,s,\sigma,r) + \frac{\partial X}{\partial s}\Delta s + \frac{\partial X}{\partial \sigma}\Delta\sigma + \frac{\partial X}{\partial r}\Delta r + \frac{\partial X}{\partial t}\Delta t \\ & + \frac{1}{2}\frac{\partial^2 X}{\partial s^2}\Delta S^2 + \frac{1}{2}\frac{\partial^2 X}{\partial \sigma^2}\Delta\sigma^2 + \frac{1}{2}\frac{\partial^2 X}{\partial r^2}\Delta r^2 + \cdots \end{aligned}$$

Example

- Denote by Γ the gamma of portfolio X, and by Γ_C the gamma of a contingent claim C. We want to buy/sell n contracts of C in order to make the portfolio gamma neutral, that is,

$$\Gamma + n\Gamma_C = 0$$

- This implies

$$n = -\frac{\Gamma}{\Gamma_C}$$

- However, taking this additional position in C will change the delta of the portfolio. We then buy/sell some shares of the underlying asset in order to make the portfolio delta-neutral. This does not change the gamma, because the underlying asset S has zero gamma. (Why?).

Example (continued)

- As an example, let's say a delta-neutral portfolio X has a gamma $\Gamma = -5,000$. A traded option has $\Delta_C = 0.4$ and $\Gamma_C = 2$. We buy $n = 5,000/2 = 2,500$ option contracts, making the portfolio gamma-neutral. This makes the delta of the portfolio equal to

$$\Delta_X = 2,500 \cdot 0.4 = 1,000 \quad .$$

- Thus, we have to sell 1,000 shares of the underlying asset to keep the portfolio delta-neutral.

Example (continued)

- If we want to make a portfolio vega-neutral, in addition to delta-neutral and gamma-neutral, then it is necessary to hold two different contingent claims written on the underlying asset. In this case we want to have

$$\Gamma + n_1\Gamma_1 + n_2\Gamma_2 = 0$$

$$\mathcal{V} + n_1\mathcal{V}_1 + n_2\mathcal{V}_2 = 0 \quad ,$$

where $n_i, \Gamma_i, \mathcal{V}_i$ are the number of contracts, the gamma and the vega of claim i.

- This is a system of two equations with two unknowns which can typically be solved. At the end, we still have to adjust the number of shares in the new portfolio in order to make it delta-neutral, similarly as above.

Portfolio insurance

- Idea: A put option with our portfolio as the underlying protects against portfolio losses
- Problem: such options are not traded.
- Solution: a synthetic put - replicating the put payoff by trading.
- However, it led to large losses in the crash of October 1987, due to loss of liquidity.

The Story of Long Term Capital Management

- Merton and Scholes were partners
- Anticipated spreads between various rates to become narrower
- “Russian crisis” pushed the spreads even wider
- LTCM was highly leveraged – margin calls forced it to start selling assets, others also did, their prices went even lower, losses huge in 1998
- Bailed out by government effort
- The reasons: high leverage and unprecedented extreme market moves