Assignment 05: Linear Systems & Least Squares

Course: Numerical Methods
Total Marks: 15
Semester: Spring 2025
Deadline: 7 May 2025
Instructions:

- Show all necessary steps. No need to show all the breakdowns.
- Use proper mathematical notation.
- Write answers clearly in the provided spaces.
- You may use a calculator for numerical answers. (Try keeping the solutions in generic formats.)

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Section: 12

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1. Linear Equations and Triangular Forms (3 Marks)

(a) Convert the following system into upper triangular form using Gaussian Elimination:

$$\begin{cases} x + 2y + 3z = 14 \\ 2x + 5y + 2z = 18 \\ 4x + 3y + z = 20 \end{cases} \quad (2)$$

Answer:

$$\left[\begin{array}{ccc|c} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{14} \\ \boxed{2} & \boxed{5} & \boxed{2} & \boxed{18} \\ \boxed{4} & \boxed{3} & \boxed{1} & \boxed{20} \end{array}\right] \quad (3)$$

Steps for triangular form:

First Row Operation: $R_2' = R_2 - \left(\frac{2}{1}\right) R_1 ; \; R_3' = R_3 - \left(\frac{4}{1}\right) R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & 1 & -4 & -10 \\ 0 & -5 & -11 & -36 \end{array}\right]$$

Second Row Operation: $R_3' = R_3 - \left(-\frac{5}{1}\right) R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & -31 & -86 \end{array}\right]$$

(b) Back-substitute to solve for (x), (y), and (z):

Answer:

$x =$ 108/31

$y =$ 34/31 (4)

$z =$ 86/31

Show the back substitution process:

matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -31 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \\ -86 \end{bmatrix}$

backward approach,

$$-31z = -86$$

$$\therefore z = \frac{86}{31}$$

$$y - 4z = -10$$

$$\therefore y = \frac{34}{31}$$

$$x + 2y + 3z = 14$$

$$\therefore x = \frac{108}{31} \quad (\text{Ans})$$

2. LU Decomposition (6 Marks)

(a) Now consider the same Linear System, and decompose the coefficient matrix into the lower and upper triangular matrix of LU decomposition. Show all the required Frobenius matrix in the process:

$$\begin{cases} x + 2y + 3z = 14 \\ 2x + 5y + 2z = 18 \\ 4x + 3y + z = 20 \end{cases} \tag{5}$$

$A^{(0)}$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

$F^{(1)} =$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$m_{21} = \frac{m_{21}}{m_{11}} = 2$

$m_{31} = \frac{m_{31}}{m_{11}} = 4$

$A^{(1)} =$ $F^{(1)} \times A$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & -5 & -11 \end{bmatrix}$$

$F^{(2)} =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

$m_{32} = \frac{m_{32}}{m_{22}}$

$= -5$

$A^{(2)}$ $= F^{(2)} \times A^{1}$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -31 \end{bmatrix}$$

$U =$ $A^{(2)}$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -31 \end{bmatrix}$$

$L =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -5 & 1 \end{bmatrix}$$

(b) Use your decomposition to solve $(Ax = b)$ where

$$b = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$$

(6)

$115/31$

$23/31$

$-2/31$

Show your working here:

$$Ly = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -5 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$$

forward substitution,

$$y_0 = 5$$

$$2y_0 + y_1 = 11 \quad \therefore y_1 = 1$$

$$4y_0 - 5y_1 + y_2 = 17$$

$$\therefore y_2 = 2$$

Again,

$$Ux = y$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -31 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$$

backward substitution,

$$-31z = 2 \quad \therefore z = -\frac{2}{31}$$

$$y - 4z = 1 \quad \therefore y = \frac{23}{31}$$

$$x + 2y + 3z = 5 \quad \therefore x = \frac{115}{31}$$

(Ans)

3. QR Decomposition (6 Marks)

Consider you are provided with the following datapoints.

Input, x		Functional Value, F(x)
3		5
2		3
1		1
-2		1

Now use QR decomposition method to get the best fit quadratic function(degree, n=2) that approximates these datapoints.

a. Calculate Q matrix. Steps provided below.

Define A:

$$\begin{bmatrix} 1 & 3 & 9 \\ 1 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{bmatrix}$$

Define the values of u_i:

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \; ; \; U_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ -2 \end{bmatrix} \; ; \; U_3 = \begin{bmatrix} 9 \\ 4 \\ 1 \\ 4 \end{bmatrix}$$

Find the values of p_i (orthogonal vectors):

$$P_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P_2 = U_2 - \frac{U_2 \cdot P_1}{P_1 \cdot P_1} \cdot P_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ -2 \end{bmatrix} - \frac{3+2+1-2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -3 \end{bmatrix}$$

$$P_3 = U_3 - \frac{U_3 \cdot P_1}{P_1 \cdot P_1} \cdot P_1 - \frac{U_3 \cdot P_2}{P_2 \cdot P_2} \cdot P_2 = \begin{bmatrix} 9 \\ 4 \\ 1 \\ 4 \end{bmatrix} - \frac{18+4-0-12}{4+1+0+9} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -3 \end{bmatrix} - \frac{9+4+1+4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 43/14 \\ -17/14 \\ -7/2 \\ 23/14 \end{bmatrix}$$

Find the values of q_i (orthogonal vectors):

$$q_1 = \frac{P_1}{|P_1|} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} ; \quad q_2 = \frac{P_2}{|P_2|} = \begin{bmatrix} 2/\sqrt{14} \\ 1/\sqrt{14} \\ 0 \\ -3/\sqrt{14} \end{bmatrix} ; \quad q_3 = \frac{P_3}{|P_3|} = \begin{bmatrix} 0.6090 \\ -0.2388 \\ -0.6883 \\ 0.3231 \end{bmatrix}$$

Define Q matrix:

$$\begin{bmatrix} 1/2 & 2/\sqrt{14} & 0.60902 \\ 1/2 & 1/\sqrt{14} & -0.2388 \\ 1/2 & 0 & -0.6883 \\ 1/2 & -3/\sqrt{14} & 0.3231 \end{bmatrix}$$

$R = Q^T A$

$$= \begin{bmatrix} 2 & 2 & 9 \\ 0 & \sqrt{14} & \frac{5\sqrt{14}}{7} \\ 0 & 0 & \frac{4525}{889} \end{bmatrix}$$

here, $Q^T =$

$$\begin{bmatrix} 1/2 & 1/2 & \frac{1}{2} & 1/2 \\ 2/\sqrt{14} & 1/\sqrt{14} & 0 & -3/\sqrt{14} \\ 0.60402 & -0.2388 & -0.6883 & 0.3231 \end{bmatrix}$$

And,

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{bmatrix}$$

b. Calculate R matrix.

$$R = \begin{bmatrix} 2 & 2 & 9 \\ 0 & \sqrt{14} & \frac{5\sqrt{14}}{7} \\ 0 & 0 & \frac{4525}{889} \end{bmatrix}$$

c. Use the QR decomposition to find the solution of the overdetermined system.

Polynomial of degree 2, $P_2(x) = \frac{62}{181} + \frac{80}{181}x + \frac{69}{181}x^2$

We know,

$$Rx = Q^T b$$

$$\Rightarrow \begin{bmatrix} 2 & 2 & 9 \\ 0 & \sqrt{14} & \frac{5\sqrt{14}}{7} \\ 0 & 0 & \frac{4525}{889} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 2/\sqrt{14} & 1/\sqrt{14} & 0 & -3/\sqrt{14} \\ 0.60402 & -0.2388 & -0.683 & 0.3231 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

Backward substitution,

$$\frac{4525}{889} a_2 = \frac{1725}{889}$$

$$\therefore a_2 = \frac{69}{181}$$

$$\sqrt{14}\, a_1 + \frac{5\sqrt{14}}{7} a_2 = \frac{5\sqrt{14}}{7}$$

$$\therefore a_1 = \frac{80}{181}$$

$$2a_0 + 2a_1 + 3a_2 = 5$$

$$\therefore a_0 = \frac{62}{181}$$

(Ans)