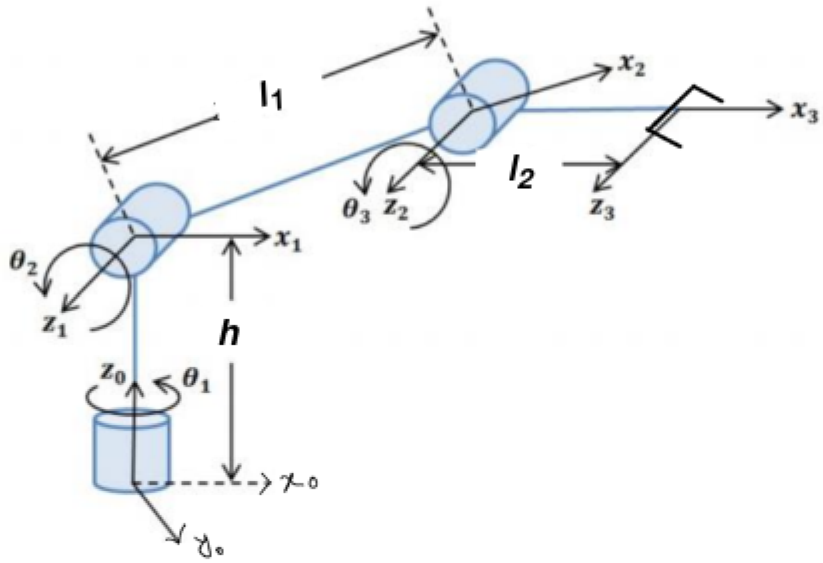
Forward and Inverse Kinematics using Trigonometry



Forward Kinematics

Given joint angles θ_1, θ_2, and θ_3, and link lengths l_1, l_2, and height h, the end-effector position (x, y, z) can be calculated using trigonometric formulas.

Step-by-Step:

1. Calculate 2D projection in rotated XZ plane:

$$x' = l_1 \cdot \cos(\theta_2) + l_2 \cdot \cos(\theta_2 + \theta_3)$$
$$z = h + l_1 \cdot \sin(\theta_2) + l_2 \cdot \sin(\theta_2 + \theta_3)$$

2. Rotate x' to global frame using θ_1:

$$x = x' \cdot \cos(\theta_1)$$
$$y = x' \cdot \sin(\theta_1)$$

Final Forward Kinematics Equations:

$x = [l_1 \cdot \cos(\theta_2) + l_2 \cdot \cos(\theta_2 + \theta_3)] \cdot \cos(\theta_1)$

$y = [l_1 \cdot \cos(\theta_2) + l_2 \cdot \cos(\theta_2 + \theta_3)] \cdot \sin(\theta_1)$

$z = h + l_1 \cdot \sin(\theta_2) + l_2 \cdot \sin(\theta_2 + \theta_3)$

Inverse Kinematics

Given the end-effector position (x, y, z), calculate joint angles θ_1, θ_2, and θ_3.

Step-by-Step:

1. θ_1 from XY projection:

$\theta_1 = \tan^{-1}(y/ x)$

2. Planar projection:

$x' = \text{sqrt}(x^2 + y^2)$,

$z' = z - h$

3. Calculate radial distance r:

$r = \text{sqrt}(x'^2 + z'^2)$

4. Use cosine law to find θ_3:

$\theta_3 = \cos^{-1}((r^2 - l_1^2 - l_2^2) / (2 \cdot l_1 \cdot l_2))$

5. Intermediate angles:

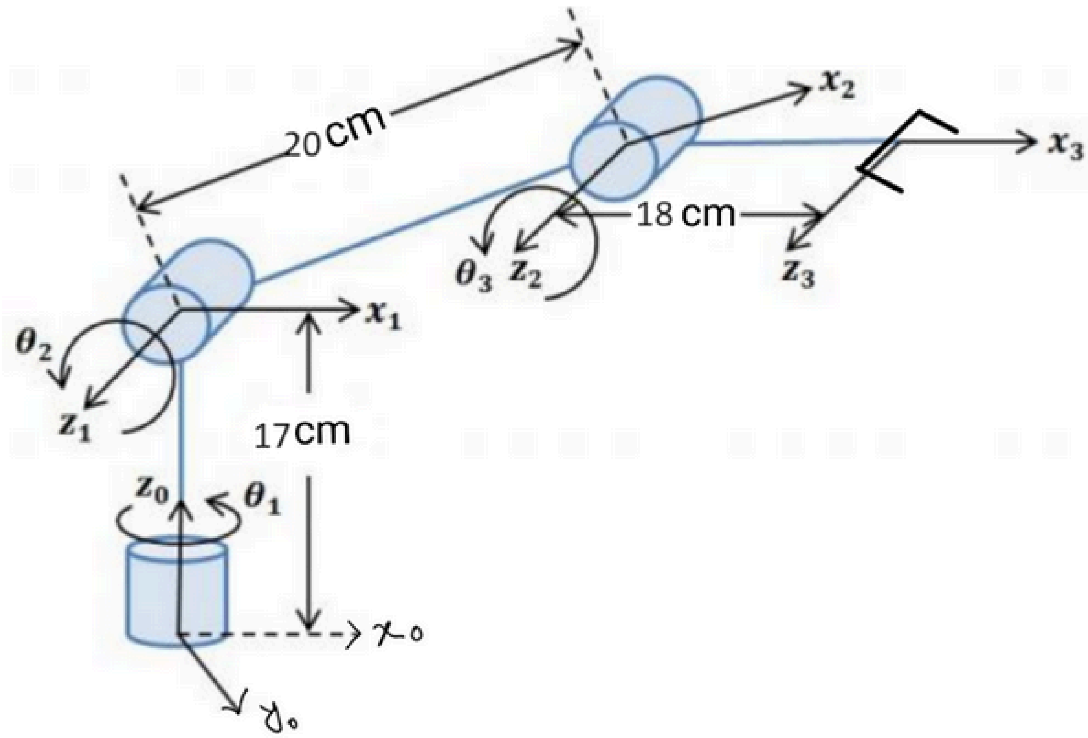
$\varphi = \tan^{-1}(z'/ x')$

$\psi = \cos^{-1}((r^2 + l_1^2 - l_2^2) / (2 \cdot r \cdot l_1))$

6. θ_2 is then:

$\theta_2 = \varphi - \psi$

For an example that I give in the suggestion:



a. To drop the object you have to move the waist for 60º , shoulder for 70º , and elbow for 58º . Calculate the coordinates (x, y, z) where the object will be dropped after the movements?
b. If the object is placed in the coordinates 25,15,30 (x, y, z). What should be the angle of waist, shoulder and elbow to pick that object?

A. Forward Kinematics

Given:

$\theta_1 = 60°$, $\theta_2 = 70°$, $\theta_3 = 58°$

Link lengths: $l_1 = 20$ cm, $l_2 = 18$ cm, Base height h = 17 cm

Step-by-step Calculation

1. Analyze the arm in the XZ plane (side view):

- Link 1 horizontal: $20 * \cos(70°) \approx 6.84$ cm

- Link 1 vertical: $20 * \sin(70°) \approx 18.79$ cm

- Link 2 angle = $70° + 58° = 128°$

- Link 2 horizontal: $18 * \cos(128°) \approx -11.08$ cm

- Link 2 vertical: $18 * \sin(128°) \approx 14.18$ cm

- Total horizontal projection (x') = $6.84 - 11.08 = -4.24$ cm

- Total vertical projection (z) = $17 + 18.79 + 14.18 = 49.97$ cm

2. Rotate into XY plane using $\theta_1 = 60°$:

$x = -4.24 * \cos(60°) = -2.12$ cm

$y = -4.24 * \sin(60°) = -3.67$ cm

Result:

End-effector position:

$x \approx -2.12$ cm

$y \approx -3.67$ cm

$z \approx 49.97$ cm

B. Inverse Kinematics

Given: End-effector position (x, y, z) = (25 cm, 15 cm, 30 cm)

Step-by-step Calculation

1. θ_1 from XY plane:

$\theta_1 = \tan^{-1}(15/25) \approx 30.96°$

2. Project into XZ arm plane:

$x' = \text{sqrt}(25^2 + 15^2) = 29.15$ cm

$z' = 30 - 17 = 13$ cm

3. Compute distance to wrist:

$r = \text{sqrt}(29.15^2 + 13^2) = 31.91$ cm

4. Use cosine law to get θ_3:

$\cos(\theta_3) = (400 + 324 - 1018.92) / (2 * 20 * 18) = -0.4096$

$\theta_3 = \cos^{-1}(-0.4096) \approx 114.2°$

5. Find intermediate angles:

$\varphi = \tan^{-1}(13/29.15) \approx 23.96°$

$\psi = \cos^{-1}((1018.92 + 400 - 324) / (2 * 31.91 * 20)) = \cos^{-1}(0.8576) \approx 30.00°$

6. Solve for θ_2 (elbow-up solution):

$\theta_2 = \varphi + \psi = 23.96° + 30.00° = 53.96°$

Result:

Joint angles:

$\theta_1 \approx 30.96°$

$\theta_2 \approx 53.96°$

$\theta_3 \approx 114.2°$