

Step 9: Form $T_n^0 = A_1 \cdots A_n$. This then gives the position and orientation of the tool frame expressed in base coordinates.

3.3 Examples

In the D-H convention the only variable angle is θ , so we simplify notation by writing c_i for $\cos \theta_i$, etc. We also denote $\theta_1 + \theta_2$ by θ_{12} , and $\cos(\theta_1 + \theta_2)$ by c_{12} , and so on. In the following examples it is important to remember that the D-H convention, while systematic, still allows considerable freedom in the choice of some of the manipulator parameters. This is particularly true in the case of parallel joint axes or when prismatic joints are involved.

Example 3.1 Planar Elbow Manipulator

Consider the two-link planar arm of Figure 3.6. The joint axes z_0 and

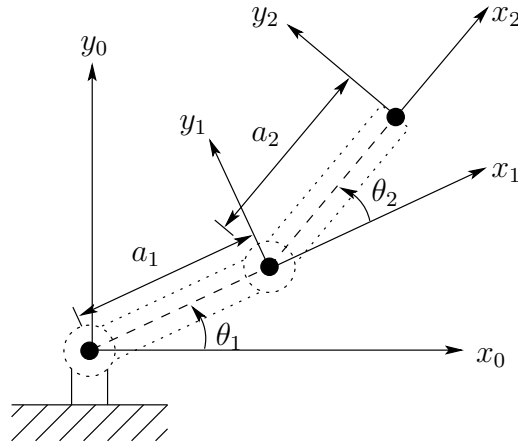


Figure 3.6: Two-link planar manipulator. The z -axes all point out of the page, and are not shown in the figure.

z_1 are normal to the page. We establish the base frame $o_0x_0y_0z_0$ as shown. The origin is chosen at the point of intersection of the z_0 axis with the page and the direction of the x_0 axis is completely arbitrary. Once the base frame is established, the $o_1x_1y_1z_1$ frame is fixed as shown by the D-H convention, where the origin O_1 has been located at the intersection of z_1 and the page. The final frame $o_2x_2y_2z_2$ is fixed by choosing the origin O_2 at the end of link 2 as shown. The link parameters are shown in Table 3.1. The A -matrices are

Table 3.1: Link parameters for 2-link planar manipulator.

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

* variable

determined from (3.10) as

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (3.22)$$

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.23)$$

The T -matrices are thus given by

$$T_1^0 = A_1. \quad (3.24)$$

$$T_2^0 = A_1 A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (3.25)$$

Notice that the first two entries of the last column of T_2^0 are the x and y components of the origin O_2 in the base frame; that is,

$$\begin{aligned} x &= a_1 c_1 + a_2 c_{12} \\ y &= a_1 s_1 + a_2 s_{12} \end{aligned} \quad (3.26)$$

are the coordinates of the end-effector in the base frame. The rotational part of T_2^0 gives the orientation of the frame $o_2 x_2 y_2 z_2$ relative to the base frame.

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Example 3.2 Three-Link Cylindrical Robot

Consider now the three-link cylindrical robot represented symbolically by Figure 3.7. We establish O_0 as shown at joint 1. Note that the placement of

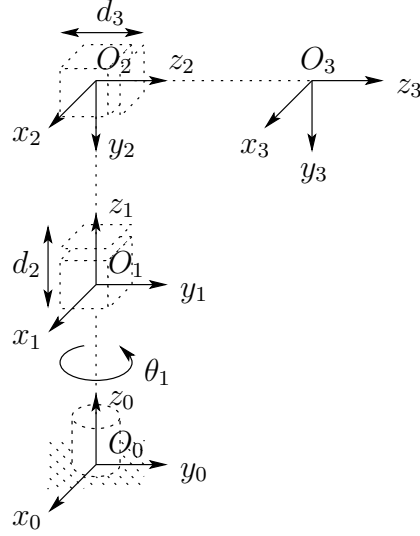


Figure 3.7: Three-link cylindrical manipulator.

Table 3.2: Link parameters for 3-link cylindrical manipulator.

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	0	-90	d_2^*	0
3	0	0	d_3^*	0

* variable

the origin O_0 along z_0 as well as the direction of the x_0 axis are arbitrary. Our choice of O_0 is the most natural, but O_0 could just as well be placed at joint 2. The axis x_0 is chosen normal to the page. Next, since z_0 and z_1 coincide, the origin O_1 is chosen at joint 1 as shown. The x_1 axis is normal to the page when $\theta_1 = 0$ but, of course its direction will change since θ_1 is variable. Since z_2 and z_1 intersect, the origin O_2 is placed at this intersection. The direction of x_2 is chosen parallel to x_1 so that θ_2 is zero. Finally, the third frame is chosen at the end of link 3 as shown.

The link parameters are now shown in Table 3.2. The corresponding A

and T matrices are

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.27)$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (3.28)$$

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Example 3.3 Spherical Wrist

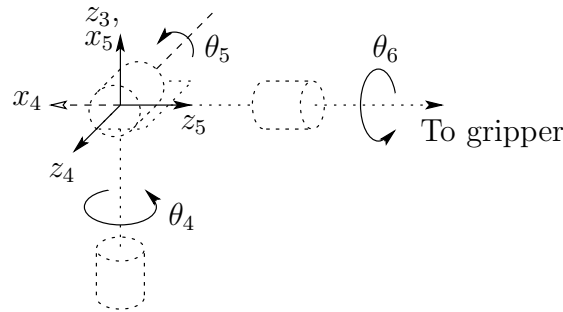


Figure 3.8: The spherical wrist frame assignment.

The spherical wrist configuration is shown in Figure 3.8, in which the joint axes z_3 , z_4 , z_5 intersect at O . The Denavit-Hartenberg parameters are shown in Table 3.3. The Stanford manipulator is an example of a manipulator that possesses a wrist of this type. In fact, the following analysis applies to virtually all spherical wrists.

Table 3.3: DH parameters for spherical wrist.

Link	a_i	α_i	d_i	θ_i
4	0	-90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

* variable

We show now that the final three joint variables, $\theta_4, \theta_5, \theta_6$ are the Euler angles ϕ, θ, ψ , respectively, with respect to the coordinate frame $o_3x_3y_3z_3$. To see this we need only compute the matrices A_4, A_5 , and A_6 using Table 3.3 and the expression (3.10). This gives

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.29)$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.30)$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (3.31)$$

Multiplying these together yields

$$\begin{aligned} T_6^3 = A_4 A_5 A_6 &= \begin{bmatrix} R_6^3 & O_6^3 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & c_4 s_5 d_6 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 & s_4 s_5 d_6 \\ -s_5 c_6 & s_5 s_6 & c_5 & c_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (3.32)$$

Comparing the rotational part R_6^3 of T_6^3 with the Euler angle transformation (2.51) shows that $\theta_4, \theta_5, \theta_6$ can indeed be identified as the Euler angles ϕ, θ and ψ with respect to the coordinate frame $o_3x_3y_3z_3$.

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Example 3.4 Cylindrical Manipulator with Spherical Wrist

Suppose that we now attach a spherical wrist to the cylindrical manipulator of Example 3.3.2 as shown in Figure 3.9. Note that the axis of rotation of

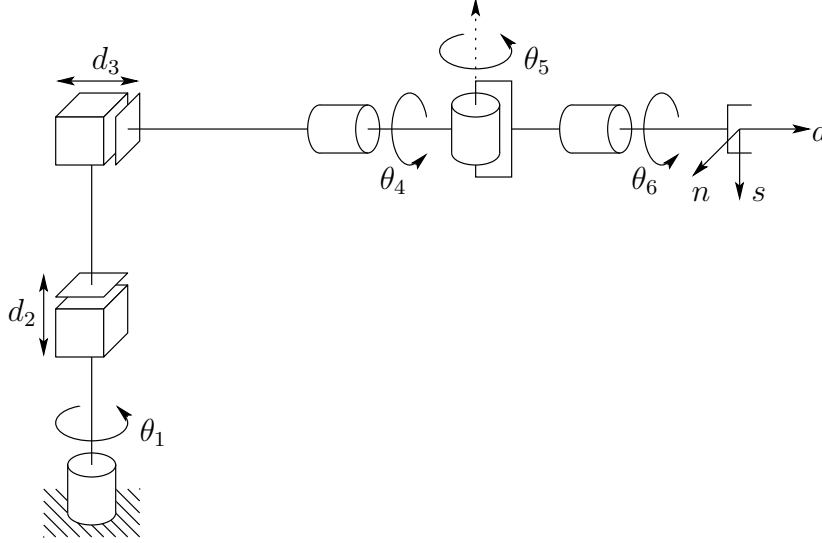


Figure 3.9: Cylindrical robot with spherical wrist.

joint 4 is parallel to z_2 and thus coincides with the axis z_3 of Example 3.3.2. The implication of this is that we can immediately combine the two previous expressions (3.28) and (3.32) to derive the forward kinematics as

$$T_6^0 = T_3^0 T_6^3 \quad (3.33)$$

with T_3^0 given by (3.28) and T_6^3 given by (3.32). Therefore the forward kinematics of this manipulator is described by

$$\begin{aligned} T_6^0 &= \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_1 \\ s_1 & 0 & c_1 & c_1 d_1 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & c_4 s_5 d_6 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 & s_4 s_5 d_6 \\ -s_5 c_6 & s_5 c_6 & c_5 & c_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.34) \\ &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned}
r_{11} &= c_1 c_4 c_5 c_6 - c_1 s_4 s_6 + s_1 s_5 c_6 \\
r_{21} &= s_1 c_4 c_5 c_6 - s_1 s_4 s_6 - c_1 s_5 c_6 \\
r_{31} &= -s_4 c_5 c_6 - c_4 s_6 \\
r_{12} &= -c_1 c_4 c_5 s_6 - c_1 s_4 c_6 - s_1 s_5 c_6 \\
r_{22} &= -s_1 c_4 c_5 s_6 - s_1 s_4 s_6 + c_1 s_5 c_6 \\
r_{32} &= s_4 c_5 c_6 - c_4 c_6 \\
r_{13} &= c_1 c_4 s_5 - s_1 c_5 \\
r_{23} &= s_1 c_4 s_5 + c_1 c_5 \\
r_{33} &= -s_4 s_5 \\
d_x &= c_1 c_4 s_5 d_6 - s_1 c_5 d_6 - s_1 d_3 \\
d_y &= s_1 c_4 s_5 d_6 + c_1 c_5 d_6 + c_1 d_3 \\
d_z &= -s_4 s_5 d_6 + d_1 + d_2.
\end{aligned}$$

Notice how most of the complexity of the forward kinematics for this manipulator results from the orientation of the end-effector while the expression for the arm position from (3.28) is fairly simple. The spherical wrist assumption not only simplifies the derivation of the forward kinematics here, but will also greatly simplify the inverse kinematics problem in the next chapter.

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Example 3.5 Stanford Manipulator

Consider now the Stanford Manipulator shown in Figure 3.10. This manipulator is an example of a spherical (RRP) manipulator with a spherical wrist. This manipulator has an offset in the shoulder joint that slightly complicates both the forward and inverse kinematics problems.

We first establish the joint coordinate frames using the D-H convention as shown. The link parameters are shown in the Table 3.4.

It is straightforward to compute the matrices A_i as

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.35)$$

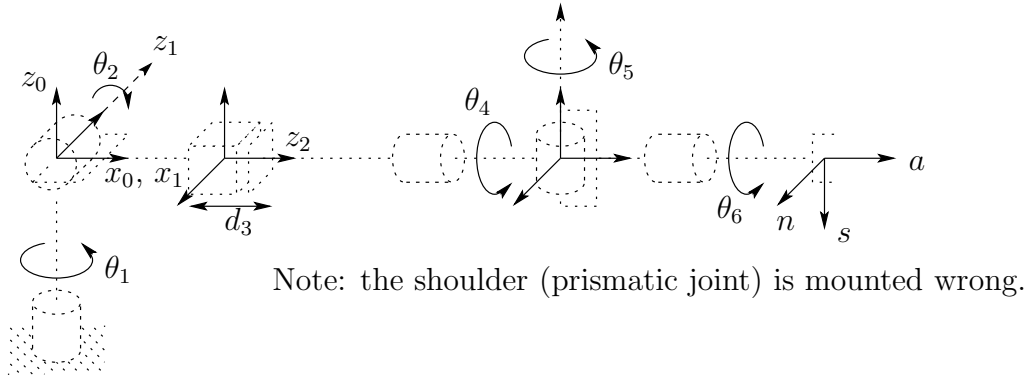


Figure 3.10: DH coordinate frame assignment for the Stanford manipulator.

Table 3.4: DH parameters for Stanford Manipulator.

Link	d_i	a_i	α_i	θ_i
1	0	0	-90	\star
2	d_2	0	$+90$	\star
3	\star	0	0	0
4	0	0	-90	\star
5	0	0	$+90$	\star
6	d_6	0	0	\star

* joint variable

$$A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.36)$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.37)$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.38)$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.39)$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.40)$$

T_6^0 is then given as

$$T_6^0 = A_1 \cdots A_6 \quad (3.41)$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.42)$$

where

$$\begin{aligned} r_{11} &= c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6) \\ r_{21} &= s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) \\ r_{31} &= -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \\ r_{12} &= c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) \\ r_{22} &= -s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) \\ r_{32} &= s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \\ r_{13} &= c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 \\ r_{23} &= s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 \\ r_{33} &= -s_2c_4s_5 + c_2c_5 \\ d_x &= c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) \\ d_y &= s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) \\ d_z &= c_2d_3 + d_6(c_2c_5 - c_4s_2s_5). \end{aligned} \quad (3.44)$$

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Example 3.6 SCARA Manipulator

As another example of the general procedure, consider the SCARA manipulator of Figure 3.11. This manipulator, which is an abstraction of the AdeptOne robot of Figure 1.11, consists of an RRP arm and a one degree-of-freedom wrist, whose motion is a roll about the vertical axis. The first

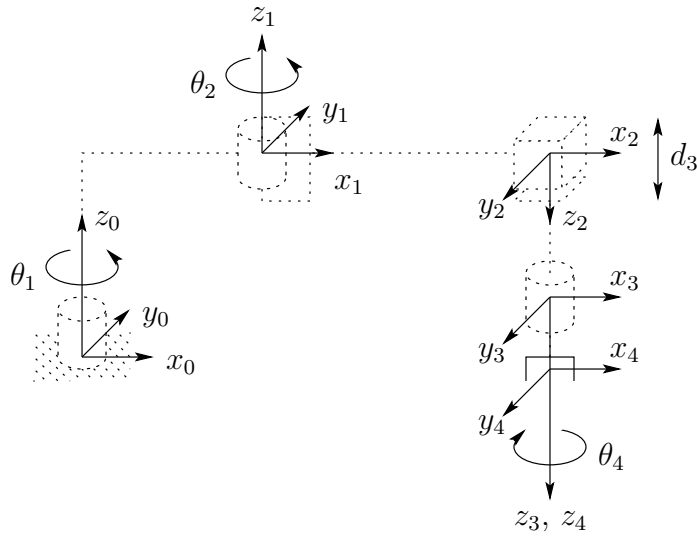


Figure 3.11: DH coordinate frame assignment for the SCARA manipulator.

Table 3.5: Joint parameters for SCARA.

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	*
2	a_2	180	0	*
3	0	0	*	0
4	0	0	d_4	*

* joint variable

step is to locate and label the joint axes as shown. Since all joint axes are parallel we have some freedom in the placement of the origins. The origins are placed as shown for convenience. We establish the x_0 axis in the plane of the page as shown. This is completely arbitrary and only affects the zero configuration of the manipulator, that is, the position of the manipulator when $\theta_1 = 0$.

The joint parameters are given in Table 3.5, and the A -matrices are as

follows.

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.45)$$

$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2 c_2 \\ s_2 & -c_2 & 0 & a_2 s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.46)$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.47)$$

$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (3.48)$$

The forward kinematic equations are therefore given by

$$T_4^0 = A_1 \cdots A_4 = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.49)$$

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