Dummy's note (5): Sound speed in relativistic fluid

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Abstract

Here we give a proper definition of sound speed in relativistic fluid.

1 Basic equations

The way we proceed is the same as in Newtonian case.

First we note that propagation of sound can be considered as a local phenomena and we can naturally assume that all the inhomogeneity of background fluid is neglected. This is particulary true for spacetime. Therefore we can safely work in locally inertial frame whose spacetime is well-described by Minkowski

$$ds^2 = \eta_{ab} dx^a dx^b = -dt^2 + dx^2 + dy^2 + dz^2. \tag{1}$$

We note that all the connection coefficients (Γ) vanishes. Take perfect fluid

$$T^{ab} = (e+p)u^a u^b + p\eta^{ab}, \tag{2}$$

where e is total energy density. Projecting conservation of stress-energy relation along fluid 4-velocity u^a, we have

$$\begin{aligned} 0 &= u_a \nabla_b T^{ab} = u_a \partial_b T^{ab} \\ &= -u^b \partial_b (e+p) - (e+p)\partial_b u^b + u^b \partial_b p = 0. \end{aligned} \tag{3}$$

While baryon conservation gives

$$\partial_b(\rho u^b) = 0, \text{ thus } \partial_b u^b - -u^b \frac{1}{\rho} \partial_b \rho \tag{4}$$

By using Eq.(4) we cast Eq.(3) as

$$0 = u^b \left(\frac{e+p}{\rho} \partial_b \rho - \partial_b e \right) \tag{5}$$

Next projecting stress-energy conservation perpendicular to 4-velocity of fluid ($h_a^b := u_a u^b + \delta_a^b$)

$$\begin{aligned} 0 &= h_a^c \nabla_b T^{ab} \\ &\quad \text{thus} \\ 0 &= (e+p)u^b \partial_b u^c + u^c u^a \partial_a p + \partial^c p \end{aligned} \tag{6}$$

2 Linear perturbation

We choose background where sound propagate to be uniform in comoving frame of the fluid

$$\rho, p = const. \quad \text{for } u^t = 1, u^j = 0. \tag{7}$$

We perform Eulerian perturbation of hydro equations. For Eulerian perturbation of physical variables, we assume plane wave that propagate in x direction (thus $\delta p = \delta p(x,t)$ and so on.). Also $\delta u^y = 0 = \delta u^z$ (thus from normalization of 4-velocity we also have $\delta u^t = 0$).

Eq.(4) reduces to

$$\partial_x \delta u^x = -\frac{1}{\rho}\partial_t \delta\rho. \tag{8}$$

Eq.(5) reduces to

$$0 = \frac{e+p}{\rho}\partial_t \delta\rho - \partial_t \delta e \tag{9}$$

Eq.(6) reduces to

$$0 = (e+p)\partial_t \delta u^x + \partial_x \delta p \tag{10}$$

From the equations above we eliminate $\delta\rho, \delta u^x$. Then we obtain

$$0 = -\partial_t^2 \delta e + \partial_x^2 \delta p. \tag{11}$$

In this case, when we **define** sound velocity c_s as

$$c_s^2 \equiv \left(\frac{dp}{de}\right)_s, \tag{12}$$

we obtain wave equation

$$(-\frac{1}{c_s^2}\partial_t^2 + \partial_x)\delta p = 0. \tag{13}$$

Therefore we can see Eq.(12) is a natural thermodynamical definition of sound speed in relativity.