二. 向量部分习题.

1. 求向量. 向量模. 与向量方向一致的单位向量

(1). 已知空间两点. A(1, 1, -1). B(-2, 1, 2). 求(1)在线段AB上求一点M. 满足 $\overrightarrow{AM}=2\overrightarrow{MB}$. (2) 向量 $\overrightarrow{OM}$. (3). 与向量 $\overrightarrow{OM}$ 方向一致的单位向量.

解: (1). 设 $M(x, y, z)$. $\overrightarrow{AM}=(x-1, y-1, z+1)$. $\overrightarrow{MB}=(-2-x, 1-y, 2-z)$

$\overrightarrow{AM}=2\overrightarrow{MB}$. $(x-1, y-1, z+1)=2(-2-x, 1-y, 2-z)$. $\therefore \begin{cases} x-1=2(-2-x) \\ y-1=2(1-y) \\ z+1=2(2-z) \end{cases}$

$\therefore x=-1$. $y=1$. $z=1$. $\therefore M(-1, 1, 1)$.

(2). $\overrightarrow{OM}=(-1, 1, 1)$

(3). $|\overrightarrow{OM}|=\sqrt{(-1)^2+1^2+1^2}=\sqrt{3}$. $\vec{e}_{\overrightarrow{OM}}=\frac{\overrightarrow{OM}}{|\overrightarrow{OM}|}=(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

2. 向量积的模的几何意义. ($S_{\square}=|\vec{a}\times\vec{b}|$).

(1). 设三角形三个顶点是A. B. C. 且 $\overrightarrow{AB}=(1, 1, 1)$. $\overrightarrow{AC}=(-1, 1, 0)$. 求三角形面积

解: $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix}=(-1, -1, 2)$.

$|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{(-1)^2+(-1)^2+2^2}=\sqrt{6}$. $S_{\triangle ABC}=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{\sqrt{6}}{2}$.

(2). 已知 M(1, 1, 1). A(2, 2, 1). B(2, 1, 2). 求 $S_{\triangle MAB}$.

解: $\overrightarrow{MA}=(2-1, 2-1, 1-1)=(1, 1, 0)$. $\overrightarrow{MB}=(1, 0, 1)$.

$\overrightarrow{MA}\times\overrightarrow{MB}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}=(1, -1, -1)$

$|\overrightarrow{MA}\times\overrightarrow{MB}|=\sqrt{1+1+1}=\sqrt{3}$

$S_{\triangle MAB}=\frac{1}{2}|\overrightarrow{MA}\times\overrightarrow{MB}|=\frac{\sqrt{3}}{2}$

(3). 设 $\vec{a}=(2,1,1)$, $\vec{b}=(1,-1,0)$. 求以 $\vec{a}$, $\vec{b}$ 为邻边的平行四边形面积.

解: $\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 1 & 1\\ 1 & -1 & 0\end{vmatrix}=(1,1,-3)$.

$S_{\square}=|\vec{a}\times\vec{b}|=\sqrt{1+1+9}=\sqrt{11}$.

(4). 已知平行四边形三个顶点 $A(1,-1,2)$, $B(2,-3,2)$, $C(1,1,-1)$. 求 $\square$ AC边上高 h.

解: $\overrightarrow{AB}=(1,-2,0)$. $\overrightarrow{AC}=(0,2,-3)$. $|\overrightarrow{AC}|=\sqrt{0+4+9}=\sqrt{13}$.

$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 0\\ 0 & 2 & -3\end{vmatrix}=(6,3,2)$. $|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{36+9+4}=7$.

$S_{\square}=|\overrightarrow{AB}\times\overrightarrow{AC}|=|\overrightarrow{AC}|\cdot h$ $\quad\therefore h=\dfrac{|\overrightarrow{AB}\times\overrightarrow{AC}|}{|\overrightarrow{AC}|}=\dfrac{7}{\sqrt{13}}=\dfrac{7}{13}\sqrt{13}$

(5). $A(1,-1,2)$, $B(2,-2,2)$, $C(1,1,-1)$. 求 $\triangle ABC$ 上以AC边为底边所应的高 h.

解, $\overrightarrow{AB}=(1,-1,0)$, $\overrightarrow{AC}=(0,2,-3)$. $|\overrightarrow{AC}|=\sqrt{4+9}=\sqrt{13}$

$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & 0\\ 0 & 2 & -3\end{vmatrix}=(3,3,2)$. $|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{9+9+4}=\sqrt{22}$.

$S_{\triangle ABC}=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}|\overrightarrow{AC}|\cdot h$, $\therefore h=\dfrac{|\overrightarrow{AB}\times\overrightarrow{AC}|}{|\overrightarrow{AC}|}=\dfrac{\sqrt{22}}{\sqrt{13}}=\dfrac{\sqrt{286}}{13}$.

(6). 已知 $\vec{a}=(2,2,1)$, $\vec{b}=(1,-1,0)$. 以 $\vec{a}+2\vec{b}$ 与 $\vec{a}-2\vec{b}$ 为边的平行四边形面积.

解: $\vec{a}+2\vec{b}=(2,2,1)+(2,-2,0)=(4,0,1)$.

$\vec{a}-2\vec{b}=(2,2,1)-(2,-2,0)=(0,4,1)$

$(\vec{a}+2\vec{b})\times(\vec{a}-2\vec{b})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & 0 & 1\\ 0 & 4 & 1\end{vmatrix}=(-4,-4,16)$

$S_{\square}=|(\vec{a}+2\vec{b})\times(\vec{a}-2\vec{b})|=\sqrt{16+16+256}=12\sqrt{2}$.

3. 求直线方程.

(1) 求过点 $(3,1,-2)$ 且与平面 $x+2z=1$ 和 $y-3z=2$ 都平行的直线方程.

解. 平面 $x+2z=1$ 的法向量记为 $\vec{n_1}=(1,0,2)$.

平面 $y-3z=2$ - - - - - - - $\vec{n_2}=(0,1,-3)$

记所求直线方向向量为 $\vec{s}$. $\vec{s}\perp\vec{n_1}$. $\vec{s}\perp\vec{n_2}$

$$\vec{s}=\vec{n_1}\times\vec{n_2}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&0&2\\0&1&-3\end{vmatrix}=(-2,3,1)$$. 所求直线过点 $(3,1,-2)$.

∴ 所求直线为: $\frac{x-3}{-2}=\frac{y-1}{3}=\frac{z+2}{1}$

(2). 求过点 $(-1,0,4)$ 且垂直于平面 $3x-4y+z-10=0$ 的直线方程.

解. 所求直线方向向量可取 $\vec{s}=(3,-4,1)$.

∴ 直线为: $\frac{x+1}{3}=\frac{y}{-4}=\frac{z-4}{1}$.

(3). 过点 $(-1,2,1)$ 且与 xoy 面垂直的直线方程. $\underline{\frac{x+1}{0}=\frac{y-2}{0}=\frac{z-1}{1}}$ 或 $\begin{cases}x=-1\\y=2\end{cases}$.

$\vec{s}=(0,0,1)$.

(4). 若直线过 $(1,1,1)$、$(2,3,4)$. 则直线方程为 $\underline{\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}}$.

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

(5). 过点 $(1,-1,1)$ 且与平面 $2x+2y-z=5$ 垂直直线方程为 $\underline{\frac{x-1}{2}=\frac{y+1}{2}=\frac{z-1}{-1}}$.

(6). 过点 $(4,-1,3)$. 且平行于直线 $\frac{x-3}{2}=\frac{y}{1}=\frac{z-1}{5}$ 的直线方程 $\underline{\frac{x-4}{2}=\frac{y+1}{1}=\frac{z-3}{5}}$

(7). 过点 $(4,-1,3)$ 且垂直于平面 $2(x-3)+y+5(z-1)=0$ 的直线方程 $\underline{\frac{x-4}{2}=\frac{y+1}{1}=\frac{z-3}{5}}$

4. 利用直线的平面束方程求平面方程.

(1). 求过直线 $\begin{cases} x+y+3z=0 \\ x-y-z=0 \end{cases}$ 且与平面 $x+y-z=0$ 垂直的平面方程.

解：过直线的平面束方程：$x+y+3z+\lambda(x-y-z)=0$.

即：$(1+\lambda)x+(1-\lambda)y+(3-\lambda)z=0$.

所求平面与 $x+y-z=0$ 垂直. $\therefore (1+\lambda)\cdot 1+(1-\lambda)\cdot 1+(3-\lambda)\cdot(-1)=0$

$\therefore \lambda=1$. $\therefore$ 所求平面为 $2x+2z=0$ 即 $x+z=0$.

(法二 略)

(2). 求过直线 $\begin{cases} x+y+z=0 \\ x-y-2z-1=0 \end{cases}$ 且与 yoz 面垂直的平面方程.

解：过直线的平面束方程：$x+y+z+\lambda(x-y-2z-1)=0$

即 $(1+\lambda)x+(1-\lambda)y+(1-2\lambda)z-\lambda=0$.

所求平面与 yoz 面垂直：$\therefore (1+\lambda, 1-\lambda, 1-2\lambda)\perp(1,0,0)$.

$\therefore 1+\lambda=0$. $\therefore \lambda=-1$. 所求平面方程为：$2y+3z+1=0$.

(法二 略)

(3). 求过直线 $\begin{cases} 3x-4y+z=0 \\ 3x-y-2z-9=0 \end{cases}$ 且与 x 轴平行的平面方程.

解：过直线的平面束方程：$3x-4y+z+\lambda(3x-y-2z-9)=0$

即. $(3+3\lambda)x-(4+\lambda)y+(1-2\lambda)z-9\lambda=0$

所求平面与 x 轴平行. $\therefore (3+3\lambda, -4-\lambda, 1-2\lambda)\perp(1,0,0)$

$\therefore 3+3\lambda=0$. $\lambda=-1$. 所求平面方程为：$-3y+3z+9=0$

即：$y-z-3=0$.

(法二). 直线方向向量 $\vec{s}=(3,-4,1)\times(3,-1,-2)=9(1,1,1)=9\vec{s}_1$.

所求平面法向量 $\vec{n}$ 满足. $\vec{n}\perp x$ 轴. 且 $\vec{n}\perp\vec{s}_1$ $\therefore \vec{n}=(1,0,0)\times(1,1,1)=(0,1,-1)$

令 $x=0$. 代入直线求出点 $(0,-1,-4)$. $\therefore$ 所求方程 $y+1-(z+4)=0$. 即 $y-z-3=0$

(4). 一平面过点 $(1,0,-1)$，且过直线 $\begin{cases} x+2y+1=0 \\ y+z-1=0 \end{cases}$，求平面方程。

解：过直线的平面束方程：$x+2y+1+\lambda(y+z-1)=0$.

即 $x+(2+\lambda)y+\lambda z+1-\lambda=0$.

平面过点 $(1,0,-1)$，代入得：$1+(2+\lambda)\cdot 0+\lambda\cdot(-1)+1-\lambda=0$

$\therefore \lambda=1$ $\therefore$ 所求平面方程为：$x+3y+z=0$.

(5). 一平面过点 $(0,0,0)$，且通过直线 $\begin{cases} x+2y-4z+7=0 \\ 3x+5y-2z-1=0 \end{cases}$. 求平面方程.

解：过直线的平面束方程：$x+2y-4z+7+\lambda(3x+5y-2z-1)=0$.

即：$(1+3\lambda)x+(2+5\lambda)y-(4+2\lambda)z+7-\lambda=0$

平面过点 $(0,0,0)$，代入得：$\lambda=7$.

所求平面方程为：$22x+37y-18z=0$

三. 多元函数微分法.

1. 多元函数的极限.（求极限和证明极限不存在）.

(1). $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{2+x^2+y^2}-\sqrt{2}}{x^2+y^2}$

$\xlongequal{令x^2+y^2=u} \lim\limits_{u\to 0}\dfrac{\sqrt{2+u}-\sqrt{2}}{u} = \lim\limits_{u\to 0}\dfrac{2+u-2}{u(\sqrt{2+u}+\sqrt{2})} = \lim\limits_{u\to 0}\dfrac{1}{\sqrt{2+u}+\sqrt{2}} = \dfrac{1}{2\sqrt{2}}$.（有理化法）

法二. $\lim\limits_{u\to 0}\dfrac{\sqrt{2+u}-\sqrt{2}}{u} \overset{\frac{0}{0}}{=} \lim\limits_{u\to 0}\dfrac{\frac{1}{2\sqrt{2+u}}-0}{1} = \lim\limits_{u\to 0}\dfrac{1}{2\sqrt{2+u}} = \dfrac{1}{2\sqrt{2}}$.（洛必达）.

(2). $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{\tan(x+y)+1}-1}{\tan(x+y)}$

$\xlongequal{令\tan(x+y)=u} \lim\limits_{u\to 0}\dfrac{\sqrt{u+1}-1}{u} = \lim\limits_{u\to 0}\dfrac{u+1-1}{u(\sqrt{u+1}+1)} = \lim\limits_{u\to 0}\dfrac{1}{\sqrt{u+1}+1} = \dfrac{1}{2}$.（有理化）

$\overset{\frac{0}{0}}{=} \lim\limits_{u\to 0}\dfrac{\frac{1}{2\sqrt{u+1}}-0}{1} = \lim\limits_{u\to 0}\dfrac{1}{2\sqrt{u+1}} = \dfrac{1}{2}$.（洛必达）

$(\sqrt{1+x}-1 \sim \frac{1}{2}x)$. $= \lim\limits_{u\to 0}\dfrac{\frac{1}{2}u}{u} = \dfrac{1}{2}$.（等价无穷小）

(3). $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$ $(1-\cos u \sim \frac{1}{2}u^2)$.

$\xlongequal{令x^2+y^2=u} \lim\limits_{u\to 0}\dfrac{1-\cos u}{u^2} = \lim\limits_{u\to 0}\dfrac{\frac{1}{2}u^2}{u^2} = \dfrac{1}{2}$（等价无穷小）

$\overset{\frac{0}{0}}{=} \lim\limits_{u\to 0}\dfrac{\sin u}{2u} = \dfrac{1}{2}$.（洛必达）

(4). $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{2-\sqrt{\sin(xy)+4}}$

$\xlongequal{令\sin(xy)=u} \lim\limits_{u\to 0}\dfrac{u}{2-\sqrt{u+4}} = \lim\limits_{u\to 0}\dfrac{u(2+\sqrt{u+4})}{4-(u+4)} = \lim\limits_{u\to 0}-(2+\sqrt{u+4}) = -4$ 有理化

$= \lim\limits_{u\to 0}\dfrac{1}{0-\frac{1}{2\sqrt{u+4}}} = \lim\limits_{u\to 0}-(2\sqrt{u+4}) = -4$. 洛必达

(5). 证明 $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x-y)}{x+y}$ 不存在.

证明: $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{\sin(x-y)}{x+y} = \lim\limits_{x\to 0} \frac{\sin(1-k)x}{(1+k)x} = \lim\limits_{x\to 0} \frac{(1-k)x}{(1+k)x} = \frac{1-k}{1+k}$ $(k\neq -1)$

当 k 取不同值时. $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x-y)}{x+y}$ 结果不同. $\therefore \lim\limits_{(x,y)\to(0,0)} \frac{\sin(x-y)}{x+y}$ 不存在.

(6). 证明 $\lim\limits_{(x,y)\to(0,0)} \frac{x+y}{x-y}$ 不存在.

证明: $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{x+y}{x-y} = \lim\limits_{x\to 0} \frac{(1+k)x}{(1-k)x} = \frac{1+k}{1-k}$ $(k\neq 1)$

当 k 取不同值时. $\lim\limits_{(x,y)\to(0,0)} \frac{x+y}{x-y}$ 取值不同. $\therefore \lim\limits_{(x,y)\to(0,0)} \frac{x+y}{x-y}$ 不存在.

2. 多元函数的偏导数.

(1). 设函数 $z=e^{-(\frac{1}{x}+\frac{1}{y})}$，证 $x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=2z$.

证明: $\frac{\partial z}{\partial x}=e^{-(\frac{1}{x}+\frac{1}{y})}[-(-\frac{1}{x^2}+0)]=\frac{1}{x^2}e^{-(\frac{1}{x}+\frac{1}{y})}$

$\frac{\partial z}{\partial y}=e^{-(\frac{1}{x}+\frac{1}{y})}[-(0-\frac{1}{y^2})]=\frac{1}{y^2}e^{-(\frac{1}{x}+\frac{1}{y})}$

$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=e^{-(\frac{1}{x}+\frac{1}{y})}+e^{-(\frac{1}{x}+\frac{1}{y})}=2z$.

(2). 证函数 $z=\ln\sqrt{x^2+y^2}$ 满足 $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$.

证明: $\frac{\partial z}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x}{x^2+y^2}$. 同理. $\frac{\partial z}{\partial y}=\frac{y}{x^2+y^2}$.

$\frac{\partial^2 z}{\partial x^2}=\frac{x^2+y^2-x\cdot 2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$. 同理. $\frac{\partial^2 z}{\partial y^2}=\frac{x^2-y^2}{(x^2+y^2)^2}$

$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{x^2-y^2}{(x^2+y^2)^2}=0$.

(3). 证 $u=\sqrt{x^2+y^2+z^2}$ 满足方程 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{2}{u}$.

证明: $\frac{\partial u}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2+z^2}}=\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{u}$

$\frac{\partial^2 u}{\partial x^2}=\frac{u-x\frac{\partial u}{\partial x}}{u^2}=\frac{u-x\cdot\frac{x}{u}}{u^2}=\frac{u^2-x^2}{u^3}=\frac{y^2+z^2}{u^3}$

同理. $\frac{\partial^2 u}{\partial y^2}=\frac{x^2+z^2}{u^3}$. $\frac{\partial^2 u}{\partial z^2}=\frac{x^2+y^2}{u^3}$

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{y^2+z^2+x^2+z^2+x^2+y^2}{u^3}=\frac{2(x^2+y^2+z^2)}{u^3}=\frac{2u^2}{u^3}=\frac{2}{u}$.

3. 隐函数的偏导数.(要求到二阶).

(1). 设 $\frac{x}{z}=\ln\frac{z}{y}$. 证: $z\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=0$.

解: 设 $F(x,y,z)=\frac{x}{z}-\ln\frac{z}{y}$

$F_x=\frac{1}{z}$. $F_y=-\frac{1}{\frac{z}{y}}\cdot\left(-\frac{z}{y^2}\right)=\frac{1}{y}$. $F_z=-\frac{x}{z^2}-\frac{1}{\frac{z}{y}}\cdot\frac{1}{y}=-\frac{x}{z^2}-\frac{1}{z}=-\frac{x+z}{z^2}$

$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{\frac{1}{z}}{-\frac{x+z}{z^2}}=\frac{z}{x+z}$. $\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{\frac{1}{y}}{-\frac{x+z}{z^2}}=\frac{z^2}{(x+z)y}$

$z\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=\frac{z^2}{x+z}-\frac{z^2}{x+z}=0$.

(2). 设 $z=z(x,y)$ 由 $e^z-xyz=1$ 所确定. 求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

解, 设 $F(x,y,z)=e^z-xyz-1$.

$F_x=-yz$. $F_y=-xz$. $F_z=e^z-xy$

$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{-yz}{e^z-xy}=\frac{yz}{e^z-xy}$. $\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{-xz}{e^z-xy}=\frac{xz}{e^z-xy}$

(3). 由方程 $x+2y-3z=\sin(x+2y-3z)$ 所确定的隐函数是 $z=z(x,y)$. 证. $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1$.

证明: $F(x,y,z)=x+2y-3z-\sin(x+2y-3z)$

$F_x=1-\cos(x+2y-3z)$. $F_y=2-2\cos(x+2y-3z)=2F_x$

$F_z=-3+3\cos(x+2y-3z)=-3F_x$

$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{F_x}{-3F_x}=\frac{1}{3}$. $\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{2F_x}{-3F_x}=\frac{2}{3}$.

$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\frac{1}{3}+\frac{2}{3}=1$.

(4). 方程 $xyz+\sqrt{x^2+y^2+z^2}=\sqrt{2}$. 确定函数 $z=z(x,y)$. 求 $\left.\frac{\partial z}{\partial y}\right|_{(1,0,-1)}$.

解: 设 $F(x,y,z)=xyz+\sqrt{x^2+y^2+z^2}-\sqrt{2}$.

$F_y=xz+\frac{y}{\sqrt{x^2+y^2+z^2}}$, $F_z=xy+\frac{z}{\sqrt{x^2+y^2+z^2}}$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{xz+\frac{y}{\sqrt{x^2+y^2+z^2}}}{xy+\frac{z}{\sqrt{x^2+y^2+z^2}}}=-\frac{xz\sqrt{x^2+y^2+z^2}+y}{xy\sqrt{x^2+y^2+z^2}+z}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,0,-1)}=-\frac{-\sqrt{2}+0}{-1}=-\sqrt{2}.$$

(5). 方程 $x^3+y^3+z^3-4z=0$. 确定了函数关系 $z=z(x,y)$. 求 $\frac{\partial z}{\partial x}$. $\frac{\partial z}{\partial y}$.

解: 设 $F(x,y,z)=x^3+y^3+z^3-4z$

$F_x=3x^2$. $F_y=3y^2$. $F_z=3z^2-4$.

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{3x^2}{3z^2-4}=\frac{3x^2}{4-3z^2}.\quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{3y^2}{3z^2-4}=\frac{3y^2}{4-3z^2}.$$

(6). 方程 $x^3+y^3+z^3-3az=0$ 确定了隐函数 $z=z(x,y)$. 求 $\frac{\partial^2 z}{\partial x\partial y}$.

解: 设 $F(x,y,z)=x^3+y^3+z^3-3az$

$F_x=3x^2$. $F_y=3y^2$. $F_z=3z^2-3a$. $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{3x^2}{3a-3z^2}=\frac{x^2}{a-z^2}$. $\frac{\partial z}{\partial y}=\frac{y^2}{a-z^2}$

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{x^2}{(a-z^2)^2}\cdot(-2z)\cdot\frac{\partial z}{\partial y}=\frac{2x^2z}{(a-z^2)^2}\cdot\frac{y^2}{a-z^2}=\frac{2x^2y^2z}{(a-z^2)^3}$$

(7). 设函数 $z=z(x,y)$ 由 $2xz+\ln(xyz)=0$ 确定的隐函数. 求 $\frac{\partial z}{\partial x}$. $\frac{\partial^2 z}{\partial x\partial y}$

解: 设 $F(x,y,z)=2xz+\ln(xyz)$

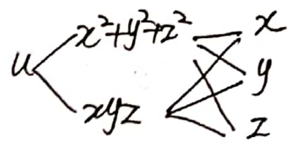
$F_x=2z+\frac{yz}{xyz}=2z+\frac{1}{x}$. $F_y=\frac{xz}{xyz}=\frac{1}{y}$. $F_z=2x+\frac{1}{z}$.

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{2z+\frac{1}{x}}{2x+\frac{1}{z}}=-\frac{(2xz+1)z}{(2xz+1)x}=-\frac{z}{x}.\quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{\frac{1}{y}}{2x+\frac{1}{z}}=-\frac{z}{(2xz+1)y}$$

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{x}\cdot\frac{\partial z}{\partial y}=-\frac{z}{xy(2xz+1)}$$

4. 多元形式复合函数求偏导.(要求到二阶).

(1). 设 $u=f(x^2+y^2+z^2, xyz)$. f有二阶连续偏导数. 求$\frac{\partial^2 u}{\partial x\partial y}$.

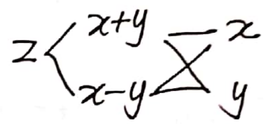


解: $\frac{\partial u}{\partial x}=f_1'\cdot 2x+f_2'\cdot yz \;=2xf_1'+yzf_2'$

$$\frac{\partial^2 u}{\partial x\partial y}=2x(f_{11}''\cdot 2y+f_{12}''\cdot xz)+zf_2'+yz(f_{21}''\cdot 2y+f_{22}''\cdot xz)$$

$$=4xyf_{11}''+2z(x^2+y^2)f_{12}''+zf_2'+xyz^2f_{22}''$$

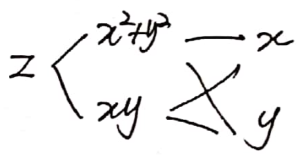
(2). 设 $z=f(x+y, x-y)$. f具有二阶连续偏导数. 求: $\frac{\partial z}{\partial x}$. $\frac{\partial^2 z}{\partial x\partial y}$.



解: $\frac{\partial z}{\partial x}=f_1'\cdot 1+f_2'\cdot 1 \;=f_1'+f_2'$

$$\frac{\partial^2 z}{\partial x\partial y}=f_{11}''\cdot 1+f_{12}''\cdot(-1)+f_{21}''\cdot 1+f_{22}''\cdot(-1)=f_{11}''-f_{22}''$$

(3). 设f具有二阶连续偏导数. $z=f(x^2+y^2, xy)$. 求$\frac{\partial^2 z}{\partial x\partial y}$.



解: $\frac{\partial z}{\partial x}=f_1'\cdot 2x+f_2'\cdot y=2xf_1'+yf_2'$

$$\frac{\partial^2 z}{\partial x\partial y}=2x(f_{11}''\cdot 2y+f_{12}''\cdot x)+f_2'+y(f_{21}''\cdot 2y+f_{22}''\cdot x)$$

$$=4xyf_{11}''+2(x^2+y^2)f_{12}''+f_2'+xyf_{22}''.$$