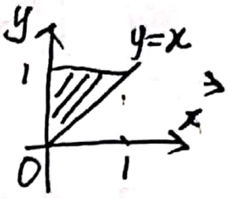
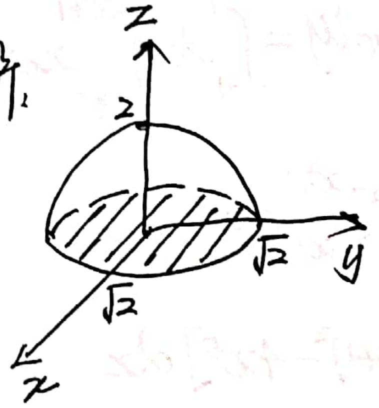
历年考题.

1. 交换积分次序. $\int_0^1 dx\int_x^1 f(x,y)dy = \underline{\int_0^1 dy\int_0^y f(x,y)dx}$



2. 求由旋转抛物面 $z=2-x^2-y^2$ 和 xoy 面所围立体体积.

解:



$D=\{(x,y)\mid x^2+y^2\le 2\}$

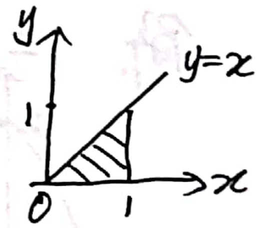
曲顶 $z=2-x^2-y^2$

$$V=\iint_D (2-x^2-y^2)\,dxdy$$

$$=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}(2-\rho^2)\rho\,d\rho$$

$$=2\pi\left[\rho^2-\frac{\rho^4}{4}\right]_0^{\sqrt{2}}=2\pi$$

1. $\int_0^1 dy\int_y^1 f(x,y)dx$ 交换积分次序. (C)

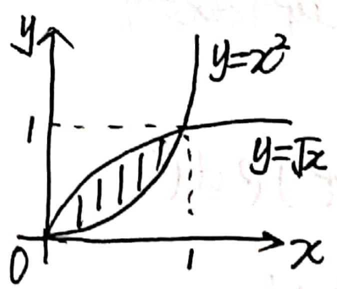


A. $\int_0^y dx\int_0^1 f(x,y)dy$. B. $\int_0^1 dx\int_0^1 f(x,y)dy$.

C. $\int_0^1 dx\int_0^x f(x,y)dy$. D. $\int_0^1 dx\int_x^1 f(x,y)dy$

2. $\iint_D xy\,dxdy$. D 由 $y=\sqrt{x}$, $y=x^2$ 所围闭区域.

解:



$$\iint_D xy\,dxdy=\int_0^1 dx\int_{x^2}^{\sqrt{x}} xy\,dy$$

$$=\int_0^1 x\left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}}dx$$

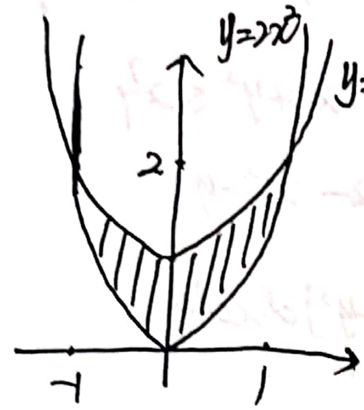
$$=\int_0^1\left(\frac{x^2}{2}-\frac{x^5}{2}\right)dx$$

$$=\left[\frac{x^3}{6}-\frac{x^6}{12}\right]_0^1=\frac{1}{12}.$$

1. $\int_0^R dx\int_0^{\sqrt{R^2-x^2}} f(\sqrt{x^2+y^2})dy$ 极坐标下二次积分 $\underline{\int_0^{\frac{\pi}{2}} d\theta\int_0^R f(r)rdr}$

2. $\iint\limits_D 2xy\,dxdy$. D由 $y=2x^2$、$y=x^2+1$ 所围.

解.



$$\iint\limits_D 2xy\,dxdy=\int_{-1}^1 dx\int_{2x^2}^{x^2+1} 2xy\,dy$$

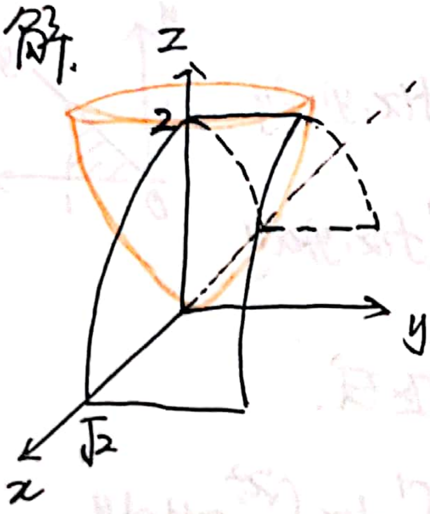
$$=\int_{-1}^1 x[y^2]_{2x^2}^{x^2+1}dx$$

$$=\int_{-1}^1 [x(x^2+1)^2-4x^5]dx$$

$$=\int_{-1}^1 [-3x^5+2x^3+x)dx=0.$$

3. 求由 $z=x^2+2y^2$ 和 $z=2-x^2$ 所围立体体积.

解.



$\begin{cases} z=x^2+2y^2 \\ z=2-x^2 \end{cases} \Rightarrow$ 投影柱面: $x^2+y^2=1$.

$\therefore D_{xy}=\{(x,y)\mid x^2+y^2\le 1\}$

$$V=\iiint\limits_\Omega dV=\iint\limits_{D_{xy}} dxdy\int_{x^2+2y^2}^{2-x^2} dz$$

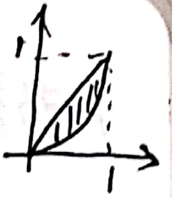
$$=\iint\limits_{D_{xy}}(2-2x^2-2y^2)dxdy$$

$$=\int_0^{2\pi} d\theta\int_0^1 (2-2\rho^2)\rho\, d\rho$$

$$=2\pi\left[\rho^2-\frac{\rho^4}{2}\right]_0^1$$

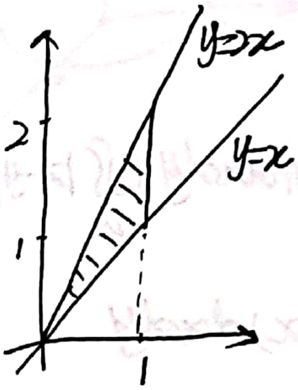
$$=\pi.$$

1. 交换积分次序 $\int_0^1 dx \int_{x^2}^{x} f(x,y)dy = \underline{\int_0^1 dy \int_y^{\sqrt{y}} f(x,y)dx}$



2. $\iint\limits_D \frac{e^x}{x} dxdy$. D由 $y=x$. $y=2x$. $x=1$ 所围.

解.



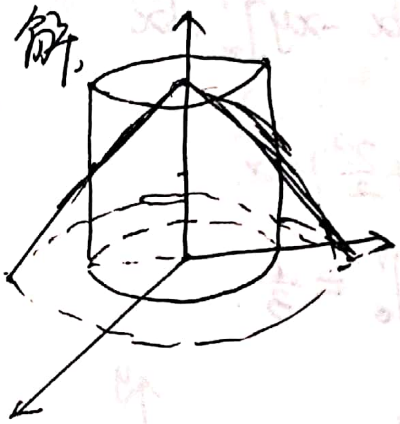
$$\iint\limits_D \frac{e^x}{x} dxdy = \int_0^1 dx \int_x^{2x} \frac{e^x}{x} dy$$

$$= \int_0^1 \frac{e^x}{x} [y]_x^{2x} dx$$

$$= \int_0^1 e^x dx = e^x \Big|_0^1 = e-1$$

3. 求由 $z=2-\sqrt{x^2+y^2}$. $x^2+y^2=1$. $z=0$ 所围立体体积.

解,



D_{xy}: $x^2+y^2 \le 1$.

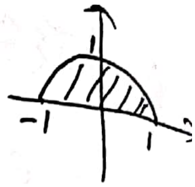
曲顶 $z=2-\sqrt{x^2+y^2}$.

$$V = \iint\limits_{D_{xy}} (2-\sqrt{x^2+y^2}) dxdy$$

$$= \int_0^{2\pi} d\theta \int_0^1 (2-\rho)\rho d\rho$$

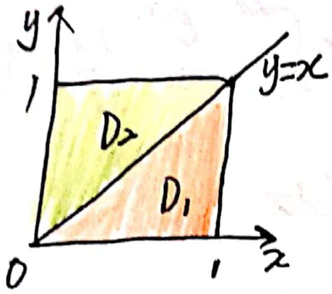
$$= 2\pi \left[\rho^2 - \frac{\rho^3}{3}\right]_0^1 = \frac{4}{3}\pi.$$

1. 改变积分次序 $\int_0^1 dy\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx = \underline{\int_{-1}^1 dx\int_0^{\sqrt{1-x^2}} f(x,y)\,dy}$



2. $\iint\limits_D |x-y|\,dxdy$. D由 $x=0$. $x=1$. $y=0$. $y=1$ 所围.

解.



D_1. D_2 如图所示、

$$\iint\limits_D |x-y|\,dxdy = \iint\limits_{D_1} |x-y|\,dxdy + \iint\limits_{D_2} |x-y|\,dxdy$$

$$= \iint\limits_{D_1} (x-y)\,dxdy + \iint\limits_{D_2} (y-x)\,dxdy$$

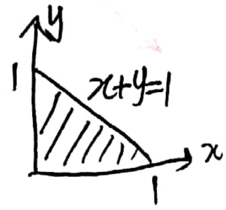
$$= \int_0^1 dx\int_0^x (x-y)\,dy + \int_0^1 dx\int_x^1 (y-x)\,dy$$

$$= \int_0^1 \left[xy-\frac{y^2}{2}\right]_0^x dx + \int_0^1 \left[\frac{y^2}{2}-xy\right]_x^1 dx$$

$$= \int_0^1 \frac{x^2}{2}\,dx + \int_0^1 \left(\frac{1}{2}-x+\frac{x^2}{2}\right)dx$$

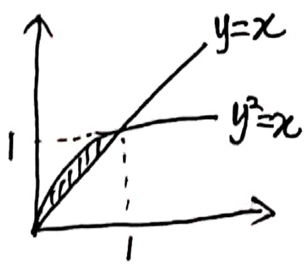
$$= \left[\frac{x^3}{6}+\frac{1}{2}x-\frac{x^2}{2}+\frac{x^3}{6}\right]_0^1 = \frac{1}{3}.$$

1. 交换积分次序. $\int_0^1 dx\int_0^{1-x} f(x,y)\,dy = \underline{\int_0^1 dy\int_0^{1-y} f(x,y)\,dx}$



2. $\iint\limits_D \frac{\sin y}{y}\,dxdy$. D由 $y=x$、$y^2=x$ 所围.

解.



$$\iint\limits_D \frac{\sin y}{y}\,dxdy = \int_0^1 dy\int_{y^2}^{y} \frac{\sin y}{y}\,dx$$

$$= \int_0^1 \frac{\sin y}{y}\, x\Big|_{y^2}^{y} dy = \int_0^1 (\sin y - y\sin y)\,dy$$

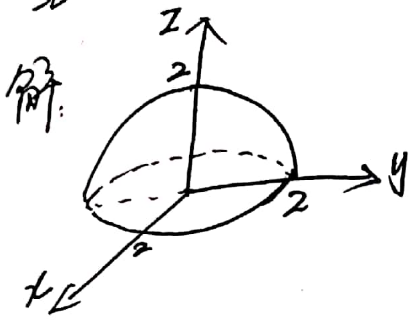
$$= -\cos y\Big|_0^1 + \int_0^1 y\,d\cos y$$

$$= 1-\cos 1 + [y\cos y]_0^1 - \int_0^1 \cos y\,dy$$

$$= 1-\cos 1+\cos 1-[\sin y]_0^1 = 1-\sin 1$$

3\. $\iiint\limits_{\Omega} z\,dxdydz$. Ω由 $z=\sqrt{4-x^2-y^2}$, $z=0$ 所围.

解:



先一后二: $0 \le z \le \sqrt{4-x^2-y^2}$.

D_{xy}: $x^2+y^2 \le 4$.

$$I=\iint\limits_{D_{xy}} dxdy \int_0^{\sqrt{4-x^2-y^2}} z\,dz = \frac{1}{2}\iint\limits_{D_{xy}} (4-x^2-y^2)\,dxdy$$

$$=\frac{1}{2}\int_0^{2\pi} d\theta \int_0^2 (4-\rho^2)\rho\,d\rho$$

$$=\frac{1}{2}\cdot 2\pi \cdot \left[2\rho^2-\frac{\rho^4}{4}\right]_0^2 = 4\pi .$$

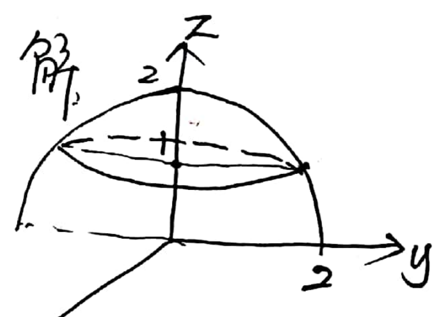
先二后一: $0 \le z \le 2$. D_z: $x^2+y^2 \le 4-z^2$

$$I=\int_0^2 z\,dz \iint\limits_{D_z} dxdy = \int_0^2 z\cdot\pi(4-z^2)\,dz$$

$$=\pi\left[2z^2-\frac{z^4}{4}\right]_0^2 = 4\pi .$$

4\. 求曲面 $z=2-x^2-y^2$ 位于 $z=1$ 上方部分曲面面积.

解:



$$z=2-x^2-y^2$$

$$z_x=-2x . \quad z_y=-2y$$

$$\sqrt{1+z_x^2+z_y^2}=\sqrt{1+4x^2+4y^2}$$

$$\begin{cases} z=2-x^2-y^2 \\ z=1 \end{cases} \Longrightarrow D: x^2+y^2 \le 1 .$$

$$S=\iint\limits_{D}\sqrt{1+4x^2+4y^2}\,dxdy = \int_0^{2\pi} d\theta \int_0^1 \sqrt{1+4\rho^2}\cdot\rho\,d\rho$$

$$=2\pi\cdot\frac{1}{8}\int_0^1\sqrt{1+4\rho^2}\,d(1+4\rho^2)=\frac{\pi}{4}\cdot\frac{2}{3}\left[(1+4\rho^2)^{\frac{3}{2}}\right]_0^1=\frac{\pi}{6}(5\sqrt{5}-1) .$$