历年考题

9.1 1. $z=f(x+y,x-y)=\frac{xy}{x^2+y^2}$. 则 $f(x,y)=$ ______

9.2 2. 设 $f(x,y)=\ln(y+\frac{x}{y})$. 则 $f_y(0,1)=$ ______

9.4 3. 设 f 具有一阶连续偏导，且 $z=f(xy,x+y)$. 则 $\frac{\partial z}{\partial x}=$ ______

9.7 4. 设 $f(x,y,z)=xyz$. 则 $\mathrm{grad}(1,-1,2)=$ ______.

9.15 5. $\lim\limits_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$

9.2 6. 试证函数 $u=\sqrt{x^2+y^2+z^2}$ 满足方程 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{2}{u}$.

9.5 7. 方程 $x^3+y^3+z^3-4z=0$ 确定了函数关系 $z=z(x,y)$. 求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

9.18 8. 函数 $z=\ln(y^2-2x+1)$ 的定义域 ______.

(6) 9. 抛物面 $z=x^2+y^2$ 上点 $(1,1,2)$ 处切平面方程是 ______.

9.1 10. $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{2-\sqrt{\sin(xy)+4}}$.

9.5 11. 由方程 $x+2y-3z=\sin(x+2y-3z)$ 所确定的隐函数是 $z=z(x,y)$.

试证. $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1$.

9.4 12. 设 f 具有二阶连续偏导数. 且 $z=f(x^2+y^2,xy)$. 求 $\frac{\partial^2 z}{\partial x\partial y}$.

$z_x=f_1'\cdot 2x+yf_2'$. $z_{xy}=2x(f_{11}''\cdot 2y+f_{12}''\cdot x)+f_2'+y(f_{21}''\cdot 2y+f_{22}''\cdot x)$

$z \begin{cases} x^2+y^2 \\ xy \end{cases}$ x, y

$=4xyf_{11}''+2(x^2+y^2)f_{12}''+f_2'+xyf_{22}''$

9.4 13. 设f具有一阶连续偏导，且$z=f(xy,x^2+y^2)$，则$\frac{\partial z}{\partial x}=$ ______

9.3 14. 设函数$z=xy+\frac{x}{y}$，则$dz=$ ______.

9.6 15. 曲面$z=x^2+y^2$在$(1,1,2)$处法线方程是 ______.

9.1 16. $\lim\limits_{(x,y)\to(0,0)}\frac{\sqrt{\tan(x+y)+1}-1}{\tan(x+y)}$

9.2 17. 试证函数$z=\ln\sqrt{x^2+y^2}$满足方程$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$.

9.5 18. 方程$x^3+y^3+z^3-3az=0$确定了隐函数关系$z=z(x,y)$，求$\frac{\partial^2 z}{\partial x\partial y}$

9.8 19. 要用铁板做一个体积为$2m^3$的有盖长方体水箱，问长、宽、高各取怎样的尺寸时，才能使用料最省？

1. 令 $x+y=u$. $x-y=v$. $\Rightarrow x=\frac{u+v}{2}$. $y=\frac{u-v}{2}$. $\Rightarrow f(u,v)=\frac{u^2-v^2}{2(u^2+v^2)}$

$\therefore f(x,y)=\frac{x^2-y^2}{2(x^2+y^2)}$

2. 法一: $f(0,y)=\ln y$. $f_y(0,y)=\frac{1}{y}$. $f_y(0,1)=1$.

法二: $f_y(x,y)=\frac{1}{y+\frac{x}{y}}(1-\frac{x}{y^2})=\cdots\cdots$

3. $\frac{\partial z}{\partial x}=yf_1'+f_2'$

4. $\mathrm{grad}f=(yz,xz,xy)$. $\mathrm{grad}(1,-1,2)=(-2,2,-1)$

5. 令 $x^2+y^2=u$. $I=\lim\limits_{u\to0}\frac{1-\cos u}{u^2}=\lim\limits_{u\to0}\frac{\frac{1}{2}u^2}{u^2}=\frac{1}{2}$.

6. $\frac{\partial u}{\partial x}=\frac{x}{\sqrt{x^2+y^2+z^2}}$. $\frac{\partial^2 u}{\partial x^2}=\frac{\sqrt{x^2+y^2+z^2}-x\frac{x}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}=\frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$.

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{y^2+z^2+x^2+z^2+x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}=\frac{2}{\sqrt{x^2+y^2+z^2}}=\frac{2}{u}$.

7. $F(x,y,z)=x^3+y^3+z^3-4z$. $F_x=3x^2$. $F_y=3y^2$. $F_z=3z^2-4$.

$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{3x^2}{4-3z^2}$. $\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{3y^2}{4-3z^2}$

8. $D=\{(x,y)\mid y^2-2x+1>0\}$

9. $z_x=2x$. $z_y=2y$. $\vec{n}=(2x,2y,-1)\big|_{(1,1,2)}=(2,2,-1)$.

$2(x-1)+2(y-1)-(z-2)=0$. $\Rightarrow 2x+2y-z=2$.

10. $\sin(xy)=u\to0$. $\lim\limits_{u\to0}\frac{u}{2-\sqrt{u+4}}=\lim\limits_{u\to0}\frac{u(2+\sqrt{u+4})}{-u}=-4$.

11. $F(x,y,z)=x+2y-3z-\sin(x+2y-3z)$. $=2F_x$ $=-3F_x$

$F_x=1-\cos(x+2y-3z)$. $F_y=2-2\cos(x+2y-3z)$. $F_z=-3+3\cos(x+2y-3z)$

$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=-\frac{F_x}{F_z}-\frac{F_y}{F_z}=-\frac{F_x}{-3F_x}-\frac{2F_x}{-3F_x}=\frac{1}{3}+\frac{2}{3}=1$.

13. $z_x = f_1' \cdot y + f_2' \cdot 2x$

14. $dz = d(xy) + d(\frac{x}{y}) = ydx + xdy + \frac{ydx - xdy}{y^2} = (y+\frac{1}{y})dx + (x - \frac{x}{y^2})dy$

法二. $z_x = y + \frac{1}{y}$. $z_y = x - \frac{x}{y^2}$. $dz = (y+\frac{1}{y})dx + (x-\frac{x}{y^2})dy$.

15. $z_x = 2x$. $z_y = 2y$. $\vec{n} = (2x, 2y, -1)\big|_{(1,1,2)} = (2, 2, -1)$.

$$\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-2}{-1}$$

16. $u = \tan(x+y) \to 0$. $\lim\limits_{u\to 0} \frac{\sqrt{u+1}-1}{u} = \lim\limits_{u\to 0} \frac{\frac{1}{2}u}{u} = \frac{1}{2}$

17. $z = \frac{1}{2}\ln(x^2+y^2)$. $z_x = \frac{x}{x^2+y^2}$. $z_{xx} = \frac{x^2+y^2 - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$z_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$. $z_{xx} + z_{yy} = 0$.

18. $F(x, y, z) = x^3 + y^3 + z^3 - 3az$

$F_x = 3x^2$. $F_y = 3y^2$. $F_z = 3z^2 - 3a$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{3x^2}{3a-3z^2} = \frac{x^2}{a-z^2}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{y^2}{a-z^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-x^2 \cdot (-2z)\frac{\partial z}{\partial y}}{(a-z^2)^2} = \frac{2x^2 z \cdot \frac{y^2}{a-z^2}}{(a-z^2)^2} = \frac{2x^2y^2z}{(a-z^2)^3}$$

19. 设长.宽.高分别为 x, y, z.

表面积 $S = 2(xy + xz + yz)$. $xyz = 2$.

作 $L = 2xy + 2xz + 2yz + \lambda(xyz - 2)$

$$\begin{cases} L_x = 2y + 2z + \lambda yz = 0 \\ L_y = 2x + 2z + \lambda xz = 0 \\ L_z = 2x + 2y + \lambda xy = 0 \\ L_\lambda = xyz - 2 = 0 \end{cases} \Longrightarrow x = y = z = \sqrt[3]{2}.$$

当长.宽.高均为 $\sqrt[3]{2}$ m时.

用料最省.

扫描全能王 创建

9-1 — 9-4 测试.

1. 讨论二元函数 $f(x,y)=\begin{cases}\dfrac{x^3y}{x^6+y^2}. & (x,y)\neq(0,0) \\ 0. & (x,y)=(0,0).\end{cases}$ 在 $(0,0)$ 点连续性.

2. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{x^2y}{x^4+y^2}$

3. $\lim\limits_{\substack{x\to 0\\ y\to 0}}(x^2+y^2)\sin\dfrac{1}{x}$.

4 $\lim\limits_{\substack{x\to 0\\ y\to 2}}\dfrac{xy}{\sqrt[3]{xy+1}-1}$

5. $z=\ln(1+xy^2)$. 求 $\left.\dfrac{\partial^2 z}{\partial x\partial y}\right|_{(0,1)}$.

6. $z=f(x,y)=\sqrt{|xy|}$, 求 $f_x'(0,0)$. $f_y'(0,0)$.

7. 设 $F(x,y)=\int_0^{xy}\dfrac{\sin t}{1+t^2}dt$. 求 $\left.\dfrac{\partial^2 F}{\partial x^2}\right|_{\substack{x=0\\ y=2}}=$ ______.

8. 已知 $(axy^3-y^2\cos x)dx+(1+by\sin x+3x^2y^2)dy$ 为某二元函数 $f(x,y)$ 全微分. 求 a. b 的值.

9. $z=\ln(u+v)+e^t$、$u=2t$、$v=t^2$. 求全导数 $\dfrac{dz}{dt}$.

10. $z=(x^2+y^2)^{xy}$. 求 $\dfrac{\partial z}{\partial x}$. $\dfrac{\partial z}{\partial y}$.

11. $u=f(x、xy、xyz)$. 求 $\dfrac{\partial u}{\partial x}$. $\dfrac{\partial u}{\partial y}$. $\dfrac{\partial u}{\partial z}$.

12. $z=f(xy,\dfrac{y}{x})$. f 是 $C^{(2)}$ 类函数. 求 $\dfrac{\partial^2 u}{\partial x^2}$.

13. 设函数 $f(u,v)$ 是有二阶连续偏导数. $y=f(e^x.\cos x)$ 求 $\left.\dfrac{dy}{dx}\right|_{x=0}$. $\left.\dfrac{d^2y}{dx^2}\right|_{x=0}$

14. $z=e^u\sin v$. $u=x+y$. $v=xy$. 利用全微分形式不变性求 z_x. z_y.

15. $z=\frac{y}{f(u)}$, $u=x^2-y^2$. $f(u)$可微. 计算 dz.

16. 设 $f(x,y)$ 有连续二阶偏导数. $z=f(x,y)-f(y,x)$. 求 $\frac{\partial^2 z}{\partial x\partial y}$.

1. $f(x,y)=\begin{cases} \dfrac{x^3y}{x^6+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

当点 $P(x,y)$ 沿 $y=kx^3$ 趋于 $(0,0)$ 时.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx^3}} f(x,y)=\lim_{x\to 0}\frac{kx^6}{(1+k^2)x^6}=\frac{k}{1+k^2}.$$

它随 k 的值的不同而改变.

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y)$ 不存在.

$\therefore f(x,y)$ 在 $(0,0)$ 点不连续.

2. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{x^2y}{x^4+y^2}$.

当 $P(x,y)$ 沿 $y=kx^2$ 趋于 $(0,0)$ 时

$$\lim_{\substack{x\to 0\\ y\to 0\\ y=kx^2}}\frac{x^2y}{x^4+y^2}=\lim_{x\to 0}\frac{kx^4}{x^4+k^2x^4}=\frac{k}{1+k^2}$$

随 k 的值的不同而改变.

$\because$ 极限不存在.

3. $\lim\limits_{\substack{x\to 0\\ y\to 0}}(x^2+y^2)\sin\frac{1}{x^2}=0$. $\because x^2+y^2\longrightarrow 0$. 而 $\sin\frac{1}{x^2}$ 有界.

4. $\lim\limits_{\substack{x\to 0\\ y\to 2}}\dfrac{xy}{\sqrt[3]{xy+1}-1}$ 令 $u=xy$.

$$=\lim_{u\to 0}\frac{u}{\sqrt[3]{u+1}-1}=\lim_{u\to 0}\frac{u}{\frac{1}{3}u}=3.$$

5. $z=\ln(1+xy^2)$.

$$\frac{\partial z}{\partial x}=\frac{y^2}{1+xy^2} \quad . \quad \left.\frac{\partial z}{\partial x}\right|_{x=0}=y^2.$$

$$\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{(0,1)}=\left.2y\right|_{(0,1)}=2$$

6. $z=f(x,y)=\sqrt{|xy|}$.

$$f_x'(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0.$$

同理. $f_y'(0,0)=0$.

7. $F(x,y)=\int_0^{xy}\frac{\sin t}{1+t^2}dt$.

$$\frac{\partial F}{\partial x}=\frac{\sin(xy)}{1+x^2y^2}\cdot y \quad . \quad \frac{\partial F(x,2)}{\partial x}=\frac{2\sin(2x)}{1+4x^2}$$

$$\frac{\partial^2 F(x,2)}{\partial x^2}=\frac{4(1+4x^2)\cos(2x)-2\sin(2x)\cdot 8x}{(1+4x^2)^2}$$

$$\left.\frac{\partial^2 F}{\partial x^2}\right|_{(0,2)}=4$$

8. $df(x,y)=(axy^3-y^2\cos x)dx+(1+by\sin x+3x^2y^2)dy$

$$\frac{\partial f}{\partial x}=axy^3-y^2\cos x \quad . \quad \frac{\partial f}{\partial y}=1+by\sin x+3x^2y^2.$$

$$\frac{\partial^2 f}{\partial x\partial y}=3axy^2-2y\cos x . \quad \frac{\partial^2 f}{\partial y\partial x}=by\cos x+6xy^2$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x} \Longrightarrow \begin{cases}3a=6\\-2=b\end{cases} \rightarrow a=2 . \ b=-2 .$$

9. $z=\ln(u+v)+e^t$. $u=2t$. $v=t^2$.

$$\frac{dz}{dt}=\frac{1}{u+v}\cdot 2+\frac{1}{u+v}\cdot 2t+e^t$$

$$\begin{tikzcd}[row sep=small] & u \arrow[dr] & \\ z \arrow[ur] \arrow[rr] \arrow[dr] & & t \\ & v \arrow[ur] & \end{tikzcd}$$

$$=\frac{2+2t}{2t+t^2}+e^t.$$

法二. $z=\ln(2t+t^2)+e^t$ 求导.

10. $z=(x^2+y^2)^{xy}$

法二. 令 $u=x^2+y^2$. $v=xy$. 求 $\frac{\partial z}{\partial x}$. $\frac{\partial z}{\partial y}$.

$$\ln z=xy\ln(x^2+y^2)$$

$$\frac{1}{z}dz=d(xy)\cdot\ln(x^2+y^2)+xy\,d\ln(x^2+y^2)$$

$$=\ln(x^2+y^2)(xdy+ydx)+xy\frac{1}{x^2+y^2}(2xdx+2ydy)$$

$$=\left[y\ln(x^2+y^2)+\frac{2x^2y}{x^2+y^2}\right]dx+\left[x\ln(x^2+y^2)+\frac{2xy^2}{x^2+y^2}\right]dy$$

$$dz=\underbrace{(x^2+y^2)^{xy}\left[y\ln(x^2+y^2)+\frac{2x^2y}{x^2+y^2}\right]}_{z_x}dx$$

$$+\underbrace{(x^2+y^2)^{xy}\left[x\ln(x^2+y^2)+\frac{2xy^2}{x^2+y^2}\right]}_{z_y}dy.$$

11. $u=f(x, xy, xyz)$

$$\frac{\partial u}{\partial x}=f_1'+yf_2'+yzf_3'$$

$$\frac{\partial u}{\partial y}=xf_2'+xzf_3'$$

$$\frac{\partial u}{\partial z}=xyf_3'.$$

12. $z=f(xy,\ \frac{y}{x})$

$$\frac{\partial z}{\partial x}=yf_1'-\frac{y}{x^2}f_2'$$

$$\frac{\partial^2 z}{\partial x^2}=y\left[f_{11}''\cdot y+f_{12}''\cdot\left(-\frac{y}{x^2}\right)\right]+\frac{2y}{x^3}f_2'-\frac{y}{x^2}\left[f_{21}''\cdot y+f_{22}''\left(-\frac{y}{x^2}\right)\right]$$

$$=y^2f_{11}''-2\frac{y^2}{x^3}f_{12}''+\frac{2y}{x^3}f_2'+\frac{y^2}{x^4}f_{22}''$$

13. $y=f(e^x,\ \cos x)$.

$$\frac{dy}{dx}=f_1'\cdot e^x+f_2'\cdot(-\sin x).\quad \left.\frac{dy}{dx}\right|_{x=0}=f_1'$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[e^xf_1'-\sin x f_2'\right]$$

$$=e^xf_1'+e^x\left[f_{11}''e^x+f_{12}''(-\sin x)\right]-\cos x f_2'-\sin x\left[f_{21}''e^x-\sin x f_{22}''\right]$$

$$=e^xf_1'+e^{2x}f_{11}''-2e^x\sin x f_{12}''-\cos x f_2'+\sin^2 x f_{22}''$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0}=f_1'+f_{11}''-f_2'$$

14. $z=e^u\sin v.\quad u=x+y,\quad v=xy$

$$dz=z_u du+z_v dv=e^u\sin v\,du+e^u\cos v\,dv$$

$$=e^u\sin v\,d(x+y)+e^u\cos v\,d(xy)$$

$$=e^u\sin v(dx+dy)+e^u\cos v(x\,dy+y\,dx)$$

$$=(e^u\sin v+ye^u\cos v)dx+(e^u\sin v+xe^u\cos v)dy$$

$z_x=e^{x+y}\sin(xy)+ye^{x+y}\cos(xy)$. $z_y=e^{x+y}\sin(xy)+xe^{x+y}\cos(xy)$.

15. $Z=\dfrac{y}{f(u)}$. $u=x^2-y^2$.

$$dZ=\frac{dy\cdot f(u)-y\,d f(u)}{f^2(u)}=\frac{f(u)dy-y\cdot f'(u)\,du}{f^2(u)}$$

$$=\frac{f(u)dy-yf'(u)[2xdx-2ydy]}{f^2(u)}$$

$$=\frac{[f(u)+2y^2f'(u)]\,dy-2xyf'(u)dx}{f^2(u)}$$

16. $Z=f(x,y)-f(y,x)$.

$Z_x=f_1'(x,y)-f_2'(y,x)$.

$Z_{xy}=f_{12}''(x,y)-f_{21}''(y,x)$.