

# Overcomplete compact representation of susceptibility at a finite bosonic frequency

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## I. THE THREE-POINT GREEN'S FUNCTIONS

We consider a three-point Green's function defined by

$$G^{3\text{pt}}(\tau_1, \tau_2, \tau_3) = \langle T_\tau A(\tau_1) B(\tau_2) C(\tau_3) \rangle, \quad (1)$$

where  $A$  and  $B$  are fermionic operators in the Heisenberg picture and  $C$  is a bosonic operator. Our expansion formula for  $G^{3\text{pt}}$  is

$$\begin{aligned} G^{3\text{pt}}(\tau_1, \tau_2, \tau_3) &= \sum_{l_1, l_2=0}^{\infty} \left\{ G_{l_1 l_2}^{(1)} U_{l_1}^{\text{F}}(\tau_{13}) U_{l_2}^{\text{F}}(\tau_{23}) \right. \\ &\quad \left. + G_{l_1 l_2}^{(2)} U_{l_1}^{\text{B}}(\tau_{13}) U_{l_2}^{\text{F}}(\tau_{21}) + G_{l_1 l_2}^{(3)} U_{l_1}^{\text{F}}(\tau_{12}) U_{l_2}^{\text{B}}(\tau_{23}) \right\}. \end{aligned} \quad (2)$$

$$\begin{aligned} G^{3\text{pt}}(i\omega_1, i\omega_2) &\equiv \int_0^\beta d\tau_{13} d\tau_{23} e^{i\omega_1 \tau_{13} + i\omega_2 \tau_{23}} G^{3\text{pt}}(\tau_1, \tau_2, \tau_3) \\ &= \sum_{l_1, l_2=0}^{\infty} \left\{ G_{l_1 l_2}^{(1)} U_{l_1}^{\text{F}}(i\omega_1) U_{l_2}^{\text{F}}(i\omega_2) \right. \\ &\quad \left. + G_{l_1 l_2}^{(2)} U_{l_1}^{\text{B}}(i\omega_1 + i\omega_2) U_{l_2}^{\text{F}}(i\omega_2) \right. \\ &\quad \left. + G_{l_1 l_2}^{(3)} U_{l_1}^{\text{F}}(i\omega_1) U_{l_2}^{\text{B}}(i\omega_1 + i\omega_2) \right\}. \end{aligned} \quad (3)$$

## II. THE FOUR-POINT GREEN'S FUNCTIONS

$$\begin{aligned} G^{4\text{pt}}(\tau_1, \tau_2, \tau_3, \tau_4) &= \sum_{l_1, l_2, l_3=0}^{\infty} \left\{ G_{l_1 l_2 l_3}^{(1)} U_{l_1}^{\text{F}}(\tau_{14}) U_{l_2}^{\text{F}}(\tau_{24}) U_{l_3}^{\text{F}}(\tau_{34}) \right. \\ &\quad \left. + \dots \right. \\ &\quad \left. + G_{l_1 l_2 l_3}^{(16)} U_{l_1}^{\text{F}}(\tau_{32}) U_{l_2}^{\text{B}}(\tau_{21}) U_{l_3}^{\text{F}}(\tau_{14}) \right\} \\ &\equiv \sum_{r=1}^{16} \sum_{l_1, l_2, l_3=0}^{\infty} G_{l_1 l_2 l_3}^{(r)} U_{l_1}^{\alpha}(\tau) U_{l_2}^{\alpha'}(\tau') U_{l_3}^{\alpha''}(\tau''), \end{aligned} \quad (4)$$

$$\chi(\tau_{12}, \tau_{34}, i\omega_m) \equiv \int_0^\beta d\tau_1 G^{4\text{pt}}(\tau_1, \tau_2, \tau_3, 0) e^{i\omega_m \tau_1} \quad (5)$$

The dependency on two fermionic frequencies at a fixed

bosonic frequency is represented as

$$\begin{aligned} \chi(\tau_{12}, \tau_{34}, i\omega_m) &= \sum_{s, s'=0,1} \sum_{l_1, l_2=0}^{\infty} \left\{ G_{ss'l_1 l_2}^{(1)} U_{sl_1}^{\text{F}}(\tau_{12}) U_{s'l_2}^{\text{F}}(\tau_{34}) \right. \\ &\quad \left. + G_{ss'l_1 l_2}^{(2)} U_{sl_1}^{\text{B}}(\tau_{12}) U_{s'l_2}^{\text{F}}((-1)^{s'} \tau_{12} + \tau_{34}) \right. \\ &\quad \left. + G_{ss'l_1 l_2}^{(3)} U_{sl_1}^{\text{B}}(\tau_{34}) U_{s'l_2}^{\text{F}}(\tau_{12} + (-1)^{s'} \tau_{34}) \right\}. \end{aligned} \quad (6)$$

$$U_{sl}^{\alpha}(\tau) \equiv e^{i s \omega_m \tau} U_l^{\alpha}(\tau). \quad (7)$$

In the Matsubara domain, this reads

$$\begin{aligned} \chi(i\omega_n, i\omega_{n'}, i\omega_m) &\equiv \int_0^\beta d\tau_{12} d\tau_{34} e^{i\omega_n \tau_{12} + i\omega_{n'} \tau_{34}} \chi(\tau_{12}, \tau_{34}, i\omega_m) \\ &\equiv \int_0^\beta d\tau_{12} d\tau_{34} d\tau_{14} e^{i\omega_n \tau_{12} + i\omega_{n'} \tau_{34} + i\omega_{n'} \tau_{14}} G^{3\text{pt}}(\tau_1, \tau_2, \tau_3, \tau_4) \\ &= \sum_{s, s'=0,1} \sum_{l_1, l_2=0}^{\infty} \left\{ G_{ss'l_1 l_2}^{(1)} U_{sl_1}^{\text{F}}(i\omega_n) U_{s'l_2}^{\text{F}}(i\omega_{n'}) \right. \\ &\quad \left. + G_{ss'l_1 l_2}^{(2)} U_{sl_1}^{\text{B}}(i\omega_n + (-1)^{s+1} i\omega_{n'}) U_{s'l_2}^{\text{F}}(i\omega_{n'}) \right. \\ &\quad \left. + G_{ss'l_1 l_2}^{(3)} U_{sl_1}^{\text{B}}(i\omega_{n'} + (-1)^{s+1} i\omega_n) U_{s'l_2}^{\text{F}}(i\omega_n) \right\}. \end{aligned} \quad (8)$$

$$U_{sl}^{\alpha}(i\omega_n) \equiv U_l^{\alpha}(i\omega_n + s i\omega_m) \quad (9)$$

## III. TENSOR REGRESSION

We decomposed the particle-hole view of the two-particle Green's function as

$$G(r, l_1, l_2) = \sum_{d=1}^D C(d, r) X_1(d, l_1) X_2(d, l_2), \quad (10)$$

where  $r$  is the index of  $12 = (3 \times 2 \times 2)$  orthogonal systems. This is referred to as a CP decomposition [canonical (CANDECOMP) / parallel factors (PARAFAC)]. We will find useful information in “Tensor Learning for Regression” written by Weiwei Guo, Irene Kotsia.

The cost function reads

$$\begin{aligned} |G(\mathbf{n}) - U(\mathbf{n}, r, l_1) U(\mathbf{n}, r, l_2) G(r, l_1, l_2)|_2^2 \\ + \alpha \{ |C|_2^2 + |X_1|_2^2 + |X_2|_2^2 \}, \end{aligned} \quad (11)$$

where  $\mathbf{n} \equiv (n, n', m)$  runs over sampling points in the Matsubara frequency domain. This cost function can be minimized using alternative projections since the penalty term is separable with respect to  $C$ ,  $X_1$  or  $X_2$ . That is,

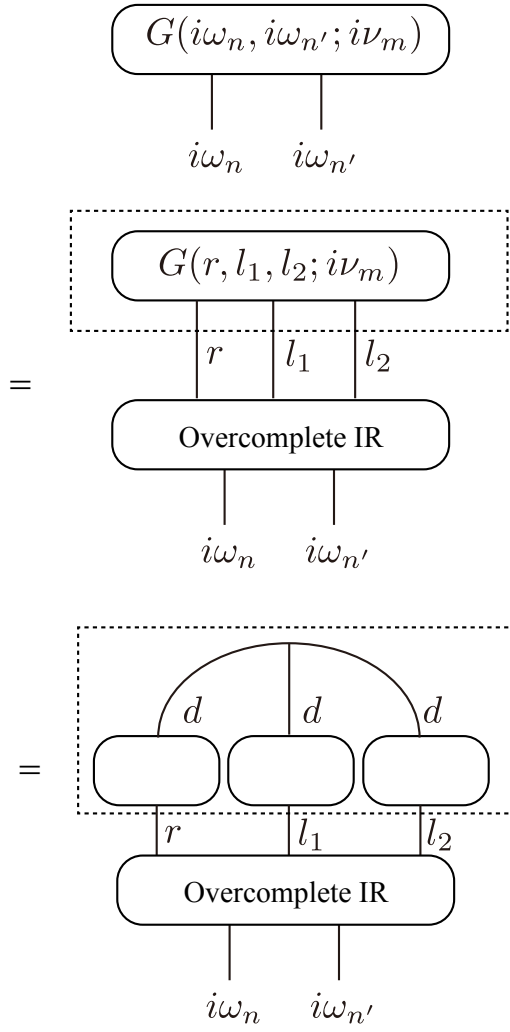


FIG. 1. Graphical representation of CP decomposition

we minimize the cost function with respect to either  $C$ ,  $X_1$  or  $X_2$  at one time. This ends up with performing many smaller Ridge regressions.