

# Overcomplete compact representation of susceptibility at a finite bosonic frequency

Hiroshi Shinaoka<sup>1</sup>

<sup>1</sup>*Department of Physics, Saitama University, 338-8570, Japan*

## I. THE THREE-POINT GREEN'S FUNCTIONS

We consider a three-point Green's function defined by

$$G^{3\text{pt}}(\tau_1, \tau_2, \tau_3) = \langle T_\tau A(\tau_1) B(\tau_2) C(\tau_3) \rangle, \quad (1)$$

where  $A$  and  $B$  are fermionic operators in the Heisenberg picture and  $C$  is a bosonic operator. Our expansion formula for  $G^{3\text{pt}}$  is

$$\begin{aligned} G^{3\text{pt}}(\tau_1, \tau_2, \tau_3) &= \sum_{l_1, l_2=0}^{\infty} \left\{ G_{l_1 l_2}^{(1)} U_{l_1}^{\text{F}}(\tau_{13}) U_{l_2}^{\text{F}}(\tau_{23}) \right. \\ &\quad \left. + G_{l_1 l_2}^{(2)} U_{l_1}^{\text{B}}(\tau_{13}) U_{l_2}^{\text{F}}(\tau_{21}) + G_{l_1 l_2}^{(3)} U_{l_1}^{\text{F}}(\tau_{12}) U_{l_2}^{\text{B}}(\tau_{23}) \right\}. \quad (2) \end{aligned}$$

$$\begin{aligned} G^{3\text{pt}}(i\omega_1, i\omega_2) &\equiv \int_0^\beta d\tau_{13} d\tau_{23} e^{i\omega_1 \tau_{13} + i\omega_2 \tau_{23}} G^{3\text{pt}}(\tau_1, \tau_2, \tau_3) \\ &= \sum_{l_1, l_2=0}^{\infty} \left\{ G_{l_1 l_2}^{(1)} U_{l_1}^{\text{F}}(i\omega_1) U_{l_2}^{\text{F}}(i\omega_2) \right. \\ &\quad \left. + G_{l_1 l_2}^{(2)} U_{l_1}^{\text{B}}(i\omega_1 + i\omega_2) U_{l_2}^{\text{F}}(i\omega_2) \right. \\ &\quad \left. + G_{l_1 l_2}^{(3)} U_{l_1}^{\text{F}}(i\omega_1) U_{l_2}^{\text{B}}(i\omega_1 + i\omega_2) \right\}. \quad (3) \end{aligned}$$

## II. THE FOUR-POINT GREEN'S FUNCTIONS

$$\begin{aligned} G^{4\text{pt}}(\tau_1, \tau_2, \tau_3, \tau_4) &= \sum_{l_1, l_2, l_3=0}^{\infty} \left\{ G_{l_1 l_2 l_3}^{(1)} U_{l_1}^{\text{F}}(\tau_{14}) U_{l_2}^{\text{F}}(\tau_{24}) U_{l_3}^{\text{F}}(\tau_{34}) \right. \\ &\quad \left. + \dots \right. \\ &\quad \left. + G_{l_1 l_2 l_3}^{(16)} U_{l_1}^{\text{F}}(\tau_{32}) U_{l_2}^{\text{B}}(\tau_{21}) U_{l_3}^{\text{F}}(\tau_{14}) \right\} \\ &\equiv \sum_{r=1}^{16} \sum_{l_1, l_2, l_3=0}^{\infty} G_{l_1 l_2 l_3}^{(r)} U_{l_1}^{\alpha}(\tau) U_{l_2}^{\alpha'}(\tau') U_{l_3}^{\alpha''}(\tau''), \quad (4) \end{aligned}$$

$$\chi(\tau_{12}, \tau_{34}, i\omega_m) \equiv \int_0^\beta d\tau_1 G^{4\text{pt}}(\tau_1, \tau_2, \tau_3, 0) e^{i\omega_m \tau_1} \quad (5)$$

The dependency on two fermionic frequencies at a fixed bosonic frequency is represented as

$$\begin{aligned} \chi(\tau_{12}, \tau_{34}, i\omega_m) &= \sum_{s, s'=0,1} \sum_{l_1, l_2=0}^{\infty} \left\{ G_{ss'l_1 l_2}^{(1)} U_{sl_1}^{\text{F}}(\tau_{12}) U_{s'l_2}^{\text{F}}(\tau_{34}) \right. \\ &\quad \left. + G_{ss'l_1 l_2}^{(2)} U_{sl_1}^{\text{B}}(\tau_{12}) U_{s'l_2}^{\text{F}}((-1)^{s'} \tau_{12} + \tau_{34}) \right. \\ &\quad \left. + G_{ss'l_1 l_2}^{(3)} U_{sl_1}^{\text{B}}(\tau_{34}) U_{s'l_2}^{\text{F}}(\tau_{12} + (-1)^{s'} \tau_{34}) \right\}. \quad (6) \end{aligned}$$

$$U_{sl}^{\alpha}(\tau) \equiv e^{i s \omega_m \tau} U_l^{\alpha}(\tau). \quad (7)$$

In the Matsubara domain, this reads

$$\begin{aligned} \chi(i\omega_n, i\omega_{n'}, i\omega_m) &\equiv \int_0^\beta d\tau_{12} d\tau_{34} e^{i\omega_n \tau_{12} + i\omega_{n'} \tau_{34}} \chi(\tau_{12}, \tau_{34}, i\omega_m) \\ &\equiv \int_0^\beta d\tau_{12} d\tau_{34} d\tau_{14} e^{i\omega_n \tau_{12} + i\omega_{n'} \tau_{34} + i\omega_{n'} \tau_{14}} G^{4\text{pt}}(\tau_1, \tau_2, \tau_3, \tau_4) \\ &= \sum_{s, s'=0,1} \sum_{l_1, l_2=0}^{\infty} \left\{ G_{ss'l_1 l_2}^{(1)} U_{sl_1}^{\text{F}}(i\omega_n) U_{s'l_2}^{\text{F}}(i\omega_{n'}) \right. \\ &\quad \left. + G_{ss'l_1 l_2}^{(2)} U_{sl_1}^{\text{B}}(i\omega_n + (-1)^{s+1} i\omega_{n'}) U_{s'l_2}^{\text{F}}(i\omega_{n'}) \right. \\ &\quad \left. + G_{ss'l_1 l_2}^{(3)} U_{sl_1}^{\text{B}}(i\omega_{n'} + (-1)^{s+1} i\omega_n) U_{s'l_2}^{\text{F}}(i\omega_n) \right\}. \quad (8) \end{aligned}$$

$$U_{sl}^{\alpha}(i\omega_n) \equiv U_l^{\alpha}(i\omega_n + s i\omega_m) \quad (9)$$