Effect of pendulum mass on the chaos motion of a double pendulum

How does the mass (5.0~30.0 kg) of primary and secondary pendulums affect the magnitude of chaos of the double pendulum system?

Jihwan Shin
02 Nov 2021

Contents

1. Introduction

1.1 Context

A double pendulum is a mechanical system where a secondary pendulum is attached to the end of the primary pendulum as its centre of pivot. Despite its simple construction as shown in figure 1, double pendulum exhibits an interesting and complex behaviour.

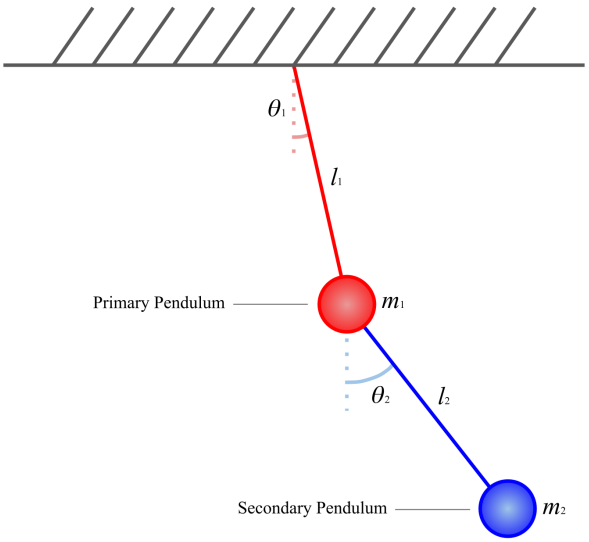

Figure 1. Diagram of a double pendulum

While the primary pendulum always follows a circular path, the secondary pendulum follows a seemingly random path in a dynamic fashion. This is well demonstrated in figure 2 which shows the path of the secondary pendulum through a long exposure photograph of a double pendulum with LED attached to the end.

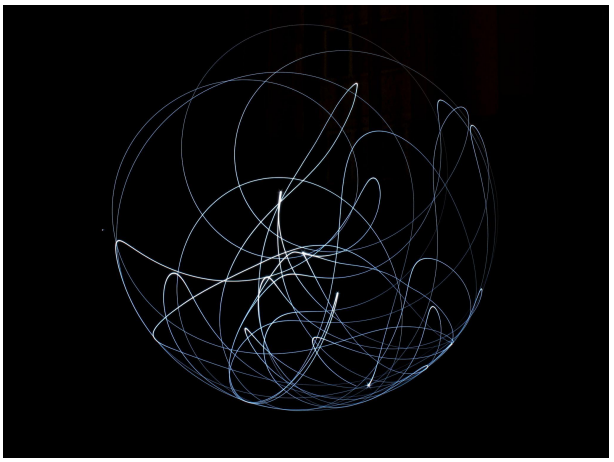

Figure 2. Path of the secondary pendulum in a double pendulum (V, Christian)

It is important to note that even though its behaviours seem random, the motion of the system is entirely predictable given its initial conditions, allowing the creation of simulations to accurately reproduce its motion. More importantly, a double pendulum is an example of a chaotic system. This means that with the slightest change to the initial conditions, the motion of the double pendulum produces drastically different outcomes. If the double pendulum in figure 2 had started at a slightly different angle, the resulting path would have been vastly different.

The aim of this extended essay is to investigate how the change of mass of primary and secondary pendulums affect the chaotic motion of the double pendulum. This investigation will allow a better understanding of the nature of chaotic systems and which variables affect its magnitude. Additionally, the results can be used to suggest ways to minimize chaos in real world applications that demonstrate similar behaviours as a double pendulum. To give an example, a golf player's arm and the club can be considered as primary and secondary pendulums to create a double pendulum system, and they can benefit from minimized chaos to produce consistent results of their swings.

1.2 Research Question

The research question of this extended essay is as follows:

How does the mass (0.5~30.0 kg) of primary and secondary pendulums affect the magnitude of chaos of the double pendulum system?

This will be answered through creating a Python program which utilizes an existing double pendulum simulation from Matplotlib to calculate the magnitude of chaos for varying masses of the primary and secondary pendulums. Average maximum Lyapunov exponent will be used as the indicator for the magnitude of chaos, due to its high interpretability and ease of implementation in Python. To analyse this data, a heatmap will be made on a 2d-matrix of each combination of mass as it has two independent variables.

2. Background Information

2.1 Pendulum Motion

A single gravity pendulum is a body suspended from a fixed point so that it can swing back and forth under the influence of gravity (Britannica). It works through converting its gravitational potential energy into kinetic energy, then back to gravitational potential energy and so on. Under ideal conditions without any kind of damping force, this will hypothetically go on forever following the law of conservation of energy. The simulation program used in this investigation will not include any damping.

Pendulums can be divided into two main types: a simple pendulum and a compound pendulum. The main difference between the two is that a simple pendulum has an additional assumption that all of its mass is located at the furthest point from the pivot, making it more ideal for calculations but impossible to perfectly recreate in real life (see figure 3). In this investigation, a simple double pendulum will be the focus.

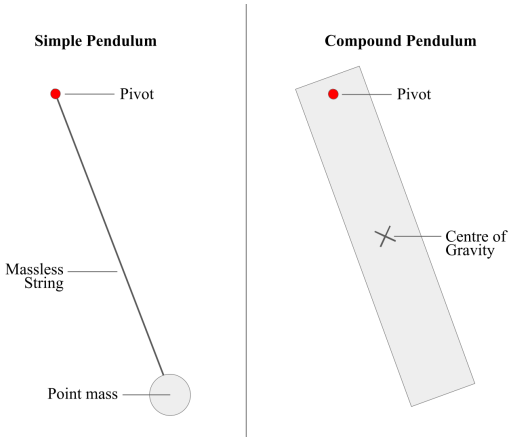

Figure 3. Comparison of simple and compound pendulum

2.2 Lagrangian Mechanics

In order to calculate the motion of the double pendulum to implement into the simulation program, Lagrangian mechanics will be used. The benefit of using Lagrangian mechanics over Newtonian mechanics when representing this kind of system is that Lagrangian equations do not need to take into account the constraint forces such as the tensions of the strings.

The *Lagrangian* is defined as the difference between the kinetic energy and the potential energy. This is shown in equation 1, where L is the Lagrangian, T is the kinetic energy and V is the potential energy (Morin 218).

$$L \equiv T - V \tag{1}$$

Using this, we can find the quantity named *action*, which is the integral of the Lagrangian over a set amount of time (Morin 221). Since the Lagrangian is found using kinetic and potential energy which depend on position (x), velocity ($\dot{x}$), and time (t), it can be expressed as a function of those variables. Therefore, action is expressed through equation 2.

$$S \equiv \int_{t_1}^{t_2} L(x, \dot{x}, t)\, \mathrm{d}t \tag{2}$$

The quantity of action follows a special principle called *the principle of stationary action*. This principle states that "the path of a particle is the one that yields a stationary value of the action" (Morin 224). What the principle indicates is that the equations of motion for the trajectory of a system is found when the action functional is made to be a stationary value with respect to its variables. The Euler-Lagrange equation (in equation 3) provides the solution to this, meaning once calculated it will provide the motion of the system (Morin 218).

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \tag{3}$$

2.3 Double Pendulum

As mentioned previously, a double pendulum is a simple mechanical system showing complex chaotic behaviours, and it is also an example of a two degrees of freedom system. According to Zhang et al., degrees of freedom is defined as the number of independent movements it has, in this case the two independent movements being each of the pendulum. Therefore, at least two equations are required to describe the motion of a double pendulum. These equations will now be found using Lagrangian mechanics explained above. Figure 4 can be used as reference for appropriate parameters during calculations.

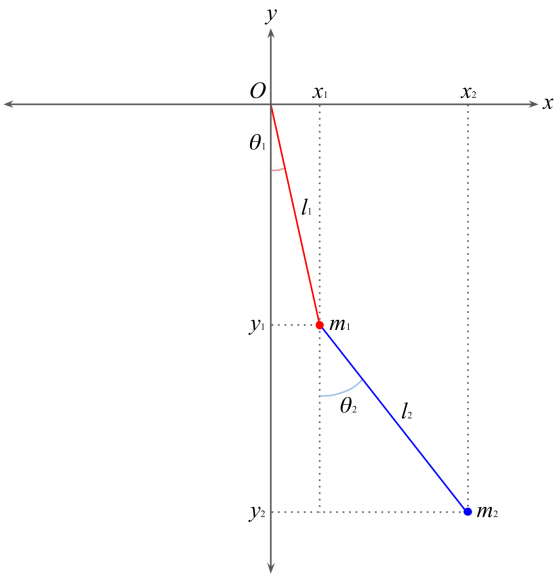

Figure 4. Double pendulum on a cartesian plane

First, the coordinates of each pendulum can be represented in terms of x and y coordinates with the origin placed at the pivot of the primary pendulum using trigonometry.

$$x_1 = l_1 \cdot \sin(\theta_1) \tag{4.1}$$

$$y_1 = -l_1 \cdot \cos(\theta_1) \tag{4.2}$$

$$x_2 = l_1 \cdot \sin(\theta_1) + l_2 \cdot \sin(\theta_2) \tag{4.3}$$

$$y_2 = -l_1 \cdot \cos(\theta_1) - l_2 \cdot \cos(\theta_2) \tag{4.4}$$

Next, equations 4 can be differentiated with respect to t in order to find their horizontal and vertical velocities. On the right hand side of the equations, chain rule will be applied as the values of θ also change with respect to t.

$$\dot{x}_1 = \dot{\theta}_1 \cdot l_1 \cdot \cos(\theta_1) \tag{5.1}$$

$$\dot{y}_1 = \dot{\theta}_1 \cdot l_1 \cdot \sin(\theta_1) \tag{5.2}$$

$$\dot{x}_2 = \dot{\theta}_1 \cdot l_1 \cdot \cos(\theta_1) + \dot{\theta}_2 \cdot l_2 \cdot \cos(\theta_2) \tag{5.3}$$

$$\dot{y}_2 = \dot{\theta}_1 \cdot l_1 \cdot \sin(\theta_1) + \dot{\theta}_2 \cdot l_2 \cdot \sin(\theta_2) \tag{5.4}$$

Using these equations, it is now possible to find the Lagrangian of the double pendulum. Equations 6 and 7 will be used to find the kinetic and potential energies.

$$T=\frac{1}{2}mv^2=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) \tag{6}$$

$$V=mgh=mgy \tag{7}$$

Equations 8 and 9 show the total kinetic and potential energies of the double pendulum system in terms of x and y. Subtracting equation 9 from equation 8 then substituting values from equations 5 gives the Lagrangian of the double pendulum in equation 10.

$$T=\frac{1}{2}m_1\left(\dot{x}_1^2+\dot{y}_1^2\right)+\frac{1}{2}m_2\left(\dot{x}_2^2+\dot{y}_2^2\right) \tag{8}$$

$$V=m_1gy_1+m_2gy_2 \tag{9}$$

$$\begin{aligned}L=&\frac{1}{2}\left(m_1+m_2\right)\cdot l_1^2\cdot\dot{\theta}_1^2+\frac{1}{2}m_2\cdot l_2^2\cdot\dot{\theta}_2^2+m_2\cdot l_1\cdot l_2\cdot\dot{\theta}_1\cdot\dot{\theta}_2\cdot\cos\left(\theta_1-\theta_2\right)\\&+g\left(m_1+m_2\right)\cdot l_1\cdot\cos\left(\theta_1\right)+g\cdot m_2\cdot l_2\cdot\cos\left(\theta_2\right)\end{aligned} \tag{10}$$

Now, it is possible to use the Euler-Lagrange equation for both θ_1 and θ_2. For θ_1, the Euler-Lagrange equation can be written as shown in equation 11.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right)=\frac{\partial L}{\partial\theta_1} \tag{11}$$

Performing the calculations on each side gives equations 12 and 13.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right)=\left(m_1+m_2\right)\cdot l_1^2\cdot\ddot{\theta}_1+m_2\cdot l_1\cdot l_2\cdot\left(\ddot{\theta}_2\cdot\cos\left(\theta_1-\theta_2\right)-\dot{\theta}_2\cdot\left(\dot{\theta}_1-\dot{\theta}_2\right)\cdot\sin\left(\theta_1-\theta_2\right)\right) \tag{12}$$

$$\frac{\partial L}{\partial\theta_1}=-m_2\cdot l_1\cdot l_2\cdot\dot{\theta}_1\cdot\dot{\theta}_2\cdot\sin\left(\theta_1-\theta_2\right)-g\left(m_1+m_2\right)\cdot l_1\cdot\sin\left(\theta_1\right) \tag{13}$$

Substituting equations 12 and 13 into equation 11 and then organizing the terms gives equation 14. By repeating the same method after replacing θ_1 with θ_2 in the Euler-Lagrange equation in equation 11, it is possible to obtain equation 15 as well. These two second order differential equations govern the motion of the double pendulum.

$$\left(m_1+m_2\right)\cdot l_1\cdot\ddot{\theta}_1+m_2\cdot l_2\cdot\ddot{\theta}_2\cdot\cos\left(\theta_1-\theta_2\right)+m_2\cdot l_2\cdot\dot{\theta}_2{}^2\cdot\sin\left(\theta_1-\theta_2\right)+g\left(m_1+m_2\right)\sin\left(\theta_1\right)=0 \quad (14)$$

$$m_2\cdot l_2\cdot\ddot{\theta}_2+m_2\cdot l_1\cdot\ddot{\theta}_1\cdot\cos\left(\theta_1-\theta_2\right)-m_2\cdot l_1\cdot\dot{\theta}_1{}^2\cdot\sin\left(\theta_1-\theta_2\right)+g\cdot m_2\cdot\sin\left(\theta_2\right)=0 \quad (15)$$

However, in order for the computer simulation to compute these equations numerically, it is necessary to rearrange these equations to have the second order variables as the subject. Additionally, θ_1-θ_2 will be replaced with $\Delta\theta$.

$$\ddot{\theta}_1=\frac{m_2\cdot l_1\cdot\dot{\theta}_1{}^2\cdot\sin(2\Delta\theta)+2m_2\cdot l_2\cdot\dot{\theta}_2{}^2\cdot\sin(\Delta\theta)+2g\cdot m_2\cdot\cos\left(\theta_2\right)\cdot\sin(\Delta\theta)+2g\cdot m_1\cdot\sin\left(\theta_1\right)}{-2l_1\cdot\left(m_1+m_2\cdot\sin^2(\Delta\theta)\right)} \quad (16)$$

$$\ddot{\theta}_2=\frac{m_2\cdot l_2\cdot\dot{\theta}_2{}^2\cdot\sin(2\Delta\theta)+2\left(m_1+m_2\right)\cdot l_1\cdot\dot{\theta}_1{}^2\cdot\sin(\Delta\theta)+2g\cdot\left(m_1+m_2\right)\cdot\cos\left(\theta_1\right)\cdot\sin(\Delta\theta)}{2l_2\cdot\left(m_1+m_2\cdot\sin^2(\Delta\theta)\right)} \quad (17)$$

Equations 16 and 17 can now be put into computer simulation to be solved using SciPy library's ordinary differential equation (ODE) solver (`scipy.integrate.odeint`), which is part of an online code library that can integrate a system of ODE given the initial conditions (SciPy).

2.4 Chaos Theory

Chaos theory is the study of apparently random or unpredictable behaviour in systems governed by deterministic laws (Britannica). A double pendulum is an example of a chaotic system, as it produces a significant change to the later state from a small change in the initial conditions.

In order to investigate this phenomenon, it is important to quantify the chaos in the system. To do so, a quantity named the *Lyapunov exponent* will be used. The following equations and explanations in this subsection were referenced from Hilborn's textbook *Chaos and Nonlinear Dynamics* (323). Suppose there are two trajectories *P* and *Q* of an identical system with a small difference to the initial conditions. In the case of a double pendulum, the trajectory will be represented using the value of θ_1 in figure 4. *P(t)* and *Q(t)* will be the value of θ_1 after time *t* from the start, and at *t=0* their θ_1 values will have a difference of $\Delta\theta$. If the difference between *P* and *Q* at time *t* is *d(t)*,

$$d(0)=\left|P(0)-Q(0)\right|=\Delta\theta \quad (18)$$

$$d(n)=\left|P(n)-Q(n)\right| \quad (19)$$

d(t) starts at the small value $\Delta\theta$, but as *t* increases trajectories *P* and *Q* diverge from one another, so *d(t)* increases. In a chaotic system, it is assumed that *d(t)* will increase exponentially. Therefore, equation 20 can be written.

$$d(n) = d(0) \cdot e^{\lambda n} \tag{20}$$

Taking the natural logarithm of both sides and rearranging the equation gives equation 21. Here, the value of λ is the Lyapunov exponent.

$$\lambda = \frac{1}{n} \cdot \ln\left(\frac{d(n)}{d(0)}\right) \tag{21}$$

The Lyapunov exponent represents the divergence rate of two neighbouring trajectories. If the value of λ is larger than 0, the system is considered chaotic, and a larger value of λ means a more chaotic system. As the value of the Lyapunov exponent changes for all values of *n*, it is often the case to use the *maximum Lyapunov exponent*, which is the maximum value of λ for a given set of *n*.

Additionally, since the value of the Lyapunov exponent often differs for different initial conditions (i.e. the value of *P(0)* and *Q(0)*), finding the mean of a range of differing initial conditions gives the *average Lyapunov exponent* which produces a more reliable result. In this investigation, a range of different initial angles will be used to find the average maximum Lyapunov exponent.

3. Data Collection

3.1 Simulation Code

To collect the data for this investigation, a simulation code has been written in Python. This can be seen in appendix A, included with inline comments for comprehensibility. The code makes use of "*The double pendulum problem*" from Matplotlib which implements the differential equations describing the motion of a double pendulum (equation 16 and 17) to make an animation. The animation aspect is omitted and an original code calculating the average maximum Lyapunov exponent (based on explanation in Section 2.4) as well as creating the heatmap visualization of the results has been added.

3.2 Preliminary Testing

For the result of the Lyapunov exponent calculation to be valid, the difference between the two neighbouring trajectories must show an exponential growth when plotted against time in a chaotic system. To confirm this assumption, the difference between θ_1 of two neighbouring trajectories has been plotted against time in figure 5. The parameters of the double pendulum are l_1=l_2=1.0m and m_1=m_2=1.0kg, and the initial conditions for each trajectory are θ_1=θ_2=90.00° and θ_1=θ_2=90.01° meaning they have a $\Delta\theta$ value of 0.01°. Data has been generated from 0.00s to 20.00s upon release with 0.02s increments.

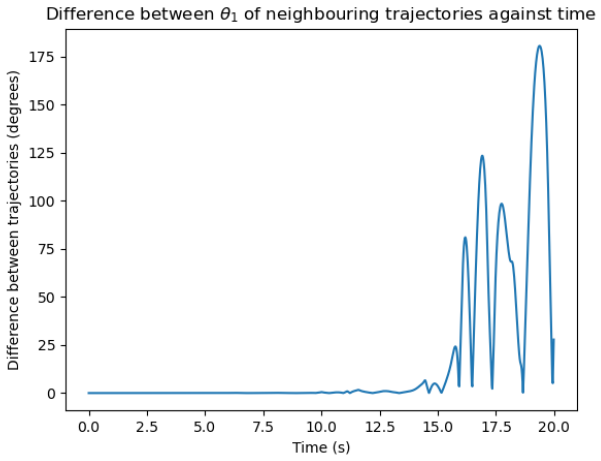

Figure 5. Graph showing difference between θ_1 of neighbouring trajectories against time

It was difficult to apply an exponential regression model to figure 5, suggesting a weakness in the assumption made. However, the relationship shown in figure 5 still appears reasonable enough to be used for Lyapunov exponent calculations due to its increasing rate of growth.

To produce more reliable results, the simulation will go through a range of initial angles and find the mean of the maximum Lyapunov exponent for each of them, giving the average maximum Lyapunov exponent. It is helpful to conduct a preliminary test to decide the values of the initial angles to be used for the best results. Figure 6 shows the maximum Lyapunov exponent values for different initial angles (0° to 180° with 5° increments) of a double pendulum system. The double pendulums were released from a straight line (θ_1=θ_2) with the neighbouring trajectory having a $\Delta\theta$ of 0.01°, and the parameters are once again l_1=l_2=1.0m and m_1=m_2=1.0kg, generated from 0.00s to 20.00s upon release with 0.02s increments.

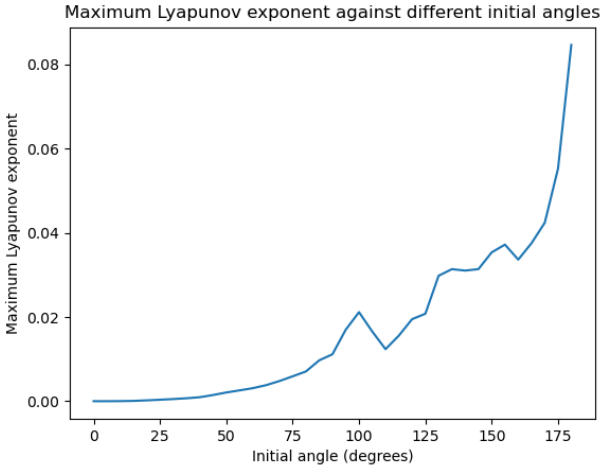

Figure 6. Graph showing the maximum Lyapunov exponent against different initial angles

The result shows a stable growth in the maximum Lyapunov exponent from 0° to 100°, but a disorderly growth after 100°. This suggests that using angles between 0° and 100° for different trials will provide a more reliable average maximum Lyapunov exponent value. It also poses an interesting relationship where a larger initial angle generally increases the chaos of the double pendulum system, which may be researched in future investigations.

Based on these preliminary results, experimental details have been decided in section 3.3 below. The range of the independent variable (mass of primary and secondary pendulum) has also been chosen to be between 0.5 and 30.0kg. This will allow a large enough dataset for trend analysis, while keeping the size manageable for an appropriate duration of the computer simulation.

3.3 Experiment Details

Independent Variable (IV)

- IV1: mass of primary pendulum (m_1) [kg]
    - Range: 0.5-30.0kg (0.5kg increments)
- IV2: mass of secondary pendulum (m_2) [kg]
    - Range: 0.5-30.0kg (0.5kg increments)

Dependent Variable (DV)

- DV: average maximum Lyapunov exponent (λ) [no units]
    - Method of collection: Python simulation
    - Result will be represented through a 2D matrix

Controlled Variable (CV)

- CV1: lengths of double pendulum (l_1, l_2) [m]
    - They will be kept constant to 1.0m
- CV2: gravitational acceleration (g) [$m \cdot s^{-2}$]
    - Standard acceleration of gravity value of 9.81$m \cdot s^{-2}$ will be used
- CV3: initial angles (θ) [°]
    - The double pendulum will be released from a straight line ($\theta_1=\theta_2=\theta$)
    - Multiple initial angles will be used for each trials, and the average maximum Lyapunov exponent will be found from these trials (θ = 10°, 20°, 30°, 40°, 50°, 60°, 70°, 80°, 90°, 100°)
- CV4: initial angular difference of neighbouring trajectories ($\Delta\theta$) [°]
    - Neighboring trajectories will have an initial angular difference of 0.01°, meaning its initial angle will be θ+0.01°
- CV5: initial angular velocity (ω) [$° \cdot s^{-1}$]
    - The double pendulum will be released from rest at $0° \cdot s^{-1}$
- CV6: simulation duration and sample frequency
    - Data will be generated from 0.00s to 30.00s upon release with 0.02s increments

4. Results

4.1 Heatmap Data

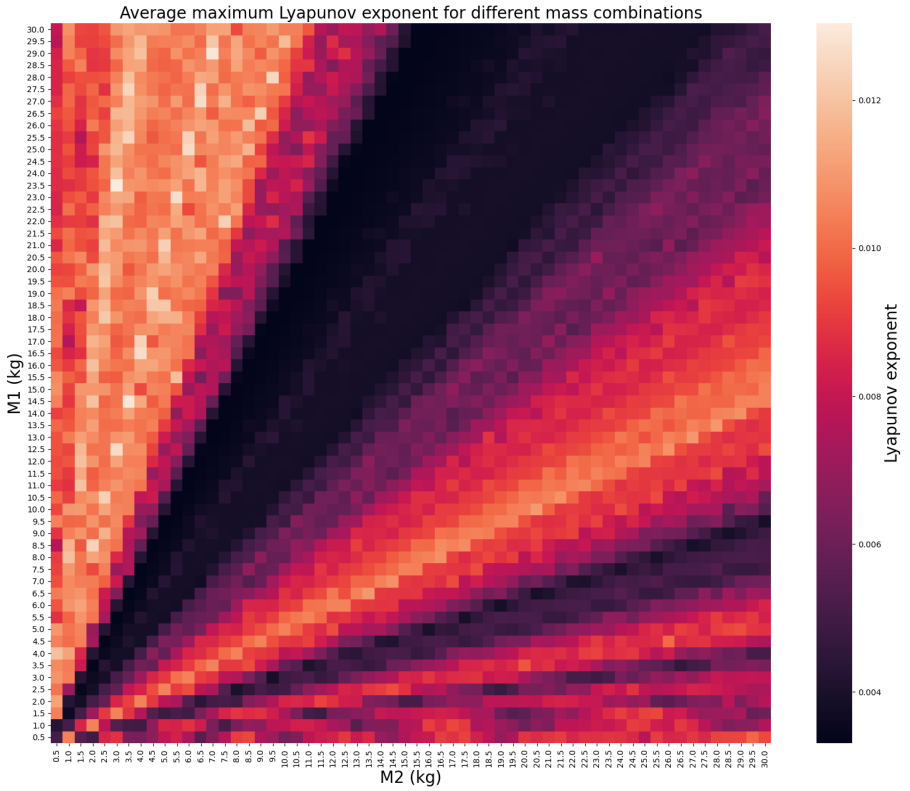

Figure 7. Heatmap of the final result

Running the Python program in appendix A gives the heatmap shown in figure 7. The full matrix table with all numerical values can optionally be downloaded as a csv file through converting the data frame *final_result* in the Python program, and a shortened version of the table with values rounded to 3 decimal points can be found in appendix B. Each tile on the heatmap shows the average maximum Lyapunov exponent at its corresponding values of m_1 and m_2 shown on the vertical and horizontal labels. The colour of the tile represents the size of the Lyapunov exponent, and it can be compared with the colour bar on the right for approximate values to ±0.001 precision (half the smallest increment).

4.2 Error Propagation

As the input and output of this experiment is made within a computer simulation unlike conventional physics experiments, the methods for calculating the uncertainties are rather different. The Python Software Foundation states that Python stores its floating point values with 16 digits of precision (53 bits in binary), so it may be reasonable to provide the absolute uncertainty of the result as $\pm 1 \times 10^{-16}$. However, as the documentation also states,

Python's floating point values involve *representation error*, which refers to the fact that "some decimal fractions cannot be represented exactly as binary fractions". As an example of this error, printing the value of 0.1 in Python would actually give 0.1000000000000000555 as its true value. While this is an extremely small difference, performing a large number of calculations (like the double pendulum simulation used) will quickly accumulate the representation error to a bigger scale. Therefore, this investigation will instead use an absolute uncertainty of $\pm 1 \times 10^{-8}$ for its results. Even though this is a rather arbitrarily chosen value, it will be able to overestimate the representation error for a more credible result while maintaining a low level of uncertainty compared to physical experiments.

However, this is only applicable to the exact values of the Lyapunov exponents calculated for each specific combination of m_1 and m_2 in appendix B. The main aim of this investigation focuses more on finding the general trend of how masses of double pendulum affect the chaos levels rather than exact Lyapunov exponent values at specific conditions. As it can be seen in figure 7, the analysis therefore uses a much less precise colour bar with an absolute uncertainty of ±0.001. Despite the fact that this is around 10% of the entire range of the dataset, this decision greatly helps to produce a more useful analysis for real world applications.

4.3 Analysis

Upon inspecting figure 7, it is apparent that the Lyapunov exponent of the double pendulum depends on the ratio of the mass rather than the individual masses. Therefore, the analysis will focus on the ratio of m_1 to m_2 (R), or the gradient in a cartesian plane where the x-axis is m_2 and the y-axis is m_1 (see equation 22). As the experiment involved a limited domain of independent variables, R's range limits from 0.017 (m_1=0.5, m_2=30) to 60 (m_1=30, m_2=0.5).

$$R = \frac{m_1}{m_2} \tag{22}$$

Figure 8 shows figure 7 transposed onto a cartesian plane, and the R values for notable gradients have been annotated on the heatmap using green lines. Table 1 shows a summary of the chaos levels shown in different R values.

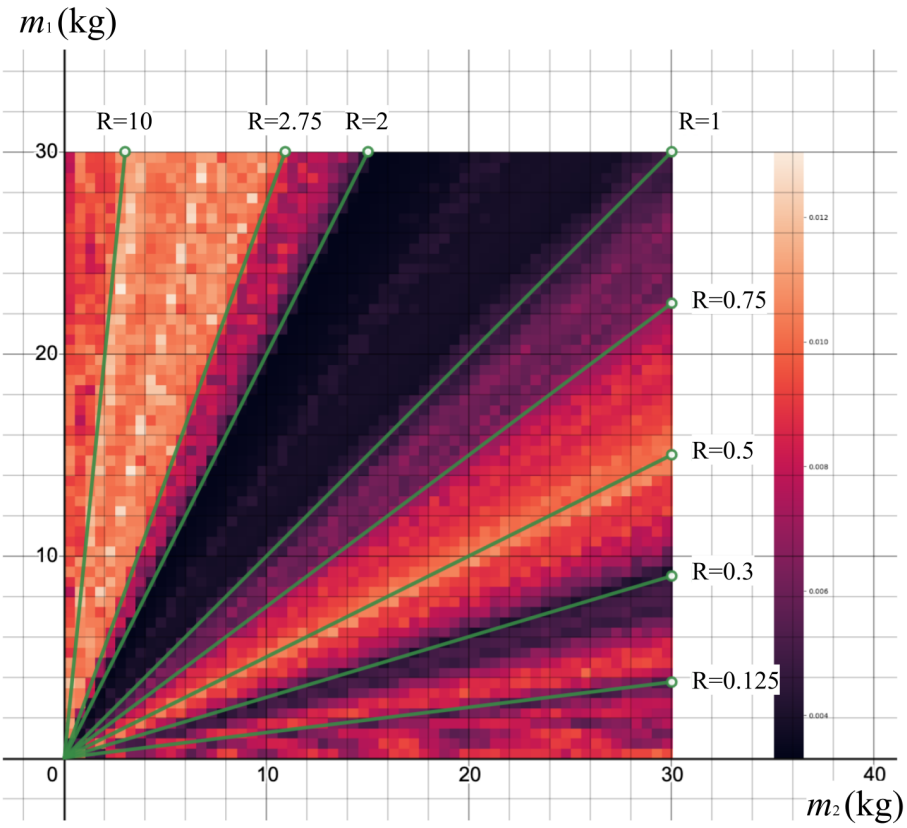

Figure 8. Heatmap on a cartesian plane with annotated R values

Table 1. Different R values and their chaos levels

R	Chaos Level	Description of Lyapunov exponent
$0.017 \leq R < 0.125$	Fluctuate	Largely fluctuates between 0.005 and 0.010 with no identifiable pattern
$R=0.125$	Low	An identifiable line can be seen with values of around 0.006
$R=0.3$	Very low	Visibly decreases as R approaches this value, going down as low as 0.004
$R=0.5$	Very high	Visibly increases as R approaches this value, going up as high as 0.011
$0.75 \leq R \leq 1$	Low	Stays stable at around 0.006
$1 < R \leq 2$	Lowest	Region showing the lowest level of chaos with values maintaining below 0.005, and even lower values below 0.004 as R value approaches 2
$2 < R < 2.75$	Mid	Stays stable at around 0.008
$2.75 \leq R \leq 10$	Highest	Region showing the highest level of chaos with values maintaining above around 0.011 in general
$R > 10$	High	Fluctuates between 0.008 and 0.011

It must be noted that the Lyapunov exponent fluctuates more heavily in some regions than others, therefore making it harder to provide exact values of the Lyapunov exponent at each region. However, comparative chaos levels are still clearly identifiable and are sufficient to provide useful conclusions for real world applications.

5. Conclusion

The research question of this essay was to find how the mass of primary and secondary pendulums affects the magnitude of chaos of the double pendulum system. To answer this, a computer simulation was made to calculate the average maximum Lyapunov exponent for a range of different masses, then visually represented through a heatmap for analysis.

It was found that chaos of a double pendulum system depends on the ratio of the masses rather than individual values for the primary and secondary pendulums, based on figure 7 showing similar Lypunov exponent values for equal gradients. Additionally, it was possible to find which ratio of primary and secondary pendulum mass results in the lowest or highest magnitude of chaos. For a double pendulum where l_1=l_2=1.0m, m_1 to m_2 ratio (R) between 1 and 2 showed the lowest level of chaos, even reaching Lyapunov exponent values lower than 0.004 in average when m_1 was twice as heavy as m_2 (R=2). The highest level of chaos was found when R was between 2.75 and 10, maintaining Lyapunov exponent values above 0.011.

The reason for R=2 having the lowest level of chaos may have connections to the torque of each pendulum. As the primary pendulum is twice as heavy as the secondary pendulum while being at half the distance from the main pivot, it is more likely for the two pendulums to maintain similar torque values for a longer period of time to delay the divergence of neighbouring trajectories. This requires further investigation through looking at different lengths for a double pendulum system in order to confirm.

For real world applications, it was found that when constructing a system similar to a double pendulum with equal lengths for the primary and secondary pendulums, it is best to make the primary pendulum twice as heavy as the secondary pendulum in order to minimize the level of chaos. This will allow more consistent results even with small differences in the initial angles.

6. Evaluation

The main strength of this essay was the use of a Python program to simulate and visualize a large set of data which may be hard in physical experiments. For the purpose of finding the Lyapunov exponent of a chaotic system, it was especially important as a slight error in setting up the initial conditions for a physical experiment will noticeably change the final results due to its chaotic nature. The heatmap made in the program also helps to easily understand the trend of chaos across combinations of different pendulum masses, which fits well with the main research question of this essay.

The preliminary testing conducted during experiment planning helped raise the reliability of the results, as parameters such as the angles used for average maximum Lyapunov exponent value were chosen based on the data obtained. However, the preliminary results also showed a limitation to the assumption that the difference between two neighbouring trajectories show an exponential growth. While the results were still suitable for yielding useful results, the experiment could be improved through making the trajectories as close to the assumption as possible. This can be done through experimenting with different parameters (e.g. $\Delta\theta$, sample frequency, etc.) to find a setup at which the difference is the closest to an exponential growth.

A weakness to this essay was that the resulting data involved a lot of fluctuation, or *noise*, in the visualization. This was especially true when R values were near the end of its domain. The noise may be considered to be a form of random error, so the solution must minimize this error. A simple solution is to increase the number of angles used when finding the average maximum Lyapunov exponent from 10 angles (increments of 10° within the range) to 20 angles (increments of 5° within the range). This will have a similar effect to conducting multiple trials which decreases the random error.

The simulation also involved a form of systematic error where it does not take into account the damping effect of the pendulum. In the real world, the double pendulum will gradually lose its mechanical energy through air resistance and friction between its components, so it is likely that the simulation yielded higher values of chaos than real double pendulums. The solution may be to multiply the velocity of the pendulum by a set percentage (e.g. 99.9%) every time frame to decrease the mechanical energy over time, but this cannot be considered to be a fully

accurate representation of the damping effect, so simulations calculating friction may need to be found and implemented to the code.

Despite the fact that these improvements aren't complicated and have a good chance of producing more accurate and precise results, the software limitations must be kept in mind. Executing the code in appendix A took 5 hours and 43 minutes to complete, as it had to calculate 1500 frames (30 seconds at 0.02s interval) of double pendulum simulation for a total of 36000 different initial conditions (60 × 60 independent variable with 10 angles for each). Adding more angles or the damping effect may increase this time even further, requiring a more powerful computing power.

As an extension to this investigation, the effect of different parameters such as length to the chaos levels may be researched in the future. Additionally, the program may be modified to calculate the chaos levels for compound pendulums instead to more closely replicate real life conditions.

7. References

Works Cited

Britannica. "Chaos theory." *Britannica School*, Encyclopædia Britannica, 24 Jan. 2018, school.ebonline.com/levels/high/article/chaos-theory/22470.

---. "Pendulum." *Britannica School*, Encyclopædia Britannica, 28 May 2020, school.ebonline.com/levels/high/article/pendulum/59060.

Hilborn, Robert C. *Chaos and Nonlinear Dynamics: An Introduction for Scientists and Engineers*. 2nd ed., Oxford UP, 2000.

Matplotlib. "The double pendulum problem." *Matplotlib*, 8 May 2021, matplotlib.org/stable/gallery/animation/double_pendulum.html.

Morin, David. *Introduction to Classical Mechanics: With Problems and Solutions*. Cambridge UP, 2008.

Python Software Foundation. "15. Floating Point Arithmetic: Issues and Limitations." *Python 3.8.11 Documentation*, 29 June 2021, docs.python.org/3.8/tutorial/floatingpoint.html.

SciPy. "Scipy.integrate.odeint — SciPy V1.7.1 Manual." *Numpy and Scipy Documentation*, docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.odeint.html.

V, Christian. "Chaos Theory & Double Pendulum - 3." Photograph. *Wikimedia Commons*, 23 Oct. 2014, commons.wikimedia.org/wiki/File:Chaos_Theory_%26_Double_Pendulum_-_3.jpg.

Zhang, Yi, et al. "Introduction to Mechanisms." *CMU School of Computer Science*, www.cs.cmu.edu/~rapidproto/mechanisms/chpt4.html.

8. Appendix

Appendix A. Python program used for data collection

```
"""
**************************************************
Calculation and visualization of average maximum lyapunov exponent for a double pendulum
system

Created for Extended Essay
"Effect of pendulum mass on the chaos motion of a double pendulum"

Original code from the source below has been used in Lines 27-64
'The double pendulum problem' by matplotlib.org
https://matplotlib.org/stable/gallery/animation/double_pendulum.html
Accessed 3 Aug 2021

Written using Python 3.8
**************************************************
"""

# import libraries required for this program
from numpy import sin, cos
from math import log
from tqdm import tqdm
import numpy as np
import pandas as pd
import seaborn as sns
import matplotlib.pyplot as plt
import scipy.integrate as integrate

# giving variable values
G = 9.81  # acceleration due to gravity, in m/s^2
L1 = 1.0  # length of pendulum 1 in m
L2 = 1.0  # length of pendulum 2 in m
L = L1 + L2  # maximal length of the combined pendulum
M1 = 1.0  # mass of pendulum 1 in kg
M2 = 1.0  # mass of pendulum 2 in kg
t_stop = 30  # how many seconds to simulate


# defining the double pendulum motion equations obtained through calculations
def derivs(state, t):
    dydx = np.zeros_like(state)
    dydx[0] = state[1]

    delta = state[2] - state[0]
    den1 = (M1 + M2) * L1 - M2 * L1 * cos(delta) * cos(delta)
    dydx[1] = ((M2 * L1 * state[1] * state[1] * sin(delta) * cos(delta)
                + M2 * G * sin(state[2]) * cos(delta)
                + M2 * L2 * state[3] * state[3] * sin(delta)
                - (M1 + M2) * G * sin(state[0]))
               / den1)
```

```
    dydx[2] = state[3]

    den2 = (L2 / L1) * den1
    dydx[3] = ((- M2 * L2 * state[3] * state[3] * sin(delta) * cos(delta)
                + (M1 + M2) * G * sin(state[0]) * cos(delta)
                - (M1 + M2) * L1 * state[1] * state[1] * sin(delta)
                - (M1 + M2) * G * sin(state[2]))
               / den2)

    return dydx


# create a time array from 0..t_stop sampled at 0.02 second steps
dt = 0.02
t = np.arange(0, t_stop, dt)


# create an array of th1 for each time frame
# th1, w1: angle (degrees) and angular velocity (degrees per second) of pendulum 1
# th2, w2: angle (degrees) and angular velocity (degrees per second) of pendulum 2
def th1_array(th1, w1, th2, w2):
    state = np.radians([th1, w1, th2, w2])  # convert degrees to radians
    y = integrate.odeint(derivs, state, t)  # integrate ODE using SciPy library
    th1_array_result = y[:, 0]  # extract only the list of th1 from the result
    return th1_array_result


# find the difference of th1 between two neighbouring trajectories
# m1, m2: mass of pendulum 1 and 2 (kg)
# dth: difference of initial angle for the two neighbouring trajectories
def trajectory_difference(th1, th2, m1, m2, dth):
    global M1, M2
    M1 = m1
    M2 = m2
    trajectory_A = th1_array(th1, 0, th2, 0)
    trajectory_B = th1_array(th1 + dth, 0, th2 + dth, 0)
    trajectory_difference_result = abs(trajectory_A - trajectory_B)
    return trajectory_difference_result


# calculate the lyapunov exponent for all values of N within the domain of simulation
def lyapunov_single(difference):
    lyapunov = [0]
    for N_value in range(1, len(difference)):
        exponent = (1 / N_value) * log((difference[N_value] / difference[0]))
        lyapunov.append(exponent)
    return lyapunov


# make arrays of M1 and M2 to iterate through for independent variable
M1_array = []
M2_array = []
for i in range(1, 61):
```

```
    M1_array.append(float(i) * 0.5)
    M2_array.append(float(i) * 0.5)

# make an array of angles to be used for average lyapunov exponent calculation
angles_array = []
for i in range(1, 11):
    angles_array.append(float(i) * 10.0)

result_2d_matrix = []

# calculate the average maximum lyapunov exponent for all initial values
for mass1 in tqdm(M1_array, desc='M1 loop'):
    temp_result = []
    for mass2 in tqdm(M2_array, desc='M2 loop', leave=False):
        lyapunov_sum = 0
        for angle in angles_array:
            diff_list = trajectory_difference(angle, angle, mass1, mass2, 0.01)
            maximum_lyapunov = max(lyapunov_single(diff_list))
            lyapunov_sum += maximum_lyapunov
        temp_result.append(lyapunov_sum / len(angles_array))
    result_2d_matrix.append(temp_result)

# convert to Pandas DataFrame
final_result = pd.DataFrame(result_2d_matrix, M1_array, M2_array)

# create heatmap
plt.subplots(figsize=(20,16))
heat_map = sns.heatmap(final_result, annot=False, square=True, cbar_kws={'label':
'Lyapunov exponent'})
heat_map.set_xticklabels(heat_map.get_xticklabels(), rotation=90,
horizontalalignment='center')
cbar_axes = heat_map.figure.axes[-1].yaxis.label.set_size(20)

plt.title("Average maximum Lyapunov exponent for different mass combinations",
fontsize=20)
plt.xlabel("M2 (kg)", fontsize=20)
plt.ylabel("M1 (kg)", fontsize=20)
plt.gca().invert_yaxis()

plt.show()
```

Appendix B. Data matrix table showing average maximum Lyapunov exponent (rounded to 3 decimal points) for each combination of pendulum masses. The full version is obtainable through using the program in appendix A.

M1 (kg) \ M2 (kg)	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0	10.5	11.0	11.5	12.0	12.5	13.0	13.5	14.0	14.5	15.0
0.5	0.005	0.010	0.005	0.005	0.009	0.009	0.006	0.006	0.007	0.008	0.008	0.007	0.006	0.006	0.008	0.009	0.009	0.007	0.007	0.006	0.006	0.008	0.008	0.008	0.009	0.008	0.009	0.007	0.007	0.008
1.0	0.004	0.005	0.009	0.010	0.008	0.005	0.005	0.005	0.007	0.009	0.008	0.009	0.008	0.006	0.005	0.006	0.007	0.007	0.008	0.008	0.008	0.008	0.008	0.007	0.007	0.006	0.006	0.006	0.006	0.008
1.5	0.010	0.004	0.005	0.007	0.009	0.010	0.009	0.008	0.005	0.005	0.005	0.005	0.006	0.007	0.009	0.009	0.009	0.009	0.008	0.007	0.006	0.005	0.005	0.006	0.006	0.007	0.007	0.008	0.008	0.008
2.0	0.011	0.004	0.004	0.005	0.006	0.009	0.009	0.010	0.009	0.008	0.007	0.005	0.004	0.005	0.005	0.005	0.007	0.007	0.008	0.009	0.009	0.008	0.009	0.009	0.008	0.008	0.007	0.006	0.006	0.005
2.5	0.010	0.007	0.003	0.004	0.005	0.006	0.008	0.008	0.009	0.010	0.009	0.009	0.008	0.007	0.005	0.004	0.005	0.005	0.005	0.005	0.006	0.007	0.007	0.008	0.009	0.008	0.009	0.009	0.009	0.009
3.0	0.011	0.010	0.004	0.004	0.004	0.005	0.006	0.007	0.009	0.009	0.010	0.010	0.009	0.009	0.008	0.008	0.006	0.005	0.004	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.007	0.007	0.008	0.009
3.5	0.011	0.010	0.008	0.003	0.004	0.004	0.005	0.006	0.006	0.008	0.008	0.009	0.010	0.010	0.009	0.009	0.009	0.008	0.007	0.006	0.005	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.006	0.006
4.0	0.012	0.011	0.009	0.004	0.004	0.004	0.004	0.005	0.007	0.006	0.007	0.009	0.009	0.009	0.010	0.010	0.009	0.009	0.009	0.008	0.008	0.007	0.006	0.005	0.005	0.004	0.005	0.005	0.005	0.005
4.5	0.010	0.010	0.010	0.007	0.003	0.004	0.004	0.004	0.005	0.006	0.006	0.007	0.009	0.009	0.009	0.009	0.010	0.010	0.009	0.009	0.009	0.009	0.008	0.008	0.006	0.006	0.005	0.004	0.004	0.005
5.0	0.011	0.010	0.011	0.007	0.004	0.003	0.004	0.004	0.004	0.005	0.006	0.006	0.006	0.008	0.009	0.008	0.009	0.009	0.010	0.010	0.010	0.009	0.009	0.009	0.008	0.008	0.007	0.007	0.007	0.005
5.5	0.009	0.010	0.011	0.010	0.007	0.003	0.004	0.004	0.004	0.004	0.005	0.006	0.006	0.006	0.007	0.008	0.009	0.009	0.009	0.010	0.010	0.010	0.010	0.009	0.009	0.009	0.008	0.008	0.008	0.007
6.0	0.010	0.011	0.011	0.010	0.008	0.004	0.003	0.004	0.004	0.004	0.004	0.005	0.006	0.006	0.006	0.007	0.009	0.009	0.008	0.009	0.009	0.010	0.010	0.010	0.011	0.009	0.009	0.009	0.009	0.008
6.5	0.010	0.011	0.011	0.010	0.008	0.006	0.003	0.003	0.004	0.004	0.004	0.004	0.005	0.006	0.006	0.006	0.007	0.007	0.008	0.009	0.008	0.009	0.009	0.010	0.009	0.010	0.011	0.009	0.009	0.009
7.0	0.009	0.011	0.010	0.010	0.010	0.008	0.004	0.003	0.004	0.004	0.004	0.004	0.004	0.005	0.006	0.006	0.006	0.006	0.007	0.008	0.009	0.008	0.009	0.009	0.009	0.010	0.010	0.010	0.011	0.009
7.5	0.009	0.011	0.010	0.009	0.010	0.007	0.007	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.005	0.006	0.007	0.006	0.006	0.007	0.008	0.008	0.009	0.008	0.009	0.009	0.009	0.010	0.010	0.010
8.0	0.009	0.012	0.010	0.011	0.011	0.009	0.008	0.004	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.007	0.006	0.006	0.007	0.007	0.008	0.009	0.008	0.009	0.009	0.009	0.010	0.010
8.5	0.008	0.012	0.010	0.012	0.011	0.010	0.008	0.006	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.006	0.006	0.007	0.007	0.008	0.008	0.008	0.009	0.009	0.009	0.009	0.009
9.0	0.009	0.010	0.011	0.010	0.012	0.010	0.008	0.007	0.004	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.006	0.006	0.006	0.006	0.007	0.007	0.009	0.009	0.009	0.008	0.009
9.5	0.010	0.010	0.010	0.011	0.010	0.011	0.009	0.006	0.006	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.007	0.006	0.006	0.006	0.007	0.007	0.008	0.009	0.009	0.008
10.0	0.009	0.011	0.011	0.010	0.011	0.011	0.010	0.007	0.007	0.004	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.007	0.008	0.008	0.009
10.5	0.010	0.009	0.011	0.011	0.012	0.010	0.010	0.008	0.008	0.005	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.006	0.006	0.006	0.007	0.006	0.007	0.007	0.008
11.0	0.009	0.009	0.012	0.010	0.010	0.011	0.010	0.010	0.008	0.007	0.004	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.006	0.006	0.006	0.007	0.006	0.007	0.007
11.5	0.009	0.009	0.011	0.010	0.010	0.010	0.011	0.011	0.007	0.008	0.006	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.006
12.0	0.009	0.010	0.012	0.011	0.010	0.011	0.010	0.010	0.009	0.008	0.007	0.004	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.006	0.007	0.006	0.006	0.006
12.5	0.010	0.009	0.012	0.010	0.010	0.013	0.011	0.011	0.009	0.007	0.008	0.006	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.006	0.006	0.006	0.006
13.0	0.010	0.010	0.011	0.011	0.010	0.011	0.011	0.010	0.010	0.008	0.007	0.006	0.004	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.006	0.006	0.006
13.5	0.010	0.010	0.010	0.011	0.010	0.010	0.010	0.010	0.010	0.009	0.008	0.007	0.006	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.006
14.0	0.009	0.009	0.010	0.011	0.010	0.010	0.011	0.010	0.010	0.010	0.007	0.008	0.007	0.004	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.006
14.5	0.010	0.009	0.011	0.012	0.010	0.010	0.013	0.011	0.010	0.010	0.008	0.008	0.008	0.006	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005
15.0	0.011	0.009	0.011	0.011	0.011	0.010	0.011	0.009	0.011	0.010	0.010	0.007	0.008	0.007	0.004	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005
15.5	0.011	0.010	0.009	0.011	0.010	0.010	0.011	0.011	0.010	0.010	0.012	0.008	0.006	0.007	0.006	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005
16.0	0.010	0.009	0.010	0.012	0.011	0.010	0.010	0.011	0.011	0.011	0.010	0.009	0.008	0.008	0.008	0.004	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004
16.5	0.009	0.009	0.009	0.011	0.011	0.010	0.011	0.013	0.011	0.011	0.010	0.010	0.007	0.008	0.007	0.005	0.004	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004
17.0	0.010	0.008	0.009	0.012	0.011	0.010	0.010	0.012	0.010	0.011	0.011	0.010	0.008	0.008	0.007	0.006	0.004	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004
17.5	0.010	0.008	0.009	0.011	0.011	0.010	0.010	0.011	0.011	0.010	0.011	0.011	0.009	0.007	0.008	0.007	0.006	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004
18.0	0.010	0.009	0.010	0.010	0.012	0.011	0.010	0.010	0.011	0.012	0.012	0.010	0.009	0.008	0.008	0.007	0.007	0.004	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004
18.5	0.010	0.009	0.008	0.011	0.012	0.010	0.010	0.010	0.012	0.012	0.010	0.010	0.010	0.009	0.008	0.008	0.007	0.006	0.004	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004
19.0	0.010	0.010	0.009	0.010	0.011	0.010	0.010	0.011	0.012	0.010	0.010	0.011	0.010	0.009	0.006	0.006	0.008	0.006	0.004	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004
19.5	0.010	0.009	0.010	0.010	0.012	0.011	0.010	0.011	0.011	0.010	0.011	0.010	0.010	0.010	0.008	0.008	0.008	0.006	0.005	0.004	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004
20.0	0.009	0.009	0.010	0.011	0.012	0.011	0.010	0.010	0.011	0.011	0.011	0.011	0.010	0.010	0.009	0.007	0.008	0.007	0.006	0.004	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004
20.5	0.010	0.010	0.011	0.009	0.011	0.011	0.010	0.010	0.010	0.012	0.010	0.011	0.011	0.010	0.010	0.007	0.008	0.008	0.007	0.006	0.004	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004
21.0	0.009	0.010	0.009	0.009	0.012	0.011	0.011	0.011	0.010	0.012	0.010	0.010	0.010	0.010	0.010	0.008	0.008	0.008	0.008	0.005	0.004	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004
21.5	0.009	0.010	0.010	0.010	0.011	0.011	0.010	0.011	0.010	0.011	0.010	0.011	0.011	0.010	0.010	0.009	0.007	0.007	0.007	0.008	0.006	0.004	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004
22.0	0.009	0.009	0.009	0.009	0.012	0.012	0.010	0.010	0.011	0.010	0.011	0.011	0.010	0.010	0.012	0.010	0.008	0.008	0.009	0.007	0.006	0.004	0.003	0.003	0.003	0.003	0.003	0.004	0.004	0.004
22.5	0.009	0.010	0.009	0.010	0.010	0.011	0.010	0.010	0.010	0.010	0.011	0.009	0.010	0.011	0.010	0.011	0.008	0.007	0.007	0.007	0.007	0.005	0.004	0.003	0.003	0.003	0.003	0.004	0.004	0.004
23.0	0.009	0.009	0.010	0.009	0.011	0.011	0.011	0.010	0.010	0.010	0.013	0.010	0.010	0.011	0.011	0.011	0.009	0.007	0.008	0.008	0.007	0.006	0.004	0.003	0.003	0.003	0.003	0.003	0.004	0.004
23.5	0.009	0.009	0.009	0.009	0.010	0.013	0.011	0.011	0.011	0.011	0.011	0.010	0.012	0.010	0.011	0.011	0.009	0.007	0.008	0.008	0.008	0.007	0.005	0.004	0.003	0.003	0.003	0.003	0.003	0.004
24.0	0.009	0.009	0.009	0.010	0.010	0.012	0.010	0.011	0.010	0.010	0.011	0.011	0.012	0.010	0.011	0.010	0.010	0.009	0.007	0.008	0.008	0.007	0.006	0.004	0.003	0.003	0.003	0.003	0.003	0.004
24.5	0.009	0.009	0.008	0.008	0.010	0.011	0.011	0.010	0.010	0.011	0.011	0.011	0.010	0.010	0.011	0.011	0.010	0.010	0.008	0.008	0.008	0.009	0.007	0.005	0.003	0.003	0.003	0.003	0.003	0.003
25.0	0.009	0.010	0.008	0.009	0.011	0.012	0.011	0.010	0.010	0.010	0.010	0.013	0.011	0.011	0.011	0.011	0.012	0.009	0.008	0.007	0.006	0.008	0.007	0.006	0.004	0.003	0.003	0.003	0.003	0.003
25.5	0.009	0.009	0.008	0.009	0.010	0.012	0.012	0.011	0.010	0.010	0.010	0.012	0.010	0.011	0.011	0.011	0.010	0.010	0.009	0.007	0.008	0.009	0.008	0.006	0.006	0.004	0.003	0.003	0.003	0.003
26.0	0.009	0.010	0.008	0.010	0.010	0.011	0.012	0.011	0.010	0.010	0.010	0.011	0.011	0.011	0.010	0.010	0.012	0.010	0.010	0.008	0.007	0.007	0.007	0.006	0.006	0.004	0.003	0.003	0.003	0.003
26.5	0.009	0.010	0.009	0.010	0.009	0.012	0.012	0.011	0.010	0.011	0.010	0.011	0.011	0.010	0.011	0.011	0.011	0.012	0.010	0.009	0.007	0.008	0.008	0.008	0.006	0.006	0.004	0.003	0.003	0.003
27.0	0.009	0.010	0.009	0.010	0.010	0.010	0.012	0.011	0.011	0.010	0.011	0.010	0.013	0.010	0.012	0.010	0.010	0.010	0.010	0.009	0.008	0.008	0.008	0.007	0.007	0.006	0.004	0.003	0.003	0.003
27.5	0.009	0.010	0.009	0.010	0.009	0.011	0.012	0.011	0.010	0.010	0.010	0.010	0.012	0.010	0.011	0.010	0.011	0.011	0.010	0.010	0.008	0.007	0.007	0.008	0.007	0.006	0.005	0.004	0.003	0.003
28.0	0.009	0.009	0.009	0.009	0.010	0.010	0.012	0.011	0.010	0.010	0.010	0.010	0.011	0.011	0.010	0.010	0.010	0.010	0.013	0.010	0.009	0.007	0.008	0.008	0.008	0.007	0.006	0.004	0.003	0.003
28.5	0.009	0.010	0.010	0.009	0.009	0.010	0.011	0.011	0.010	0.010	0.010	0.011	0.011	0.011	0.010	0.011	0.010	0.011	0.010	0.010	0.009	0.009	0.007	0.006	0.008	0.008	0.006	0.005	0.004	0.003
29.0	0.008	0.010	0.010	0.009	0.009	0.011	0.011	0.012	0.010	0.010	0.011	0.010	0.010	0.013	0.010	0.011	0.011	0.010	0.011	0.010	0.009	0.008	0.007	0.008	0.009	0.008	0.007	0.006	0.004	0.003
29.5	0.008	0.011	0.009	0.009	0.009	0.010	0.011	0.012	0.010	0.010	0.010	0.011	0.010	0.012	0.010	0.011	0.010	0.011	0.010	0.012	0.010	0.008	0.007	0.009	0.007	0.007	0.007	0.006	0.005	0.004
30.0	0.008	0.011	0.009	0.009	0.010	0.011	0.011	0.011	0.011	0.011	0.010	0.010	0.010	0.011	0.011	0.009	0.010	0.011	0.011	0.010	0.010	0.010	0.008	0.007	0.008	0.008	0.007	0.007	0.006	0.004

M1 (kg) \ M2 (kg)	15.5	16.0	16.5	17.0	17.5	18.0	18.5	19.0	19.5	20.0	20.5	21.0	21.5	22.0	22.5	23.0	23.5	24.0	24.5	25.0	25.5	26.0	26.5	27.0	27.5	28.0	28.5	29.0	29.5	30.0
0.5	0.008	0.008	0.009	0.009	0.009	0.008	0.008	0.007	0.007	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.008	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.010	0.009
1.0	0.008	0.009	0.009	0.009	0.009	0.007	0.008	0.007	0.007	0.006	0.007	0.006	0.008	0.008	0.008	0.008	0.008	0.008	0.009	0.009	0.008	0.008	0.009	0.009	0.008	0.007	0.006	0.007	0.007	0.008
1.5	0.008	0.008	0.008	0.008	0.008	0.007	0.007	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.008	0.008	0.009	0.009	0.009	0.009	0.009	0.008	0.008	0.007	0.007	0.007	0.007	0.007	0.006	0.006
2.0	0.005	0.006	0.006	0.007	0.007	0.007	0.008	0.008	0.008	0.008	0.008	0.008	0.008	0.008	0.008	0.008	0.007	0.007	0.007	0.007	0.006	0.006	0.006	0.006	0.007	0.006	0.008	0.006	0.007	0.008
2.5	0.008	0.008	0.008	0.007	0.006	0.006	0.005	0.005	0.005	0.006	0.006	0.007	0.006	0.007	0.007	0.008	0.008	0.008	0.009	0.008	0.008	0.008	0.008	0.008	0.008	0.008	0.008	0.008	0.007	0.007
3.0	0.008	0.009	0.008	0.009	0.009	0.009	0.008	0.008	0.008	0.007	0.007	0.006	0.006	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.007	0.007	0.007	0.007	0.008	0.008	0.008	0.008	0.008	0.008
3.5	0.007	0.007	0.007	0.008	0.009	0.009	0.009	0.008	0.009	0.010	0.009	0.009	0.008	0.008	0.008	0.007	0.007	0.007	0.006	0.006	0.006	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006
4.0	0.005	0.005	0.006	0.007	0.007	0.007	0.007	0.008	0.008	0.009	0.009	0.009	0.009	0.008	0.009	0.009	0.009	0.009	0.009	0.008	0.008	0.008	0.007	0.007	0.007	0.006	0.006	0.006	0.005	0.005
4.5	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.007	0.007	0.008	0.008	0.009	0.009	0.008	0.009	0.009	0.009	0.009	0.010	0.009	0.009	0.009	0.008	0.008	0.008	0.008	0.007
5.0	0.005	0.004	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.007	0.007	0.007	0.007	0.008	0.008	0.009	0.009	0.008	0.009	0.009	0.008	0.009	0.009	0.009	0.009	0.009
5.5	0.007	0.006	0.005	0.005	0.004	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.007	0.007	0.006	0.007	0.007	0.007	0.008	0.008	0.009	0.009	0.008	0.009	0.009	0.008
6.0	0.008	0.008	0.007	0.006	0.006	0.005	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.007	0.006	0.007	0.007	0.007	0.007	0.008	0.008	0.008	0.009
6.5	0.009	0.008	0.008	0.008	0.008	0.007	0.006	0.006	0.005	0.005	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.007	0.006	0.007	0.006	0.007	0.007
7.0	0.009	0.009	0.009	0.009	0.008	0.008	0.008	0.007	0.007	0.006	0.006	0.005	0.005	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006
7.5	0.010	0.009	0.009	0.009	0.009	0.009	0.008	0.008	0.008	0.008	0.007	0.007	0.006	0.007	0.005	0.005	0.005	0.004	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
8.0	0.010	0.010	0.011	0.009	0.009	0.009	0.009	0.009	0.009	0.008	0.009	0.008	0.007	0.007	0.007	0.006	0.006	0.005	0.005	0.005	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
8.5	0.010	0.010	0.011	0.010	0.011	0.009	0.009	0.009	0.009	0.009	0.009	0.008	0.008	0.008	0.008	0.007	0.007	0.006	0.006	0.006	0.005	0.005	0.005	0.004	0.004	0.005	0.005	0.005	0.005	0.005
9.0	0.009	0.009	0.010	0.010	0.010	0.010	0.010	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.008	0.008	0.008	0.008	0.007	0.006	0.006	0.006	0.006	0.005	0.005	0.004	0.004	0.004	0.005	0.005
9.5	0.009	0.009	0.009	0.009	0.010	0.010	0.010	0.010	0.011	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.008	0.008	0.009	0.008	0.007	0.007	0.007	0.006	0.006	0.007	0.005	0.005	0.005	0.004
10.0	0.009	0.008	0.009	0.009	0.009	0.009	0.010	0.010	0.010	0.010	0.011	0.010	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.008	0.008	0.008	0.008	0.007	0.007	0.007	0.006	0.007	0.006	0.005
10.5	0.009	0.009	0.008	0.009	0.009	0.009	0.009	0.010	0.010	0.010	0.010	0.010	0.011	0.010	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.008	0.008	0.008	0.008	0.008	0.007	0.007	0.008	0.006
11.0	0.008	0.008	0.009	0.009	0.008	0.009	0.009	0.009	0.009	0.010	0.010	0.010	0.010	0.010	0.011	0.010	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.008	0.008	0.008	0.008	0.008	0.008	0.007
11.5	0.007	0.008	0.008	0.008	0.009	0.008	0.008	0.009	0.009	0.009	0.009	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.008	0.008	0.009	0.008
12.0	0.006	0.007	0.007	0.009	0.008	0.009	0.009	0.008	0.009	0.009	0.009	0.009	0.009	0.010	0.010	0.010	0.010	0.010	0.010	0.011	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.008	0.008
12.5	0.007	0.006	0.007	0.007	0.008	0.008	0.009	0.009	0.008	0.008	0.009	0.009	0.009	0.009	0.009	0.010	0.010	0.010	0.010	0.010	0.010	0.011	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.009
13.0	0.006	0.006	0.006	0.007	0.007	0.007	0.009	0.008	0.009	0.009	0.009	0.008	0.009	0.009	0.009	0.009	0.010	0.010	0.010	0.009	0.010	0.010	0.010	0.011	0.009	0.009	0.009	0.009	0.009	0.009
13.5	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.008	0.009	0.009	0.009	0.009	0.008	0.009	0.009	0.009	0.009	0.009	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.011	0.009	0.009	0.009	0.009
14.0	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.008	0.008	0.008	0.009	0.009	0.008	0.008	0.009	0.009	0.009	0.009	0.009	0.010	0.010	0.010	0.010	0.010	0.010	0.011	0.011	0.009	0.009
14.5	0.006	0.006	0.006	0.006	0.006	0.007	0.006	0.007	0.007	0.007	0.008	0.008	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.010
15.0	0.005	0.006	0.006	0.007	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.008	0.008	0.008	0.009	0.009	0.008	0.008	0.009	0.009	0.009	0.009	0.009	0.009	0.010	0.010	0.010	0.010	0.010	0.010
15.5	0.005	0.005	0.005	0.006	0.007	0.006	0.006	0.006	0.006	0.007	0.007	0.007	0.007	0.009	0.008	0.009	0.009	0.009	0.009	0.008	0.009	0.009	0.009	0.009	0.009	0.010	0.010	0.010	0.010	0.010
16.0	0.005	0.005	0.004	0.005	0.006	0.007	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.007	0.008	0.008	0.008	0.009	0.009	0.008	0.008	0.009	0.009	0.009	0.009	0.009	0.009	0.010	0.010	0.010
16.5	0.004	0.004	0.005	0.005	0.005	0.006	0.007	0.006	0.006	0.006	0.007	0.006	0.007	0.007	0.007	0.008	0.008	0.008	0.009	0.009	0.009	0.008	0.008	0.009	0.009	0.009	0.009	0.009	0.009	0.010
17.0	0.004	0.004	0.004	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.007	0.006	0.007	0.007	0.008	0.008	0.008	0.008	0.008	0.009	0.009	0.008	0.009	0.009	0.009	0.010	0.009	0.009	0.009
17.5	0.004	0.004	0.004	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.005	0.006	0.007	0.007	0.007	0.008	0.008	0.008	0.009	0.009	0.009	0.008	0.008	0.009	0.009	0.009	0.009
18.0	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.007	0.009	0.009	0.008	0.009	0.009	0.009	0.008	0.008	0.009	0.009
18.5	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.007	0.008	0.008	0.008	0.009	0.009	0.009	0.008	0.008	0.009
19.0	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.006	0.007	0.007	0.006	0.006	0.006	0.007	0.006	0.007	0.007	0.007	0.007	0.008	0.008	0.009	0.009	0.009	0.009	0.009	0.008
19.5	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.007	0.007	0.008	0.009	0.008	0.009	0.009	0.009
20.0	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.006	0.006	0.007	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.007	0.008	0.008	0.008	0.009	0.009
20.5	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.007	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.007	0.007	0.008	0.008	0.008
21.0	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.006	0.006	0.007	0.007	0.007	0.007	0.008	0.008
21.5	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.007	0.008
22.0	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.006	0.006	0.006	0.007	0.007	0.007
22.5	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.004	0.005	0.006	0.006	0.006	0.007	0.006	0.006	0.006	0.006	0.007	0.006	0.007	0.007	0.007
23.0	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.006	0.006	0.006
23.5	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006
24.0	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.007	0.006	0.006	0.006	0.006	0.007	0.006
24.5	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.006	0.005	0.006	0.007	0.006	0.006	0.006	0.006	0.006
25.0	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.006	0.006
25.5	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.006
26.0	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.007	0.006
26.5	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.007
27.0	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006
27.5	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.005	0.006
28.0	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.005	0.006
28.5	0.003	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.005	0.005
29.0	0.003	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.005
29.5	0.003	0.003	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.005	0.005
30.0	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005