Summer Problems

A pdf version is available at https://shinchanmath.github.io/summer-problems/summer.pdf

Here are some problems for the summer. Classified by topics ("ACGN" - Algebra, Combinatorics, Geometry, Number Theory). The number theory ones are relatively simpler. The geometry ones are harder. The combinatorics ones are reasonably hard - with patience one can make some progress without knowing much theory.

Problem 1 (A1): Find all positive integers n such that

$$\sum_{1\le i,j\le n}\left[\frac{ij}{n+1}\right]=\frac{n^2(n-1)}{4},$$

where $[-]$ is the floor function.

Problem 2 (A2): It is known that (if you don't, it's a good exercise) if $p>3$ is a prime, then $\binom{2p}{p}=2 \bmod p^3$, or equivalently, $\binom{2p-1}{p-1}=1 \bmod p^3$. Show that for $p>3$ a prime number, we have

$$\binom{2p}{p}=2 \bmod p^4$$

if and only if

$$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{p-1}\equiv 0 \bmod p^3,$$

i.e. the numerator of left hand side as a reduced fraction is divisible by p^3.

Problem 3 (C1): Let G be a simple[1] graph with $2n+1$ vertices and n^2+n+1 edges, show that there must be a triangle in G, i.e. three vertices that's connected to each other.

Definition 1: A convex polyhedron is the set of convex combination of finite number of points in $\mathbb{R}^3$. Roughly speaking, it is a three-dimensional convex shape with flat polygonal faces, straight edges and sharp corners or vertices.

Problem 4 (C2): Let P be a convex polyhedron, such that every vertex belongs to three faces, and there's a way to color the vertices in black and white such that the two ends of an edge are of different color. Show that we can color the interior of the edges by three colors red, yellow and blue, such that

1. the three edges connected to every vertex are of different color.
2. every face only have edges of two colors.

Comment: It's not difficult to see if there's a way then it's unique in some sense - if one colors the edges connected to one vertex the rest are determined. The difficult part is to show that such a way exists and describe it properly. Once we color the edges connected to one vertex the colors of rest are dictated by the rules, but are they consistent? Could there be an edge that has to be yellow and blue at the same time? In this case there won't exist a solution. □

Problem 5 (G1): In triangle ABC, let I be the incenter, D be the tangent point of the incircle on BC, E be a point on segment AD. BE, CE intersect the incircle at F, G respectively, show that AD, BG, CF are concurrent.

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[1]"Simple" means every two nodes are connected by at most one edge.

Problem 6 (G2): In triangle ABC, let I be the incenter, D be the tangent point of the incircle on BC. Let M, N be the midpoints of BC, CI respectively. Let S be a point on the circumcircle such that $\angle ASI = 90°$. Let E be the intersection of line AM and DI, and K be the intersection of IM and SE. Show that D, M, N, K are concyclic.

Problem 7 (N1): We say a number is "good" if there are three distint factors of it that sums to 2022. Find the smallest good number.

Problem 8 (N2): Let p be a prime such that there exists a square root of $5 \bmod p$, i.e. there exists x such that $x^2 = 5 \bmod p$. Show that $p \mid F_{p-1}$, where F_n is the fibonacci number, $F_0 = 0, F_1 = 1$.

Problem 9 (N3, couldn't resist to share this, may be hard): Let p be a prime such that there does **not** exist a square root of $5 \bmod p$, i.e. there are no x such that $x^2 = 5 \bmod p$. Show that $p \mid F_{p+1}$, where F_n is the fibonacci number, $F_0 = 0, F_1 = 1$.