

May 11, 2016

0.0.1 Size of sets

The size of a *finite* set is the number of elements it contains. Sets can also be infinite.

Examples:

- Natural numbers $\{0, 1, 2, \dots\}$
 - Integers
 - Rational, real, complex numbers
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Def: A set A is *countable* if and only if:

1. A is empty, or
2. there is an onto (surjective) function $f : \mathbb{N} \rightarrow A$ (can have multiple numbers mapped to same element in A)
 - then we just iterate through \mathbb{N}

These statements are equivalent for any set A :

1. A is countable
 2. There exists an injective function $g : A \rightarrow \mathbb{N}$ (no 2 numbers mapped from the same element in A)
 3. Either A is finite, or there exists a bijective (injective and surjective) function $h : \mathbb{N} \rightarrow A$
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Examples

- \mathbb{N} is countable by choosing $f(n) = n$
- \mathbb{Z} is countable by choosing $f(n) = (n + 1)/2$ if n is odd, else $f(n) = -n/2$ if n is even

To show \mathbb{Q} is countable:

List 0: 0

List 1: -1, 1

List 2: -2, -1/2, 1/2, 2

List 3: -3, -3/2, -2/3, -1/3, 1/3, 2/3, 3/2, 3 (denominator between 1 and 3, min = -3, max = 3)

Not every set is countable.

Def: a *power set* of any set A (denoted by $P(A)$) is the set of all subsets of A

Thm: *Cantor*

The set $P(\mathbb{N})$ is *not* countable

Proof: by contradiction.

Assume it is countable. Then there is an onto function $f : \mathbb{N} \rightarrow P(\mathbb{N})$. Let $S = \{n \in \mathbb{N} : n \notin f(n)\}$.

We have $S \subseteq \mathbb{N}$, so $S \in P(\mathbb{N})$.

Because f is onto, there must exist $m \in \mathbb{N}$ such that $f(m) = S$.

Is $m \in S$?

$m \in S \iff m \in f(m)$ as $f(m) = S$.

$m \in S \iff m \notin f(m)$ by definition of S .

Hence contradiction.

0.0.2 Alphabets, strings, languages

An alphabet is a finite and nonempty set.

We think about the elements of alphabets as symbols.

Common names: Σ, Γ, Δ

Common symbols: $0, 1, a, b, c$

A *string* over an alphabet Σ is a finite sequence of symbols from Σ .

Examples:

011010 is a string over the binary alphabet $\Sigma = \{0, 1\}$. Also a string over $\{0, 1, 2\}$.

The empty string, ϵ , has no symbols in it.

Strings must be finite. $|\epsilon| = 0$, $|01101| = 5$.

A language over an alphabet Σ is any set of strings, each being over Σ .

Example: $\Sigma = \{0, 1\}$.

The language Σ^* is the set of all strings over Σ .