

## 2

May 6, 2016

### 0.0.1 Languages

$A, B, C$  typical names for languages

$\Sigma^*$  language of *all* strings over alphabet  $\Sigma$

$\emptyset$  empty language

$A = \{\omega \in \Sigma^* : |\omega| \text{ is even}\}$

Q: For an alphabet  $\Sigma$  (let us take  $\Sigma = \{0, 1\}$ ). Is the language  $\Sigma^*$  countable or uncountable?

A: Countable.

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### 0.0.2 Lexicographic ordering

The *lexicographic ordering* of  $\Sigma^*$  is the ordering of strings as follows: - order by increasing length - dictionary ordering among strings of the same length

For  $\Sigma = \{0, 1\}$ , we have:  $\epsilon, 0, 1, 00, 01, 10, 11, 000, \dots$

Now define a 1-to-1 and onto function  $f : \mathbb{N} \rightarrow \Sigma^*$  in the natural way:

$0 \rightarrow \epsilon, 1 \rightarrow 0, 2 \rightarrow 1, 3 \rightarrow 00, \dots$

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Proposition: for any alphabet  $\Sigma$ , the language  $\Sigma^*$  is countable.

Q: now consider the set of all languages over  $\Sigma = \{0, 1\}$ . Is  $P(\Sigma^*)$  countable or uncountable?

A: uncountable.

Proof: we know that there exists a bijective function  $f : \mathbb{N} \rightarrow \Sigma^*$ .

We want a bijective function  $g : P(\mathbb{N}) \rightarrow P(\Sigma^*)$ .

$S \subseteq \mathbb{N}$ . Let  $g(S) = \{f(n) : n \in S\}$ .

But that contradicts Cantor's theorem as if there is a bijective function  $h : \mathbb{N} \rightarrow P(\Sigma^*)$ , then  $g^{-1} \circ h$  is also bijective.

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### 0.0.3 Deterministic Finite Automata (DFA)

Things to keep in mind as we discuss the definition. 1. Definition is based on sets (and functions, etc). 2. It is a simple and not very powerful mode - but we're starting simple

Def: a DFA is a 5-tuple:  $m = (Q, \Sigma, \delta, q_0, F)$  where: 1.  $Q$  = finite and nonempty set (whose elements we call *states*) 2.  $\Sigma$  = alphabet 3.  $\delta$  is a transition function of form  $\delta : Q \times \Sigma \rightarrow Q$  4.  $q_0 \in Q$  (*start state*) 5.  $F \subseteq Q$  (set of *accept states*)

*State diagrams*: the circle-y pictures

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Def: suppose  $m = (Q, \Sigma, \delta, q_0, F)$  is a DFA and  $\omega \in \Sigma^*$  is a string. The DFA  $m$  *accepts*  $\omega$  if: 1.  $\omega = \epsilon$  and  $q_0 \in F$ , or 2.  $\omega = \sigma_1 \dots \sigma_n$  where  $\sigma_i \in \Sigma$ , and there exists states  $r_0, \dots, r_n \in Q$  such that  $r_0 = q_0, r_n \in F$ , and  $r_{k+1} = \delta(r_k, \sigma_{k+1})$  for  $k = 0, \dots, n-1$

Remark:  $m$  rejects  $\omega$  if  $m$  does not accept  $\omega$ .

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Another way to define acceptance/rejection is to define a function

$$\delta^* : Q \times \Sigma^* \rightarrow Q$$

$\delta^*(q, \omega)$  is the state we land on if we start at  $q$  and read string  $\omega$ . It is defined as:

$$\delta^*(q, \omega) = q$$

$$\delta^*(q, \omega\sigma) = \delta(\delta^*(q, \omega), \sigma)$$

$$\forall q \in Q, \omega \in \Sigma^*, \sigma \in \Sigma$$


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The language *recognized* by a DFA  $m$  is  $L(m) = \{\omega \in \Sigma^* : m \text{ accepts } \omega\}$ .

$$A = \{\omega \in \Sigma^* : |\omega| \text{ is prime}\}$$

Def: Let  $\Sigma$  be an alphabet and let  $A \subseteq \Sigma^*$ . The language  $A$  is *regular* if there exists a DFA  $m$  such that  $A = L(m)$ .

Q: Let  $\Sigma = \{0, 1\}$ . How many regular languages over the alphabet  $\Sigma$  are there?

A: Countably many

Proof: For any  $n \geq 1$ , there are at most finitely many DFAs with  $n$  states (up to renaming the states)  $\leq n^{2n}n2^n$ .

Now create a sequence of lists:

1  $\rightarrow$  all DFAs with 1 state,

2  $\rightarrow$  all DFAs with 2 states...

The set of regular languages (over any alphabet) is countable  $\implies$  there are some languages that are not regular.