1

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0.0.1 Size of sets

The size of a *finite* set is the number of elements it contains. Sets can also be infinite.
Examples:

- Natural numbers $\{0, 1, 2, ...\}$
- Integers
- Rational, real, complex numbers

Def: A set A is *countable* if and only if:

1. A is empty, or
2. there is an onto (surjective) function $f : \mathbb{N} \to A$ (can have multiple numbers mapped to same element in A)
   - then we just iterate through $\mathbb{N}$

These statements are equivalent for any set A:

1. A is countable
2. There exists an injective function $g : A \to \mathbb{N}$ (no 2 numbers map to the same element in A)
3. Either A is finite, or there exists an bijective (injective and surjective) function $h : \mathbb{N} \to A$

Examples

- $\mathbb{N}$ is countable by choosing $f(n) = n$
- $\mathbb{Z}$ is countable by choosing $f(n) = (n+1)/2$ if n is odd, else $f(n) = -n/2$ if n is even

To show $\mathbb{Q}$ is countable:
List 0: 0
List 1: -1, 1
List 2: -2, -1/2, 1/2, 2
List 3: -3, -3/2, -2/3, -1/3, 1/3, 2/3, 3/2, 3 (denominator between 1 and 3, min = -3, max = 3)
Not every set is countable.
Def: a *power set* of any set A (denoted by $P(A)$) is the set of all subsets of A
Thm: *Cantor*
The set $P(\mathbb{N})$ is *not* countable
Proof: by contradiction.
Assume it is countable. Then there is an onto function $f : \mathbb{N} \to P(\mathbb{N})$. Let $S = \{n \in \mathbb{N} : n \notin f(n)\}$.

We have $S \subseteq \mathbb{N}$, so $S \in P(\mathbb{N})$.
Because f is onto, there must exist $m \in \mathbb{N}$ such that $f(m) = S$.
Is $m \in S$?
$m \in S \iff m \in f(m)$ as $f(m) = S$.
$m \in S \iff m \notin f(m)$ by definition of S.
Hence contradiction.

0.0.2 Alphabets, strings, languages

An alphabet is a finite and nonempty set.
We think about the elements of alphabets as symbols.
Common names: Σ, Γ, Δ
Common symbols: $0, 1, a, b, c$
A *string* over an alphabet Σ is a finite sequence of symbols from Σ.
Examples:
011010 is a string over the binary alphabet $\Sigma = \{0, 1\}$. Also a string over $\{0, 1, 2\}$.
The empty string, ϵ, has no symbols in it.
Strings must be finite. $|\epsilon| = 0$, $|01101| = 5$.
A language over an alphabet Σ is any set of strings, each being over Σ.
Example: $\Sigma = \{0, 1\}$.
The language Σ^* is the set of all strings over Σ.