Calculate $p(x;\theta)$ in terms of Gaussian distribution, with known σ^2

$$
\begin{aligned}
P\left(x;\mu,\sigma^2\right) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{\ln\frac{1}{\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\ln(\sigma)-\frac{(x-\mu)^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}+\frac{\mu x}{\sigma^2}-\frac{\mu^2}{2\sigma^2}+\ln\left(\frac{1}{\sigma}\right)} \\
&= \frac{1}{\sqrt{2\pi}} e^{\begin{bmatrix}\frac{\mu}{\sigma^2} & -\frac{1}{2\sigma^2}\end{bmatrix}\begin{bmatrix}x & x^2\end{bmatrix}^T-\left(\frac{\mu^2}{2\sigma^2}+\ln\sigma\right)}
\end{aligned}
$$

Therefore

$$
P\left(x;\mu,\sigma^2\right) = \frac{1}{\sqrt{2\pi}}\exp\left(\begin{bmatrix}\frac{\mu}{\sigma^2} \\ -\frac{1}{2\sigma^2}\end{bmatrix}\cdot\begin{bmatrix}x \\ x^2\end{bmatrix}-\left(\frac{\mu^2}{2\sigma^2}+\ln\sigma\right)\right) \tag{1}
$$

In Eq (1), counting measure is $1/\sqrt{2\pi}$, natural parameter η is $\begin{bmatrix}\frac{\mu}{\sigma^2} \\ -\frac{1}{2\sigma^2}\end{bmatrix}$, sufficient statistic $t(x)$ is $\begin{bmatrix}x \\ x^2\end{bmatrix}$, log-partition function $a(\eta)$ is $\left(\frac{\mu^2}{2\sigma^2}+\ln\sigma\right)$. Since

$$
\eta_1 = \frac{\mu}{\sigma^2} \qquad \eta_2 = -\frac{1}{2\sigma^2},
$$

thus,

$$
\mu = \frac{\eta_1}{-2\eta_2}.
$$

And,

$$
\begin{aligned}
a(\eta) &= \left[\frac{\left(\frac{\eta_1}{2\eta_2}\right)^2}{2\left(\frac{1}{-2\eta_2}\right)^2}+\ln\sqrt{\frac{1}{-2\eta_2}}\right] = -\frac{1}{2}\eta_1^2\frac{1}{2\eta_2}-\frac{1}{2}\ln\left(-2\eta_2\right) \\
&= \frac{\eta_1^2}{4\eta_2}-\frac{1}{2}\ln\left(-2\eta_2\right)
\end{aligned}
$$

Calculate $p(\theta; b_0)$ in terms of Gaussian distribution, with known σ^2

In this setting, the prior becomes $P\left(\mu; m, v^2\right)$ since σ is known. We have

$$\begin{aligned}
P\left(\mu; m, v^2\right) &= \frac{1}{\sqrt{2\pi} v} \exp\left(-\frac{(\mu - m)^2}{2v^2}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\ln v - \frac{(\mu - m)^2}{2v^2}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\ln v - \frac{\mu^2}{2v^2} + \frac{m}{v^2}\mu - \frac{m^2}{2v^2}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2}{v^2} \begin{bmatrix} -m \\ -m^2 \end{bmatrix} \cdot \begin{bmatrix} \frac{\mu}{\sigma^2} \\ -\frac{1}{2\sigma^2} \end{bmatrix} + \left(-\frac{\sigma^2}{v^2}\right)\left(\frac{\mu^2}{2\sigma^2} + \ln\sigma\right) + \frac{\sigma^2}{v^2}\ln\sigma - \ln v\right)
\end{aligned}$$

Therefore we have

$$P\left(\mu; m, v^2\right) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2}{v^2} \begin{bmatrix} -m \\ -m^2 \end{bmatrix} \cdot \begin{bmatrix} \frac{\mu}{\sigma^2} \\ -\frac{1}{2\sigma^2} \end{bmatrix} + \left(-\frac{\sigma^2}{v^2}\right)\left(\frac{\mu^2}{2\sigma^2} + \ln\sigma\right) + \frac{\sigma^2}{v^2}\ln\sigma - \ln v\right).$$

Since $P(\mu; m, v^2)$ can also be parametrized by

$$P(\mu; n, \nu) = \exp\left(\langle n\nu, \theta\rangle + nT(\theta) - \psi(\nu, n)\right),$$

we can see that

$$\begin{aligned}
n &= -\frac{\sigma^2}{v^2}, \\
\nu &= \begin{bmatrix} -m \\ -m^2 \end{bmatrix}, \\
\psi(\nu, n) &= \ln v - \frac{\sigma^2}{v^2}\ln\sigma.
\end{aligned}$$

Questions and Concerns. According to the derivation, the 2nd entry of ν (which in this case is $-m^2$) can literally be anything since $\psi(\nu, n)$ can be anything. How can we determine what the entry should be? Is it really $-m^2$?