1 Bayesian Market Maker (BMM)

1.1 Prof Kutty's View

Mathematical Background

How do we solve the problem that the eventual price only reflects the expectation of the last agent?

1. For the general form of the exponential family distribution:

$$p(x;\theta) = \exp\left(\langle\theta, \phi(x)\rangle - T(\theta)\right),$$

   the conjugate prior has the form:

$$p(\theta; b_0) = \exp(\langle n\nu, \theta\rangle + nT(\theta) - \psi(\nu, n)).$$

   - **Random variable:** θ
   - **Natural parameter:** $b_0 = (n\nu, n)$

2. Since it can be verified that

$$\mathbf{E}_{\theta\sim b_0}\mathbf{E}_{x\sim\theta}[\phi(x)] = \nu,$$

   meaning that $\nu = n\nu/n$ is the posterior mean (**Why?**), we can think of the initial belief of the market maker based on a "virtual" dataset with size n and mean ν.

Simulation

1. The market maker maintains an initial belief $p(x;\theta)$ parametrized by prior $p(\theta; b_0)$, with $b_0 = \begin{bmatrix} n\nu \\ n \end{bmatrix}$.

2. Each incoming agent has his own belief $\tilde{p}(x;\tilde{\theta})$ parametrized by prior $p(\tilde{\theta}; \tilde{b_0})$, with $\tilde{b_0} = \begin{bmatrix} m\hat{\mu} \\ m \end{bmatrix}$.

3. With each agent completing trade in the market, the posterior of the market maker's belief now has the updated expectation

$$\nu' = \frac{n\nu + m\hat{\mu}}{n+m}.$$

   This can be viewed as a combination of the two dataset, one of mean ν and size n, and the other of mean $\hat{\mu}$ and size m, resulting in a bigger dataset of mean ν' and size $n+m$.

4. Since $\nabla C(\theta) = \mathbf{E}_p[\phi(x)] = \mathbf{E}_{\theta\sim b_0}\mathbf{E}_{x\sim\theta}[\phi(x)]$, we know that at any instant, we have the price

$$\nabla C(\theta) = \nu' = \frac{n\nu + m\hat{\mu}}{n+m}.$$

1.2 Brahma, et al's View [with reference to Das and Magdon-Ismail, 2008]

Assumptions

- The random variable of an event v can take on any value between $[0, V_{max}]$, and the actual outcome is V.
- The security in the market has a payoff V_{max} if the event is realized. Combined with the first assumption, in this case, *the price $p_t(v)$ will be the belief for the outcome of the event.*
- Each trader t possesses a noisy estimate w_t of V, where $w_t = V + \epsilon_t$, and the random variable ϵ_t has:
  - zero mean. Namely, $\mathbf{E}[\epsilon_t] = 0$.
  - cumulative distribution function $F_\epsilon(x)$. That is,

$$\epsilon_t \sim f_\epsilon(x), \quad F_\epsilon(x) = \int_{-\infty}^{x} f_\epsilon(t)\mathrm{d}t.$$

- $F_\epsilon(x)$ is symmetric, which means $F_\epsilon(-x) = 1 - F_\epsilon(x)$.

Simulation

1. At each time $t > 0$, the market maker has a belief for the price $p_t(v)$ (and thus the outcome, according to *assumption 1*), and set bid and ask prices $b_t \leq a_t$.

2. Each agent will buy at a_t if $w_t > a_t$ (he thinks the security is undervalued) and sell at b_t if $w_t < b_t$ (he thinks the security is overvalued), otherwise he does nothing.

3. If the agent buys, he sends a signal $x_t = +1$; if he does nothing, $x_t = 0$; if he sells, $x_t = -1$.

4. Assume $q_t(v; b_t, a_t)$ to be the probability of receiving signal x_t given bid and ask prices (b_t, a_t) (**Why??**), and a straightforward calculation yields (**Why???**)

$$q_t(v; b_t, a_t) = \begin{cases} 1 - F_\epsilon(a_t - v) & x_t = +1, \\ F_\epsilon(a_t - v) - F_\epsilon(b_t - v) & x_t = 0, \\ F_\epsilon(b_t - v) & x_t = -1. \end{cases}$$

5. The market maker updates its belief $p_t(v)$ for the price based on the following formula:

$$p_{t+1}(v) = \frac{1}{\mathcal{A}_t} p_t(v) q_t(v; b_t, a_t)$$

   where the normalizing constant $\mathcal{A}_t = \int_{-\infty}^{\infty} p_t(v) q_t(v; b_t, a_t) \mathrm{d}v$.

6. For zero-profit (ZP) market maker, the formula for the new bid and ask prices b_{t+1}, a_{t+1} are:

$$b_{t+1} = \frac{\int_{-\infty}^{\infty} v p_t(v) F_\epsilon(b_{t+1} - v) \mathrm{d}v}{\int_{-\infty}^{\infty} p_t(v) F_\epsilon(b_{t+1} - v) \mathrm{d}v}, \quad a_{t+1} = \frac{\int_{-\infty}^{\infty} v p_t(v) F_\epsilon(v - a_{t+1}) \mathrm{d}v}{\int_{-\infty}^{\infty} p_t(v) F_\epsilon(v - a_{t+1}) \mathrm{d}v}.$$

   Solve the two fixed point equations for b_{t+1}, a_{t+1} and we can get the new prices (**Why????**).

2 Prediction of the occurrence of one event

Suppose the occurrence of one event is E. Conversely, $\overline{E}$ means that an event will not happen.

The *sufficient statistic* $\phi(x)$ in this case is $[[x = 1], [\overline{x} = 1]]$. The two values in this vector should add up to 1.

From our last discussion we can see that this is the sufficient statistic of a categorical distribution (with two buckets), and we can easily find its log-partition function. In this section we introduce two parameters, b and δ. The cost function is now

$$C(\mathbf{q}_t) = (1 + \delta) \cdot b \cdot \ln(e^{q_{E,t}/b} + e^{q_{\overline{E},t}/b})$$

In this section we are also introducing two different kinds of prices ASK_t and BID_t.

$$ASK - BID = \delta$$

$$ASK_t = \frac{(1 + \delta) \exp q_{E,t}/b}{\exp q_{E,t}/b + \exp q_{\overline{E},t}/b}$$

$$\overline{ASK}_t = \frac{(1 + \delta) \exp q_{\overline{E},t}/b}{\exp q_{E,t}/b + \exp q_{\overline{E},t}/b}$$

$$BID_t = \frac{(1 - \delta) \exp q_{E,t}/b}{\exp q_{E,t}/b + \exp q_{\overline{E},t}/b}$$

$$\overline{BID}_t = \frac{(1 - \delta) \exp q_{\overline{E},t}/b}{\exp q_{E,t}/b + \exp q_{\overline{E},t}/b}$$

The agent will have a belief s as to what the price should be. If $s > ASK$, then the risk-neutral agent will be inclined to buy. If $s < BID$, then the risk-neutral agent will be inclined to buy. If s is in between these values, then the agent will not do anything.

Upon entering the prediction market, the agent j will have a belief $w_{j,t}$ (in this case the probability that a certain event will occur at time t). However this belief will be adjusted by the prices in the market that he or she observes.

$$x_{j,t} = (1-\theta)\left(\frac{ASK_t - BID_t}{2}\right) + \theta w_{j,t}$$

A risk neutral agent will have the incentive to move the price in the prediction market to match his or her own expectation. However, the trader is also bounded by his or her own budget.

$$q_j^{optimal} = b\ln\left(\frac{x_{j,t}}{1+\delta-x_{j,t}}\right) + q_{\overline{E},t} - q_{E,t}$$

$$q_j^{budget} = b\ln\left[\exp\left(\frac{c+C(\mathbf{q}_t)}{b\cdot(1+\delta)}\right) - \exp\left(\frac{q_{\overline{E},t}}{b}\right)\right] - q_{E,t}$$

The agent will choose $q = \min(q_j^{optimal}, q_j^{budget})$ (*given he or she knows how the prediction market works :/*).

Here is a simple example as to how the model works. Suppose

$$\delta = 0.1$$

$$b = 1$$

Initially $ASK = (1+\delta)/2 = 1.1/2 = 0.55$, $BID = (1-\delta)/2 = 0.45$. Now the first agent comes into the market. He thinks the event will happen at a probability of 0.7, and his belief will be adjusted by weight $\theta = 0.5$. He, (like all the other agents) has a budget 0.50 dollars.

$$x_{j,t} = 0.5 \times \left(\frac{0.55-0.45}{2}\right) + 0.5\times 0.7 = 0.375$$

$$q_j^{optimal} = b\ln\left(\frac{x_{j,t}}{1+\delta-x_{j,t}}\right) + q_{\overline{E},t} - q_{E,t} = \ln\left(\frac{0.375}{1+0.1-0.375}\right) + 0 - 0 = -0.659$$

$$q_j^{budget} = b\ln\left[\exp\left(\frac{c+C(\mathbf{q}_t)}{b\cdot(1+\delta)}\right) - \exp\left(\frac{q_{\overline{E},t}}{b}\right)\right] - q_{E,t} = \ln\left[\exp\left(\frac{0.50+1.1\ln(2)}{1+0.1}\right) - 1\right] - 0 = 0.766$$

There the agent will sell 0.659 shares of the E. **But wait he doesn't have any shares yet oh no!**