

Calculate $p(x; \theta)$ in terms of Gaussian distribution, with known σ^2

$$\begin{aligned}
 P(x; \mu, \sigma^2) &= \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
 &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
 &= \frac{1}{\sqrt{2\pi}} e^{\ln \frac{1}{\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
 &= \frac{1}{\sqrt{2\pi}} e^{-\ln(\sigma) - \frac{(x-\mu)^2}{2\sigma^2}} \\
 &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2} - \frac{\mu^2}{2\sigma^2} + \ln(\frac{1}{\sigma})} \\
 &= \frac{1}{\sqrt{2\pi}} e^{\left[\frac{\mu}{\sigma^2} \quad -\frac{1}{2\sigma^2} \right] \begin{bmatrix} x \\ x^2 \end{bmatrix}^T - \left(\frac{\mu^2}{2\sigma^2} + \ln \sigma \right)}
 \end{aligned}$$

Therefore

$$P(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \exp \left(\left[\begin{array}{c} \frac{\mu}{\sigma^2} \\ -\frac{1}{2\sigma^2} \end{array} \right] \cdot \begin{bmatrix} x \\ x^2 \end{bmatrix} - \left(\frac{\mu^2}{2\sigma^2} + \ln \sigma \right) \right) \quad (1)$$

In Eq (1), counting measure is $1/\sqrt{2\pi}$, natural parameter η is $\begin{bmatrix} \frac{\mu}{\sigma^2} \\ -\frac{1}{2\sigma^2} \end{bmatrix}$, sufficient statistic $t(x)$ is $\begin{bmatrix} x \\ x^2 \end{bmatrix}$, log-partition function $a(\eta)$ is $\left(\frac{\mu^2}{2\sigma^2} + \ln \sigma \right)$. Since

$$\eta_1 = \frac{\mu}{\sigma^2} \quad \eta_2 = -\frac{1}{2\sigma^2},$$

thus,

$$\mu = \frac{\eta_1}{-2\eta_2}.$$

And,

$$\begin{aligned}
 a(\eta) &= \left[\frac{\left(\frac{\eta_1}{2\eta_2} \right)^2}{2 \left(\frac{1}{-2\eta_2} \right)^2} + \ln \sqrt{\frac{1}{-2\eta_2}} \right] = -\frac{1}{2} \eta_1^2 \frac{1}{2\eta_2} - \frac{1}{2} \ln(-2\eta_2) \\
 &= \frac{\eta_1^2}{4\eta_2} - \frac{1}{2} \ln(-2\eta_2)
 \end{aligned}$$

Calculate $p(\theta; b_0)$ in terms of Gaussian distribution, with known σ^2

In this setting, the prior becomes $P(\mu; m, v^2)$ since σ is known. We have

$$\begin{aligned} P(\mu; m, v^2) &= \frac{1}{\sqrt{2\pi}v} \exp\left(-\frac{(\mu - m)^2}{2v^2}\right) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\ln v - \frac{(\mu - m)^2}{2v^2}\right) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\ln v - \frac{\mu^2}{2v^2} + \frac{m}{v^2}\mu - \frac{m^2}{2v^2}\right) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(\left[\begin{array}{c} \frac{m}{v^2} \\ -\frac{1}{2v^2} \end{array}\right] \cdot \left[\begin{array}{c} \mu \\ \mu^2 \end{array}\right] - \left(\ln v + \frac{m^2}{2v^2}\right)\right) \end{aligned}$$

Therefore we have

$$P(\mu; m, v^2) = \frac{1}{\sqrt{2\pi}} \exp\left(\left[\begin{array}{c} \frac{m}{v^2} \\ -\frac{1}{2v^2} \end{array}\right] \cdot \left[\begin{array}{c} \mu \\ \mu^2 \end{array}\right] - \left(\ln v + \frac{m^2}{2v^2}\right)\right)$$

where $h(x) = \frac{1}{\sqrt{2\pi}}$ is the counting measure. As we're not interested in this coefficient, we will ignore it in the following derivation.

Since $P(\mu; m, v^2)$ can also be parametrized by

$$\begin{aligned} P(\mu; n, \nu) &= \exp(\langle n\nu, \theta \rangle + nT(\theta) - \psi(\nu, n)) \\ &= \exp\left(\left[\begin{array}{c} n\nu \\ n \end{array}\right] \cdot \left[\begin{array}{c} \theta \\ T(\theta) \end{array}\right] - \psi(\nu, n)\right), \end{aligned}$$

we can see that

$$\begin{aligned} n &= -\frac{1}{2v^2}, \\ \nu &= -2m \\ \psi(\nu, n) &= \ln v + \frac{m^2}{2v^2}. \end{aligned}$$

Questions and concerns

1. Why is n still not a positive integer?

Ryan: I doubt that $\exp(\langle n\nu, \theta \rangle + nT(\theta) - \psi(\nu, n))$ works only in the case where both μ and σ are unknown since it's the most general form. As long as we fix σ , we should compare the formula we got above with another expression, rather than this general form, to extract the correct n and ν .

2. How does $\left[\begin{array}{c} \theta \\ T(\theta) \end{array}\right]$ map to $\left[\begin{array}{c} \mu \\ \mu^2 \end{array}\right]$?