Conjugate Prior for Univariate Gaussian Distribution with Fixed Variance

Problem Statement

Considering a univariate Gaussian distribution with fixed variance ($\sigma = 1$) that:

$$p(x; \mu) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2}\right),$$

we have already known that its conjugate prior must be

$$p(\mu; m, v) = \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(\mu-m)^2}{2v^2}\right).$$

Since it is an exponential family distribution of general form $p(x;\theta) = \exp\left(\langle\theta, x\rangle - T(\theta)\right)$, its conjugate prior must have the geneal form of $p(\theta; n, \nu) = \exp\left(\langle n\nu, \mu\rangle + nT(\theta) - \psi(\nu, n)\right)$. We wish to calculate n and ν in terms of m and v, in order to finish our simulation of prediction market activities in terms of a univariate Gaussian event with fixed variance.

Derivations

Rewrite Gaussian distribution in exponential family form.

From the traditional Gaussian form we have:

$$\begin{aligned}
p(x; \mu) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2} + \mu x - \frac{\mu^2}{2}\right) \\
&= \frac{\exp\left(-x^2/2\right)}{\sqrt{2\pi}} \exp\left(\mu x - \frac{\mu^2}{2}\right) \\
&= h(x) \exp\left(\langle\mu, x\rangle - T(\mu)\right).
\end{aligned}$$

Therefore we have the base measure and the log-partition function:

$$\begin{aligned}
h(x) &= \frac{\exp(-x^2/2)}{\sqrt{2\pi}}, \\
T(\mu) &= \frac{\mu^2}{2}.
\end{aligned}$$

Decide each element in the prior from general form.

From the traditional form of prior we know:

$$\begin{aligned}
p(\mu; m, v) &= \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(\mu - m)^2}{2v^2}\right) \\
&= \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{1}{2v^2}\mu^2 + \frac{m}{v^2}\mu - \frac{m^2}{2v^2}\right) \\
&= \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{1}{v^2}T(\mu) + \frac{m}{v^2}\mu - \frac{m^2}{2v^2}\right) \\
&= h(\mu) \exp\left(\langle n\nu, \mu\rangle + nT(\mu) - \psi(n, \nu)\right).
\end{aligned}$$

Therefore we know that

$$\begin{aligned}
n &= -\frac{1}{v^2} \\
\nu &= -m \\
\psi(\nu, n) &= \frac{m^2}{2v^2}.
\end{aligned}$$

Validation

We can validate our results by deriving the posterior with respect to N data points from a Gaussian distribution.

Assume that there is a set of N points $S = \{x_i\}_{i=1}^N$ such that

$$p(x_i; \mu) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}} \exp\left(\mu x_i - \frac{\mu^2}{2}\right)$$

and μ follows the exact prior mentioned above. Then we will have the posterior:

$$\begin{aligned}
p(\mu; m, v | x_1, \ldots, x_N) &= \frac{1}{\sqrt{2\pi v}} \exp\left(\begin{bmatrix} m/v^2 \\ -1/v^2 \end{bmatrix} \cdot \begin{bmatrix} \mu \\ T(\mu) \end{bmatrix} - \frac{m^2}{2v^2}\right) \prod_{i=1}^N \frac{\exp(-x^2/2)}{\sqrt{2\pi}} \exp\left(\mu x_i - \frac{\mu^2}{2}\right) \\
&= h(\mu | x_1, \ldots, x_N) \exp\left(\begin{bmatrix} m/v^2 \\ -1/v^2 \end{bmatrix} \cdot \begin{bmatrix} \mu \\ T(\mu) \end{bmatrix} - \frac{m^2}{2v^2}\right) \exp\left(\mu \sum_{i=1}^N x_i - \frac{N}{2}\mu^2\right) \\
&= h(\mu | x_1, \ldots, x_N) \exp\left(\begin{bmatrix} \frac{m}{v^2} + \sum_{i=1}^N x_i \\ \\ -\frac{1}{v^2} - N \end{bmatrix} \cdot \begin{bmatrix} \mu \\ \\ T(\mu) \end{bmatrix} - \frac{m^2}{2v^2}\right) \\
&= h(\mu | x_1, \ldots, x_N) \exp\left(\begin{bmatrix} \frac{m}{v^2} + \sum_{i=1}^N x_i \\ \\ -\frac{1}{v^2} - N \end{bmatrix} \cdot \begin{bmatrix} \mu \\ \\ T(\mu) \end{bmatrix} - \psi(\nu, n)\right)
\end{aligned}$$

which is still a Gaussian distribution.

Therefore from our derivation we can see that for a univariate Gaussian distribution with $\sigma^2 = 1$,

$$n = -\frac{1}{v^2}, \qquad \nu = -m, \qquad \psi(\nu, n) = \frac{m^2}{2v^2}.$$