Oscillations Problem Set

1 Part A

Part A problems are hands-on application problems for students to practice directly applying the concepts and formulas to mathematically understand simple physical scenarios.

Problem A.1. Sinusoid

Verify that $x(t) = A\cos(\omega t) + B\sin(\omega t)$ is a solution to $\ddot{x} = -\omega^2 x$. Find the amplitude of its motion in terms of A and B.

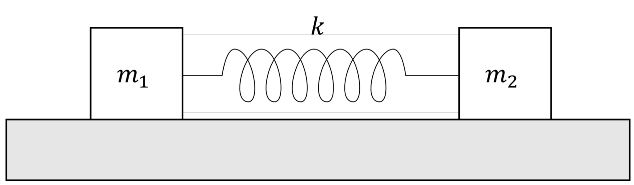
Problem A.2. Max Vibration

A horizontal plate is vertically in simple harmonic motion (SHM) with frequency 20Hz. What is the maximum amplitude of vibration so that the fine sand on the plate always remains in contact with it?

Problem A.3. Pendulum Period

(a) Show that the period of a simple pendulum of length l is $T = 2\pi\sqrt{L/g}$.
(b) Now we consider a physical pendulum. Consider a mass m attached to the end of a uniform rod of mass M and length L. Find the period of its oscillations.

Problem A.4. Unequal Spring Mass



Two masses m_1 and m_2 oscillate with their center of mass stationary. Find the period of the oscillation.

Problem A.5. Floating Oscillations

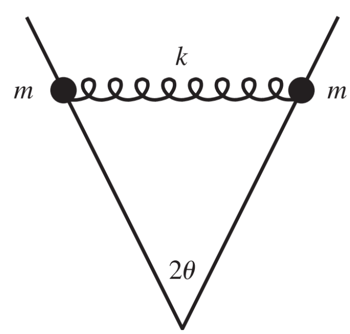
A cube of side length $a = 1$cm and density 500kgm^{-3} is floating on water surface. Given a vertical push, the cube begins exhibiting simple harmonic motion. Find the period of its oscillations.

2 Part B

Intermediate problems like those in Part B tend to be more subtle, but these are the ones, when practiced in bulk, develop physical intuition and a tighter grasp on the abstract formulas and equations.

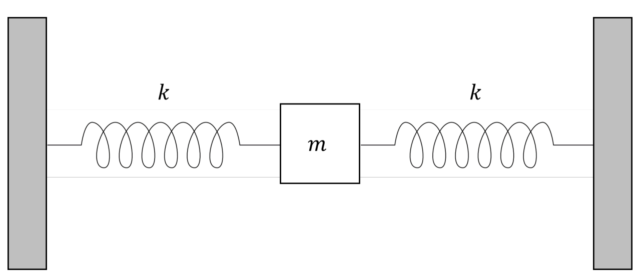
Problem B.1. Falling through the Earth
A tunnel is straight dug from Singapore to New York (which you may assume to be on the diametrically opposite point), through the center of the Earth. Find the time it takes to fall through the length of the tunnel. What if we dug a straight tunnel from Singapore to Malaysia instead?

Problem B.2. Angled Rails



Two particles of mass m are constrained to move along two horizontal frictionless rails that make an angle 2θ with respect to each other. They are connected by a spring with spring constant k, whose relaxed length is at the position shown in the figure above. What is the frequency of oscillations where the spring remains parallel to its initial position shown?

Problem B.3. Removing the spring



The springs in the figure above are at their equilibrium length. The mass oscillates along the line of the springs with amplitude d. At the moment (let this be $t = 0$) when the mass is at position $x = d/2$ (and moving to the right), the right spring is removed. What is the resulting $x(t)$? What is the amplitude of the new oscillation?

Problem B.4. Ratio of Maxima

A mass on the end of a spring is released from rest at position x_0. The experiment is repeated, but now with the system immersed in a fluid that causes the motion to be critically damped. Show that the maximum speed of the mass in the first case is e times the maximum speed in the second case.

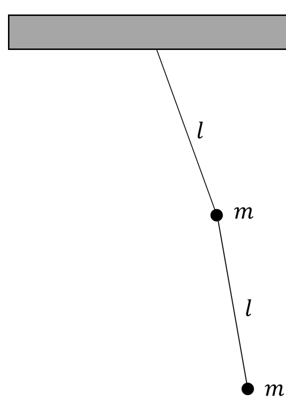
Problem B.5. Rolling around

A solid cylinder of radius r rolls on the inner surface of the bottom half of a solid cylinder with radius R. Find the period of oscillations.

3 Part C

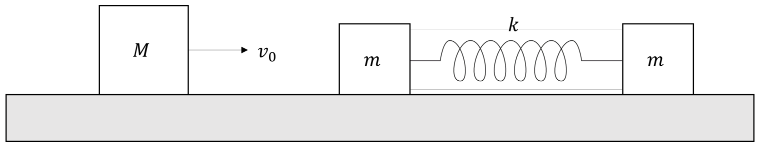
Here in Part C you should find the most conceptually complex and mathematically intensive problems, but you should still savour the process of carefully thinking through questions like these, for this is the essence which makes problem-solving in physics enjoyable. And don't forget to give yourself a pat on the back after solving these!

Problem C.1. Double Pendulum



Find the normal mode frequencies of the double pendulum system above, assuming small angle oscillations. For each frequency, find the behaviour of the normal modes.

Problem C.2. Double Collision

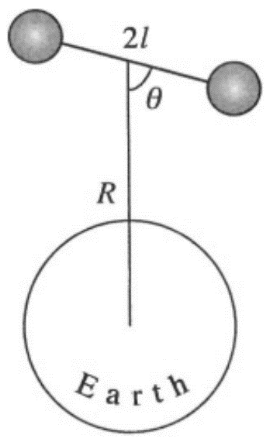


A ball of mass M moving with velocity on a frictionless plane strikes the first of two identical balls, each of mass $m = 2\text{kg}$ connected by a massless spring with spring constant $k = 1\text{Nm}^{-1}$. Consider the collision to be central and elastic and essentially instantaneous.

(a) Find the minimum value of the mass M for the incident ball to strike the system of two balls again.

(b) How much time will elapse between the two collisions?

Problem C.3. Dumbbell Satellite



Consider a dumbbell-shaped satellite consisting of two point masses m connected by a massless rod of length $2l$, much less than R where the rod lies in the plane of the orbit. The orientation of the satellite relative to the direction toward the Earth is measured by angle θ.

(a) Determine the value of θ for the stable orientation of the satellite.

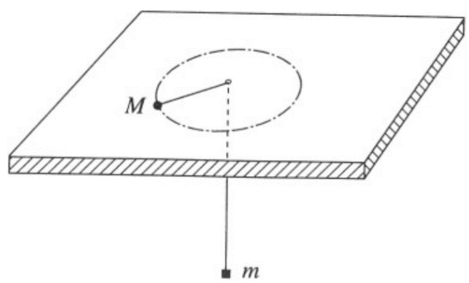
(b) Show that the angular frequency of small-angle oscillations of the satellite about its stable orientation is $\sqrt{3}$ times the orbital angular velocity of the satellite.

Problem C.4. Spring Projectile

A projectile of mass m is fired from the origin at speed v_0 and angle θ. It is attached to the origin by a spring with spring constant k and relaxed length zero.

(a) Find $x(t)$ and $y(t)$
(b) Show that for small $\omega = \sqrt{k/m}$, the trajectory reduces to normal projectile motion. And show that for large ω, the trajectory reduces to simple harmonic motion.
(c) What value should ω take so that the projectile hits the ground when it is moving straight downwards?

Problem C.5. String orbit



A particle of mass M is constrained to move on a horizontal plane. A second particle of mass m is constrained to a vertical line. The two particles are connected by a massless string which passes through a hole in plane. Show that the orbit is stable with respect to small changes in the radius, and find the frequency of small oscillations. Assume that the motion is frictionless.

4 Part D

**Bonus Part D problems span a variety: from quick brain teasers to challenging ones employing even more advanced techniques than those found in Part C. They are usually out of the scope of even the most accelerated high school curriculums, and will likely not be tested for olympiads either. But nonetheless, here's one such problem to keep you entertained if you are done with all the above.*

Problem D.1. Non-linear pendulum
Show that, to second order expansion, the period of oscillations of a simple pendulum is given approximately by:

$$T \approx 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\theta_0^2}{16}\right)$$

.

Problem D.2. Springs on a circle

$N = 2$

$N = 3$

N identical masses m are connected around in a circle with N identical springs with spring constant k. Find the normal modes.

5 Part E

Supplementary problems entirely for your own practice and will not be covered in class.

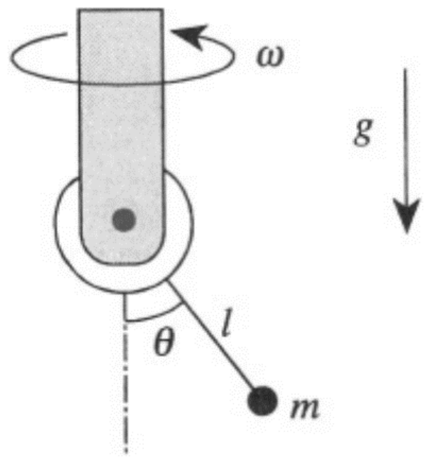
Problem E.1. Child on swing

A child of mass m on a swing raises her center of mass by a small distance b every time the swing passes the vertical position, and lowers her mass by the same amount at each extremal position. Assuming small oscillations, calculate the work done by the child per period of oscillation. Show that the energy of the swing grows exponentially according to $dE/dt = \alpha E$ and determine the constant α.

Problem E.2. Solid vs fluid friction

Consider a brick of mass m lying on a horizontal table attached to a fixed end by a spring with spring constant k. We assume that the spring is elongated enough to set the brick into motion and we denote by $x_0 > 0$ the initial extension of the spring with respect to its equilibrium position. The brick starts at rest from $t = 0$. Both static and kinetic coefficients of friction are μ. Sketch a phase portrait of the ensuing motion, with displacement x on the x-axis and 'velocity' v/ω where $\omega = \sqrt{k/m}$ on the y-axis. Compare this to the phase portrait of an oscillator damped with a fluid friction force: a friction force $f = -\mu_f \dot{x}$ where μ_f is the damping constant.

Problem E.3. Rotating Pendulum



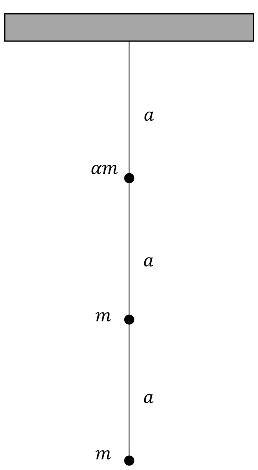
The bearing of a rigid pendulum of mass is forced to rotate uniformly with angular velocity (see figure). The angle between the rotation axis and the pendulum is called θ. Neglect the inertia of the bearing and of the rod connecting it to the mass. Neglect friction. Include the effects of the uniform force of gravity.

(a) Find the differential equation for θ.

(b) At what angular speed ω_c does the stationary point at $\theta = 0$ become unstable?

(c) For $\omega > \omega_c$ what is the stable equilibrium value of θ?

(d) What is the frequency Ω of small oscillations about this point?



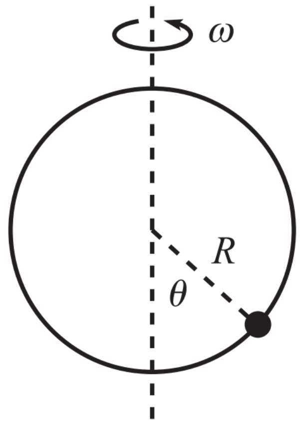
Problem E.4. Triple Pendulum

A triple pendulum consists of masses αm, m and m attached to a single light string at distances $a, 2a$ and $3a$ respectively from its point of suspension.

(a) Determine the value of α such that one of the normal frequencies of this system will equal the frequency of a simple pendulum of length $a/2$ and mass m. You may assume the displacements of the masses from equilibrium to be small.

(b) Find the mode corresponding to this frequency and sketch it.

Problem E.5. Bead on Rotating Hoop



A bead is free to slide along a frictionless hoop of radius R. The hoop rotates with constant angular speed ω around a vertical diameter (see figure).

(a) Find the equation of motion for the angle θ shown.

(b) What are the equilibrium positions?

(c) What is the frequency of small oscillations about the stable equilibrium?

(d) There is one value of ω that is rather special; what is it, and why is it special?