Coadjoint orbits of $\mathrm{SL}(2, \mathbb{C})$

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Let $\mathrm{SL}(2, \mathbb{C})$ denote the Lie group of 2×2 complex matrices with determinant 1, and $\mathfrak{sl}(2, \mathbb{C})$ its Lie algebra. The Killing form of $\mathfrak{sl}(2, \mathbb{C})$ is

$$\kappa(A, B) = \mathrm{tr}(AB)$$

which is nondegenerate. Therefore, if we define $R : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{sl}(2, \mathbb{C})^*$ by

$$R(A)(B) = \kappa(A, B) = \mathrm{tr}(AB)$$

Then R is injective, therefore, it is an isomorphism of vector spaces. We can use R to identify the adjoint and coadjoint orbits. In particular, if $\alpha = R(A)$, then

$$\mathrm{Ad}^*_g(\alpha)(B) = \alpha(\mathrm{Ad}_{g^{-1}} B) = \mathrm{tr}\left(Ag^{-1}Bg\right) = \mathrm{tr}\left(gAg^{-1}B\right) = R(gAg^{-1})(B) = R(\mathrm{Ad}_g(A))(B)$$

That is, $\mathrm{Ad}^*_g \circ R = R \circ \mathrm{Ad}_g$. Therefore, up to identification by R, the adjoint and coadjoint orbits are the same.

Let $A \in \mathfrak{sl}(2, \mathbb{C})$. Note that we can put A into Jordan normal form by conjugating it by some $g \in \mathrm{SL}(2, \mathbb{C})$. Therefore, we can assume that A is in Jordan normal form. Suppose $A \neq 0$. Then we must have

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$$

1 Nilpotent orbit

We will focus on the first case, i.e. the nilpotent orbit. In this case, if

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

we must have $\det(A) = 0$, i.e. $\alpha^2 + \beta\gamma = 0$. Define

$$M = \left\{x^2 + yz = 0\right\} \setminus \{0\} \subseteq \mathbb{C}^3$$

We restrict to the open submanifold, since the origin is a singular point. By the implicit function theorem, we can see that M is a complex surface. We want to study the topology of M. First, notice that M is conical, that is, for $t \in \mathbb{R}$, $t > 0$, $tM = M$. Therefore, we want to first study

$$Z = M \cap S^5$$

where $S^5 \subseteq \mathbb{C}^3$ is the unit ball, since topologically, we have that $M \cong Z \times (0, \infty) \cong Z \times \mathbb{R}$.

Let $W = \mathbb{C}^2 \setminus 0 \subseteq \mathbb{C}^2$, and define $\phi : W \to M$ by

$$\phi(u, v) = (iuv, u^2, v^2)$$

Note that ϕ is a homogeneous polynomial of degree 2, therefore, for $t \in \mathbb{C}$, we have that

$$\phi(tu, tv) = t^2\phi(u, v) \tag{*}$$

Therefore, suffices to consider the restriction of ϕ to S^3, since $\phi(S^3)$ is homeomorphic to Z. By (*), $\phi(-u, -v) = \phi(u, v)$, and ϕ is otherwise injective. Therefore, we get a bijective continuous map

$$\tilde{\phi} : \mathbb{RP}^3 \to Z$$

from a compact space to a Hausdorff space, and so $\tilde{\phi}$ is a homeomorphism. In particular, this means that $M \cong \mathbb{RP}^3 \times \mathbb{R}$.

2 Regular semisimple orbit

Now suppose we have $A = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$, with $\alpha \neq 0$. In this case, we have that the stabiliser of A is the torus

$$T = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \,\middle|\, \lambda \in \mathbb{C}^* \right\}$$

In this case, it is easier to compute the orbit of A by computing the orbit space $\mathrm{SL}(2, \mathbb{C})/T$. Now

$$\begin{pmatrix} z_0 & z_1 \\ z_2 & z_3 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \lambda z_0 & \lambda^{-1} z_1 \\ \lambda z_2 & \lambda^{-1} z_3 \end{pmatrix}$$

Therefore, we see that the ratios $z_0 : z_2$ and $z_1 : z_3$ are fixed. Therefore, we can define a map $\phi : \mathrm{SL}(2, \mathbb{C})/T \to \mathbb{CP}^1 \times \mathbb{CP}^1$ by

$$\phi\left(\begin{pmatrix} z_0 & z_1 \\ z_2 & z_3 \end{pmatrix}\right) = ((z_0 : z_2), (z_1 : z_3))$$

which has inverse on the open submanifold $z_0 z_3 - z_1 z_2 \neq 0$ (which is well defined since this is a bihomogeneous polynomial) given by

$$\psi((z_0 : z_2), (z_1 : z_3)) = \frac{1}{\sqrt{z_0 z_3 - z_1 z_2}} \begin{pmatrix} z_0 & z_1 \\ z_2 & z_3 \end{pmatrix} T$$

which is well defined, since

$$\psi((a z_0 : a z_2), (b z_1 : b z_3)) = \frac{1}{\sqrt{ab(z_0 z_3 - z_1 z_2)}} \begin{pmatrix} a z_0 & b z_1 \\ a z_2 & b z_3 \end{pmatrix} T = \psi((z_0 : z_2), (z_1 : z_3))$$

since it is just multiplication the element of T given by $\lambda = \sqrt{a/b}$. Hence $\mathrm{SL}(2, \mathbb{C})$ is homeomorphic to $\{z_0 z_3 - z_1 z_2 \neq 0\} \subseteq \mathbb{CP}^1 \times \mathbb{CP}^1$, which by the Segre embedding[1], is homeomorphic to an open submanifold of a projective quadric surface.

$$\{Z_0 Z_3 - Z_1 Z_2 = 0, Z_1 - Z_2 \neq 0\} \subseteq \mathbb{CP}^3$$

[1] Which we take to be

$$(Z_0 : Z_1 : Z_2 : Z_3) = (z_0 z_1 : z_0 z_3 : z_2 z_1 : z_2 z_3)$$