Length reduction in diagrams in “Classical nilpotent orbits as hyperkähler quotients”

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August 2, 2023

On page 23 of [1], there is an example of constructing a two step diagram from a three step one, and showing that the first is a quotient of the second. In this note, we will go through the details of this construction.

For simplicity, let us first consider the case of $\mathrm{SL}(n, \mathbb{C})$ orbits. We start off with a diagram of the form

$$\mathbb{C}^a \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \mathbb{C}^{a+b} \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \mathbb{C}^{a+b+c} \tag{1}$$

We can then use this diagram to construct a two step diagram

$$\mathbb{C}^{a+b} \underset{\begin{pmatrix} -\alpha_1 & \beta_2 \end{pmatrix}}{\overset{\begin{pmatrix} \beta_1 \\ \alpha_2 \end{pmatrix}}{\rightleftarrows}} \mathbb{C}^a \oplus \mathbb{C}^{a+b+c} \tag{2}$$

Dimensions

The dimension of the flat space (as a complex vector space) in the first case is

$$2a(a+b) + 2(a+b)(a+b+c)$$

and in the second case, it is

$$2(a + (a+b+c))(a+b)$$

and these are equal.

Group actions

In eq. (1), we have a $\mathrm{U(a)} \times \mathrm{U(a+b)}$ action, by

$$(g_1, g_2) \cdot (\alpha_1, \beta_1, \alpha_2, \beta_2) = (g_2 \alpha_1 g_1^{-1}, g_1 \beta_1 g_2^{-1}, \alpha_2 g_2^{-1}, g_1 \beta_2) \tag{3}$$

and the corresponding moment map is given by

$$\begin{aligned} \mu_c &= (-\beta_1 \alpha_1, \alpha_1 \beta_1 - \beta_2 \alpha_2) \\ \mu_r &= (\beta_1 \beta_1^* - \alpha_1^* \alpha_1, \alpha_1 \alpha_1^* - \beta_1^* \beta_1 + \beta_2 \beta_2^* - \alpha_2^* \alpha_2) \end{aligned}$$

In eq. (2), we have a $\mathrm{U(a+b)}$ action, by

$$g \cdot \left(\begin{pmatrix} \beta_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} -\alpha_1 & \beta_2 \end{pmatrix} \right) = \left(\begin{pmatrix} \beta_1 \\ \alpha_2 \end{pmatrix} \cdot g^{-1}, g \cdot \begin{pmatrix} -\alpha_1 & \beta_2 \end{pmatrix} \right) = \left(\begin{pmatrix} \beta_1 g^{-1} \\ \alpha_2 g^{-1} \end{pmatrix}, \begin{pmatrix} -g\alpha_1 & g\beta_2 \end{pmatrix} \right) \tag{4}$$

Which we can also see is eq. (3) with $g_1 = 1, g_2 = g$. In this case, the moment map is

$$\tilde{\mu}_c = -\begin{pmatrix} -\alpha_1 & \beta_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_2 \end{pmatrix} = \alpha_1\beta_2 - \beta_2\alpha_2$$
$$\tilde{\mu}_r = \begin{pmatrix} -\alpha_1 & \beta_2 \end{pmatrix} \begin{pmatrix} -\alpha_1 & \beta_2 \end{pmatrix}^* - \begin{pmatrix} \beta_1 \\ \alpha_2 \end{pmatrix}^* \begin{pmatrix} \beta_1 \\ \alpha_2 \end{pmatrix} = \alpha_1\alpha_1^* - \beta_1^*\beta_1 + \beta_2\beta_2^* - \alpha_2^*\alpha_2$$

Moreover, we can consider the U(a) action coming from eq. (3), i.e.

$$g \cdot \left(\begin{pmatrix} \beta_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} -\alpha_1 & \beta_2 \end{pmatrix} \right) = \left(\begin{pmatrix} g\beta_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} -\alpha_1 g^{-1} & \beta_2 \end{pmatrix} \right)$$

and the moment map is

$$\overline{\mu}_c = -\beta_1\alpha_1$$
$$\overline{\mu}_r = \beta_1\beta_1^* - \alpha_1^*\alpha_1$$

With all of these, we can see that

$$\frac{\mu^{-1}(0)}{\mathrm{U(a)} \times \mathrm{U(a+b)}} \cong \frac{\tilde{\mu}^{-1}(0)/\mathrm{U(a+b)}}{\mathrm{U(a)}}$$

i.e. the hyperKähler quotient of eq. (1) is the hyperKähler quotient by U(a) of the hyperKähler quotient of eq. (2).

References

[1] Piotr Z. Kobak and Andrew Swann. "Classical nilpotent orbits as hyperkähler quotients". In: *Int. J. Math.* 07.02 (Apr. 1996). Publisher: World Scientific Publishing Co., pp. 193–210. ISSN: 0129-167X. DOI: 10.1142/S0129167X96000116. URL: https://www.worldscientific.com/doi/10.1142/S0129167X96000116 (visited on 07/27/2023).