Killing form

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1 Solvability

In this section, let $\mathfrak{g}$ be a real Lie algebra.

Definition 1.1 (ideal)
A subspace $I \subseteq \mathfrak{g}$ is an *ideal* if $[I, \mathfrak{g}] \subseteq I$.

Remark 1.2. Every ideal is a Lie subalgebra of $\mathfrak{g}$.

Definition 1.3 (simple)
$\mathfrak{g}$ is simple if $\mathfrak{g} \neq 0$, and the only ideals of $\mathfrak{g}$ are 0 and $\mathfrak{g}$.

Definition 1.4 (derived series, solvable)
The *derived series* of $\mathfrak{g}$ is

$$\mathfrak{g}^{(0)} = \mathfrak{g} \quad \text{and} \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$$

Each $\mathfrak{g}^{(i)}$ is an ideal of $\mathfrak{g}$. We say that $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^{(n)} = 0$ for some n.

Definition 1.5 (radical, semisimple)
$\mathfrak{g}$ has a unique maximal solvable ideal, called the *radical* of $\mathfrak{g}$, and denoted $\operatorname{rad}(\mathfrak{g})$. We say that $\mathfrak{g}$ is *semisimple* if $\operatorname{rad}(\mathfrak{g}) = 0$.

Lemma 1.6. Suppose $\mathfrak{g}$ is a complex Lie algebra, $\operatorname{tr}\left(\operatorname{ad}_x \operatorname{ad}_y\right) = 0$ for all $x \in \mathfrak{g}, y \in [\mathfrak{g}, \mathfrak{g}]$. Then $\mathfrak{g}$ is solvable.

2 Killing form

In this section, $\mathfrak{g}$ is a finite dimensional complex Lie algebra.

Definition 2.1 (Killing form)
The *Killing form* of $\mathfrak{g}$ is

$$\kappa(x, y) = \operatorname{tr}\left(\operatorname{ad}_x \operatorname{ad}_y\right)$$

where $\mathrm{ad}_x(y) = [x, y]$, $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the adjoint representation of $\mathfrak{g}$.

Lemma 2.2. κ defines a symmetric bilinear form on $\mathfrak{g}$. Moreover,

$$\kappa([x, y], z) = \kappa(x, [y, z])$$

Definition 2.3 (radical)
The radical of κ is the ideal

$$\mathrm{rad}(\kappa) = \{x \in \mathfrak{g} \mid \kappa(x, y) = 0 \text{ for all } y \in \mathfrak{g}\}$$

Theorem 2.4. The following are equivalent:

(i) $\mathfrak{g}$ is semisimple,

(ii) κ is non-degenerate, that is, $\mathrm{rad}(\kappa) = 0$,

(iii) if $x_1, \ldots, x_n$ is a basis of $\mathfrak{g}$, then $\det\bigl(\kappa(x_i, x_j)\bigr) \neq 0$.

Theorem 2.5. Suppose $\mathfrak{g}$ is semisimple. Then there exists ideals $I_1, \ldots, I_t$ of $\mathfrak{g}$ which are simple (as Lie algebras), such that

$$\mathfrak{g} = I_1 \oplus \cdots \oplus I_t$$

Moreover, each simple ideal of $\mathfrak{g}$ is one of the I_j, and the Killing form of I_j is $\kappa|_{I_j}$.

2.1 Killing form over $\mathbb{R}$

Now suppose instead that $\mathfrak{g}$ is a real Lie algebra.

Definition 2.6 (abelian)
$\mathfrak{g}$ is abelian if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

Lemma 2.7. $\mathfrak{g}$ is semisimple if and only if there are no non-zero abelian ideals of $\mathfrak{g}$.

Lemma 2.8. Any abelian ideal of $\mathfrak{g}$ is contained in $\mathrm{rad}(\kappa)$.

Proof. Let $I \trianglelefteq \mathfrak{g}$ be an abelian ideal, $x \in I$, $y \in \mathfrak{g}$. We want to show that $\kappa(x, y) = 0$. First, note that we have

$$\mathfrak{g} \xrightarrow{\mathrm{ad}_y} \mathfrak{g} \xrightarrow{\mathrm{ad}_x} I \xrightarrow{\mathrm{ad}_y} I \xrightarrow{\mathrm{ad}_x} 0$$

as I is an ideal, and I is abelian. Therefore, we have that $(\mathrm{ad}_x \, \mathrm{ad}_y)^2 = 0$. As any nilpotent endomorphism is tracefree, we must have that $\kappa(x, y) = 0$. $\square$

Theorem 2.9. $\mathfrak{g}$ is semisimple if and only if κ is non-degenerate.

Proof. Suppose κ is non-degenerate. Then we've shown any abelian ideal is contained in $\mathrm{rad}(\kappa) = 0$, therefore we must have that $\mathrm{rad}(\mathfrak{g}) = 0$, i.e. $\mathfrak{g}$ is semisimple.

On the other hand, suppose $\mathrm{rad}(\kappa) \neq 0$. Let $\mathfrak{h}$ be any real Lie algebra. $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{h}$. We can make $\mathfrak{h}_{\mathbb{C}}$ into a complex Lie algebra via

$$[v \otimes \lambda, w \otimes \mu] = [v, w] \otimes (\lambda\mu)$$

With this, we can see that $\mathfrak{h}$ is abelian if and only if $\mathfrak{h}_{\mathbb{C}}$ is abelian, and as $[\mathfrak{h}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}] = [\mathfrak{h}, \mathfrak{h}]_{\mathbb{C}}$, $\mathfrak{h}$ is solvable if and only if $\mathfrak{h}_{\mathbb{C}}$ is solvable. Moreover, by the above definition of the Lie bracket, we can see that the Killing form of $\mathfrak{g}_{\mathbb{C}}$ is the complexification of the Killing form of $\mathfrak{g}$, i.e.

$$\kappa_{\mathbb{C}}(v \otimes \lambda, w \otimes \mu) = \lambda\mu \cdot \kappa(v, w)$$

Therefore, we get that $\mathrm{rad}(\kappa_{\mathbb{C}}) = \mathrm{rad}(\kappa)_{\mathbb{C}}$. In particular, by lemma 1.6, we see that $\mathrm{rad}(\kappa)_{\mathbb{C}}$ is solvable, hence $\mathrm{rad}(\kappa)$ is solvable. Therefore, $\mathfrak{g}$ is not semisimple. □

Moreover, we have a similar result to the complex case, in

Theorem 2.10. Suppose $\mathfrak{g}$ is semisimple. Then there exists ideals $I_1, \dots, I_t$ of $\mathfrak{g}$ which are simple (as Lie algebras), such that

$$\mathfrak{g} = I_1 \oplus \cdots \oplus I_t$$

Moreover, each simple ideal of $\mathfrak{g}$ is one of the I_j, and the Killing form of I_j is $\kappa|_{I_j}$.

2.2 Diagonalisation

Recall Sylvester's law of inertia:

Theorem 2.11 (Sylvester's law of inertia)**.** Let A be a symmetric bilinear form on a finite dimensional real vector space V. Then there exists a basis of V such that

$$[A] = \begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0 \end{pmatrix}$$

In the complex case, we can get

Corollary 2.12. Let A be a symmetric bilinear form on a finite dimensional complex vector space V. Then there exists a basis of V such that

$$[A] = \begin{pmatrix} I_{p+q} & \\ & 0 \end{pmatrix}$$

Note however a general symmetric bilinear form on a complex vector space will *not* be positive definite, since $A(iv, iv) = -A(v, v)$.