Finite and hyperKähler quotients in "Classical nilpotent orbits as hyperkähler quotients"

Shing Tak Lam

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In this note, we study in detail sections 3.2 and 3.3 of "Classical nilpotent orbits as hyperkähler quotients" [1].

Lemma ([1, Lemma 3.1])**.** Let H be a Lie group acting on $\mathbb{H}^N$ preserving the complex structures. Let $\mathbb{H}^*$ act on $\mathbb{H}^N$ by right multiplication, and $\operatorname{Im}(\mathbb{H})$ by conjugation. Then the map

$$\mu : \mathbb{H}^N \to \mathfrak{h}^* \otimes \operatorname{Im}(\mathbb{H}) \mu^X(q) = -\overline{q}^t X q$$

is the unique moment map for this action which is equivariant with respect to the $\mathbb{H}^*$ action.

We will use the notation from [1, p. 19] that for Lie groups G, H, $G \sim H$ if some finite covers of G, H are isomorphic Lie groups.

O(2) **as a double cover of** U(1)

First of all, notice that we can always assume without loss of generality that $n_1 < \cdots < n_k$. In this case, U(1) and O(2) can only occur as the smallest groups in the diagrams for orbits. Moreover, we never have U(m) $\sim$ Sp(n), so the diagrams have length 2. That is, we have the diagram

$$\mathbb{C} \rightleftarrows \mathbb{C}^m$$

for an $\mathfrak{sl}(m, \mathbb{C})$ orbit, and the diagram

$$\mathbb{C}^2 \rightleftarrows \mathbb{C}^{2n}$$

for an $\mathfrak{sp}(n, \mathbb{C})$ orbit. We also want the flat spaces to have the same dimension, which is $2m$ in the $\mathfrak{sl}(M, \mathbb{C})$ case, and $4n$ in the $\mathfrak{sp}(N, \mathbb{C})$ case. Therefore, we must have that $m = 2n$.

The smallest nontrivial nilpotent orbit in $\mathfrak{sl}(2n, \mathbb{C})$ is the orbit with Jordan type $[2, 1^{2n-2}]$, which has rank 1. This is given by the $U(1)$ quotient of the diagram

$$\mathbb{C} \rightleftarrows \mathbb{C}^{2n}$$

Similarly, if we take the O(2) quotient of the diagram

$$\mathbb{C}^2 \rightleftarrows \mathbb{C}^{2n}$$

we get a nilpotent orbit in $\mathfrak{sp}(n, \mathbb{C})$. As groups acting on $\mathbb{C} \cong \mathbb{R}^2$, we have that U(1) $\cong$ SO(2) $\leq$ O(2).

References

[1] Piotr Z. Kobak and Andrew Swann. "Classical nilpotent orbits as hyperkähler quotients". In: *Int. J. Math.* 07.02 (Apr. 1996). Publisher: World Scientific Publishing Co., pp. 193–210. ISSN: 0129-167X. DOI: 10.1142/S0129167X96000116. URL: https://www.worldscientific.com/doi/10.1142/S0129167X96000116 (visited on 07/27/2023).