

# Kähler structures on coadjoint orbits of $SU(n)$ and $SL(n, \mathbb{C})$

Shing Tak Lam

August 9, 2023

In this note, we will consider coadjoint orbits of  $SU(n)$  and  $SL(n, \mathbb{C})$ , and show that they are Kähler manifolds.

# Contents

<b>1</b>	<b>Preliminaries</b>	<b>3</b>
1.1	Adjoint and Coadjoint Representations . . . . .	3
1.2	Kähler manifolds . . . . .	4
1.2.1	Cotangent space, forms and exterior derivative . . . . .	4
1.2.2	Exterior derivative . . . . .	5
1.2.3	Riemannian metric and symplectic form . . . . .	6
1.2.4	Complex structure . . . . .	6
1.2.5	Kähler structure . . . . .	6
1.3	Quotient manifolds . . . . .	7
1.3.1	Matrix Lie Groups . . . . .	7
1.3.2	Orbits . . . . .	7
<b>2</b>	<b><math>SU(n)</math> orbits</b>	<b>9</b>
2.1	Root decomposition . . . . .	9
2.2	Tangent space and Diagonalisation . . . . .	9
2.2.1	Diagonalisation and Stabilisers of the adjoint action . . . . .	9
2.2.2	Tangent space . . . . .	10
2.2.3	General case . . . . .	10
2.3	Kirillov-Kostant-Souriau symplectic form . . . . .	10
2.3.1	$\omega$ is well defined . . . . .	11
2.3.2	$\omega$ is non-degenerate . . . . .	11
2.3.3	$\omega$ is closed . . . . .	11
2.4	Kähler structure . . . . .	12
2.4.1	Complex quotient . . . . .	12
2.4.2	Reduction to a diagonal element . . . . .	13
2.4.3	At a diagonal element . . . . .	14
2.4.4	General case . . . . .	15
<b>3</b>	<b><math>SL(n, \mathbb{C})</math> orbits</b>	<b>16</b>
3.1	HyperKähler manifolds . . . . .	16
3.2	Flat hyperKähler space . . . . .	16

# Chapter 1

## Preliminaries

### 1.1 Adjoint and Coadjoint Representations

Define the Lie algebra

$$\mathfrak{su}(n) = \{X \in \text{Mat}(n, \mathbb{C}) \mid X^* + X = 0, \text{tr}(X) = 0\} \quad (1.1)$$

where  $X^*$  is the conjugate transpose of  $X$ , and with the Lie bracket being the matrix commutator. We can define the adjoint representation of  $\text{SU}(n)$  as

$$\begin{aligned} \text{Ad} : \text{SU}(n) &\rightarrow \text{GL}(\mathfrak{su}(n)) \\ \text{Ad}_a(X) &= aXa^{-1} \end{aligned}$$

Taking the dual representation, we get the coadjoint representation, which is

$$\begin{aligned} \text{Ad}^* : \text{SU}(n) &\rightarrow \text{GL}(\mathfrak{su}(n)^*) \\ \text{Ad}_a^*(\beta)(X) &= \langle \beta, \text{Ad}_{a^{-1}}(X) \rangle \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  is used here to denote the pairing  $\mathfrak{su}(n)^* \times \mathfrak{su}(n) \rightarrow \mathbb{R}$ . The definition above is used to ensure that  $\text{Ad}^*$  is a group homomorphism, i.e.

$$\text{Ad}_{ab}^* = \text{Ad}_a^* \circ \text{Ad}_b^*$$

whereas if we simply take the dual map, we would get  $(\text{Ad}_{ab})^* = (\text{Ad}_b)^* \circ (\text{Ad}_a)^*$ .

We will use the same notation for the inner product on  $\mathfrak{su}(n)$ , which should not be an issue as the inner product defines a natural isomorphism. Now note that  $-\kappa$ , where  $\kappa$  is the Killing form, defines an inner product

$$\langle X, Y \rangle = -\text{tr}(XY) = \text{tr}(XY^*)$$

on  $\mathfrak{su}(n)^1$ , which means that we have a natural isomorphism

$$\begin{aligned} \Phi : \mathfrak{su}(n) &\rightarrow \mathfrak{su}(n)^* \\ X &\mapsto \langle X, \cdot \rangle \end{aligned}$$

With this, suppose  $\beta = \Phi(B)$ , then

$$\text{Ad}_a^*(\beta)(X) = \langle B, \text{Ad}_{a^{-1}}(X) \rangle = -\text{tr}(Ba^{-1}Xa) = -\text{tr}(aBa^{-1}X) = \Phi(\text{Ad}_a(B))(X)$$

Therefore, the following diagram commutes

---

<sup>1</sup>In fact,  $\langle A, B \rangle = \text{tr}(AB^*)$  defines a Hermitian inner product on the space of complex matrices.

$$\begin{array}{ccc}
\mathfrak{su}(n) & \xrightarrow{\text{Ad}_g} & \mathfrak{su}(n) \\
\downarrow \Phi & & \downarrow \Phi \\
\mathfrak{su}(n)^* & \xrightarrow{\text{Ad}_g^*} & \mathfrak{su}(n)^*
\end{array}$$

or equivalently,  $\Phi$  defines an isomorphism of representations between  $\text{Ad}$  and  $\text{Ad}^*$ . This means that in the remainder of this note, we will focus on the adjoint representation instead.

If we differentiate  $\text{Ad}$  at the identity, we get the representation

$$\begin{aligned}
\text{ad} : \mathfrak{su}(n) &\rightarrow \mathfrak{gl}(\mathfrak{su}(n)) \\
\text{ad}_X(Y) &= [X, Y]
\end{aligned}$$

Some properties of  $\text{Ad}$ ,  $\text{ad}$  and the inner product which we will need are:

- $\text{Ad}_g$  is an isometry, that is, for all  $X, Y \in \mathfrak{su}(n)$ ,

$$\langle \text{Ad}_g(X), \text{Ad}_g(Y) \rangle = \langle X, Y \rangle$$

- The Jacobi identity: if  $X, Y, Z \in \mathfrak{su}(n)$ , then

$$[X, [Y, Z]] + [Y, [X, Z]] + [Z, [X, Y]] = 0$$

- Associativity: for  $X, Y, Z \in \mathfrak{su}(n)$ ,  $\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle$ .

## 1.2 Kähler manifolds

Let  $M$  be a manifold. In this section, we will define a Kähler structure on  $M$ .

Throughout, if  $V, W$  are vector spaces,

1.  $V \otimes W$  is the tensor product of  $V$  with  $W$ ,
2.  $v \cdot w := v \otimes w + w \otimes v$  is the symmetric product of  $v$  and  $w$ ,  $S^2V$  is the subspace of  $V \otimes V$  spanned by  $\{v \cdot w \mid v, w \in V\}$ ,
3.  $v \wedge w := v \otimes w - w \otimes v$  is the exterior product of  $v$  and  $w$ ,  $\Lambda^2V$  is the subspace of  $V \otimes V$  spanned by  $\{v \wedge w \mid v, w \in V\}$ ,
4.  $V^* \otimes W^*$  defines a bilinear form on  $V \times W$ , via

$$(\alpha \otimes \beta)(v, w) = \alpha(v)\beta(w)$$

In particular,  $S^2V^*$  is the space of symmetric bilinear forms,  $\Lambda^2V^*$  is the space of alternating bilinear forms.

### 1.2.1 Cotangent space, forms and exterior derivative

Let  $p \in M$  be a point,  $x_1, \dots, x_n$  be local coordinates near  $p$ . Then we have the tangent space  $T_pM$ , which has basis

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$$

In particular, if  $\phi$  is a parametrisation of  $M$  with local coordinates  $x_1, \dots, x_n$ , then we have

$$\frac{\partial}{\partial x_j} := \frac{\partial \phi}{\partial x_j}$$

**Definition 1.2.1** (cotangent space)

The cotangent space of  $M$  at  $p$  is  $T_p^*M$ , which is the dual space to  $T_pM$ . We will write  $dx^j$  for the dual basis to  $\frac{\partial}{\partial x_j}$ .

**Definition 1.2.2** ( $k$ -form)

A 1-form  $\alpha$  on  $M$  is a smooth field of cotangent vectors. That is, for each  $p \in M$ , we have  $\alpha_p \in T_p^*M$ . More generally, a  $k$ -form  $\alpha$  has  $\alpha_p \in \wedge^k T_p^*M$ .

If  $\alpha$  is a 1 form,  $V$  is a vector field on  $M$ , then we define the smooth function  $\alpha(V) : M \rightarrow \mathbb{R}$  by

$$(\alpha(V))(p) = \alpha_p(V_p)$$

and we can make a similar definition with  $k$ -forms and  $k$  vector fields.

**1.2.2 Exterior derivative**

In terms of local coordinates, a 1-form  $\alpha$  can be written as

$$\alpha_p = \sum_{j=1}^n f_j(p) dx_j \quad (1.2)$$

where  $f_j : M \rightarrow \mathbb{R}$  are smooth. In the remainder of this subsection, we will define objects in terms of local coordinates. We will omit the proofs that these are independent of the choice of coordinates.

Let  $f : M \rightarrow \mathbb{R}$  be a smooth map. Then the exterior derivative of  $f$  is the 1-form  $df$ , given in local coordinates by

$$(df)_p = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j$$

Suppose  $X$  is a vector field. Then we write  $X(f) := df(X)$  for the smooth function  $M \rightarrow \mathbb{R}$ . If  $\alpha$  is a 1-form as in eq. (1.2), then  $d\alpha$  is the 2-form defined by

$$(d\alpha)_p = \sum_{j=1}^n \sum_{k=1}^n \frac{\partial f_j}{\partial x_k} dx_k \wedge dx_j$$

We will also need the exterior derivative of a 2-form, which is defined similarly to the above, but we will not need it explicitly.

**Definition 1.2.3** (Lie bracket of vector fields)

Suppose we have vector fields  $V, W$ , which is given by

$$[V, W](f) = V(W(f)) - W(V(f))$$

for all  $f : M \rightarrow \mathbb{R}$  smooth.

We will only need this definition for the following result.

**Lemma 1.2.4.** Let  $\alpha$  be a 2 form on  $M$ ,  $U, V, W$  vector fields on  $M$ . Then at all  $p \in M$ ,

$$\begin{aligned} d\alpha(U, V, W) &= U(\alpha(V, W)) - V(\alpha(U, W)) + W(\alpha(U, V)) \\ &\quad - \alpha([U, V], W) + \alpha([U, W], V) - \alpha([V, W], U) \end{aligned}$$

### 1.2.3 Riemannian metric and symplectic form

#### Definition 1.2.5 (Riemannian metric)

A Riemannian metric  $g$  on  $M$  is given on each  $T_p M$  by a positive definite symmetric bilinear form

$$g_p \in S^2 T_p^* M$$

#### Definition 1.2.6 (Symplectic form)

A symplectic form  $\omega$  on  $M$  is a non-degenerate 2-form, i.e. on each  $T_p M$ , we have a non-degenerate alternating bilinear form

$$\omega_p \in \Lambda^2 T_p^* M$$

Moreover, we require that  $\omega$  is closed, i.e.  $d\omega = 0$ .

### 1.2.4 Complex structure

Now suppose in addition that  $M$  is even dimensional, that we have local coordinates  $x_1, \dots, x_n, y_1, \dots, y_n$ , such that if we have a change of coordinates

$$(\tilde{x}_j, \tilde{y}_j) = F(x_j, y_j)$$

Then  $F$  is a holomorphic function in terms of the complex coordinates

$$z_j = x_j + iy_j \quad \tilde{z}_j = \tilde{x}_j + i\tilde{y}_j$$

Equivalently, the change of coordinates satisfies the Cauchy-Riemann equations

$$\frac{\partial \tilde{x}_j}{\partial x_k} = \frac{\partial \tilde{y}_j}{\partial y_k} \quad \frac{\partial \tilde{x}_j}{\partial y_k} = -\frac{\partial \tilde{y}_j}{\partial x_k}$$

for all  $j, k = 1, \dots, n$ . In this case, we call  $M$  a *complex manifold*, and  $z_j$  the local complex coordinates.

#### Definition 1.2.7 (almost complex structure)

The almost complex structure  $J$  is for each  $p \in M$ , a linear map  $J_p : T_p M \rightarrow T_p M$  for each  $p \in M$ , sending

$$J_p \left( \frac{\partial}{\partial x_j} \right) = \frac{\partial}{\partial y_j} \quad J_p \left( \frac{\partial}{\partial y_j} \right) = -\frac{\partial}{\partial x_j}$$

In fact,  $J$  is independent of the choice of local coordinates.

### 1.2.5 Kähler structure

#### Definition 1.2.8 (Kähler manifold)

A Kähler manifold  $M$  is a complex manifold, with a Riemannian metric  $g$ , symplectic form  $\omega$  and almost complex structure  $J$ , such that

$$\omega_p(u, v) = g_p(J_p(u), v)$$

for all  $p \in M, u, v \in T_p M$ .

## 1.3 Quotient manifolds

Let  $G \subseteq \text{GL}(n, \mathbb{C})$  be a matrix Lie group,  $P$  a closed Lie subgroup. We would like to show that  $G/P$ , with the quotient topology, is a manifold.

Choose local coordinates  $x_1, \dots, x_k$  for  $P$  near  $I$ , and extend this to local coordinates  $x_1, \dots, x_k, y_1, \dots, y_\ell$  for  $G$  near  $I$ . Let  $Y = \{x_1 = \dots = x_k = 0\}$ . Then  $Y$  is a submanifold of  $G$ , with local coordinates  $y_1, \dots, y_\ell$ . We will show that there exists a neighbourhood  $V$  of  $I$  in  $Y$ , such that  $VP = \{vP \mid v \in V\}$  is open in  $G$ .

For this, consider the map

$$\begin{aligned} m : Y \times P &\rightarrow G \\ m(y, p) &= yp \end{aligned}$$

Then

$$dm_{(I, I)}(X, Y) = (Y, X)$$

where we use the isomorphism

$$T_I G = T_I P \oplus T_I Y$$

Hence by the inverse function theorem, we have an open neighbourhood  $V \times W$  of  $(I, I)$  in  $Y \times P$ , such that  $m$  is a diffeomorphism onto its image. But then this means that

$$VP = \bigcup_{p \in P} m(V \times W)p$$

is an open subset of  $G$ .

Let  $\pi : G \rightarrow G/P$  denote the quotient map. Let  $\tilde{V} = \pi(V)$ . Then  $\tilde{V}$  is open, and  $\pi : V \rightarrow \tilde{V}$  is a homeomorphism. In particular, this means that  $y_1, \dots, y_\ell$  induce local coordinates on  $G/P$  on  $\tilde{V} \ni P$ .

Next, notice that  $G$  acts on  $G/P$  by left multiplication, i.e.  $h \cdot (gP) = hgP$ . The  $G$  action is by homeomorphisms, therefore by left translation we can define local coordinates on  $G/P$  on  $g\tilde{V} \ni gP$ . We need to show that the transition maps are smooth.

We can assume without loss of generality that one of the charts is  $\tilde{V}$ . Let  $g \in G$ , and suppose  $g\tilde{V} \cap \tilde{V}$  is nonempty. But in this case, by the definition of the charts, we have that the transition map is the same as the local coordinate representation of left multiplication by  $g$ , which is smooth. An immediate result from the definition of the charts is

**Proposition 1.3.1.** The action of  $G$  on  $G/P$  by left multiplication is by diffeomorphisms.

### 1.3.1 Matrix Lie Groups

Moreover, suppose  $\mathfrak{g}$  is the Lie algebra of  $G$ ,  $\mathfrak{p}$  the Lie algebra of  $P$ . Choose a vector space complement  $Z$  of  $\mathfrak{p}$  in  $\mathfrak{g}$ , i.e.  $\mathfrak{g} = \mathfrak{p} \oplus Z$ . Then

$$\begin{aligned} \exp : \mathfrak{g} &\rightarrow G \\ X &\mapsto \sum_{k=0}^{\infty} \frac{X^k}{k!} \end{aligned}$$

is a diffeomorphism between neighbourhoods of  $0 \in \mathfrak{g}$  and  $I \in G$ . By the above, this means that  $\pi \circ \exp : Z \rightarrow G/P$  defines local coordinates on  $G/P$  near  $P$ .

### 1.3.2 Orbits

Now suppose  $G$  acts on a manifold  $M$  by diffeomorphisms, such that  $\text{Stab}(m)$  is a Lie subgroup of  $G$ . Then by the orbit stabiliser theorem, we have a bijection

$$\text{Orb}(m) \leftrightarrow G/\text{Stab}(m)$$



Let  $\alpha(g) = g \cdot m$  be the map  $G \rightarrow M$  from the  $G$  action, and  $\tilde{\alpha} : G/\text{Stab}(m) \rightarrow M$  the induced map. We would like to show  $\ker(d\alpha_I) = \mathfrak{s}$ , where  $\mathfrak{s}$  is the Lie algebra for  $\text{Stab}(m)$ . This would then imply that  $\tilde{\alpha}$  is an immersion.  $\mathfrak{s} \subseteq \ker(d\alpha_I)$  is clear, since  $\alpha|_{\text{Stab}(m)}$  is constant.

Now suppose  $X \in \ker(d\alpha_I)$ . Consider  $f(t) = \alpha(\exp(tX))$ . Then  $df_0 = d\alpha_I(X) = 0$ , and so  $f(t) = m$  for all  $t$ , i.e.  $\exp(tX) \in \text{Stab}(m)$  for all  $t$ . So  $X \in \mathfrak{s}$ . As  $\tilde{\alpha}$  is also a bijection, it must then be a diffeomorphism onto its image.

Using the same notation as in section 1.3.1, this means that we have a (local) parametrisation of the orbits, given by

$$\tilde{\alpha} \circ \pi \circ \exp = \alpha \circ \exp : Z \rightarrow \text{Orb}(m) \tag{1.3}$$

## Chapter 2

# SU(n) orbits

### 2.1 Root decomposition

In this section, we will consider the root decomposition of  $\mathfrak{sl}(n, \mathbb{C})$  and use this to derive the root decomposition of  $\mathfrak{su}(n)$ . Humphreys [2, §8] contains this, and much more.

Consider the Lie algebra  $\mathfrak{sl}(n, \mathbb{C})$  of trace free  $n \times n$  complex matrices. Then we have the Cartan subalgebra  $\mathfrak{t}$  of diagonal matrices. Let  $E_{ij}$  be the standard basis matrices for  $\text{Mat}(n, \mathbb{C})$ ,  $B \in \mathfrak{t}$ . Say

$$B = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix}$$

Then  $[B, E_{ij}] = (b_i - b_j)E_{ij}$ . This means that we have the eigendecomposition

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{t} \oplus \bigoplus_{1 \leq i, j \leq n, i \neq j} \mathbb{C}E_{ij} \quad (2.1)$$

In particular, if we restrict this to the subalgebra  $\mathfrak{su}(n)$ , we get the decomposition

$$\mathfrak{su}(n) = \tilde{\mathfrak{t}} \oplus \bigoplus_{1 \leq i < j \leq n} (\mathbb{R}(E_{ij} - E_{ji}) \oplus i\mathbb{R}(E_{ij} + E_{ji})) \quad (2.2)$$

where  $\tilde{\mathfrak{t}} = \mathfrak{t} \cap \mathfrak{su}(n)$  is the subalgebra of  $\mathfrak{su}(n)$  of diagonal matrices.

### 2.2 Tangent space and Diagonalisation

#### 2.2.1 Diagonalisation and Stabilisers of the adjoint action

First of all, we note that elements of  $\mathfrak{su}(n)$  are skew-hermitian, hence diagonalisable by an element of  $\text{SU}(n)$ <sup>1</sup>. With this, we can classify the coadjoint orbits based off a diagonal element in the orbit. Consider

$$A = \begin{pmatrix} i\lambda_1 I_{m_1} & & \\ & \ddots & \\ & & i\lambda_k I_{m_k} \end{pmatrix}$$

where  $I_m$  is the  $m \times m$  identity matrix,  $\lambda_j \in \mathbb{R}$ , with  $\lambda_1 > \lambda_2 > \dots > \lambda_k$ ,  $m_1 + \dots + m_k = n$  and  $m_1\lambda_1 + \dots + m_k\lambda_k = 0$ . In this case, by the orbit stabiliser theorem, we have a bijection

$$\text{Orb}(A) \leftrightarrow \text{SU}(n)/\text{Stab}(A)$$

where  $\text{Stab}(A)$  is the stabiliser of  $A$  under the adjoint action. In this case, we have that the stabiliser is the block diagonal subgroup

$$\text{Stab}(A) = \text{S}(\text{U}(m_1) \times \dots \times \text{U}(m_k))$$

---

<sup>1</sup>From standard linear algebra arguments, we know that they are  $\text{U}(n)$ -diagonalisable. But if  $PBP^{-1}$  is diagonal, then so is  $(\lambda P)B(\lambda P)^{-1}$ , and by choosing  $\lambda$  appropriately,  $\lambda \in \text{SU}(n)$ .

where we consider  $U(m_1) \times \cdots \times U(m_k) \leq SU(n)$  as the block diagonal subgroup, and

$$S(U(m_1) \times \cdots \times U(m_k)) = (U(m_1) \times \cdots \times U(m_k)) \cap SU(n)$$

the subgroup with determinant 1.

## 2.2.2 Tangent space

Let  $M$  be an adjoint orbit. We will now focus on the generic case, that is, the eigenvalues of  $A$  are distinct. In this case, we have that

$$\text{Stab}(A) \cong T^{n-1}$$

is the torus of diagonal matrices in  $SU(n)$ . In this case, the parametrisation eq. (1.3) is given by

$$\phi(X) = \exp(X)A\exp(X)^{-1}$$

and we find that  $d\phi_0(X) = XA - AX = -\text{ad}_A(X)$ . Using the root decomposition from eq. (2.2), we have that

$$\text{ad}_A(\tilde{\mathfrak{t}}) = 0 \quad \text{ad}_A(E_{ij}) = i(\lambda_i - \lambda_j)E_{ij}$$

and so, the tangent space is given by

$$T_A M = \frac{\mathfrak{su}(n)}{\tilde{\mathfrak{t}}} = \bigoplus_{1 \leq i < j \leq n} (\mathbb{R}(E_{ij} - E_{ji}) \oplus i\mathbb{R}(E_{ij} + E_{ji})) = \{\text{ad}_A(X) \mid X \in \mathfrak{su}(n)\}$$

For  $a \in SU(n)$ ,  $\text{Ad}_a : \mathfrak{su}(n) \rightarrow \mathfrak{su}(n)$  is a diffeomorphism. Therefore, if  $B = \text{Ad}_a(A)$ , then

$$T_B M = \text{Ad}_a(T_A M) = \{\text{ad}_B(X) \mid X \in \mathfrak{su}(n)\}$$

where we use the fact that

$$\text{ad}_{\text{Ad}_a(A)} = \text{Ad}_a \circ \text{ad}_A \circ \text{Ad}_{a^{-1}}$$

and that  $\text{Ad}_{a^{-1}}$  is a bijection.

## 2.2.3 General case

If we do not assume the eigenvalues are distinct, then  $\text{Stab}(A)$  will have Lie algebra

$$\mathfrak{s} = (\mathfrak{u}(m_1) \oplus \cdots \oplus \mathfrak{u}(m_k)) \cap \mathfrak{su}(n)$$

In particular, we will still have that

$$T_A M = \frac{\mathfrak{su}(n)}{\mathfrak{s}}$$

and in general,

$$T_B M = \{\text{ad}_B(X) \mid X \in \mathfrak{su}(n)\}$$

## 2.3 Kirillov-Kostant-Souriau symplectic form

The statement here is from [4] Chapter 14, although modified to define  $\omega$  on an adjoint orbit, instead of a coadjoint orbit. The proof is from [1, Section II.1.d], where we use the Jacobi identity to show that the form is closed.

**Theorem 2.3.1** (Kirillov-Kostant-Souriau, [4, Theorem 14.4.1]). Let  $M \subseteq \mathfrak{su}(n)$  be an adjoint orbit. Define the 2-form  $\omega$  on  $M$  by

$$\omega_B(\text{ad}_X(B), \text{ad}_Y(B)) = -\langle B, [X, Y] \rangle = \text{tr}(B[X, Y]) \quad (2.3)$$

for all  $B \in M, X, Y \in \mathfrak{su}(n)$ . Then  $\omega$  is a symplectic form on  $M$ .

From the definition, we can see that  $\omega_B$  is antisymmetric, so all we need to check is that it is well defined, non-degenerate and closed.

### 2.3.1 $\omega$ is well defined

The adjoint representation  $\text{ad}$  of  $\mathfrak{su}(n)$  may have a nontrivial kernel, which means that we need to check that eq. (2.3) is independent of the choice of  $X, Y$ . Suppose we have  $Z \in \mathfrak{su}(n)$  such that  $\text{ad}_X(B) = \text{ad}_Z(B)$ , i.e.  $[X, B] = [Z, B]$ . Then for all  $Y \in \mathfrak{su}(n)$ , we have that

$$\langle B, [X, Y] \rangle = \langle [B, X], Y \rangle = \langle [B, Z], Y \rangle = \langle B, [Z, Y] \rangle$$

and so  $\omega$  is independent of the choice of  $X$ .

### 2.3.2 $\omega$ is non-degenerate

Now suppose  $X \in \mathfrak{su}(n)$  is such that  $\omega_B(\text{ad}_X(B), \text{ad}_Y(B)) = 0$  for all  $Y \in \mathfrak{su}(n)$ . That is,

$$\langle B, [X, Y] \rangle = 0$$

for all  $Y \in \mathfrak{su}(n)$ . But by associativity,

$$\langle B, [X, Y] \rangle = \langle [B, X], Y \rangle = -\langle \text{ad}_X(B), Y \rangle$$

for all  $Y \in \mathfrak{su}(n)$ . Therefore, we must then have that  $\text{ad}_X(B) = 0$ .

### 2.3.3 $\omega$ is closed

For  $X \in \mathfrak{su}(n)$ , we will write  $X_B^\# = \text{ad}_X(B)$  for the vector field generated by  $X$ .

**Lemma 2.3.2.**

$$[X, Y]^\# = [X^\#, Y^\#]$$

*Proof.* To show that the two vector fields are the same, suffices to show they act on functions in the same way. Moreover, for  $C \in \mathfrak{su}(n)$ , we can define  $f : \mathfrak{su}(n) \rightarrow \mathbb{R}$ , by

$$f(B) = \langle C, B \rangle$$

In this case,  $\text{grad}(f) = C$ . In particular, this means that we only need to check that the vector fields act the same way on functions of the above form.  $f : \mathfrak{su}(n) \rightarrow \mathbb{R}$  is linear, therefore we have that

$$df_B(H) = \langle C, H \rangle$$

and so  $X^\#(f)(B) = \langle C, [X, B] \rangle$ . Moreover, the function

$$B \mapsto \langle C, [X, B] \rangle$$

is also linear, so we have that

$$d(X^\#(f))_B(H) = \langle C, [X, H] \rangle$$

Combining these, we need to show

$$\langle C, [[X, Y], B] \rangle = \langle C, [[Y, X], B] \rangle - \langle C, [[X, Y], B] \rangle$$

But

$$[Y, [X, B]] - [X, [Y, B]] = [Y, [X, B]] + [X, [B, Y]] \stackrel{(*)}{=} -[B, [Y, X]] = [B, [X, Y]]$$

where  $(*)$  follows from the Jacobi identity. □

Using this, and lemma 1.2.4, we have that

$$\begin{aligned} d\omega_B(X^\#, Y^\#, Z^\#) &= X^\#(\omega_B(Y^\#, Z^\#)) - Y^\#(\omega_B(X^\#, Z^\#)) + Z^\#(\omega_B(X^\#, Y^\#)) \\ &\quad - \omega_B([X^\#, Y^\#], Z^\#) + \omega_B([X^\#, Z^\#], Y^\#) - \omega_B([Y^\#, Z^\#], X^\#) \end{aligned}$$

For the first line, notice that the function

$$\begin{aligned} \mathfrak{g} &\rightarrow \mathbb{R} \\ B &\mapsto \omega_B(Y^\#, Z^\#) = -\langle B, [Y, Z] \rangle \end{aligned}$$

is linear, and so its derivative is itself. Hence this means that

$$\begin{aligned} X^\#(\omega_B(Y^\#, Z^\#)) &= -\langle X^\#, [Y, Z] \rangle \\ &= -\langle [X, B], [Y, Z] \rangle \\ &= -\langle -[B, X], [Y, Z] \rangle \\ &= -\langle B, [-X, [Y, Z]] \rangle \\ &= -\langle B, [[Y, Z], X] \rangle \end{aligned}$$

For the second line, we have that

$$\omega_B([X^\#, Y^\#], Z^\#) = \omega_B([X, Y]^\#, Z^\#) = -\langle B, [[X, Y], Z] \rangle$$

With all of these, we get

$$\begin{aligned} d\omega_B(X^\#, Y^\#, Z^\#) &= -\langle B, [[Y, Z], X] - [[X, Z], Y] + [[X, Y], Z] \rangle \\ &\quad + \langle B, [[X, Y], Z] - [[X, Z], Y] + [[Y, Z], X] \rangle \end{aligned}$$

Both lines are zero by the Jacobi identity, hence  $d\omega_B = 0$ .

## 2.4 Kähler structure

In this section, we will construct the Kähler structure on the adjoint orbits of  $SU(n)$ . We have already constructed the symplectic form  $\omega$  in theorem 2.3.1. We will now construct the Riemannian metric  $g$  and the almost complex structure  $J$ .

### 2.4.1 Complex quotient

First, we note that  $SL(n, \mathbb{C})$  is a complex manifold, and if  $P$  is the subgroup of upper triangular matrices, a variant of the proof in section 1.3 shows that  $SL(n, \mathbb{C})/P$  is a complex manifold, with complex coordinates given by the exponential map.

Consider the composition  $\varphi : SU(n) \rightarrow SL(n, \mathbb{C})/P$  given by the composition

$$SU(n) \hookrightarrow SL(n) \twoheadrightarrow SL(n, \mathbb{C})/P$$

Suppose  $\varphi(g) = \varphi(h)$ . That is,  $gP = hP$ . This is true if and only if there exists  $p \in P$ , such that  $h = gp$ . In this case,  $p = g^{-1}h \in SU(n)$ , therefore,  $p \in SU(n) \cap P = T$ , since  $p^* = p^{-1}$  is also lower triangular. This means that  $\varphi$  induces a homeomorphism  $\tilde{\varphi} : SU(n)/T \rightarrow SL(n, \mathbb{C})/P$ , as it is a continuous bijection from a compact space to a Hausdorff space. Moreover, this is in fact a diffeomorphism.

To see this, consider the natural embedding  $\psi : SU(n) \hookrightarrow SL(n, \mathbb{C})$ . The derivative at the identity gives a linear map

$$d\psi_I : \mathfrak{su}(n) \rightarrow \mathfrak{sl}(n, \mathbb{C})$$

By [5, Theorem 3.32], the following diagram commutes

$$\begin{array}{ccc}
\mathrm{SU}(n) & \xrightarrow{\psi} & \mathrm{SL}(n, \mathbb{C}) \\
\uparrow \exp & & \uparrow \exp \\
\mathfrak{su}(n) & \xrightarrow{d\psi_I} & \mathfrak{sl}(n, \mathbb{C})
\end{array}$$

Therefore, if  $\mathfrak{t}$  is the Lie algebra of  $T$  and  $\mathfrak{p}$  the Lie algebra of  $P$ , then  $d\psi_I$  induces a linear isomorphism  $\mathfrak{su}(n)/\mathfrak{t} \rightarrow \mathfrak{sl}(n, \mathbb{C})/\mathfrak{p}$ . Therefore,  $\tilde{\varphi}$  is a diffeomorphism near  $I$ . But  $\mathrm{SU}(n)$  acts on both spaces transitively by diffeomorphisms, and so  $\tilde{\varphi}$  is a diffeomorphism everywhere.

Using the above, we can get a complex structure on  $\mathrm{SU}(n)/T \cong M$ .

## 2.4.2 Reduction to a diagonal element

We will now show that it suffices to show that  $\omega$  and  $J$  are compatible at  $A \in M$ .  $\mathrm{SU}(n)$  acts transitively via left multiplication on both  $\mathrm{SU}(n)/T$  and  $\mathrm{SL}(n, \mathbb{C})/P$  by diffeomorphisms. Let  $\ell_a$  denote the left multiplication map in both cases, and suppose  $B = \mathrm{Ad}_a(A)$ .

If  $\tilde{J}$  denotes the complex structure on  $\mathrm{SL}(n, \mathbb{C})$ , note that  $d\ell_a(\tilde{J}v) = \tilde{J}d\ell_a(v)$  and we have that

$$\tilde{J}_a = d\ell_a \circ \tilde{J}_I \circ d\ell_{a^{-1}}$$

$\ell_g$  descends to a biholomorphism on  $\mathrm{SL}(n, \mathbb{C})/P \cong \mathrm{SU}(n)/T$ , and so the corresponding almost complex structure  $\bar{J}$  on  $\mathrm{SU}(n)/T$  is given by

$$\bar{J}_{[a]} = d\ell_a \circ \bar{J}_{[I]} \circ d\ell_{a^*} \quad (2.4)$$

Since the diffeomorphism  $\mathrm{SU}(n)/T \cong M$  is induced by the map  $\pi(g) = \mathrm{Ad}_g(A)$ , we have that

$$\begin{aligned}
J_B &= d\pi \circ \bar{J}_{[a]} \circ d\pi^{-1} \\
&= d\pi \circ d\ell_a \circ \bar{J}_{[I]} \circ d\ell_{a^*} \circ d\pi^{-1} \\
&= d(\pi \circ \ell_a \circ \pi^{-1}) \circ J_A \circ d(\pi \circ \ell_{a^*} \circ \pi^{-1})
\end{aligned}$$

It is easy to see that  $\pi \circ \ell_a = \mathrm{Ad}_a \circ \pi$ . Moreover, since  $\mathrm{Ad}_a : \mathfrak{su}(n) \rightarrow \mathfrak{su}(n)$  is a linear map, its restriction to  $M$  has  $d\mathrm{Ad}_a = \mathrm{Ad}_a$ . Therefore, we have that the almost complex structure is given by

$$J_B = \mathrm{Ad}_a \circ J_A \circ \mathrm{Ad}_{a^*}$$

We want to show that this is compatible with the Kirillov-Kostant-Souriau symplectic form. Recall from that

$$\mathrm{T}_B M = \mathrm{Ad}_a(\mathrm{T}_A M)$$

Then we have that

$$\begin{aligned}
\omega_B([B, X], [B, Y]) &= -\langle B, [X, Y] \rangle \\
&= -\langle \mathrm{Ad}_a(A), \mathrm{Ad}_a([\mathrm{Ad}_{a^*}(X), \mathrm{Ad}_{a^*}(Y)]) \rangle \\
&= -\langle A, [\mathrm{Ad}_{a^*}(X), \mathrm{Ad}_{a^*}(Y)] \rangle \\
&= \omega_A([A, \mathrm{Ad}_{a^*}(X)], [A, \mathrm{Ad}_{a^*}(Y)]) \\
&= \omega_A(\mathrm{Ad}_{a^*}([B, X]), \mathrm{Ad}_{a^*}([B, Y]))
\end{aligned}$$

Therefore,  $\omega_B$  is  $\mathrm{Ad}$ -invariant, and thus compatible with  $J$ . The Riemannian metric is given by

$$g_B(X, Y) = g_A(\mathrm{Ad}_{a^*}(X), \mathrm{Ad}_{a^*}(Y))$$

### 2.4.3 At a diagonal element

Recall from section 2.2 that the stabiliser of  $A$  is the torus  $T \cong T^{n-1}$  of diagonal matrices in  $SU(n)$ .

$$T_A M = \bigoplus_{1 \leq i < j \leq n} (\mathbb{R}(E_{ij} - E_{ji}) \oplus i\mathbb{R}(E_{ij} + E_{ji})) \cong \frac{\mathfrak{su}(n)}{\mathfrak{t}} \cong T_{[I]} \left( \frac{SU(n)}{T^{n-1}} \right)$$

where the isomorphism is induced by the quotient map

$$\begin{aligned} \pi : SU(n) &\rightarrow M \\ a &\mapsto \text{Ad}_a(A) = a A a^{-1} \end{aligned}$$

Let  $e_{ij} = E_{ij} - E_{ji}$  and  $f_{ij} = i(E_{ij} + E_{ji})$ . We will show that with respect to this basis,  $\omega_A$  is block diagonal.

#### Proposition 2.4.1.

$$\begin{aligned} \omega_A(e_{ij}, e_{kl}) &= 0 \\ \omega_A(f_{ij}, f_{kl}) &= 0 \\ \omega_A(e_{ij}, f_{ij}) &= \frac{2}{\lambda_i - \lambda_j} \\ \omega_A(e_{ij}, f_{kl}) &= 0 \text{ for } (i, j) \neq (k, l) \end{aligned}$$

One can show that the Lie algebra of  $P$  is the Lie algebra of upper triangular matrices, with trace zero

$$\mathfrak{p} = \mathfrak{sl}(n, \mathbb{C}) \cap \bigoplus_{i \leq j} \mathbb{C} E_{ij}$$

Which gives the decomposition

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{p} \oplus \bigoplus_{i > j} \mathbb{C} E_{ij}$$

Therefore, we have an isomorphism

$$T_{[I]}(SL(n, \mathbb{C})/P) = \bigoplus_{i > j} \mathbb{C} E_{ij} = \bigoplus_{i > j} (\mathbb{R} E_{ij} \oplus i\mathbb{R} E_{ij})$$

With respect to this basis, and the basis  $e_{ij}, f_{ij}$  for  $T_A M$ , we have that the isomorphism  $d\tilde{\varphi}_{[I]}$  is given by

$$d\tilde{\varphi}(e_{ij}) = -E_{ji} \quad d\tilde{\varphi}(f_{ij}) = iE_{ji}$$

Using this, we can define a complex structure on  $T_A M$  by multiplication by  $-i^2$ . That is,

$$J(e_{ij}) = f_{ij} \quad J(f_{ij}) = -e_{ij}$$

Moreover, we can define an inner product on  $T_A M$  by

$$g_A(e_{ij}, e_{ij}) = g_A(f_{ij}, f_{ij}) = \frac{2}{\lambda_i - \lambda_j}$$

and requiring  $e_{ij}, f_{ij}$  to form an orthogonal basis. This is positive definite since we required  $\lambda_i > \lambda_j$  for  $i < j$ . Using this, we find that

$$\omega_A(e_{ij}, f_{ij}) = \frac{2}{\lambda_i - \lambda_j} = g_A(f_{ij}, f_{ij}) = g_A(JA(e_{ij}), f_{ij})$$

and that  $J$  defines an isometry.

---

<sup>2</sup>Throughout there were occasions where we had a choice of signs. If we chose  $-\omega$  instead of  $\omega$ , then the complex structure would be the one coming from multiplication by  $i$ . Similarly, if we required the eigenvalues to be decreasing, rather than increasing this would flip the signs as well.

*Proof of proposition 2.4.1.*

$$\begin{aligned}
\langle A, [E_{ij}, E_{kl}] \rangle &= \langle [A, E_{ij}], E_{kl} \rangle \\
&= i(\lambda_i - \lambda_j) \langle E_{ij}, E_{kl} \rangle \\
&= -i(\lambda_i - \lambda_j) \delta_{jk} \text{tr}(E_{il}) \\
&= -i(\lambda_i - \lambda_j) \delta_{jk} \delta_{il} \\
&= i(\lambda_j - \lambda_i) \delta_{il} \delta_{jk}
\end{aligned}$$

and so,

$$\begin{aligned}
\langle A, [e_{ij}, e_{kl}] \rangle &= \langle A, [E_{ij}, E_{kl}] \rangle - \langle A, [E_{ji}, E_{kl}] \rangle - \langle A, [E_{ij}, E_{lk}] \rangle + \langle A, [E_{ji}, E_{lk}] \rangle \\
&= i(\lambda_j - \lambda_i) \delta_{il} \delta_{jk} - i(\lambda_i - \lambda_j) \delta_{jl} \delta_{ik} - i(\lambda_j - \lambda_i) \delta_{ik} \delta_{jl} + i(\lambda_i - \lambda_j) \delta_{jk} \delta_{il} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
-\langle A, [f_{ij}, f_{kl}] \rangle &= \langle A, [E_{ij}, E_{kl}] \rangle + \langle A, [E_{ji}, E_{kl}] \rangle + \langle A, [E_{ij}, E_{lk}] \rangle + \langle A, [E_{ji}, E_{lk}] \rangle \\
&= i(\lambda_j - \lambda_i) \delta_{il} \delta_{jk} + i(\lambda_i - \lambda_j) \delta_{jl} \delta_{ik} + i(\lambda_j - \lambda_i) \delta_{ik} \delta_{jl} + i(\lambda_i - \lambda_j) \delta_{jk} \delta_{il} \\
&= 0
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
-i \langle A, [e_{ij}, f_{kl}] \rangle &= \langle A, [E_{ij}, E_{kl}] \rangle - \langle A, [E_{ji}, E_{kl}] \rangle + \langle A, [E_{ij}, E_{lk}] \rangle - \langle A, [E_{ji}, E_{lk}] \rangle \\
&= i(\lambda_j - \lambda_i) \delta_{il} \delta_{jk} - i(\lambda_i - \lambda_j) \delta_{jl} \delta_{ik} + i(\lambda_j - \lambda_i) \delta_{ik} \delta_{jl} - i(\lambda_i - \lambda_j) \delta_{jk} \delta_{il} \\
&= 2i(\lambda_j - \lambda_i) (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl}) \\
&= 2i(\lambda_j - \lambda_i) \delta_{ik} \delta_{jl}
\end{aligned}$$

where the last line is because we require  $i < j$  and  $k < l$ . Finally, note that

$$\begin{aligned}
\text{ad}_{e_{ij}}(A) &= -[A, E_{ij} + E_{ji}] = (\lambda_j - \lambda_i) f_{ij} \\
\text{ad}_{f_{ij}}(A) &= -i[A, E_{ij} - E_{ji}] = (\lambda_i - \lambda_j) e_{ij}
\end{aligned}$$

Using this, we get that

$$\begin{aligned}
\omega_A(e_{ij}, f_{ij}) &= \frac{1}{(\lambda_j - \lambda_i)^2} \omega_A(\text{ad}_{e_{ij}}(A), \text{ad}_{f_{ij}}(A)) \\
&= -\frac{1}{(\lambda_j - \lambda_i)^2} \langle A, [e_{ij}, f_{ij}] \rangle \\
&= -\frac{1}{(\lambda_j - \lambda_i)^2} 2(\lambda_j - \lambda_i) \\
&= -\frac{2}{\lambda_j - \lambda_i} \\
&= \frac{2}{\lambda_i - \lambda_j}
\end{aligned}$$

□

## 2.4.4 General case

In this case, we can write

$$\mathfrak{su}(n) = \mathfrak{s} \oplus \bigoplus_{(i,j) \in S} (\mathbb{R} e_{ij} \oplus \mathbb{R} f_{ij})$$

where  $S = \{(i, j) \mid 1 \leq i < j \leq n, \lambda_i \neq \lambda_j\}$ . Then the same formulae as above hold, and defines a Kähler structure on  $M$ .



## Chapter 3

# SL(n, ℂ) orbits

In this section, we will follow [3] and show that the nilpotent orbits of  $SL(n, \mathbb{C})$  are hyperKähler manifolds.

### 3.1 HyperKähler manifolds

#### Definition 3.1.1 (hyperKähler manifold)

A hyperKähler manifold  $M$  is a manifold, with

- (almost) complex structures  $I, J, K$  satisfying the quaternionic relations  $I^2 = J^2 = K^2 = -1$ ,
- a Riemannian metric  $g$ ,
- symplectic forms  $\omega_I, \omega_J, \omega_K$

such that  $(M, g, \omega_I, I), (M, g, \omega_J, J), (M, g, \omega_K, K)$  are Kähler manifolds.

TODO: Add hyperkähler quotients.

### 3.2 Flat hyperKähler space

First choose a sequence  $(V_0, \dots, V_k)$  of Hermitian vector spaces, with  $\dim_{\mathbb{C}} V_i = n_i$ ,  $n_0 = 0$ ,  $n_k = n$ . Define

$$M = \bigoplus_{i=0}^{k-1} (\text{Hom}_{\mathbb{C}}(V_i, V_{i+1}) \oplus \text{Hom}_{\mathbb{C}}(V_{i+1}, V_i))$$

We will write a point in  $M$  as  $(\alpha_j, \beta_j)$ , where

$$V_0 \xrightleftharpoons[\beta_0]{\alpha_0} V_1 \xrightleftharpoons[\beta_1]{\alpha_1} \dots \xrightleftharpoons[\beta_{k-1}]{\alpha_{k-1}} V_k$$

The inner product on  $M$  is given by

$$g((\alpha_j, \beta_j), (\tilde{\alpha}_j, \tilde{\beta}_j)) = \sum_{j=0}^{k-1} \left( \text{Re tr}(\alpha_j^* \tilde{\alpha}_j) + \text{Re tr}(\beta_j^* \tilde{\beta}_j) \right)$$

and we can make  $M$  into a flat hyperKähler manifold, using

$$I(\alpha_j, \beta_j) = (i\alpha_j, i\beta_j) \quad J(\alpha_j, \beta_j) = (-\beta_j^*, \alpha_j^*)$$

where  $\alpha^*$  denotes the Hermitian adjoint.

The Lie group  $G = U(n_1) \times \dots \times U(n_{k-1})$  acts on  $M$ , via

$$\begin{aligned} \alpha_j &\mapsto g_{j+1} \alpha_j g_j^{-1} \\ \beta_j &\mapsto g_j \beta_j g_{j+1}^{-1} \end{aligned} \tag{3.1}$$

where  $g_j \in U(n_j)$ . In this case, the hyperKähler moment map is

# Bibliography

- [1] Michèle Audin. *Torus Actions on Symplectic Manifolds*. Publisher: Birkhäuser Basel. Sept. 2004. doi: 10.1007/978-3-0348-7960-6.
- [2] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Publisher: Springer New York. Jan. 1973. doi: 10.1007/978-1-4612-6398-2.
- [3] Piotr Z. Kobak and Andrew Swann. "Classical nilpotent orbits as hyperkähler quotients". In: *Int. J. Math.* 07.02 (Apr. 1996). Publisher: World Scientific Publishing Co., pp. 193–210. issn: 0129-167X. doi: 10.1142/S0129167X96000116. URL: <https://www.worldscientific.com/doi/10.1142/S0129167X96000116> (visited on 07/27/2023).
- [4] Jerrold E. Marsden and Tudor S. Ratiu. *Introduction to Mechanics and Symmetry*. Publisher: Springer New York. Apr. 1999. doi: 10.1007/978-0-387-21792-5.
- [5] Frank W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Publisher: Springer New York. Oct. 1983. doi: 10.1007/978-1-4757-1799-0.