KKS forms on Lie groups

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We compute some examples of the KKS form on some Lie groups.

1 SU(2)

Define the following complex matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{j} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad \mathbf{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Let $\mathbb{H} = \operatorname{span}_{\mathbb{R}} \{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be the *quaternions*. For $q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, define $\overline{q} = w - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$ for its *conjugate*, and $|q| = \sqrt{q\overline{q}} = \sqrt{w^2 + x^2 + y^2 + z^2}$ for its absolute value.

With this in mind, we have that SU(2) is the unit ball in $\mathbb{H}$, i.e.

$$\mathrm{SU}(2) = \{q \in \mathbb{H} \mid |q| = 1\} = \left\{\cos(\theta) + \sin(\theta)(x\mathbf{i} + \mathbf{y}\mathbf{j} + \mathbf{z}\mathbf{k}) \mid (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathsf{S}^2\right\}$$

Let $\mathfrak{su}(2)$ be the Lie algebra of SU(2). In fact,

$$\mathfrak{su}(2) = \operatorname{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$$

with the Lie bracket given by $[A, B] = AB - BA$. We can define an isomorphism $\varphi : \mathbb{R}^3 \to \mathfrak{su}(2)$ of vector spaces, by

$$\varphi(x, y, z) = x\mathbf{i} + \mathbf{y}\mathbf{j} + \mathbf{z}\mathbf{k}$$

Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$, and $\wedge$ the standard vector product in $\mathbb{R}^3$.

Proposition 1.1.

1. For $u, v \in \mathbb{R}^3$, $[\varphi(u), \varphi(v)] = 2\varphi(u \wedge v)$.
2. For $u \in \mathbb{R}^3$, $\|u\|^2 = \det(\varphi(u))$,
3. For $u, v \in \mathbb{R}^3$, with the standard Euclidean inner product, $\langle u, v\rangle = \frac{1}{2}\operatorname{tr}(\varphi(u)^*\varphi(v))$.

Proof. By standard quaternion relations. □

In particular, if we define an inner product on $\mathfrak{su}(2)$ by

$$(A, B) = \frac{1}{2}\operatorname{tr}(A^*B)$$

Then this defines an inner product, with the same norm as the one from the quaternions. Moreover, this makes φ into an isometry. Therefore, by the Riesz representation theorem (or equivalently, the dual space of a finite dimensional vector space), we know that the map $R : \mathfrak{su}(2) \to \mathfrak{su}(2)^*$ given by

$$R(A)(B) = (A, B)$$

is a vector space isomorphism.

With these identifications, we make the following convention: We use upper case letters $(A, B, C, \dots)$ for elements of $\mathfrak{su}(2)$, lower case letters $(a, b, c, \dots)$ for the corresponding (under φ) elements of $\mathbb{R}^3$, and Greek letters $(\alpha, \beta, \gamma, \dots)$ for elements of $\mathfrak{su}(2)^*$. If we have an equation where the left and right hand side belong two different spaces, we implicitly use the isomorphisms R and φ to identify them. *However, within each formula, all the objects will belong to the same space, and we will use the isomorphisms explicitly.*

g will denote an element of $\mathrm{SU}(2)$.

Next, we want to compute the coadjoint action of $\mathrm{SU}(2)$ on $\mathfrak{su}(2)^*$. We know that the adjoint action is $\mathrm{Ad}_g(A) = gA\overline{g}$. Given $\alpha \in \mathfrak{su}(2)^*$, $g \in \mathrm{SU}(2)$, $B \in \mathfrak{su}(2)$, we have

$$\mathrm{Ad}^*_g(\alpha)(B) = \alpha(\mathrm{Ad}_{g^{-1}}(B)) = (A, \mathrm{Ad}_{\overline{g}}(B)) = (A, \overline{g}Bg)$$

But

$$(A, \overline{g}Bg) = \frac{1}{2}\operatorname{tr}(A^*\overline{g}Bg) = \frac{1}{2}\operatorname{tr}(gA^*\overline{g}B) = \frac{1}{2}\operatorname{tr}((gA\overline{g})^*B) = (\mathrm{Ad}_g(A), B) = R(\mathrm{Ad}_g(A))(B)$$

That is, $\mathrm{Ad}^*_g(\alpha) = R(\mathrm{Ad}_g(A))$, or $\mathrm{Ad}^*_g = R \circ \mathrm{Ad}_g \circ R^{-1}$. Hence the coadjoint orbits and the adjoint orbits in this case are the same, up to identification by R.

Proposition 1.2. For $B \in \mathfrak{su}(2)$, the (adjoint) orbit is

$$\mathcal{O}_B = \mathrm{Orb}(B) = \{C \in \mathfrak{su}(2) \mid \det(B) = \det(C)\}$$

i.e. a sphere in $\mathfrak{su}(2) \simeq \mathbb{R}^3$.

Proof. Omitted. $\subseteq$ is easy, for $\supseteq$, we need some geometry about conjugation by quaternions and rotations. ∎

Lemma 1.3. For $A, B \in \mathfrak{su}(2)$, $\mathrm{ad}_A(B) = [A, B] = 2a \wedge b$.

Moreover, for $A \in \mathfrak{su}(2), \beta \in \mathfrak{su}(2)^*$, $\mathrm{ad}^*_A(\beta)(C) = \beta(-\mathrm{ad}_A(C)) = \beta(-[A, C])$. Therefore, if $A \in \mathfrak{su}(2), \beta \in \mathfrak{su}(2)^*$, we have

$$\mathrm{ad}^*_A(\beta) = (B, -[A, C]) = ([A, B], C) = R([A, B])(C)$$

Hence if we define $\mathrm{ad}^*_A(B) := \mathrm{ad}^*_A(\beta)$, then $\mathrm{ad}^*_A(B) = [A, B] = 2a \wedge b$ as well.

Proposition 1.4. Let $A \in \mathfrak{su}(2) \setminus 0$, $r = |A| = \|a\|$. Let ω be the KKS 2-form on $\mathcal{O}_A$. That is, for $B \in \mathcal{O}_A$, $U, V \in T_B\mathcal{O}_A \subseteq \mathfrak{su}(2)$,

$$\omega_B(\mathrm{ad}^*_U(B), \mathrm{ad}^*_V(B)) = (B, [U, V]) \tag{*}$$

Then in fact,

$$\omega_B(U, V) = -\frac{1}{2r}u \wedge v$$

Proof. Let $B \in \mathcal{O}_A$. For $U, V \in \mathfrak{su}(2)$, (*) becomes

$$4\omega_B(u \wedge b, v \wedge b) = (B, [U, V]) = 2\langle b, u \wedge v\rangle \implies \omega_B(u \wedge b, v \wedge b) = \frac{1}{2}\langle b, u \wedge v\rangle$$

Let $e_r = \frac{1}{r}b$ and e_θ, e_ϕ such that e_r, e_θ, e_ϕ is a positively oriented orthonormal basis of $\mathbb{R}^3$. Now for $U \in \mathfrak{su}(2)$, $\mathrm{ad}^*_U(B) = 2u \wedge b = 2ru \wedge e_r$. Therefore, we have that $T_B\mathcal{O}_A = \mathrm{span}\{E_\theta, E_\phi\}$. Let $U, V \in T_B\mathcal{O}_A$, where $U = u_\theta E_\theta + u_\phi E_\phi$ and $V = v_\theta E_\theta + v_\phi E_\phi$. Then let

$$\tilde{u} = -\frac{u_\phi}{r}e_\theta + \frac{u_\theta}{r}e_\phi \quad \text{and} \quad \tilde{v} = -\frac{v_\phi}{r}e_\theta + \frac{v_\theta}{r}e_\phi$$

With this, $\tilde{u} \wedge b = u$ and $\tilde{v} \wedge b = v$. Hence

$$\omega_B(u, v) = \frac{1}{2}\langle b, \tilde{u} \wedge \tilde{v}\rangle = r\frac{-u_\phi v_\theta + u_\theta v_\phi}{2r^2} = -\frac{1}{2r}u \wedge v$$

∎

2 $\mathrm{SL}_2(\mathbb{R})$

Let $\mathrm{SL}_2(\mathbb{R})$ be the space of 2×2 matrices with determinant 1. The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is the space of 2×2 matrices with trace 0, with Lie bracket $[A, B] = AB - BA$. Define

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then X, Y, Z is a basis for $\mathfrak{sl}_2(\mathbb{R})$. Define the isomorphism $\varphi : \mathbb{R}^3 \to \mathfrak{sl}_2(\mathbb{R})$ by

$$\varphi(a, b, c) = aX + bY + cZ$$

Lemma 2.1. 1. $XY = -YX = Z, YZ = -ZY = X, ZX = -XZ = -Y, [X, Y] = 2Z, [Y, Z] = -2X, [Z, X] = -2Y$,

2. for all $u, v \in \mathbb{R}^3$, $\langle u, v\rangle = \frac{1}{2}\operatorname{tr}\left(\varphi(u)^{\mathsf{T}}\varphi(v)\right)$.

Therefore, we can define an inner product on $\mathfrak{sl}_2(\mathbb{R})$ by

$$(A, B) = \frac{1}{2}\operatorname{tr}\left(A^{\mathsf{T}} B\right)$$

With this, φ becomes an isometry. Again by Riesz, we define the isomorphism $R : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_2(\mathbb{R})^*$ by

$$R(A)(B) = (A, B)$$

With all of this, we will use the same convention as above. That is, A, B, C are elements of $\mathfrak{sl}_2(\mathbb{R})$, a, b, c the corresponding elements (under φ) of $\mathbb{R}^3$, and α, β, γ, the corresponding elements of $\mathfrak{sl}_2(\mathbb{R})^*$. g will be an element of $\mathrm{SL}_2(\mathbb{R})$.

Let $g \in \mathrm{SL}_2(\mathbb{R})$. Then

$$\mathrm{Ad}^*_g(\alpha)(B) = \alpha(\mathrm{Ad}_{g^{-1}}(B)) = (A, \mathrm{Ad}_{g^{-1}}(B)) = (A, g^{-1}Bg)$$

But

$$(A, g^{-1}Bg) = \frac{1}{2}\operatorname{tr}\left(A^{\mathsf{T}} g^{-1}Bg\right) = \frac{1}{2}\operatorname{tr}\left(gA^{\mathsf{T}} g^{-1}B\right) = \frac{1}{2}\operatorname{tr}\left(((g^{\mathsf{T}})^{-1}Ag^{\mathsf{T}})^{\mathsf{T}} B\right) = (\mathrm{Ad}_{(g^{\mathsf{T}})^{-1}}(A), B)$$

So we have that

$$\mathrm{Ad}^*_g = R \circ \mathrm{Ad}_{(g^{\mathsf{T}})^{-1}} \circ R^{-1}$$

Hence the adjoint and coadjoint orbits are the same in this case.

Lemma 2.2. 1. For all $A \in \mathfrak{sl}_2(\mathbb{R})$, the (adjoint) orbit is

$$\mathcal{O}_A = \left\{gAg^{-1} \mid g \in \mathrm{SL}_2(\mathbb{R})\right\}$$

2. for $x, y, z \in \mathbb{R}^3$,

$$\det(xX + yY + zZ) = z^2 - x^2 - y^2$$

3. for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$,

$$\begin{aligned} gXg^{-1} &= (ad + cb)X + (cd - ab)Y - (ab + cd)Z \\ gYg^{-1} &= (-ac + bd)X + \frac{a^2 - b^2 - c^2 + d^2}{2}Y + \frac{a^2 + c^2 - b^2 - d^2}{2}Z \\ gZg^{-1} &= -(ac + bd)X + \frac{a^2 + b^2 - c^2 - d^2}{2}Y + \frac{a^2 + b^2 + c^2 + d^2}{2}Z \end{aligned}$$

Proposition 2.3. The coadjoint orbits (up to identification by φ) are

1. $$\{x^2 + y^2 - z^2 = \lambda^2\} \quad \lambda > 0$$
2. $$\{x^2 + y^2 - z^2 = -\lambda^2, z > 0\} \quad \lambda \geq 0$$
3. $$\{x^2 + y^2 - z^2 = -\lambda^2, z < 0\} \quad \lambda \geq 0$$
4. $$\{0\}$$

Proposition 2.4. Let $A = xX + yY + zZ \in \mathfrak{sl}_2(\mathbb{R})$, and G_A be the stabiliser of the coadjoint action of $\mathrm{SL}_2(\mathbb{R})$. Then

1. If $\det(A) < 0$, then
$$G_A \simeq \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \middle| \, r \in \mathbb{R}^\times \right\} \simeq \mathbb{R}^\times$$
2. If $\det(A) = 0$ and $z = 0$, then $G_A = \mathrm{SL}_2(\mathbb{R})$,
3. If $\det(A) = 0$ and $z \neq 0$, then
$$G_A = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{R} \right\} \cup \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix} \simeq \{\pm 1\} \times \mathbb{R}$$
4. if $\det(A) > 0$, then
$$G_A = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \right\} \simeq S^1$$

Note if $U = xX + yY + zZ$, then $U^\mathsf{T} = xX + yY - zZ$. Thus, for $u = (x, y, z) \in \mathbb{R}^3$, we define $u^t = (x, y, -z)$.

Lemma 2.5. For $A, B \in \mathfrak{sl}_2(\mathbb{R})$, $\mathrm{ad}_A(B) = 2a^t \wedge b^t$, and $\mathrm{ad}^*_A(\beta) = \mathrm{ad}_B(A^\mathsf{T}) = 2b^t \wedge a$.

We will abuse notation and define $\mathrm{ad}^*_A(B) = \mathrm{ad}^*_A(\beta)$. Therefore, let $A \in \mathfrak{sl}_2(\mathbb{R}) \setminus 0$, and ω be the KKS symplectic form, that is, for $B \in \mathcal{O}_A$ and $u, v \in T_B\mathcal{O}_A$,

$$\omega_B(\mathrm{ad}^*_U(B), \mathrm{ad}^*_V(B)) = \langle B, [U, V] \rangle$$

Say $B = xX + yY + zZ$. Let $r = \sqrt{x^2 + y^2}$, e_x, e_y, e_z the standard basis of $\mathbb{R}^3$.

If $r \neq 0$, let e_r, e_θ be such that e_r, e_θ, e_z is a positively oriented orthonormal basis of $\mathbb{R}^3$, and with $B = re_r + ze_z$. Let

$$\mathrm{d}x = \langle e_x, \cdot \rangle, \mathrm{d}y = \langle e_y, \cdot \rangle, \mathrm{d}z = \langle e_z, \cdot \rangle, \mathrm{d}r = \langle e_r, \cdot \rangle, \mathrm{d}\theta = \frac{1}{r} \langle e_\theta, \cdot \rangle$$

Then by a computation,

$$\omega_B = \begin{cases} \frac{1}{2}\mathrm{d}z \wedge \mathrm{d}\theta & \text{if } r \neq 0 \\ \frac{1}{2z}\mathrm{d}x \wedge \mathrm{d}y & \text{if } z \neq 0 \end{cases}$$

This is well defined, since we know that on the coadjoint orbits, $r^2 - z^2 = c$, where c is a constant. Differentiating this, we get that, $r\mathrm{d}r = z\mathrm{d}z$. Moreover, if we set

$$x = r\cos(\theta + \theta_0) \quad y = r\sin(\theta + \theta_0) \quad z = z$$

Then we get that $\mathrm{d}x \wedge \mathrm{d}y = r\mathrm{d}r \wedge \mathrm{d}\theta = z\mathrm{d}z \wedge \mathrm{d}\theta$, so the definitions of ω_B agree.

3 GA($\mathbb{R}$)

Let GA($\mathbb{R}$) be the affine group of $\mathbb{R}$, that is, transformations of the form $x \mapsto ax + b$, where $a \neq 0$. We can identify GA($\mathbb{R}$) with $\{(a, b) \in \mathbb{R}^2 \mid a \neq 0\}$, with group operations

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_1, a_1 b_2 + b_1) \quad (a, b)^{-1} = \left(\frac{1}{a}, -b\frac{b}{a}\right)$$

The Lie algebra is $\mathfrak{ga}(\mathbb{R}) = \mathbb{R}^2$. Conjugation is given by

$$C_{(a,b)}(c, d) = (c, ad - bc + b)$$

Differentiating with respect to (c, d) at $(1, 0)$ in the direction $(u, v) \in \mathfrak{ga}(\mathbb{R})$ gives the adjoint representation

$$\mathrm{Ad}_{(a,b)}(u, v) = (u, av - bu)$$

Differentiation this with respect to (a, b) in the direction (r, s) gives the Lie bracket

$$[(r, s), (u, v)] = (0, rv - su)$$

The adjoint orbit through (u, v) is

$$\begin{cases} \{u\} \times \mathbb{R} & \text{if } (u, v) \neq (0, 0) \\ \{(0, 0)\} & \text{if } (u, v) = (0, 0) \end{cases}$$

The adjoint orbit $\{u\} \times \mathbb{R}$ can't be symplectic, since it is one-dimensional. Let e_1, e_2 be the standard basis of $\mathfrak{ga}(\mathbb{R}) = \mathbb{R}^2$, and e^1, e^2 be the dual basis. Computing the coadjoint orbits, we have[1]

$$\begin{aligned} \mathrm{Ad}^*_{(a,b)^{-1}}(\alpha)(ue_1 + ve_2) &= \alpha(\mathrm{Ad}_{(a,b)}(u, v)) \\ &= \alpha(u, av - bu) \end{aligned}$$

Setting $\alpha = e^1$ and $\alpha = e^2$, we get that

$$\mathrm{Ad}^*_{(a,b)^{-1}}(\alpha e^1 + \beta e^2) = (\alpha - \beta b)e^1 + \beta a e^2$$

Therefore, if $\beta = 0$, then the coadjoint orbit through $u = \alpha e^1 + \beta e^2$ is a since point. Otherwise, it is $\mathbb{R}^2$ minus the e^2-axis.

For $\beta \neq 0$, $\mu = \alpha e^1 + \beta e^2$, with $\beta \neq 0$, the KKS formula gives[2]

$$\omega_\mu((r, s)_{\mathfrak{ga}(\mathbb{R})^*}(\mu), (u, v)_{\mathfrak{ga}(\mathbb{R})^*}(\mu)) = \mu([(r, s), (u, v)]) = -\beta(rv - su)$$

In terms of local coordinates (q, p) given by ad^* (i.e. coordinates $(r, s) \mapsto (r, s)_{\mathfrak{ga}(\mathbb{R})^*}(\mu)$[3]), we have that

$$\omega = \beta \mathrm{d}q \wedge \mathrm{d}p$$

[1] The inverses are by the definition of the coadjoint representation. Since we are interested in the coadjoint orbits, it won't affect anything.

[2] Note the $-$-sign.

[3] Computing, we find that

$$(r, s)_{\mathfrak{ga}(\mathbb{R})^*}(\mu) = \beta r e^2 - \beta s e^1$$