# Coadjoint Orbits of SU($n$)

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In this note, we will consider the coadjoint orbits of SU($n$), and show that they are Kähler manifolds.

## Contents



## 1 Adjoint and Coadjoint Orbits

Define the Lie algebra

$$\mathfrak{su}(n) = \{A \in \mathrm{Mat}(n, \mathbb{C}) \mid A^* + A = 0, \mathrm{tr}(A) = 0\}$$

where $A^*$ is the conjugate transpose of $A$, and with the Lie bracket being the matrix commutator. We can define the adjoint representation of SU($n$) as

$$\begin{aligned} \mathrm{Ad} &: \mathrm{SU}(n) \to \mathrm{GL}(\mathfrak{su}(n)) \\ \mathrm{Ad}_g(X) &= gXg^\dagger \end{aligned}$$

Taking the dual representation, we get the coadjoint representation, which is

$$\begin{aligned} \mathrm{Ad}^* &: \mathrm{SU}(n) \to \mathrm{GL}(\mathfrak{su}(n)^*) \\ \mathrm{Ad}^*_g(\alpha)(X) &= \alpha(\mathrm{Ad}_{g^{-1}}(X)) \end{aligned}$$

Now note that

$$\langle A, B\rangle = -\mathrm{tr}(AB) = \mathrm{tr}(AB^*)$$

defines an inner product on $\mathfrak{su}(n)$[1], which means that we have a natural isomorphism

[1] In fact, $\langle A, B\rangle = \mathrm{tr}(AB^*)$ defines a Hermitian inner product on the space of complex matrices.

$$\Phi : \mathfrak{su}(n) \to \mathfrak{su}(n)^*$$
$$A \mapsto \langle A, \cdot \rangle$$

With this, suppose $\alpha = \Phi(A)$, then

$$\mathrm{Ad}_g^*(\alpha)(X) = \left\langle A, \mathrm{Ad}_{g^{-1}}(X)\right\rangle = -\operatorname{tr}\left(Ag^{-1}Xg\right) = -\operatorname{tr}\left(gAg^{-1}X\right) = \Phi(\mathrm{Ad}_g(A))(X)$$

Therefore, up to identification by $\Phi$, the adjoint and coadjoint representations are the same.

## 2 Tangent space

Let $\mathcal{O}^*$ be a coadjoint orbit. For $X \in \mathfrak{su}(n)$, consider the curve $g(t) = \exp(tX)$ in $G$. This has $g'(0) = X$, and we have a curve

$$\mu(t) = \mathrm{Ad}^*_{g(t)}(\mu)$$

through $\mu \in \mathcal{O}^*$. In particular, we have that for $Y \in \mathfrak{su}(n)$,

$$\langle \mu(t), Y \rangle = \left\langle \mu, \mathrm{Ad}_{g(t)^{-1}}(Y)\right\rangle$$

Differentiating this at $t = 0$, we get

$$\left\langle \mu'(0), Y \right\rangle = -\langle \mu, \mathrm{ad}_X(Y)\rangle = -\langle (\mathrm{ad}_X)^*(\mu), Y\rangle$$

That is, $\mu'(0) = -(\mathrm{ad}_X)^*(\mu)$. Hence we have that

$$\mathrm{T}_\mu\left(\mathcal{O}^*\right) = \left\{(\mathrm{ad}_X)^*(\mu) \mid X \in \mathfrak{su}(n)\right\}$$

If $\alpha = \Phi(A)$, then

$$\begin{aligned}
\langle (\mathrm{ad}_X)^*(\alpha), Y \rangle &= \langle \alpha, \mathrm{ad}_X(Y)\rangle \\
&= \langle A, [X, Y]\rangle \\
&= -\operatorname{tr}(AXY - AYX) \\
&= -\operatorname{tr}(AXY - XAY) \\
&= \langle [A, X], Y\rangle \\
&= \langle -\mathrm{ad}_X(A), Y\rangle
\end{aligned}$$

Hence by identification with $\Phi$, $(\mathrm{ad}_X)^* = -\mathrm{ad}_X$. Thus, in this case we have the tangent space to the corresponding adjoint orbit as

$$T_A\mathcal{O} = \{\mathrm{ad}_X(A) \mid X \in \mathfrak{su}(n)\}$$

## 3 Diagonalisation

Since all elements in $\mathcal{O}$ are diagonalisable by $\mathrm{SU}(n)$, we can choose $A \in \mathcal{O}$ to be diagonal. Moreover, the eigenvalues are pure imaginary, so we can assume

$$A = \begin{pmatrix} i\lambda_1 & & \\ & \ddots & \\ & & i\lambda_n \end{pmatrix}$$

with $\lambda_1 \geq \cdots \geq \lambda_n$. Then define the projection map $\pi : \mathrm{SU}(n) \to \mathcal{O}$ by

$$\pi(g) = \mathrm{Ad}_g(A) = gAg^\dagger$$

and it is easy to check that $\pi$ is a surjective submersion. In particular, if $\alpha$ is any form on $\mathcal{O}$, then $\alpha$ is determined by its pullback $\pi^*\alpha$.

## 4 Symplectic structure

For details in this section, see `kirillov-kostant.pdf` or Marsden-Ratiu Chapter 14.

The Kirillov-Kostant-Souriau form on $\mathcal{O}^*$ is given by

$$\omega_\mu(-(\mathrm{ad}_X)^*(\mu), -(\mathrm{ad}_Y)^*(\mu)) = -\langle \mu, [X, Y] \rangle$$

Therefore, using the isomorphism $\Phi$, we have a corresponding symplectic form on $\mathcal{O}$ given by

$$\omega_B(\mathrm{ad}_X(B), \mathrm{ad}_Y(B)) = -\langle B, [X, Y] \rangle \tag{1}$$

Then $\pi^*\omega = \mathrm{d}\alpha_\ell$, where $\alpha = \Phi(A)$, and $\alpha_\ell$ is the left-invariant 1-form on $\mathrm{SU}(n)$ taking value $\alpha$ at 1. That is,

$$\alpha_\ell(g) = ((\mathrm{d}\ell_{g^{-1}})_g)^*(\alpha)$$

Using this, we get that $\pi^*\mathrm{d}\omega = \mathrm{d}\pi^*\omega = \mathrm{d}^2\alpha_\ell = 0$, and as $\pi$ is a surjective submersion, $\mathrm{d}\omega = 0$. The facts that eq. (1) is well defined and non-degenerate are easy to check as well.

## 5 Complex structure

From §3, we assumed that $A$ is diagonal. Suppose the eigenvalues have algebraic multiplicities $n_1, \ldots, n_k$. Then we have that the stabiliser of $A$ is the block diagonal subgroup

$$\mathrm{Stab}(A) = \mathrm{S}(\mathrm{U}(\mathrm{n_1}) \times \cdots \times \mathrm{U}(\mathrm{n_k}))$$

and the quotient space is the flag manifold

$$\mathcal{F}(n_1, \ldots, n_k) = \frac{\mathrm{SU}(n)}{\mathrm{Stab}(A)}$$

We won't discuss this any further, see `coadjoint-orbits-sln.pdf` for details. We will now focus on the generic case, where $A$ has distinct eigenvalues. In this case, the stabiliser is the diagonal subgroup $T$, which is isomorphic to the torus $T^{n-1} = (S^1)^{n-1}$.

Let $P$ be the subgroup of lower triangular matrices in $\mathrm{SL}(n, \mathbb{C})$. Consider the composition $\varphi : \mathrm{SU}(n) \to \mathrm{SL}(n, \mathbb{C})/P$ given by the composition

$$\mathrm{SU}(n) \hookrightarrow \mathrm{SL}(n) \twoheadrightarrow \mathrm{SL}(n, \mathbb{C})/P$$

Suppose $\varphi(g) = \varphi(h)$. That is, $gP = hP$. This is true if and only if there exists $p \in P$, such that $h = gp$. In this case, $p = g^{-1}h \in \mathrm{SU}(n)$, therefore, $p \in \mathrm{SU}(n) \cap P = T$, since $p^\dagger = p^{-1}$ is also lower triangular. This means that $\varphi$ induces a homeomorphism $\mathrm{SU}(n)/T \cong \mathrm{SL}(n, \mathbb{C})/P$. The right hand side is a complex manifold ($\mathrm{SL}(n, \mathbb{C})$) quotiented by a complex Lie group $P$, so it is a complex manifold. Using the above, we can get a complex structure on $\mathrm{SU}(n)/T \cong \mathcal{O}$.

## 6 Local coordinates

This section is from `un_kks.pdf`, which follows an exercise from Prof. Bryant's notes. The computations to verify these expressions can be found in `un_kks.pdf`, and have been omitted.

let $\theta$ be the Maurer-Cartan form on $\mathrm{SU}(n)$. That is, it is the $\mathfrak{su}(n)$-valued 1-form on $\mathrm{SU}(n)$ given by

$$\theta_g(u) = \mathrm{d}\left(\ell_{g^{-1}}\right)_g(u) \in \mathrm{T}_e\,\mathrm{SU}(n) = \mathfrak{su}(n)$$

Writing $\theta = \sum_{j,k} \theta_{jk} \mathrm{d}g^{jk}$ where $(g^{jk})$ are the matrix entries on $\mathrm{SU}(n)$. Then we have that

$$\pi^*\omega = i \sum_{k>j} (\lambda_j - \lambda_k)\theta_{kj} \wedge \overline{\theta_{kj}}$$

and the Hermitian metric $h$ on $\mathcal{O}$ is given by

$$\pi^*h = \sum_{k>j} 2(\lambda_j - \lambda_k)\theta_{kj}\overline{\theta_{kj}}$$

Using the fact that

$$\mathrm{T}_g \,\mathrm{SU}(n) = g\mathfrak{su}(n)$$

and that

$$\mathrm{d}\pi_g(h) = hAg^\dagger + gAh^\dagger$$

we get that

$$\mathrm{d}\pi_g(h) = [B, hg^\dagger]$$

where $B = \pi(g) = gAg^\dagger$. With this, we can recover the hermitian metric, as

$$h_B([B, C], [B, D]) = \sum_{k>j} 2(\lambda_j - \lambda_k)(g^\dagger Cg)_{kj}\overline{(g^\dagger Dg)_{kj}}$$

With this, we can see that $\omega = -\operatorname{Im}(h)$, and that the real part $g = \operatorname{Re}(h)$[2] defines a Riemannian metric.

Moreover, we can recover the almost complex structure from the symplectic form and the Riemannian metric, via

$$J = \widetilde{g}^{-1} \circ \widetilde{\omega}$$

where $\widetilde{\omega}, \widetilde{g} : \mathrm{T}\,\mathrm{SU}(n) \to \mathrm{T}^*\,\mathrm{SU}(n)$ are linear isomorphisms given by

$$\widetilde{\omega}(u)(v) = \omega(u, v)$$
$$\widetilde{g}(u)(v) = g(u, v)$$

This follows as we have that $g(u, v) = \omega(u, Jv)$.

## 7 Root decomposition

As the expressions we have above for $\pi^*\omega$ and $\pi^*h$ are left invariant, suffices to consider them at $A$, since we can then use left translation in $\mathrm{SU}(n)$ to get the expression at a different tangent space. This section should connect the results in the previous two sections.

Consider the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. Then we have the Cartan subalgebra $\mathfrak{t}$ of diagonal matrices. Let $E_{ij}$ be the standard basis matrices for $\operatorname{Mat}(n, \mathbb{C})$, $B \in \mathfrak{t}$. Say

$$B = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix}$$

Then $[B, E_{ij}] = (b_i - b_j)E_{ij}$. This means that we have the eigendecomposition

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{t} \oplus \bigoplus_{1\le i,j\le n, i\ne j} \mathbb{C}E_{ij} \tag{2}$$

In particular, if we restrict this to the subalgebra $\mathfrak{su}(n)$, we get the decomposition

$$\mathfrak{su}(n) = \widetilde{\mathfrak{t}} \oplus \bigoplus_{1\le i<j\le n} \big(\mathbb{R}(E_{ij} - E_{ji}) \oplus i\mathbb{R}(E_{ij} + E_{ji})\big)$$

where $\widetilde{\mathfrak{t}} = \mathfrak{t} \cap \mathfrak{su}(n)$ is the subalgebra of $\mathfrak{su}(n)$ of diagonal matrices. In particular, we have that (assuming the eigenvalues of $A$ are distinct)

$$\mathrm{T}_A\mathcal{O} \cong \mathrm{T}_{[1]}\left(\frac{\mathrm{SU}(n)}{T^{n-1}}\right) \cong \frac{\mathfrak{su}(n)}{\widetilde{t}} = \bigoplus_{1\le i<j\le n} \big(\mathbb{R}(E_{ij} - E_{ji}) \oplus i\mathbb{R}(E_{ij} + E_{ji})\big)$$

Then the almost complex structure is given by

[2] The usual notation for a Riemannian metric is $g$, but I've also used $g$ as the element of $\mathrm{SU}(n)$. Hopefully it should be clear from context which one is which.

$$\begin{aligned}\mathbb{R}(E_{ij} - E_{ji}) &\mapsto i\mathbb{R}(E_{ij} + E_{ji}) \\ i\mathbb{R}(E_{ij} + E_{ji}) &\mapsto i\mathbb{R}(E_{ij} - E_{ji})\end{aligned}$$

is

1. The action of multiplication by $i$ in eq. (2),
2. The complex structure linking our expressions for $\pi^*\omega$ and $\pi^*h$.

In particular, this means that the coadjoint orbits are Kähler.

# 8 Another proof

**This section contains details which I have not checked. But assuming the claims are true, this should give another construction for the Kähler structure on coadjoint orbits of $\mathrm{SU}(n)$.**

The idea here is that we can make $\mathrm{T}^*\,\mathrm{SU}(n)$ into a Kähler manifold, and if we define an appropriate moment map, then we can use Kähler reduction (see appendix) to get a Kähler structure on coadjoint orbits.

First of all, any cotangent bundle $\mathrm{T}^*M$ has a symplectic form, called the *canonical symplectic form*

$$\omega = -\mathrm{d}\alpha$$

where $\alpha_p = (\mathrm{d}\pi_p)^*\xi \in \mathrm{T}_p^*M$, $p = (x, \xi)$. In local cotangent coordinates,

$$\omega = \sum_i \mathrm{d}x^i \wedge \mathrm{d}\xi^i$$

See da Silva §2.3 for details.

In this case, $\mathrm{SU}(n)$ is a Lie group, so we can left trivialise the cotangent bundle, via the map

$$\begin{aligned}\mathrm{T}^*\,\mathrm{SU}(n) &\to \mathrm{SU}(n) \times \mathfrak{su}(n)^* \\ (g, \alpha) &\mapsto (g, ((\mathrm{d}\ell_g)_e)^*\alpha)\end{aligned}$$

where $\ell_g(x) = gx$ is the left multiplication map on $\mathrm{SU}(n)$. Using the isomorphism $\Phi : \mathfrak{su}(n) \to \mathfrak{su}(n)^*$ from above, we get an diffeomorphism

$$\mathrm{T}^*\,\mathrm{SU}(n) \cong \mathrm{SU}(n) \times \mathfrak{su}(n)$$

Moreover, the polar decomposition

$$\begin{aligned}\mathrm{U(n)} \times \mathfrak{u}(\mathrm{n}) &\to \mathrm{GL}(n, \mathbb{C}) \\ (U, X) &\mapsto U \exp(X)\end{aligned}$$

Gives a diffeomorphism $\mathrm{SU}(n) \times \mathfrak{su}(n) \cong \mathrm{SL}(n, \mathbb{C})$. Therefore, we also have a complex structure on $\mathrm{T}^*\,\mathrm{SU}(n)$.

**Unchecked claim: This is compatible with the symplectic structure, making $\mathrm{T}^*\,\mathrm{SU}(n)$ into a Kähler manifold.**

Assuming the claim, let $\mathrm{SU}(n)$ act on $\mathrm{SU}(n) \times \mathfrak{su}(n)^*$ by

$$g \cdot (h, \xi) = (h, \mathrm{Ad}_g^*(\xi))$$

and define a map $\mu : \mathrm{SU}(n) \times \mathfrak{su}(n)^*$ by $\mu(h, \xi) = \xi$. Then $\mu$ is equivariant, and

**Unchecked claim: $\mu$ defines a moment map.**

Assuming this claim, then for $\xi \in \mathfrak{su}(n)^*$, $\mu^{-1}(\xi)$ is a copy of $\mathrm{SU}(n)$, and so if we perform Kähler reduction with momentum $\xi$, then we get a Kähler structure on the coadjoint orbit

$$\frac{\mathrm{SU}(n)}{\mathrm{Stab}(\xi)} \cong \mathrm{Orb}(\xi)$$

# A Kähler reduction

This section is copied from `kahler-reduction.tex`. It is included for completeness as the "proof" from §8 uses the Kähler reduction.

Throughout,

1. $(M, \omega, g, I)$ is a Kähler manifold,
2. $G$ is a compact Lie group acting on $M$,
3. $(M, \omega, G, \mu)$ is a Hamiltonian $G$-space,
4. $G$ acts by biholomorphisms on $M$,
5. $G$ acts freely on $\mu^{-1}(0)$.

In particular, as $\omega(u, v) = g(I(u), v)$, $I$ is an isometry on $M$, $G$ acts by isometries on $M$. Let $Z = \mu^{-1}(0)/G$. Let

$$\begin{tikzcd} \mu^{-1}(0) \arrow[r, hook, "i"] \arrow[d, two heads, "\pi"'] & M \\ Z = \mu^{-1}(0)/G & \end{tikzcd}$$

be the natural inclusion and quotient maps. In particular, note that $\pi$ is a surjective submersion. Therefore, any tensor $\alpha$ of type $(0, r)$ on $Z$ is determined by its pullback $\pi^*\alpha$.

The Marsden-Weinstein reduction theorem from symplectic geometry states that there exists a symplectic form $\widetilde{\omega}$ on $Z$, such that

$$\pi^*\widetilde{\omega} = i^*\omega$$

We will now construct the almost complex structure and Riemannian metric on $Z$.

Since $\pi$ is a submersion, for $p \in \mu^{-1}(0)$, $z = \pi(p)$, we have

$$\mathrm{d}\pi_p : \mathrm{T}_p\mu^{-1}(0) \twoheadrightarrow \mathrm{T}_z Z$$

Let $V_p = \ker(\mathrm{d}\pi_p)$ be the vertical bundle, and $H_p = V_p^{\perp} \leq \mathrm{T}_p\mu^{-1}(0)$ be the horizontal bundle. Therefore, we have an isomorphism

$$\mathrm{d}\pi_p|_{H_p} : H_p \cong \mathrm{T}_z Z$$

For brevity, we write this isomorphism as

$$\begin{aligned} \mathrm{d}\pi_p|_{H_p} : H_p &\to \mathrm{T}_z Z \\ v &\mapsto v_* \\ w^* &\leftarrow\!\shortmid w \end{aligned}$$

With this, we can see that

$$\widetilde{\omega}(u, v) = \omega(u^*, v^*)$$

and that

$$\widetilde{g}(u, v) = g(u^*, v^*)$$

defines a Riemannian metric on $Z$. Therefore, the almost complex structure we want must be given by

$$\widetilde{I}(u) = I(u^*)_*$$

Assuming this is well defined, then we have that

$$\begin{aligned}\widetilde{\omega}(u, v) &= \omega(u^*, v^*) \\ &= g(I(u^*), v^*) \\ &= g(\widetilde{I}(u)^*, v^*) \\ &= \widetilde{g}(\widetilde{I}(u), v)\end{aligned}$$

so $(\widetilde{\omega}, \widetilde{g}, \widetilde{I})$ is a compatible triple.

**Lemma.** $I$ restricts to a map $H_p \to H_p$.

*Proof.* Let $N_p = \left(\mathrm{T}_p\mu^{-1}(0)\right)^\perp \leq \mathrm{T}_pM$ be the normal bundle of $\mu^{-1}(0) \subseteq M$. This gives us an orthogonal direct sum

$$\mathrm{T}_pM = N_p \oplus V_p \oplus H_p$$

Fix $X \in \mathfrak{g}$. Then for $v \in \mathrm{T}_pM$,

$$g(\mathrm{grad}(\mu^X), v) = \mathrm{d}\mu^X(v) = \omega(X^\#, v) = g(I(X^\#), v)$$

where $\mathrm{grad}(f)$ is the $g$-dual of $\mathrm{d}f$. In particular, this means that $\mathrm{grad}(\mu^X) = I(X^\#)$. Let $X_1, \ldots, X_k$ be a basis of $\mathfrak{g}$, with corresponding dual basis $\xi^1, \ldots, \xi^k$. Then the moment map can be written as

$$\mu(p) = \mu^{X_1}(p)\xi^1 + \cdots + \mu^{X_k}(p)\xi^k$$

But this means that

$$\left\{\mathrm{grad}(\mu^{X_1}), \ldots, \mathrm{grad}(\mu^{X_k})\right\} = \left\{I(X_1^\#), \ldots, I(X_k^\#)\right\}$$

is a basis of $N_p$. As $X_1^\#, \ldots, X_k^\#$ is a basis for $V_p$, we have that $I$ restricts to a map $N_p \oplus V_p \to N_p \oplus V_p$. By orthogonality, this means that $I$ restricts to a map $H_p \to H_p$. □

Therefore, the map $\widetilde{I}$ as above is well defined. Finally, we need to show that we have a Kähler structure. That is, $I$ is integrable.

**Lemma.** Let $M$ be a manifold, $(\omega, g, I)$ a compatible triple on $M$. Then $(M, \omega, g, I)$ is a Kähler manifold if and only if $\nabla I = 0$, where $\nabla$ is the Levi-Civita connection induced by $g$.

Moreover, we have the expression

$$\nabla I(u) = \nabla(I(u)) - I(\nabla u)$$

and so $\nabla I = 0$ if and only if $\nabla(I(u)) = I(\nabla u)$ for all vector fields $u$.

*Proof.* See Huybrechts, §4.A. for the first part. For the second part, see Nicolaescu page 96. □

**Lemma.** The Levi-Civita connection induced by $\widetilde{g}$ is

$$\widetilde{\nabla}_X Y = \mathrm{pr}_H(\nabla_{X^*} Y^*)_*$$

for vector fields $X, Y$ on $Z$, and we extend $X^*, Y^*$ arbitrarily to a neighbourhood of $\mu^{-1}(0) \subseteq M$. In addition, $\mathrm{pr}_H : \mathrm{T}_pM \to H_p$ is the orthogonal projection.

*Proof.* Omitted. □

Finally, we note that since $I$ respects the orthogonal decomposition

$$\mathrm{T}_p M = \left(N_p \oplus V_p\right) \oplus H_p$$

$\mathrm{pr}_H$ and $I$ commute. With this, we can now compute $\widetilde{\nabla}\widetilde{I}$.

$$\begin{aligned}
\left(\widetilde{\nabla}_X \widetilde{I}(Y)\right)^* &= \mathrm{pr}_H \left(\nabla_{X^*} \widetilde{I}(Y)^*)\right) \\
&= \mathrm{pr}_H \left(\nabla_{X^*} I(Y^*)\right) \\
&= \mathrm{pr}_H \left(I(\nabla_{X^*} Y^*)\right) \\
&= I \left(\mathrm{pr}_H(\nabla_{X^*} Y^*)\right) \\
&= \widetilde{I}(\widetilde{\nabla}_X Y)^*
\end{aligned}$$

Hence we have that

$$\widetilde{\nabla}_X \widetilde{I}(Y) = \widetilde{I}(\widetilde{\nabla}_X Y)$$

for any vector fields $X, Y$ on $Z$, and so $\widetilde{\nabla}\widetilde{I} = 0$, and $(Z, \widetilde{\omega}, \widetilde{g}, \widetilde{I})$ is a Kähler manifold.