

# Semisimple Lie Algebras

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This document sketches the definition of semisimple Lie algebras and the Killing form. In addition, we will look at representations, and the root space decomposition. The main reference is Humphreys' *Introduction to Lie Algebras and Representation Theory*, where all of the skipped proofs can be found.

Throughout, let  $F$  be an algebraically closed field, with  $\text{char}(F) = 0$ ,  $\mathfrak{g}$  is a Lie algebra over  $F$ .

## 1 Definitions

### Definition 1.1 (Lie Algebra)

Let  $\mathfrak{g}$  be a vector space over  $F$ . Suppose  $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  is a bilinear form such that

1.  $[x, x] = 0$  for all  $x \in \mathfrak{g}$ ,
2. (Jacobi identity)  $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$  for all  $x, y, z \in \mathfrak{g}$ .

Then  $(\mathfrak{g}, [\cdot, \cdot])$  is called a *Lie algebra* over  $F$ .

### Definition 1.2 (abelian)

$\mathfrak{g}$  is called *abelian* if  $[x, y] = 0$  for all  $x, y \in \mathfrak{g}$ .

### Definition 1.3 (subalgebra)

A subspace  $\mathfrak{h} \subseteq \mathfrak{g}$  is called a (Lie) *subalgebra* if for all  $x, y \in \mathfrak{h}$ ,  $[x, y] \in \mathfrak{h}$ .

### Definition 1.4 (Ideal)

A subspace  $I$  of  $\mathfrak{g}$  is called an *ideal* if for all  $x \in \mathfrak{g}$  and  $y \in I$ ,  $[x, y] \in I$ .

**Proposition 1.5.** Every ideal is a subalgebra.

### Definition 1.6 (centre)

The *centre* of  $\mathfrak{g}$  is the ideal

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, z] = 0 \text{ for all } z \in \mathfrak{g}\}$$

### Definition 1.7 (derived algebra)

The *derived algebra* of  $\mathfrak{g}$  is the ideal  $[\mathfrak{g}, \mathfrak{g}]$ .

**Proposition 1.8.** The following are equivalent:

1.  $\mathfrak{g}$  is abelian,
2.  $Z(\mathfrak{g}) = \mathfrak{g}$ ,
3.  $[\mathfrak{g}, \mathfrak{g}] = 0$ .

**Definition 1.9** (simple)

Suppose  $\mathfrak{g}$  is not simple, and the only ideals are 0 and  $\mathfrak{g}$ . Then we say  $\mathfrak{g}$  is *simple*.

**Definition 1.10** (homomorphism)

Suppose  $\mathfrak{g}, \mathfrak{h}$  are Lie algebras over  $F$ . Then a linear map  $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$  is called a *homomorphism* if for all  $x, y \in \mathfrak{g}$ ,  $\phi([x, y]) = [\phi(x), \phi(y)]$ .

**Proposition 1.11.** Suppose  $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$  is a homomorphism. Then  $\ker(\phi)$  is an ideal, and  $\text{im}(\phi)$  is a subalgebra. Moreover,  $\mathfrak{g}/\ker(\phi) \cong \text{im}(\phi)$ .

## 2 Solvable Lie algebras

**Definition 2.1** (Derived series, solvable)

Define the sequence of ideals by

$$\mathfrak{g}^{(0)} = \mathfrak{g} \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$$

The sequence  $\mathfrak{g}^{(i)}$  is called the *derived series* of  $\mathfrak{g}$ . We say that  $\mathfrak{g}$  is solvable if  $\mathfrak{g}^{(n)} = 0$  for some  $n$ .

**Proposition 2.2.**  $\mathfrak{g}$  has a unique maximal solvable ideal.

*Proof.* Existence follows by Zorn's lemma. Suppose  $S$  is a maximal solvable ideal for  $\mathfrak{g}$ ,  $I$  is any solvable ideal. Then  $S + I$  is solvable, and  $S + I \supseteq S$ , so  $S + I = S$ . Thus  $I \subseteq S$ , so  $S$  is unique.  $\square$

**Definition 2.3** (radical, semisimple)

The unique maximal solvable ideal of  $\mathfrak{g}$  is called the *radical* of  $\mathfrak{g}$ , denoted  $\text{rad}(\mathfrak{g})$ . We say that  $\mathfrak{g}$  is *semisimple* if  $\text{rad}(\mathfrak{g}) = 0$ .

**Lemma 2.4.**  $\mathfrak{g}/\text{rad}(\mathfrak{g})$  is semisimple.

**Proposition 2.5.**  $\mathfrak{g}$  is semisimple if and only if it has no nonzero abelian ideals.

*Proof.* Any nonzero abelian ideal would be contained in  $\text{rad}(\mathfrak{g})$ , as any abelian Lie algebra is solvable. Conversely, suppose  $\text{rad}(\mathfrak{g})$  is nonzero. Then the last nonzero term  $\text{rad}(\mathfrak{g})^{(n-1)}$  of the derived series satisfies

$$[\text{rad}(\mathfrak{g})^{(n-1)}, \text{rad}(\mathfrak{g})^{(n-1)}] = \text{rad}(\mathfrak{g})^{(n)} = 0$$

Moreover,  $\text{rad}(\mathfrak{g})^{(n-1)}$  is an ideal of  $\mathfrak{g}$ . □

### 3 Killing form

Suppose in addition that  $\mathfrak{g}$  is finite dimensional.

**Definition 3.1** (adjoint representation)

The *adjoint representation* is the homomorphism  $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  defined by  $\text{ad}(x)(y) = [x, y]$ .

**Definition 3.2** (Killing form)

The *Killing form* of  $\mathfrak{g}$  is a symmetric bilinear form on  $\mathfrak{g}$ , defined by

$$\kappa(x, y) = \text{tr}(\text{ad}(x) \cdot \text{ad}(y))$$

**Proposition 3.3.**  $\kappa$  is associative, that is,

$$\kappa([x, y], z) = \kappa(x, [y, z])$$

**Lemma 3.4.** Let  $I$  be an ideal of  $\mathfrak{g}$ . Then  $I$  is a Lie algebra, with Killing form  $\kappa_I$ . Then  $\kappa_I = \kappa|_{I \times I}$ .

**Definition 3.5** (radical, nondegenerate)

Suppose  $\beta$  is a symmetric bilinear form on  $\mathfrak{g}$ . Define the *radical* of  $\beta$  to be

$$\text{rad}(\beta) = \{x \in \mathfrak{g} \mid \beta(x, y) = 0 \text{ for all } y \in \mathfrak{g}\}$$

Then  $\text{rad}(\beta)$  is a subspace of  $\mathfrak{g}$ . We say that  $\beta$  is *nondegenerate* if  $\text{rad}(\beta) = 0$ .

**Proposition 3.6.**  $\text{rad}(\kappa)$  is an ideal of  $\mathfrak{g}$ .

**Lemma 3.7.** Let  $x_1, \dots, x_n$  be a basis of  $\mathfrak{g}$ . Then  $\kappa$  is nondegenerate if and only if the matrix  $(\kappa(x_i, x_j))$  is invertible.

**Theorem 3.8.** Let  $\mathfrak{g}$  be a Lie algebra. Then  $\mathfrak{g}$  is semisimple if and only if  $\kappa$  is nondegenerate.

**Theorem 3.9.** Let  $\mathfrak{g}$  be a semisimple Lie algebra. Then there exists ideals  $I_1, \dots, I_r$  of  $\mathfrak{g}$ , such that

$$\mathfrak{g} = I_1 \oplus \dots \oplus I_r$$

as vector spaces. Every simple ideal of  $\mathfrak{g}$  is one of the  $I_j$ , and the Killing form of  $I_j$  is  $\kappa|_{I_j \times I_j}$ .

**Corollary 3.10.** If  $\mathfrak{g}$  is semisimple, then  $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ , and all ideals and quotients of  $\mathfrak{g}$  are semisimple. Moreover, each ideal of  $\mathfrak{g}$  is a direct sum of simple ideals in  $\mathfrak{g}$ .

## 4 Representations

**Definition 4.1** (representation)

A *representation* of  $\mathfrak{g}$  is a homomorphism  $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ , where  $V$  is a vector space over  $F$ .

**Definition 4.2** ( $\mathfrak{g}$ -module)

Let  $V$  be a vector space, then  $V$  is a  $\mathfrak{g}$ -module if there exists a map  $\mathfrak{g} \times V \rightarrow V$ ,  $(x, v) \mapsto x \cdot v$ , such that

1.  $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$
2.  $x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w)$
3.  $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$

**Proposition 4.3.** Suppose  $(\phi, V)$  is a representation of  $\mathfrak{g}$ . Then

$$x \cdot v = \phi(x)(v) \tag{*}$$

makes  $V$  into a  $\mathfrak{g}$ -module. Conversely, if  $V$  is a  $\mathfrak{g}$ -module, then  $(*)$  defines a representation of  $\mathfrak{g}$  on  $V$ .

**Definition 4.4** (homomorphism)

A *homomorphism* of  $\mathfrak{g}$ -modules is a linear map  $\phi : V \rightarrow W$  such that  $x \cdot \phi(v) = \phi(x \cdot v)$ .

**Definition 4.5** (irreducible)

A  $\mathfrak{g}$ -module  $V$  is *irreducible* if it has precisely two  $\mathfrak{g}$ -submodules,  $0$  and  $V$ . Note in particular  $0$  is not irreducible.

**Definition 4.6** (completely reducible)

A  $\mathfrak{g}$ -module  $V$  is *completely reducible* if it is a direct sum of irreducible  $\mathfrak{g}$ -modules. Equivalently, for each  $\mathfrak{g}$ -submodule  $W$  of  $V$ , there exists a  $\mathfrak{g}$ -submodule  $W'$  of  $V$  such that  $V = W \oplus W'$ .

**Lemma 4.7** (Schur). Suppose  $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  is an irreducible representation, then the only endomorphisms of  $V$  commuting with all  $\phi(x)$  are scalar multiples of the identity.

**Definition 4.8** (dual module)

Let  $V$  be a  $\mathfrak{g}$ -module. Then the dual vector space  $V^*$  is an  $\mathfrak{g}$ -module, with action defined by

$$(x \cdot f)(v) = -f(x \cdot v)$$

**Definition 4.9** (tensor module)

Let  $V, W$  be  $\mathfrak{g}$ -modules. Then  $V \otimes W$  is a  $\mathfrak{g}$ -module, with action defined by

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w)$$

**Definition 4.10** (Hom module)

Let  $V, W$  be  $\mathfrak{g}$ -modules. Then  $\text{Hom}(V, W) \simeq V^* \otimes W$  is a  $\mathfrak{g}$ -module, with  $\mathfrak{g}$  action defined by

$$(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v)$$

## 4.1 Weyl's theorem

**Lemma 4.11.** Let  $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  be a representation of a semisimple Lie algebra. Then  $\phi(\mathfrak{g}) \subseteq \mathfrak{sl}(V)$ . In particular,  $\mathfrak{g}$  acts trivially on any one-dimensional  $\mathfrak{g}$ -module.

**Theorem 4.12** (Weyl). Let  $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  be a (f.d.) representation of a semisimple Lie algebra. Then  $\phi$  is completely reducible.

## 5 Representations of $\mathfrak{sl}_2(F)$

In this section, we will write  $\mathfrak{sl}_2$  instead of  $\mathfrak{sl}_2(F)$ . Let

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the standard basis of  $\mathfrak{sl}_2$ . Then  $[h, x] = 2x$ ,  $[h, y] = -2y$ ,  $[x, y] = h$ .