"Classical nilpotent orbits as hyperkähler quotients" by Kobak and Swann

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In this note, we follow [2] and construct a hyperKähler metric on the adjoint orbits of $SL(n, \mathbb{C})$ using a finite dimensional hyperKähler quotient.

1 Flat HyperKähler space

First, choose a sequence $(V_0, \ldots, V_k)$ of Hermitian vector spaces, with $\dim_{\mathbb{C}}(V_i) = n_i$, $n_0 = 0$ and $n_k = n$. Define

$$M = \bigoplus_{j=0}^{k-1} \left(\operatorname{Hom}_{\mathbb{C}}(V_j, V_{j+1}) \oplus \operatorname{Hom}_{\mathbb{C}}(V_{j+1}, V_j) \right)$$

We will write a point in M in terms of linear maps (α_j, β_j), where $\alpha_j : V_j \to V_{j+1}$ and $\beta_j : V_{j+1} \to V_j$. The quaternionic structure on M is given by

$$I(\alpha_j, \beta_j) = (i\alpha_j, i\beta_j) \qquad J(\alpha_j, \beta_j) = (-\beta_j^*, \alpha_j^*)$$

where * denotes the Hermitian adjoint. That is, if we choose a basis for each V_j, the conjugate transpose. The metric on M is given by the standard Euclidean inner product, that is,

$$\left\| (\alpha_j, \beta_j) \right\|^2 = \sum_{j=0}^{k-1} \operatorname{Re} \operatorname{tr} \left(\alpha_j^* \alpha_j + \beta_j \beta_j^* \right)$$

This metric, along with the complex structures $I, J, K = IJ$, makes M into a flat hyperKähler manifold.

2 HyperKähler quotient

Now define the Lie group

$$G = U(n_1) \times \cdots \times U(n_{k-1})$$

This acts on M by

$$g \cdot (\alpha_j, \beta_j) = (g_{j+1} \alpha_j g_j^{-1}, g_j \beta_j g_{j+1}^{-1}) \tag{1}$$

where $g = (g_0, \ldots, g_k)$, $g_0 = g_k = 1$, $g_j \in U(n_j)$ for $1 \leq j \leq k-1$. The corresponding hyperKähler moment map for this action is

$$\mu = i\mu_r + 2k\mu_c : M \to \mathfrak{g}^* \otimes \operatorname{Im}(\mathbb{H})$$

where $\mathfrak{g} \cong \mathfrak{g}^* \cong \mathfrak{u}(n_1) \oplus \cdots \oplus \mathfrak{u}(n_{k-1})$. The real moment map is

$$\mu_r(\alpha_j, \beta_j) = (\alpha_{j-1} \alpha_{j-1}^* - \beta_{j-1}^* \beta_{j-1} + \beta_j \beta_j^* - \alpha_j^* \alpha_j) \in \mathfrak{g} \otimes i\mathbb{R} \cong i\mathfrak{g}$$

and the complex moment map is

$$\mu_c(\alpha_j, \beta_j) = (\alpha_{j-1} \beta_{j-1} - \beta_j \alpha_j) \in \mathfrak{g} \otimes \mathbb{C}$$

Using this, we can consider various (hyper)Kähler quotients. In particular, we would like to show that the Kähler quotient $\mu_r^{-1}(0)/G$ is homeomorphic to the complex quotient $G^{\mathbb{C}}\mu_r^{-1}(0)/\!\!/G^{\mathbb{C}}$, where $G^{\mathbb{C}} = \mathrm{GL}(n_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(n_{k-1}, \mathbb{C})$ is the complexification of G, and $G^{\mathbb{C}}\mu_r^{-1}(0)/\!\!/G^{\mathbb{C}}$ is the set of closed $G^{\mathbb{C}}$ orbits in $G^{\mathbb{C}}\mu_r^{-1}(0)$.

Define $f : M \to \mathbb{R}$, $f(x) = \|\mu_r\|^2$. For a fixed point $x_0 \in M$, the path of steepest descent of x under f is a function $x : [0, \infty) \to M$, such that

$$x(0) = x_0$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\nabla f(x(t))$$

From results of Kirwan[1], see also [2, Theorem 2.2], it suffices to show that for all $x_0 \in M$, the path of steepest descent of x under f is bounded, and the limit points lie in $\mu_r^{-1}(0)$. Noting that $f(x) \leq \|x\|^4$, we have thet the paths of steepest descent for f are bounded. To show that the limit points lie in $\mu_r^{-1}(0)$, we note that $\mu_r^* = \mu_r$, and so

$$\nabla f = 2(\mathrm{d}\mu_r)\mu_r$$

and hence the critical points of μ_r are when $\mu_r = 0$. With all of this in mind, the natural map

$$\mu_r^{-1}(0)/G \to G^{\mathbb{C}}\mu_r^{-1}(0)/\!\!/G^{\mathbb{C}}$$
$$[x] \mapsto [x]$$

is a homeomorphism. Now consider $\mu^{-1}(0) = \mu_r^{-1}(0) \cap \mu_c^{-1}(0)$, then the hyperKähler quotient $\mu^{-1}(0)/G$ is a submanifold of $\mu_r^{-1}(0)/G$, which using the above identification, corresponding to closed $G^{\mathbb{C}}$ orbits of $\mu_c^{-1}(0)$.

Therefore, in the next section, we will consider closed $G^{\mathbb{C}}$ orbits in the complex quotient $\mu_c^{-1}(0)/G^{\mathbb{C}}$.

3 Complex quotient

Define a map

$$X : M \to \mathrm{End}(\mathbb{C}^n)$$
$$X(\alpha_j, \beta_j) = \alpha_{k-1}\beta_{k-1}$$

If $\mu_c(\alpha_j, \beta_j) = 0$, then

$$X^2 = \alpha_{k-1}\beta_{k-1}\alpha_{k-1}\beta_{k-1} = \alpha_{k-1}\alpha_{k-2}\beta_{k-2}\beta_{k-1}$$

and in general,

$$X^k = \alpha_{k-1}\cdots\alpha_0\beta_0\cdots\beta_{k-1} = 0$$

Therefore, X is a nilpotent matrix. Moreover, let $G^{\mathbb{C}}$ act via eq. (1). Then this action preserves X, and also preserves $\mu_c^{-1}(0)$ setwise. Therefore, we have a well defined map

$$\Phi^c : \mu_c^{-1}(0)/G^{\mathbb{C}} \to \mathcal{N}$$
$$\Phi^c([\alpha_j, \beta_j]) = \alpha_{k-1}\beta_{k-1}$$

where $\mathcal{N} \subseteq \mathfrak{sl}(n, \mathbb{C})$ is the variety of nilpotent matrices. The main theorem is

Theorem 3.1. The map Φ^c, restricted to the set of closed $G^{\mathbb{C}}$ orbits, is injective. Furthermore, its image consistes of a union of closures of nilpotent orbits in $\mathfrak{sl}(n, \mathbb{C})$. If there exists $X \in \mathfrak{sl}(n, \mathbb{C})$ such that $\mathrm{rank}(X^i) = n_{k-i}$ for $i = 0, \ldots, k$, then the image is precisely the closure of the nilpotent orbit containing X.

Proof. First of all, notice that we have a $\mathrm{GL}(n, \mathbb{C})$ action, using eq. (1), setting $g_0 = \cdots = g_{k-1} = 1$ and $g_k \in \mathrm{GL}(n, \mathbb{C})$. In this case, the action preserves the set $\mu_c^{-1}(0)$, and we have that

$$X(g \cdot (\alpha_j, \beta_j)) = gX(\alpha_j, \beta_j)g^{-1}$$

Therefore, the image of Φ^c is a union of nilpotent orbits.

3.1   Injectivity

Let X be a point in the image, (α_j, β_j) be a point in a closed $G^{\mathbb{C}}$ orbit, with $X(\alpha_j, \beta_j) = X$. We will show that this orbit is unique.

Jordan Normal Form. Using the $\mathrm{GL}(n, \mathbb{C})$ action, we may assume without loss of generality that X is in Jordan Normal Form, and we can write

$$V_k = V_k^1 \oplus \cdots \oplus V_k^r$$

where each V_k^j is a cyclic subspace for X. Set $V_{k-1}^i = \beta_{k-1}(V_k^i) \subseteq V_{k-1}$. Since $X = \alpha_{k-1}\beta_{k-1}$ preserves V_k^i, we must have that $\alpha(V_{k-1}^i) \subseteq V_k^i$, and by considering the action of a nilpotent Jordan block, we must also have that

$$\dim(V_{k-1}^i) \geq \dim(V_k^i) - 1$$

More generally, if we set

$$V_j^i = \beta_j(V_{j+1}^i)$$

and assume inductively that $\alpha_{j+1}(V_{j+1}^i) \subseteq V_{j+2}^i$, then

$$\alpha_j(V_j^i) = \alpha_j\beta_j(V_{j+1}^i) = \beta_{j+1}\alpha_{j+1}(V_{j+1}^i) \subseteq \beta_{j+1}(V_{j+2}^i) = V_{j+1}^i$$

Therefore, we can now assume without loss of generality that $r = 1$, i.e. X is a single nilpotent Jordan block.

β_j **surjective.** We will now show that we can assume that each β_j is surjective. For each j, ket V_i^0 be such that

$$V_j = V_j^0 \oplus \mathrm{Im}(\beta_j)$$

We will use the $G^{\mathbb{C}}$ action to modify (α_j, β_j) such that $\alpha_{k-1}|_{V_{k-1}^0} = 0$. Let $g_j \in \mathrm{GL}(n_j, \mathbb{C})$ be multiplication by λ on V_j^0, and id on $\mathrm{Im}(\beta_j)$. Then with respect to the splitting, $g = (g_1, \dots, g_{k-1})$ acts, via eq. (1) as

$$\alpha_{k-1} = \begin{pmatrix} A_{11} & A_{12} \end{pmatrix} \mapsto \begin{pmatrix} \lambda^{-1}A_{11} & A_{12} \end{pmatrix}$$
$$\beta_{k-1} = \begin{pmatrix} 0 \\ B_{21} \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ B_{21} \end{pmatrix}$$
$$\alpha_j = \begin{pmatrix} A_{j11} & 0 \\ A_{j21} & A_{j22} \end{pmatrix} \mapsto \begin{pmatrix} A_{j11} & 0 \\ \lambda^{-1}A_{j21} & A_{j22} \end{pmatrix}$$
$$\beta_j = \begin{pmatrix} 0 & 0 \\ B_{j21} & B_{j22} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ \lambda^{-1}B_{j21} & B_{j22} \end{pmatrix}$$

Note that the top left entry of α_j is zero, since

$$\alpha_j(\beta_j(v)) = \beta_{j+1}\alpha_{j+1}(v) \in \mathrm{Im}(\beta_{j+1})$$

Since we assumed the $G^{\mathbb{C}}$ orbit is closed, letting $|\lambda| \to \infty$ gives us a new point with $A_{11} = 0$. By repeating the process above, using $g = (g_1, \dots, g_{k-2}, 1), \dots, g = (g_1, 1, \dots, 1)$, we can get

$$\alpha_j|_{V_j^0} = 0$$

as required.

α_j **injective.** A very similar argument to the above shows that we can assume that each α_j is injective. This then means that $\dim(V_i) = \dim(V_{i-1}) + 1$ for all i, by considering the rank of X^i.

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Standard form for β_j Using the $G^{\mathbb{C}}$ action, we can consider a change of basis, so that each β_j is of the form

$$
\beta_j = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}
$$

which follows by induction.

Each α_j is upper triangular. We will prove this by induction. The complex moment map equations state that

$$
\beta_j \alpha_j = \alpha_{j-1} \beta_{j-1}
$$

In particular, we have that

$$
0 = \beta_1 \alpha_1 = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{111} \\ \alpha_{121} \end{pmatrix} = \begin{pmatrix} \alpha_{121} \end{pmatrix}
$$

Therefore, α_1 is upper triangular. Now suppose α_{j-1} is upper triangular, i.e. $\alpha_{j-1,a,b} = 0$ for $a > b$. Then

$$
(\beta_{j-1} \alpha_{j-1})_{a,b} = \alpha_{j-1,a+1,b}
$$

which is zero if $a \geq b$, and $(\alpha_j \beta_j)_{a,b} = \alpha_{j,a,b-1}$. Therefore, $\alpha_{j,a,b} = 0$ for $a > b$, and the nonzero entries of α_{j-1} are determined by the nonzero entries of α_j.

Uniqueness. Now $X = \alpha_{k-1} \beta_{k-1}$ has (a, b) entry $\alpha_{k-1,a,b-1}$. Therefore, α_{k-1}, and thus all the other α_i, are uniquely determined by X. Thus, the orbit is unique.

3.2 Closures of nilpotent orbits

We will first need the following lemmas.

> **Lemma 3.2.** Let X be a nilpotent matrix. Then the numbers $\operatorname{rank}(X^i)$ determine the nilpotent orbit of X in $\mathfrak{sl}(n, \mathbb{C})$.

Proof. We may put X into Jordan normal form. In this case, the nilpotent orbits are determined by the sizes of the Jordan blocks. Using some combinatorics, we can recover the sizes of the Jordan blocks from the numbers $\operatorname{rank}(X^i)$. $\qquad\square$

> **Lemma 3.3.** Suppose X, Y are nilpotent $n \times n$ matrices. Then Y lies in the closure of the nilpotent orbit containing X if and only if $\operatorname{rank}(Y^i) \leq \operatorname{rank}(X^i)$ for all i.

Proof. Again using Jordan normal form, we can reduce to the case where X is a Jordan block. Noting that for all $\lambda \neq 0$,

$$
\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ \vdots & & & \ddots & \lambda \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}
$$

has Jordan normal form

$$
\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}
$$

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Taking the limit as $\lambda \to 0$ gives a Jordan block of size one smaller. Repeating this process, for any of the nonzero entries, we can show that any $n \times n$ nilpotent matrix is in the closure of the orbit of X. The rank condition just gives us the sizes of the Jordan blocks, and shows that this process works for any such X, Y. $\quad\square$

Note that if $X = \alpha_{k-1}\beta_{k-1}$, then we have an upper bound

$$\text{rank}(X^i) \leq n_{k-i} \tag{2}$$

and in fact (apart from the fact that X has to be nilpotent), this is the only constraint. Therefore, the image of Φ^c is a union of closures of nilpotent orbits, or equivalently, the union of orbits which satisfy eq. (2). Moreover, if there exists X such that $\text{rank}(X^i) = n_{k-i}$, then the image must be the closure of its nilpotent orbit. $\quad\square$

Remark 3.4. From the proof, we can see that in fact, we could have assumed $n_0 < n_1 < \cdots < n_k$ without loss of generality.

Combining this result with the discussion in section 2, we have the following:

Theorem 3.5. The hyperKähler quotient of M by G is a union of closures of nilpotent orbits in $\mathfrak{sl}(n, \mathbb{C})$. If there is a nilpotent element $X \in \mathfrak{sl}(n, \mathbb{C})$ with $\text{rank}(X^i) = n_{k-i}$ for all i, then the quotient is isomorphic to the closure of the nilpotent orbit containing X.

References

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[2] Piotr Z. Kobak and Andrew Swann. "Classical nilpotent orbits as hyperkähler quotients". In: *Int. J. Math.* 07.02 (Apr. 1996). Publisher: World Scientific Publishing Co., pp. 193–210. ISSN: 0129-167X. DOI: `10.1142/S0129167X96000116`. URL: `https://www.worldscientific.com/doi/10.1142/S0129167X96000116` (visited on 07/27/2023).