Kähler structure on T^*G

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Let G be a compact connected Lie group. We will show that T^*G is a Kähler manifold.

1 Tautological form and canonical symplectic form

Let M be any manifold. Suppose we have local coordinates $x_1, \dots, x_n$ on M. Then at each point $p \in M$, $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ is a basis for T^*_pM. In particular, for $\xi \in \mathrm{T}^*_pM$, we have that

$$\xi = \sum_i \xi_i \mathrm{d}x_i$$

The coodinates $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ are called *cotangent coordinates* on T^*M. With this, define

$$\alpha = \sum_i \xi_i \mathrm{d}x_i$$

which is a 1-form, and let

$$\omega = \sum_i \mathrm{d}x_i \wedge \mathrm{d}\xi_i = -\mathrm{d}\alpha$$

α and ω are intrinsically defined, that is, they are independent of the choice of coordinates. α is called the *tautological* 1*-form* and ω is called the *canonical symplectic form* on T^*M.

2 Complex structure on T^*G

First, we show that every Lie group is parallelisable, that is, we have a bundle isomorphism

$$\mathrm{T}G \cong G \times \mathfrak{g}$$

In particular, a bundle is trivial if and only if a collection of sections forming a fibrewise basis exists.

Let $x_1, \dots, x_n$ be local coordinates near $e \in G$. So

$$v_1 = \frac{\partial}{\partial x_1}, \dots, v_n = \frac{\partial}{\partial x_n}$$

is a basis for $\mathfrak{g}$. Moreover, by considering the left translations $\ell_g^* v_j$, we can see that these form a global frame for $\mathrm{T}G$. Hence we have a bundle isomorphism $\mathrm{T}G \cong \mathfrak{g} \times G$.

Next, we want to show that T^*G is also trivial. As G is compact, we can choose a bi-invariant Riemannian metric γ on G. Suppose without loss of generality that $v_1, \dots, v_n$ is an orthonormal basis of $\mathfrak{g}$. Then by bi-invariance, we have that $\ell_g^* v_1, \dots, \ell_g^* v_n$ is an orthonormal basis of $\mathrm{T}_{g^{-1}}G$.

Using the bi-invariant metric, we define

$$\eta_j = \gamma(\ell_g^* v, \cdot)$$

This is an orthonormal basis with respect to the induced metric on T^*G, and so it defines an isomorphism $\mathrm{T}^*G \cong \mathrm{T}G$. In particular, this gives us an isomorphism $\mathrm{T}^*G \cong G \times \mathfrak{g}$.

Finally, we note that we can put a complex structure on $G \times \mathfrak{g}$, by

$$J(x, y) = (-\mathrm{d}\ell_g y, \mathrm{d}\ell_{g^{-1}} x)$$

for $x \in \mathrm{T}_gG$, $y \in \mathrm{T}_z\mathfrak{g} = \mathfrak{g}$.

2.1 In cotangent coordinates

In cotangent coordinates about $e \in G$, where we assume as above that the v_i are orthonormal, we find that $\eta_j = \mathrm{d}x_j$. Moreover, the isomorphism above is given by

$$\mathrm{T}^*G \ni (p, \xi) \mapsto (p, (\xi_1, \ldots, \xi_n)) \in G \times \mathfrak{g}$$

Therefore, given $v \in \mathrm{T}_{(p,\xi)}\mathrm{T}^*G$, say $v = a + b$, where

$$v = \left(\sum_i a_i \frac{\partial}{\partial x_i}\right) + \left(\sum_j b_j \frac{\partial}{\partial \xi_j}\right)$$

we have that $J(a, b) = (-b, a)$.