Kähler reduction

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1 Preliminaries

Throughout, let (M, ω) be a symplectic manifold, G a compact Lie group, $\psi : G \to \mathrm{Sympl}(M, \omega)$ a symplectic (left) action of G on M.

Let $\psi(g)(p) = \psi_g(p) = \psi^p(g) = g \cdot p$. We call $\psi^p : G \to M$ the orbit map, the orbit of p is $G \cdot p$, and the stabiliser, or isotropy subgroup of G is denoted by G_p.

For $X \in \mathfrak{g}$, define the vector field $X^\#$ on M by

$$X^\#_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi(\exp(tX), p) = (\mathrm{d}\psi^p)_e(X_e)$$

and write $\hat{\psi} : \mathfrak{g} \to \mathfrak{X}(M)$ for the map $X \mapsto X^\#$.

1.1 Orbit space

In this subsection, we list some theorems, and basic facts about orbit spaces, which we shall not prove.

Theorem 1.1. Suppose G is a compact Lie group acting smoothly, freely and properly on a smooth manifold M. Then the quotient space M/G is a manifold, and has a unique smooth structure such that the natural projection $\pi : M \to M/G$ is a smooth submersion. Moreover, $\pi : M \to M/G$ has the structure of a principal G-bundle.

Proposition 1.2. The stabiliser G_p of a point p is a closed Lie subgroup of G. Moreover, the right action of G_p on G by right multiplication gives a compact smooth manifold G/G_p.

Proposition 1.3. The orbit $G \cdot p$ is a properly embedded submanifold of M, diffeomorphic to G/G_p. Moreover, the restriction of the orbit map to $G \cdot p$,

$$\psi^p : G \to G \cdot p$$

is a surjective smooth submersion, and its derivative at e induces a surjective linear map

$$\mathfrak{g} \to \mathrm{T}_p(G \cdot p)$$

given by $X \mapsto X^\#_p$, with kernel $\mathfrak{g}_p = \mathrm{Lie}(G_p)$. Therefore, we have that

$$\mathrm{T}_p(G \cdot p) \cong \mathfrak{g}/\mathfrak{g}_p$$

Proposition 1.4. Let $\pi : M \to M/G$ be the projection map. Then

$$\mathrm{T}_p(G \cdot p) = \ker(\mathrm{d}\pi_p)$$

and

$$\mathrm{T}_{\pi(p)}(M/G) \simeq \frac{\mathrm{T}_p M}{\mathrm{T}_p(G \cdot p)}$$

1.2 Hamiltonian vector fields

Let $H : M \to \mathbb{R}$ be a smooth function. Then

$$\mathrm{d}H = \frac{\partial H}{\partial x^j}\mathrm{d}x^j \in \Omega^1(M)$$

is a 1-form. On the other hand, if V is any vector field, then we can contract V and ω to get a 1-form

$$\iota_V \omega = V^i \omega_{ij} \mathrm{d}x^j$$

Since ω is non-degenerate, there exists a unique vector field X_H such that

$$\iota_{X_H} \omega = \mathrm{d}H$$

Definition 1.5 (Hamiltonian vector field)
X_H is called a *Hamiltonian vector field* with *Hamiltonian function H*.

1.3 Moment maps

We say that the action ψ is *Hamiltonian* if there exists a map

$$\mu : M \to \mathfrak{g}^*$$

such that

1. For each $X \in \mathfrak{g}$, let
   - $\mu^X : M \to \mathbb{R}$, $\mu^X(p) = \langle \mu(p), X \rangle$ be the component of μ along X,
   - $X^\#$ the vector field on M generated by the one-parameter subgroup $\{\exp(tX) : t \in \mathbb{R}\} \leq G$, that is,

   $$X^\#_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX) \cdot p$$

   Then

   $$\iota_{X^\#} \omega = \mathrm{d}\mu^X$$

   That is, $X^\#$ is a Hamiltonian vector field, with Hamiltonian function μ^X.

2. μ is equivariant with respect to the G action on M and the Ad^* action on $\mathfrak{g}^*$. That is,

   $$\begin{tikzcd} M \arrow[r, "\psi_g"] \arrow[d, "\mu"'] & M \arrow[d, "\mu"] \\ \mathfrak{g}^* \arrow[r, "\mathrm{Ad}^*_g"'] & \mathfrak{g}^* \end{tikzcd}$$

   commutes for all $g \in G$.

Definition 1.6 (Hamiltonian G-space, moment map)
(M, ω, G, μ) is called a *Hamiltonian G-space*, μ is called a *moment map*.

1.4 Prinical bundles

Definition 1.7 (vertical bundle, (Ehresmann) connection)
Let G be a Lie group, $\pi : M \to M/G$ a principal G-bundle. The *vertical bundle* $V \subseteq \mathrm{T}M$ is

$$V_p = \ker(\mathrm{d}\pi_p)$$

V is a G-invariant subbundle of $\mathrm{T}M$, and an *Ehresmann connection* on M is a G-invariant subbundle $H \subseteq \mathrm{T}M$ such that

$$\mathrm{T}_pM = V_p \oplus H_p$$

for all $p \in M$. We call H the *horizontal bundle*.

Lemma 1.8. $\mathrm{d}\pi : \mathrm{T}M \to \mathrm{T}(M/G)$ restricts to isomorphisms

$$\mathrm{d}\pi_p : H_p \cong \mathrm{T}_{\pi(p)}(M/G)$$

Proposition 1.9 (horizontal lift)**.** Let $\pi : M \to M/G$ be a principal G-bundle, $\mathrm{T}M = V \oplus H$ a connection. Every vector field $X \in \mathfrak{X}(M/G)$ has a unique *horizontal lift*. That is, a unique smooth G-invariant vector field $X^* \in \mathfrak{X}(M)$ such that

1. $X^*_p \in H_p$
2. $\mathrm{d}\pi_p(X^*_p) = X_{\pi(p)}$

for all $p \in M$. Conversely, given a G-invariant section Y of H, there exists a unique vector field $X \in \mathfrak{X}(M/G)$ such that $Y = X^*$. We write $X = \pi_*(Y)$. Moreover,

$$(fX)^* = (f \circ \pi)X^* \quad (X+Y)^* = X^* + Y^* \quad \pi_*((f\circ\pi)Y) = f\pi_*(Y) \quad \pi_*(X+Y) = \pi_*(X) + \pi_*(Y)$$

Proposition 1.10. Let ω be a G-invariant k-tensor on M. Then there exists a unique covariant k-tensor η on M/G such that

$$\eta(X_1, \dots, X_k) \circ \pi = \omega(X_1^*, \dots, X_k^*)$$

for all smooth vector fields $X_1, \dots, X_k$ on M/G.

2 Symplectic reduction

Proposition 2.1. Let $\mathfrak{g}_p$ be the Lie algebra of G_p for some $p \in M$. Then $\mathrm{d}\mu_p : \mathrm{T}_pM \to \mathfrak{g}^*$ has

$$\begin{aligned}\ker(\mathrm{d}\mu_p) &= \big(\mathrm{T}_p(G\cdot p)\big)^{\omega_p} = \{v \in \mathrm{T}_pM \mid \omega_p(u,v) = 0 \text{ for all } u \in \mathrm{T}_p(G\cdot p) \leq \mathrm{T}_pM\}\\ \mathrm{im}(\mathrm{d}\mu_p) &= \mathrm{Ann}(\mathfrak{g}_p) = \{\xi \in \mathfrak{g}^* \mid \langle \xi, X\rangle = 0 \text{ for all } X \in \mathfrak{g}_p\}\end{aligned}$$

Proof. First note that $G \cdot p$ is a properly embedded submanifold of M. For $X \in \mathfrak{g}$, we get a linear map $X : \mathfrak{g}^* \to \mathbb{R}$, so its derivative is itself. Therefore, we have that

$$\omega_p(X_p^\#, v) = \mathrm{d}\mu_p^X(v) = \mathrm{d}(X \circ \mu)_p\,(v) = X(\mathrm{d}\mu_p(v)) = \langle \mathrm{d}\mu_p(v), X\rangle$$

Moreover, we have that

$$\mathrm{T}_p(G \cdot p) = \left\{X_p^\# \mid X \in \mathfrak{g}\right\}$$

which means that

$$\begin{aligned}
\ker(\mathrm{d}\mu_p) &= \{v \in \mathrm{T}_pM \mid \langle \mathrm{d}\mu_p(v), X\rangle = 0 \text{ for all } X \in \mathfrak{g}\} \\
&= \{v \in \mathrm{T}_pM \mid \omega_p(X_p^\#, v) = 0 \text{ for all } X \in \mathfrak{g}\} \\
&= \{v \in \mathrm{T}_pM \mid \omega_p(w, v) = 0 \text{ for all } w \in \mathrm{T}_p(G \cdot p)\} \\
&= \left(\mathrm{T}_p(G \cdot p)\right)^{\omega_p}
\end{aligned}$$

Now let $v \in \mathrm{T}_pM$ be arbitrary. We know that the kernel of the map $X \mapsto X_p^\#$ is $\mathfrak{g}_p$, so for all $X \in \mathfrak{g}_p$,

$$\langle \mathrm{d}\nu_p(v), X\rangle = \omega_p(X_p^\#, v) = \omega_p(0, v) = 0$$

which means that $\mathrm{im}(\mathrm{d}\mu_p) \subseteq \mathrm{Ann}(\mathfrak{g}_p)$. Moreover,

$$\begin{aligned}
\dim(\mathrm{im}(\mathrm{d}\mu_p)) &= \dim(\mathrm{T}_pM) - \dim(\ker(\mathrm{d}\mu_p)) \\
&= \dim(\mathrm{T}_pM) - \dim(\left(\mathrm{T}_p(G \cdot p)\right)^{\omega_p}) \\
&= \dim(\mathrm{T}_p(G \cdot p)) \\
&= \dim(\mathfrak{g}/\mathfrak{g}_p) \\
&= \dim(\mathrm{Ann}(\mathfrak{g}_p))
\end{aligned}$$

So equality holds. □

Lemma 2.2. For $\xi \in \mathfrak{g}^*$, the stabiliser G_ξ of ξ, with the coadjoint action, acts freely on $\mu^{-1}(\xi)$ if and only if $G_p = \{e\}$ for all $p \in \mu^{-1}(\xi)$.

Proof. G_ξ acts freely on $\mu^{-1}(\xi)$ if and only if $(G_\xi)_p = \{e\}$ for all $p \in \mu^{-1}(\xi)$. Therefore, suffices to show that $(G_\xi)_p = G_\xi \cap G_p = G_p$. For one inclusion,

$$(G_\xi)_p = \{g \in G \mid \mathrm{Ad}_g^*(\xi) = \xi \text{ and } g \cdot p = p\} = G_\xi \cap G_p \subseteq G_p$$

Conversely, for the other inclusion, let $g \in G_p$, by equivariance,

$$\mathrm{Ad}_g^*(\xi) = \mathrm{Ad}_g^*\,\mu(p) = \mu(g \cdot p) = \mu(p) = \xi$$

Hence $g \in G_p \cap G_\xi$. □

Lemma 2.3. Suppose G_ξ acts freely on $\mu^{-1}(\xi)$. Then ξ is a regular value of μ, and $\mu^{-1}(\xi)/G_\xi$ is a smooth manifold.

Proof. In this case, we have that $G_p = \{e\}$, hence $\mathfrak{g}_p = 0$. This means that $\mathrm{im}(\mathrm{d}\mu_p) = \mathrm{Ann}(\mathfrak{g}_p) = \mathrm{Ann}(0) = \mathfrak{g}^*$, hence ξ is a regular value for μ. This means that $\mu^{-1}(\xi)$ is a submanifold of M, and so the action of G_ξ on M gives a smooth action on $\mu^{-1}(\xi)$. Since the action is free and G_ξ is compact, the quotient $\mu^{-1}(\xi)/G_\xi$ is a smooth manifold. □

Proposition 2.4. Suppose $\xi \in \mathfrak{g}^*$ is such that G_ξ acts freely on $\mu^{-1}(\xi)$. Then

(i) $\mu^{-1}(\xi)$ is a properly embedded submanifold of M, with

$$\mathrm{T}_p\mu^{-1}(\xi) = \left(\mathrm{T}_p(G \cdot p)\right)^{\omega_p}$$

(ii) for each $p \in \mu^{-1}(\xi)$, the orbit $G_\xi \cdot p$ is a properly embedded submanifold of $\mu^{-1}(\xi)$, and

$$\mathrm{T}_p(G_\xi \cdot p) = \left(\mathrm{T}_p\mu^{-1}(\xi)\right)^{\omega_p} \cap \mathrm{T}_p\mu^{-1}(\xi)$$

Proof. (i) follows from the fact that ξ is a regular value of μ, and we have that

$$\mathrm{T}_p(G_\xi \cdot p) = \ker(\mathrm{d}\mu_p) = \left(\mathrm{T}_p(G \cdot p)\right)^{\omega_p}$$

For (ii), let $p \in \mu^{-1}(\xi)$. Then we have that $G_\xi \cdot p \subseteq \mu^{-1}(\xi)$. Hence $G_\xi \cdot p$ is a properly embedded submanifold of $\mu^{-1}(\xi)$. In particular, $\mathrm{T}_p(G_\xi \cdot p) \subseteq \mathrm{T}_p\mu^{-1}(\xi)$. Combining this with (i) gives one inclusion.

Conversely, suppose

$$v \in \left(\mathrm{T}_p\mu^{-1}(\xi)\right)^{\omega_p} \cap \mathrm{T}_p\mu^{-1}(\xi) = \mathrm{T}_p(G \cdot p) \cap \ker(\mathrm{d}\mu_p)$$

Since $v \in \mathrm{T}_p(G \cdot p)$, we have that $v = X_p^\# = (\mathrm{d}\psi^p)_e(X)$ for some $X \in \mathfrak{g}$. Let $\widehat{\mathrm{Ad}^*}(X) \in \mathfrak{X}(\mathfrak{g}^*)$ be the vector field associated with the coadjoint action. Then we have that

$$(\mathrm{Ad}^*)^\xi(g) = \mathrm{Ad}_g^*\,\xi = \mathrm{Ad}_g^*\,\mu(p) = \mu(\psi_g(p)) = (\mu \circ \phi^p)(g)$$

for all $g \in G$. Hence

$$\widehat{\mathrm{Ad}^*}(X)_\xi = \mathrm{d}\left((\mathrm{Ad}^*)^\xi\right)_e(X) = \mathrm{d}(\mu \circ \psi^p)_e(X) = \mathrm{d}\mu_p(X_p^\#) = \mathrm{d}\mu_p(v) = 0$$

This means that $X \in \mathfrak{g}_\xi = \mathrm{Lie}(G_\xi)$. But then $X_p^\# \in \mathrm{T}_p(G_\xi \cdot p)$, and so $v = X_p^\# \in \mathrm{T}_p(G\xi \cdot p)$. □

Theorem 2.5 (Symplectic reduction). Let (M, ω, G, μ) be a Hamiltonian G-space, with G compact. Suppose $\xi \in \mathfrak{g}^*$ is such that $G_p = \{e\}$ for all $p \in \mu^{-1}(\xi)$. Then the orbit space $\mu^{-1}(\xi)/G_\xi$ is a smooth manifold, and there exists a unique symplectic form ω_ξ on $\mu^{-1}(\xi)/G_\xi$, such that $\pi^*\omega_\xi = i^*\omega$, where $\pi : \mu^{-1}(\xi) \to \mu^{-1}(\xi)/G_\xi$ is the quotient map, and $i : \mu^{-1}(\xi) \hookrightarrow M$ is the inclusion map.

We write $M/\!\!/_\xi G_\xi$ for this symplectic space.

Proof. We know that $\mu^{-1}(\xi)$ is a smooth manifold, on which the compact Lie group G_ξ acts smoothly and freely, which gives us a smooth manifold $\mu^{-1}(\xi)/G_\xi$, where $\pi : \mu^{-1}(\xi) \to \mu^{-1}(\xi)/G_\xi$ is a principal G_ξ bundle.

Let $\tilde{\psi} : G_\xi \times \mu^{-1}(\xi) \to \mu^{-1}(\xi)$ denote the action map, then $i \circ \tilde{\psi}_g = \psi_g \circ i$ for $g \in G_\xi$. Consider the 2-form $i^*\omega$ on $\mu^{-1}(\xi)$. We have that $i^*\omega$ is G_ξ invariant, since

$$\tilde{\psi}_g^*(i^*\omega) = (i \circ \tilde{\psi}_g)^*\omega = (\psi_g \circ i)^*\omega = i^*\psi_g^*\omega = i^*\omega$$

where $\psi_g^*\omega = \omega$ since the action of G on M is symplectic. Therefore, there exists a unique 2-form ω_ξ on $\mu^{-1}(\xi)/G_\xi$, such that $\pi^*\omega_\xi = i^*\omega$. Moreover, since π is a surjective submersion, π^* is injective. In particular, as

$$\pi^*(\mathrm{d}\omega_\xi) = \mathrm{d}\left(\pi^*\omega_\xi\right) = \mathrm{d}(i^*\omega) = i^*(\mathrm{d}\omega) = 0$$

we have that ω_ξ is closed. To show that it is non-degenerate, let $p \in \mu^{-1}(\xi)$, and $x = \pi(p) \in \pi^{-1}(\xi)/G_\xi$. Suppose $v \in \mathrm{T}_x\left(\mu^{-1}(\xi)/G_\xi\right)$ is such that $(\omega_\xi)_x(v, w) = 0$ for all $w \in \mathrm{T}_x\left(\mu^{-1}(\xi)/G_\xi\right)$. As π is a submersion, $v = \mathrm{d}\pi_p(\hat{v})$ for some $\hat{v} \in \mathrm{T}_p\mu^{-1}(\xi)$. For any $\hat{w} \in \mathrm{T}_p\mu^{-1}(\xi)$,

$$(i^*\omega)_p(\hat{v}, \hat{w}) = (\pi^*\omega_\xi)_p(\hat{v}, \hat{w}) = \omega_\xi(\mathrm{d}\pi_p(\hat{v}), \mathrm{d}\pi_p(\hat{w})) = 0$$

Therefore,

$$\hat{v} \in \left(\mathrm{T}_p\mu^{-1}(\xi)\right)^{\omega_p} \cap \mathrm{T}_p\mu^{-1}(\xi) = \mathrm{T}_p(G_\xi \cdot p) = \ker(\mathrm{d}\pi_p)$$

This means that $v = \mathrm{d}\pi_p(\hat{v}) = 0$, and so ω_ξ is non-degenerate. □

3 Riemannian quotient

Throughout, let g be a Riemannian metric on M, and with associated Levi-Civita connection ∇ on $\mathrm{T}M$ and tensor/duals of this. Suppose E is a subbundle of $(\mathrm{T}M)^{\otimes p} \otimes (\mathrm{T}^*M)^{\otimes q}$, with the Levi-Civita connection ∇ on E. Then for a section $s \in \Gamma(E)$, and a vector field X on M, we write

$$\nabla_X s = (\nabla s)(X)$$

where $\nabla s \in \mathrm{T}^*M \otimes E$, and so we contract the $\mathrm{T}M \times \mathrm{T}^*M$ factors.

Definition 3.1 (orthogonal connection)
Suppose M has a G-invariant Riemannian metric. Then we can define the *orthogonal connection* as

$$H_p = V_p^\perp$$

Throughout, H will be the orthogonal connection, and $\mathrm{pr}_H : V \oplus H \to H$ is the orthogonal projection.

Theorem 3.2 (metric on quotient manifold). Suppose $\pi : M \to M/G$ is a principal G-bundle, g is a G-invariant Riemannian metric on M. Then there exists a unique Riemannian metric $\overline{g}$ on M/G, such that $\overline{g}(X, Y) \circ \pi = g(X^*, Y^*)$ for all smooth vector fields X, Y on M/G. Moreover, the Levi-Civita connection $\overline{\nabla}$ of $\overline{g}$ is given by

$$\overline{\nabla}_X Y = \pi_* \left(\mathrm{pr}_H(\nabla_{X^*} Y^*)\right)$$

Theorem 3.3 (metric on submanifold). Let (M, g) be a Riemannian manifold, $\tilde{M} \subseteq M$ an embedded submanifold with the induced metric $\tilde{g} = i^*g$. Then the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$ is given by

$$\tilde{\nabla}_X Y = \mathrm{pr}(\nabla_X Y)$$

where the vector fields on the right are extensions of $X, Y \in \mathfrak{X}(\tilde{M})$ to a neighbourhood of $\tilde{M}$, and $\mathrm{pr} : \mathrm{T}M \to \mathrm{T}\tilde{M}$ is the orthogonal projection.

4 Kähler reduction

For simplicity, we will assume throughout that $\xi = 0$. We want to show

Theorem 4.1 (Kähler reduction). Let (M, ω, g, I) be a Kähler manifold, and G a compact Lie group acting on M isometrically and in a Hamiltonian way. Let μ be the moment map for the action. Suppose G acts freely on $\mu^{-1}(0)$.

Then the symplectic reduction $(M/\!\!/_0 G, \omega_0)$ and the quotient metric g_0 induced by i^*g, where $i : \mu^{-1}(0) \hookrightarrow M$ is the inclusion map, make $M/\!\!/G := M/\!\!/_0 G = \mu^{-1}(0)/G$ into a Kähler manifold.

4.1 Almost complex structure

Since we already have the symplectic form and the Riemannian metric, we need to define the almost complex structure.

Lemma 4.2. Let (M, g) be a Riemannian manifold, $f = (f_1, \ldots, f_k) : M \to \mathbb{R}^k$ be a smooth function with regular value c. Let $\tilde{M} = f^{-1}(c)$, then $\{\mathrm{grad}(f_1), \ldots, \mathrm{grad}(f_k)\}$ is a smooth global frame for the normal bundle $\mathrm{N}\tilde{M}$ over $\tilde{M}$.

Let $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ be the quotient map, V be the vertical bundle of this principal G-bundle, and H its orthogonal complement with respect to the metric i^*g on $\mu^{-1}(0) \subseteq M$. Then for $p \in \mu^{-1}(0)$, we have an orthogonal decomposition

$$\mathrm{T}_pM = N_p \oplus V_p \oplus H_p$$

where $N_p = (\mathrm{N}\mu^{-1}(0))_p$ is the normal bundle of $\mu^{-1}(0)$.

Lemma 4.3. The horizonal bundle H is invariant under I.

Proof. First note that for $X \in \mathfrak{g}$ and $Y \in \mathrm{T}_pM$, we have that

$$g(\mathrm{grad}(\mu^X), Y) = \mathrm{d}\mu^X(Y) = \omega(X^\#, Y) = g(IX^\#, Y)$$

where we use the fact that $g(\mathrm{grad}(f), \cdot) = \mathrm{d}f$ by the tangent-cotangent isomorphism given by g, and that μ^X is a Hamiltonian function for $X^\#$. Therefore, this means that

$$\nabla\left(\mu^X\right) = IX^\#$$

since g is non-degenerate. Now let $X_1, \dots, X_k \in \mathfrak{g}$ be a basis, with dual basis $\xi^1, \dots, \xi^k$ for $\mathfrak{g}^*$. Then we have that

$$\mu(p) = \mu^{X_1}(p)\xi^1 + \dots + \mu^{X_k}(p)\xi^k$$

Hence by the previous lemma, a global frame for N is

$$\{\mathrm{grad}(\mu^{X_1}), \dots, \mathrm{grad}(\mu^{X_k})\} = \{IX_1^\#, \dots, IX_k^\#\}$$

Moreover, since G acts freely on $\mu^{-1}(0)$, the map

$$\mathfrak{g} \to \mathrm{T}_p(G \cdot p) = \ker(\mathrm{d}\pi_p) = V_p$$

sending $X \to X_p^\#$ is an isomorphism. Therefore, $\{X_1^\#, \dots, X_k^\#\}$ is a basis for V_p. Hence

$$\{X_1^\#, \dots, X_k^\#, IX_1^\#, \dots, IX_k^\#\}$$

is a basis for $N_p \oplus V_p$, and so $N_p \oplus V_p$ is invariant under I. Now for $v \in H_p$, $w \in N_p \oplus V_p$, we have that

$$g(I(v), w) = -g(v, I(w)) = 0$$

Hence $I(v) \in (N_p \oplus V_p)^\perp = H_p$. □

Lemma 4.4. Let X be a smooth vector field on $M /\!\!/ G$, then the map $\mu^{-1}(0) \to H$, given by

$$p \mapsto I_p(X_p^*)$$

is a smooth G-invariant section of H, which we denote by IX^*.

Proof. Since I preserves H, it defines a bundle isomorphism $\bar{I} : H \to H$, and the map above is the composition $\bar{I} \circ X^*$. This is G-invariant since G perserves I and X^* is G-invariant. □

With this, we can define an almost complex structure on $M /\!\!/ G$ by

$$I_0X = \pi_*(IX^*)$$

for all smooth vector fields X on $M /\!\!/ G$.

Lemma 4.5. I defines a $(1,1)$-tensor field, and $I_0^2 = -\mathrm{id}$.

Proof. We need to show that I is C^∞ linear. But

$$I_0(fX) = \pi_*(I(fX)^*) = \pi_*(I((f\circ\pi)X)^*) = \pi_*((f\circ\pi)I_X^*) = f\pi_*(IX^*) = fI_0(X)$$

Moreover, by definition, we have that $(I_0X)^* = IX^*$, so

$$(I_0^2X)^* = (I_0(I_0(X)))^* = I(I_0(X))^* = I(I(X^*)) = -X^* = (-X)^*$$

and so $I_0^2X = -X$. □

4.2 Levi-Civita connection

Lemma 4.6. The Levi-Civita connection of g_0 is given by

$$\overline{\nabla}_X Y = \pi_*\left(\mathrm{proj}_H(\nabla_{X^*}Y^*)\right)$$

where X^*, Y^* are extensions of X, Y to a neighbourhood of $\mu^{-1}(0)$, ∇ is the Levi-Civita connection of g, and $\mathrm{proj}_H : \mathrm{T}M \to H$ is the orthogonal projection.

Proof. Let $\widetilde{\nabla}$ be the Levi-Civita connection on $\mu^{-1}(0)$. Then we have that

$$\widetilde{\nabla}_X Y = \mathrm{proj}_{\mathrm{T}\mu^{-1}(0)}\left(\nabla_X Y\right)$$

where on the right hand side, we extend X, Y to a neighbourhood of $\mu^{-1}(0)$. With this, we get that the Levi-Civita connection for g_0 is given by

$$\overline{\nabla}_X Y = \pi_*\left(\mathrm{proj}_H(\widetilde{\nabla}_{X^*}Y^*)\right) = \pi_*\left(\mathrm{proj}_H\left(\mathrm{proj}_{\mathrm{T}\mu^{-1}(0)}(\nabla_{X^*}Y^*)\right)\right) = \pi_*\left(\mathrm{proj}_H(\nabla_{X^*}Y^*)\right)$$

□

Lemma 4.7.

$$\overline{\nabla} I_0 = 0$$

Proof. Let X, Y be smooth vector fields on $M/\!\!/G$. Since I preserves H, it commutes with the projection $\mathrm{proj}_H : \mathrm{T}M \to H$. Moreover, we have that $(I_0Y)^* = IY^*$, hence

$$(\overline{\nabla}_X I_0 Y)^* = \mathrm{proj}_H(\nabla_{X^*}IY^*) = \mathrm{proj}_H(I\nabla_{X^*}Y^*) = I\,\mathrm{proj}_H(\nabla_{X^*}Y^*) = I(\overline{\nabla}_X Y)^*$$

Therefore, $\overline{\nabla}_X I_0 Y = I_0(\overline{\nabla}_X Y)$, and so $\overline{\nabla} I_0 = 0$. □

4.3 Compatibility

Proof of Kähler reduction. With the above, all thet remains is to show that (I_0, g_0, ω_0) are compatible. Let X, Y be smooth vector fields on $\mu^{-1}(0)/G$, $p \in \mu^{-1}(0)$, we have that

$$\begin{aligned}
\omega_0(X_{\pi(p)}, Y_{\pi(p)}) &= \omega_0(\mathrm{d}\pi_p(X_p^*), \mathrm{d}\pi_p(Y_p^*)) \\
&= \pi^*\omega_0(X_p^*, Y_p^*) \\
&= i^*\omega(X_p^*, Y_p^*) \\
&= i^*g(I(X_p^*), Y_p^*) \\
&= i^*g((I_0X)_p^*, Y_p^*) \\
&= g_0((I_0X)_{\pi(p)}, Y_{\pi(p)})
\end{aligned}$$

Hence $\omega_0(X, Y) = g_0(I_0(X), Y)$. □