Tangent spaces

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Let M be as in [1], and write a generic point as (α_j, β_j). Then we have the real and complex moment maps, which are

$$\begin{aligned}
\mu_r(\alpha_j, \beta_j) &= (\alpha_{j-1}\alpha_{j-1}^* - \beta_{j-1}^*\beta_{j-1} + \beta_j\beta_j^* - \alpha_j^*\alpha_j)_{j=1}^{k-1} \\
\mu_c(\alpha_j, \beta_j) &= (\alpha_{j-1}\beta_{j-1} - \beta_j\alpha_j)_{j=1}^{k-1}
\end{aligned}$$

First of all, we want to compute the tangent space to $\mu_c^{-1}(0)$. By standard arguments, we have that

$$\mathrm{T}_{(\alpha_j,\beta_j)}\mu_c^{-1}(0) = \ker\left((\mathrm{d}\mu_c)_{(\alpha_j,\beta_j)}\right)$$

We can compute the derivative, since

$$\begin{aligned}
\mu_c(\alpha_j + \delta_j, \beta_j + \varepsilon_j) &= (\alpha_{j-1} + \delta_{j-1})(\beta_{j-1} + \varepsilon_{j-1}) - (\beta_j + \varepsilon_j)(\alpha_j + \delta_j) \\
&= \alpha_{j-1}\beta_{j-1} - \beta_j\alpha_j + \delta_{j-1}\beta_{j-1} + \alpha_{j-1}\varepsilon_{j-1} - \beta_j\delta_j - \varepsilon_j\alpha_j + \text{ higher order terms} \\
&= \mu_c(\alpha_j, \beta_j) + \delta_{j-1}\beta_{j-1} + \alpha_{j-1}\varepsilon_{j-1} - \beta_j\delta_j - \varepsilon_j\alpha_j + \text{ higher order terms}
\end{aligned}$$

Hence we have that

$$\mathrm{T}_{(\alpha,\beta)}\mu_c^{-1}(0) = \left\{(\delta_j, \varepsilon_j) \mid \delta_{j-1}\beta_{j-1} + \alpha_{j-1}\varepsilon_{j-1} - \beta_j\delta_j - \varepsilon_j\alpha_j = 0\right\}$$

Next, we have the map $\Phi^c : \mu_c^{-1}(0) \to \mathcal{N}$, given by $\Phi^c(\alpha, \beta) = \alpha_{k-1}\beta_{k-1}$. The derivative of this map is given by

$$\Phi^c(\alpha + \delta, \beta + \varepsilon) = (\alpha_{k-1} + \delta_{k-1})(\beta_{k-1} + \varepsilon_{k-1}) = \Phi^c(\alpha, \beta) + \delta_{k-1}\beta_{k-1} + \alpha_{k-1}\varepsilon_{k-1} + \text{ higher order terms}$$

Therefore, the map $\mathrm{d}\Phi^c$ is given by

$$\mathrm{d}\Phi^c(\delta, \varepsilon) = \delta_{k-1}\beta_{k-1} + \alpha_{k-1}\varepsilon_{k-1}$$

Restricting to an open subset, giving us the top nilpotent orbit N given by M, Φ^c is a submersion.

The complex structure I acts on the tangent space by

$$I(\delta, \varepsilon) = (i\delta, i\varepsilon)$$

and so,

$$\mathrm{d}\Phi^c(I(\delta, \varepsilon)) = i\mathrm{d}\Phi^c(\delta, \varepsilon)$$

Hence, we must have that Φ^c on $\mu_c^{-1}(0)/G^{\mathbb{C}}$ is a biholomorphism. Fix the point (α, β) and let $X = \alpha_{k-1}\beta_{k-1}$. In this case, we must have that

$$\mathrm{d}\Phi^c(\delta, \varepsilon) \in \mathrm{T}_X N = \{[X, Y] \mid Y \in \mathfrak{sl}(n, \mathbb{C})\}$$

Using the biholomorphism, and the fact that

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{su}(n) \oplus i\mathfrak{su}(n)$$

We just need to find δ, ε such that for a fixed $Y \in \mathfrak{su}(n)$, $\mathrm{d}\Phi^c(\delta, \varepsilon) = [X, Y]$.

One choice would be

$$\delta_j^0 = \begin{cases} -Y\alpha_{k-1} & j = k-1 \\ 0 & j < k-1 \end{cases}$$
$$\varepsilon_j^0 = \begin{cases} \beta_{k-1}Y & j = k-1 \\ 0 & j < k-1 \end{cases}$$

Define

$$V^{\mathbb{C}} = \left\{ (X_{j+1}\alpha_j - \alpha_j X_j, X_j\beta_j - \beta_j X_{j+1}) \mid X_j \in \mathfrak{gl}(n_j, \mathbb{C}) \right\}$$

for the subspace given by the $G^{\mathbb{C}}$ action. This is also the kernel of $\mathrm{d}\Phi^c$. Hence the choices of (δ, ε) is the affine space

$$(\delta^0, \varepsilon^0) + V^{\mathbb{C}}$$

References

[1] Piotr Z. Kobak and Andrew Swann. "Classical nilpotent orbits as hyperkähler quotients". In: *Int. J. Math.* 07.02 (Apr. 1996). Publisher: World Scientific Publishing Co., pp. 193–210. ISSN: 0129-167X. DOI: 10.1142/S0129167X96000116. URL: https://www.worldscientific.com/doi/10.1142/S0129167X96000116 (visited on 07/27/2023).