“Instantons and the geometry of the nilpotent variety” by Kronheimer

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In this document, we will discuss the paper [1]. For concreteness, instead of general Lie groups and Lie algebras, we will focus on the case

$$G = \mathrm{SU}(n) \qquad \mathfrak{g} = \mathfrak{su}(n)$$

which has complexification

$$G^c = \mathrm{SL}(n, \mathbb{C}) \qquad \mathfrak{g}^c = \mathfrak{sl}(n, \mathbb{C})$$

1 Introduction

The inner product on $\mathfrak{su}(n)$ is given by $-\kappa$, where κ is the Killing form. That is,

$$\langle A, B \rangle = -\operatorname{tr}(AB)$$

Define

$$\begin{aligned} \varphi &: \mathfrak{su}(n) \times \mathfrak{su}(n) \times \mathfrak{su}(n) \to \mathbb{R} \\ \varphi(A_1, A_2, A_3) &= \sum_{j=1}^{3} \langle A_j, A_j \rangle + \langle A_1, [A_2, A_3] \rangle \end{aligned}$$

We are interested in studying the gradient flow of φ. That is, $A_1, A_2, A_3 : I \to \mathfrak{su}(n)$ such that

$$(\dot{A}_1, \dot{A}_2, \dot{A}_3) = -\nabla \varphi(A_1, A_2, A_3) \tag{1}$$

First of all, notice that

$$\varphi(A_1 + H_1, A_2, A_3) = \varphi(A_1, A_2, A_3) + 2 \langle H_1, A_1 \rangle + \langle H_1, [A_2, A_3] \rangle$$

and that $\langle A_1, [A_2, A_3] \rangle = \langle A_2, [A_3, A_1] \rangle = \langle A_3, [A_1, A_2] \rangle$. Therefore, eq. (1) becomes

$$\begin{aligned} \dot{A}_1 &= -2A_1 - [A_2, A_3] \\ \dot{A}_2 &= -2A_2 - [A_3, A_1] \\ \dot{A}_3 &= -2A_3 - [A_1, A_2] \end{aligned} \tag{2}$$

The critical points of eq. (2) are triples (A_1, A_2, A_3) satisfying

$$[A_1, A_2] = -2A_3 \quad [A_2, A_3] = -A_1 \quad [A_3, A_1] = -2A_2$$

Recall that the Lie algebra $\mathfrak{su}(2)$ has basis

$$e_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

satifying the above relations. Therefore, critical points of eq. (2) correspond to Lie algebra homomorphisms $\rho : \mathfrak{su}(2) \to \mathfrak{su}(n)$. From this, we see that at all critical points of eq. (2), φ is nonnegative, and it is zero only at $(0,0,0)$.

Next, we will identify $\mathfrak{su}(n) \times \mathfrak{su}(n) \times \mathfrak{su}(n) \cong \mathrm{L}(\mathfrak{su}(2), \mathfrak{su}(\mathrm{n}))$, the space of linear maps $\mathfrak{su}(2) \to \mathfrak{su}(n)$, sending (A_1, A_2, A_3) to the linear map A given by $e_i \mapsto A_i$.

The adjoint action of $\mathrm{SU}(n)$ on $\mathfrak{su}(n)$ is given by

$$\mathrm{Ad}_g(A) = gAg^{-1}$$

and this induces an action on $\mathrm{L}(\mathfrak{su}(2), \mathfrak{su}(\mathrm{n}))$ by

$$g \cdot A : e_i \mapsto gA_i g^{-1}$$

For any Lie algebra homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{su}(n)$, define

$$C(\rho) = \{g \cdot \rho \mid g \in \mathrm{SU}(n)\}$$

for the critical manifold of all homomorphisms which are conjugate to ρ via the adjoint action. For Lie algebra homomorphisms $\rho_-, \rho_+ : \mathfrak{su}(2) \to \mathfrak{su}(n)$, define $M(\rho_-, \rho_+)$ for the space of solutions $A(t)$ to eq. (2), with boundary conditions

$$\begin{aligned} \lim_{t \to -\infty} A(t) &\in C(\rho_-) \\ \lim_{t \to \infty} A(t) &= \rho_+ \end{aligned} \tag{3}$$

Note that we are considering parametrised trajectories, therefore there is a natural $\mathbb{R}$-action sending $A(t)$ to $A(t+c)$.

For a Lie algebra homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{su}(n)$, we can extend it to a Lie algebra homomorphism $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{sl}(n, \mathbb{C})$, and define

$$H = \rho \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \rho \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

We will then define $\mathcal{N}(\rho)$ for the nilpotent orbit of Y in $\mathfrak{sl}(n, \mathbb{C})$, and the affine subspace

$$S(\rho) = Y + Z(X)$$

where $Z(X) = \{A \in \mathfrak{sl}(n, \mathbb{C}) \mid [A, X] = 0\}$. Using this, we have

Theorem 1.1. For any pair of homomorphisms ρ_-, ρ_+, there is a diffeomorphism

$$M(\rho_-, \rho_+) \cong \mathcal{N}(\rho_-) \cap S(\rho_+)$$

If $\rho_+ = 0$, then $S(\rho_+) = \mathfrak{sl}(n, \mathbb{C})$, and in this case, we have a diffeomorphism

$$M(\rho_-, 0) \cong \mathcal{N}(\rho_-)$$

Moreover, every nilpotent orbit is $\mathcal{N}(\rho)$ for some homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{su}(n)$, which means that we have a description of all nilpotent orbits in $\mathfrak{sl}(n, \mathbb{C})$.

2 Complex trajectories

2.1 Nahm's equations

Consider the change of variables

$$T_i = e^{2t} A_i \qquad s = -\frac{1}{2} e^{-2t}$$

Using this, eq. (2) becomes

$$\frac{dT_1}{ds} = -[T_2, T_3]$$
$$\frac{dT_2}{ds} = -[T_3, T_1]$$
$$\frac{dT_3}{ds} = -[T_1, T_2]$$

which are Nahm's equations.

2.2 Gauge group

First of all, we will extend eq. (2) by considering $A_0, \dots, A_3 : \mathbb{R} \to \mathfrak{su}(n)$, satisfying the equations

$$\begin{aligned}\dot{A}_1 &= -2A_1 - [A_0, A_1] - [A_2, A_3] \\ \dot{A}_2 &= -2A_2 - [A_0, A_2] + [A_1, A_3] \\ \dot{A}_3 &= -2A_3 - [A_0, A_3] - [A_1, A_2]\end{aligned} \tag{4}$$

Define the group

$$\mathcal{G} = \{g : \mathbb{R} \to \mathrm{SU}(n)\}$$

with pointwise operations. Then $\mathcal{G}$ acts $A = (A_0, \dots, A_3)$ by

$$(g \cdot A)(t) = \left(g(t)A_0(t)g(t)^{-1} - \frac{dg}{dt} \cdot g(t)^{-1}, g(t)A_1(t)g(t)^{-1}, g(t)A_2(t)g(t)^{-1}, g(t)A_3(t)g(t)^{-1}\right) \tag{5}$$

For brevity, when clear, we will write this as

$$g \cdot A = (gA_0g^{-1} - \dot{g}g^{-1}, gA_1g^{-1}, gA_2g^{-2}, gA_3g^{-1})$$

Note that $\dot{g}(t) \in \mathrm{T}_{g(t)}\,\mathrm{SU}(n) = g(t)\mathfrak{su}(n)$, and so $\dot{g}(t)g(t)^{-1} \in g(t)\mathfrak{su}(n)g(t)^{-1} = \mathfrak{su}(n)$. First, we will show that eq. (4) is invariant under the action eq. (5). To see this, the transformed right hand side (for the first equation) is

$$\begin{aligned}-2gA_1g^{-1} - [gA_0g^{-1} - \dot{g}g^{-1}, gA_1g^{-1}] - [gA_2g^{-1}, gA_3g^{-1}] &= g(-2A_1 - [A_0, A_1] - [A_2, A_3])g^{-1} + [\dot{g}g^{-1}, gA_1g^{-1}] \\ &= g\dot{A}_1g^{-1} + \dot{g}A_1g^{-1} - gA_1g^{-1}\dot{g}g^{-1}\end{aligned}$$

which is precisely $\frac{d}{dt}\left(gA_1g^{-1}\right)$. Moreover, in eq. (5), we can always choose g to make $A_0 = 0$, by considering the linear ODE

$$\dot{g} = gA_0$$

Therefore, we don't change the problem much by considering eq. (4).

2.3 Complex equations

Next, we will break the symmetry in the equations, by choosing A_1 to be 'special'. More precisely, we will consider $\alpha, \beta : \mathbb{R} \to \mathfrak{sl}(n, \mathbb{C})$, defined by

$$\alpha = \frac{1}{2}(A_0 + iA_1) \qquad \beta = \frac{1}{2}(A_2 + iA_3)$$

In this case, we have the followiing expressions:

$$
\begin{aligned}
\alpha^* &= \frac{1}{2}(-A_0 + iA_1)\\
\alpha + \alpha^* &= iA_1\\
[\alpha, \alpha^*] &= \frac{1}{2}i[A_0, A_1]\\
[\beta, \beta^*] &= \frac{1}{2}i[A_2, A_3]
\end{aligned}
$$

and so the first equation in eq. (4) can be written as the *real equation*

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\alpha + \alpha^*) + 2(\alpha + \alpha^*) + 2([\alpha, \alpha^*] + [\beta, \beta^*]) = 0 \tag{6}
$$

and using

$$
[\alpha, \beta] = \frac{1}{4}\left([A_0, A_2] + [A_3, A_1]\right) + \frac{1}{4}i\left([A_0, A_3] + [A_1, A_2]\right)
$$

the second equation in eq. (4) becomes the *complex equation*

$$
\frac{\mathrm{d}\beta}{\mathrm{d}t} + 2\beta + 2[\alpha, \beta] = 0 \tag{7}
$$

As above, the real equation is invariant under the action of $\mathcal{G}$. But in this case, the complex equation is invariant under the action of the complex gauge group

$$
\mathcal{G}^c = \{\mathbb{R} \to \mathrm{SL}(n, \mathbb{C})\}
$$

via eq. (5). In particular, the action is given by

$$
g \cdot (\alpha, \beta) = \left(g\alpha g^{-1} - \frac{1}{2}\dot{g}g^{-1}, g\beta g^{-1}\right)
$$

and so substituting into eq. (7), we get

$$
\dot{g}\beta g^{-1} + g\dot{\beta}g^{-1} - g\beta g^{-1}\dot{g}g^{-1} + 2g\beta g^{-1} + 2g[\alpha, \beta]g^{-1} - [\dot{g}g^{-1}, g\beta g^{-1}] = g\left(\dot{\beta} + 2\beta + 2[\alpha, \beta]\right)g^{-1}
$$

2.4 Complex trajectories

Let $\rho_+, \rho_- : \mathfrak{su}(2) \to \mathfrak{su}(n)$ be Lie algebra homomorphisms. Extend them to Lie algebra homomorphisms $\mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{sl}(n, \mathbb{C})$, and define

$$
H_\pm = \rho_\pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X_\pm = \rho_\pm \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y_\pm = \rho_\pm \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
$$

Definition 2.1 (complex trajectory)

A *complex trajectory* associated to ρ_+, ρ_- is a pair of smooth functions $\alpha, \beta : \mathbb{R} \to \mathfrak{sl}(n, \mathbb{C})$, which satisfy the complex equation eq. (7), and the boundary conditions

$$
\begin{aligned}
\lim_{t\to\infty} 2\alpha(t) &= H_+\\
\lim_{t\to-\infty} 2\alpha(t) &= gH_-g^{-1}\\
\lim_{t\to\infty} \beta(t) &= Y_+\\
\lim_{t\to-\infty} \beta(t) &= gY_-g^{-1}
\end{aligned} \tag{8}
$$

for some $g \in \mathrm{SU}(n)$. Moreover, we require that the convergence in eq. (8) is exponential, that is,

$$
\left\|2\alpha(t) - H_+\right\| < Ke^{-\eta t}
$$

for some $\eta, K > 0$ and so on.

Now define the subgroup $\mathcal{G}_0^c$ of $\mathcal{G}^c$ by

$$\mathcal{G}_0^c = \left\{ g \in \mathcal{G}^c \mid g \text{ bounded}, \lim_{t\to\infty} g(t) = 1 \right\}$$

Using the operator norm, which satisfies $\|gh\| \leq \|g\|\|h\|$, it is clear that $\mathcal{G}_0^c$ is closed under multiplication. Therefore, all we need to show is that it is closed under inverses. One proof is as follows:

By Cayley-Hamilton, we have coefficients $c_1(t), \ldots, c_{n-1}(t)$ such that

$$g(t)^n + c_{n-1}g(t)^{n-1} + \cdots + c_1(t)g(t) + 1 = 0$$

Multiplying by $g(t)^{-1}$, we get

$$g(t)^{-1} = -\left(g(t)^{n-1} + c_{n-1}g(t)^{n-2} + \cdots + c_1(t)\right)$$

The $c_i(t)$ are the elementary symmetric functions in the eigenvalues of $g(t)$, and the eigenvalues of $g(t)$ are bounded, since any eigenvalue λ of $g(t)$ necessarily satisfies $|\lambda| \leq \|g(t)\|$. Therefore, the coefficients on the right hand side are bounded. Hence by the triangle inequality, we have a bound on $\|g(t)^{-1}\|$.

Definition 2.2 (equivalent)
We say that two complex trajectories (α, β) and (α', β') are *equivalent* if there exists $g \in \mathcal{G}_0^c$ such that

$$(\alpha', \beta') = g \cdot (\alpha, \beta)$$

i.e. they are in the same $\mathcal{G}_0^c$ orbit.

2.5 Classification of complex trajectories

First of all, note that under the $\mathcal{G}^c$ action, we can always make $\alpha = 0$. In particular, we need

$$\dot{g} = 2g\alpha$$

Assuming this, the complex equation eq. (7) becomes

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} + 2\beta = 0$$

which has solution

$$\beta(t) = e^{-2t}\beta_0$$

for some β_0. Therefore, the only local invariant under the $\mathcal{G}^c$ (and $\mathcal{G}_0^c$) action is the conjugacy class of β_0. Reversing the $\mathcal{G}^c$ action, we find that a generic local solution is

$$\alpha = \frac{1}{2}g^{-1}\dot{g}$$
$$\beta = e^{-2t}g^{-1}\beta_0 g$$

As a consequence of this, we have

Lemma 2.3. If (α, β) and (α', β') are complex trajectories which are equal outside of some compact set $K \subseteq \mathbb{R}$, then (α, β) and (α', β') are equivalent.

Proof. Without loss of generality, we may assume $K = [-M, M]$ for some $M > 0$. Using the $\mathcal{G}^c$ action, we may assume that

$$\alpha(t) = 0 \qquad \beta(t) = e^{-2t}\beta_0$$

Now let $g \in \mathcal{G}_0^c$ be such that

$$g \cdot (\alpha', \beta') = (0, e^{-2t}\beta_0')$$

In particular, as

$$\dot{g} = 2g\alpha'$$

$\dot{g} = 0$ for $t \notin [-M, M]$, and so g is constant outside of $[-M, M]$. Say $g = g_-$ for $t < -M$ and $g = g_+$ for $t > M$. By the boundary condition $g(t) \to 1$ as $t \to \infty$, $g_+ = 1$. This means that for $t > M$, $\beta'(t) = e^{-2t}\beta_0'$. But in this case $\beta = \beta'$, so $\beta_0 = \beta_0'$. Hence $g \cdot (\alpha', \beta') = (\alpha, \beta)$, and so they are equivalent. □

References

[1] P. B. Kronheimer. "Instantons and the geometry of the nilpotent variety". In: *Journal of Differential Geometry* 32.2 (Jan. 1990). Publisher: Lehigh University, pp. 473–490. ISSN: 0022-040X. DOI: 10.4310/jdg/1214445316. URL: https://projecteuclid.org/journals/journal-of-differential-geometry/volume-32/issue-2/Instantons-and-the-geometry-of-the-nilpotent-variety/10.4310/jdg/1214445316.full (visited on 03/15/2023).