Inclusion map as a moment map

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Let G be a Lie group, acting on $\mathfrak{g}^*$ by the coadjoint action. Fix $\xi \in \mathfrak{g}^*$, and let $\mathcal{O} = G \cdot \xi$ be its orbit. Recall we have the Kirillov-Kostant symplectic form

$$\omega_\xi(\mathrm{ad}^*_X(\xi), \mathrm{ad}^*_Y(\xi)) = \langle \xi, [X, Y] \rangle$$

for $X, Y \in \mathfrak{g}$. We will show that the inclusion map $i : \mathcal{O} \to \mathfrak{g}^*$ is a moment map for the action of G on $\mathcal{O}$. Equivariance is clear since the action is just the restriction of the coadjoint action.

For $X \in \mathfrak{g}$, define a vector field $X^\#$ by

$$X^\#_\mu = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}^*_{\exp(tX)} \mu$$

and the component of the inclusion map i is

$$i^X(\mu) = \langle i(\mu), X \rangle = \langle \mu, X \rangle$$

We want to show that $\mathrm{d}i^X = \iota_{X^\#}\omega$. For $Y \in \mathfrak{g}$, $\mu \in \mathcal{O}$, we have that

$$\begin{aligned}
\mathrm{d}i^X(Y^\#)(\mu) &= Y^\#_\mu \left(i^X\right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mathrm{Ad}^*_{\exp(tY)} \mu\right) i^X \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left\langle \mathrm{Ad}^*_{\exp(tY)} \mu, X \right\rangle \\
&= \left\langle Y^\#_\mu, X \right\rangle
\end{aligned}$$

Lemma.

$$\left\langle Y^\#_\mu, X \right\rangle = \langle \mu, [X, Y] \rangle$$

for all $X, Y \in \mathfrak{g}$.

Proof.

$$\begin{aligned}
\left\langle Y^\#_\mu, X \right\rangle &= \left\langle \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}^*_{\exp(tY)} \mu, X \right\rangle \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left\langle \xi, \mathrm{Ad}_{\exp(-tY)} X \right\rangle \\
&= \left\langle \xi, \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{\exp(-tY)} X \right\rangle \\
&= \langle \xi, [X, Y] \rangle
\end{aligned}$$

$\square$

Therefore, we have that

$$\mathrm{d}i^X(Y^\#)(\mu) = \langle \mu, [X, Y] \rangle$$

On the other hand, we have that

$$\iota_{X^\#}\omega(Y^\#)(\mu) = \omega_\mu(X^\#, Y^\#) = \langle \mu, [X, Y] \rangle$$