$\mathrm{SU}(n)/T \cong \mathrm{SL}(n, \mathbb{C})/P$

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Consider the following Lie groups:

$$\mathrm{SU}(n) \quad \text{with} \quad T = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \right\} \leq \mathrm{SU}(n)$$

and

$$\mathrm{SL}(n, \mathbb{C}) \quad \text{with} \quad P = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ * & & \lambda_n \end{pmatrix} \right\} \leq \mathrm{SL}(n, \mathbb{C})$$

where P is the parabolic subgroup of lower triangular matrices. Consider the composition $\varphi : \mathrm{SU}(n) \to \mathrm{SL}(n, \mathbb{C})/P$ given by the composition

$$\mathrm{SU}(n) \hookrightarrow \mathrm{SL}(n) \twoheadrightarrow \mathrm{SL}(n, \mathbb{C})/P$$

Suppose $\varphi(g) = \varphi(h)$. That is, $gP = hP$. This is true if and only if there exists $p \in P$, such that $h = gp$. In this case, $p = g^{-1}h \in \mathrm{SU}(n)$, therefore, $p \in \mathrm{SU}(n) \cap P = T$, since $p^\dagger = p^{-1}$ is also lower triangular. This means that φ induces a homeomorphism $\mathrm{SU}(n)/T \cong \mathrm{SL}(n, \mathbb{C})/P$.