Coadjoint Orbits of SU(n)

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In this note, we will consider the coadjoint orbits of SU(n), and show that they are Kähler manifolds.

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1 Adjoint and Coadjoint Orbits

Define the Lie algebra

$$\mathfrak{su}(n) = \{A \in \mathrm{Mat}(n, \mathbb{C}) \mid A^* + A = 0, \mathrm{tr}(A) = 0\}$$

where A^* is the conjugate transpose of A, and with the Lie bracket being the matrix commutator. We can define the adjoint representation of SU(n) as

$$\begin{gathered}\mathrm{Ad} : \mathrm{SU}(n) \to \mathrm{GL}(\mathfrak{su}(n)) \\ \mathrm{Ad}_g(X) = gXg^*\end{gathered}$$

Taking the dual representation, we get the coadjoint representation, which is

$$\begin{gathered}\mathrm{Ad}^* : \mathrm{SU}(n) \to \mathrm{GL}(\mathfrak{su}(n)^*) \\ \mathrm{Ad}^*_g(\alpha)(X) = \left\langle \alpha, \mathrm{Ad}_{g^{-1}}(X) \right\rangle\end{gathered}$$

where $\langle \cdot, \cdot \rangle$ is used here to denote the pairing $\mathfrak{su}(n)^* \times \mathfrak{su}(n) \to \mathbb{R}$. We will use the same notation for the inner product on $\mathfrak{su}(n)$, which should not be an issue as the inner product defines a natural isomorphism. Now note that $-\kappa$, where κ is the Killing form, defines an inner product

$$\langle A, B \rangle = -\mathrm{tr}(AB) = \mathrm{tr}(AB^*)$$

defines an inner product on $\mathfrak{su}(n)$[1], which means that we have a natural isomorphism

$$\begin{aligned}\Phi : \mathfrak{su}(n) &\to \mathfrak{su}(n)^* \\ A &\mapsto \langle A, \cdot \rangle\end{aligned}$$

With this, suppose $\alpha = \Phi(A)$, then

$$\mathrm{Ad}_g^*(\alpha)(X) = \left\langle A, \mathrm{Ad}_{g^{-1}}(X) \right\rangle = -\operatorname{tr}\left(Ag^{-1}Xg\right) = -\operatorname{tr}\left(gAg^{-1}X\right) = \Phi(\mathrm{Ad}_g(A))(X)$$

Therefore, the following diagram commutes

$$\begin{tikzcd} \mathfrak{su}(n) \arrow[r, "\mathrm{Ad}_g"] \arrow[d, "\Phi"'] & \mathfrak{su}(n) \arrow[d, "\Phi"] \\ \mathfrak{su}(n)^* \arrow[r, "\mathrm{Ad}_g^*"] & \mathfrak{su}(n)^* \end{tikzcd}$$

or equivalently, Φ defines an isomorphism of representations between Ad and Ad^*.

2 Tangent space and Diagonalisation

2.1 Tangent space

Let M be a coadjoint orbit. For $X \in \mathfrak{su}(n)$, consider the curve $g(t) = \exp(tX)$ in $\mathrm{SU}(n)$. This has $g'(0) = X$, and we have a curve

$$\mu(t) = \mathrm{Ad}_{g(t)}^*(\mu)$$

through $\mu \in M$. In particular, we have that for $Y \in \mathfrak{su}(n)$,

$$\langle \mu(t), Y \rangle = \left\langle \mu, \mathrm{Ad}_{g(t)^{-1}}(Y) \right\rangle$$

Differentiating this at $t = 0$, we get

$$\left\langle \mu'(0), Y \right\rangle = -\left\langle \mu, \mathrm{ad}_X(Y) \right\rangle = -\left\langle (\mathrm{ad}_X)^*(\mu), Y \right\rangle$$

That is, $\mu'(0) = -(\mathrm{ad}_X)^*(\mu)$. Hence we have that

$$\mathrm{T}_\mu(M) = \{(\mathrm{ad}_X)^*(\mu) \mid X \in \mathfrak{su}(n)\}$$

If $\alpha = \Phi(A)$, then

$$\begin{aligned}\langle (\mathrm{ad}_X)^*(\alpha), Y \rangle &= \langle \alpha, \mathrm{ad}_X(Y) \rangle \\ &= \langle A, [X, Y] \rangle \\ &= -\operatorname{tr}(AXY - AYX) \\ &= -\operatorname{tr}(AXY - XAY) \\ &= \langle [A, X], Y \rangle \\ &= \langle -\mathrm{ad}_X(A), Y \rangle\end{aligned}$$

Therefore, we have that

$$\begin{tikzcd} \mathfrak{su}(n) \arrow[r, "\mathrm{ad}_X"] \arrow[d, "\Phi"'] & \mathfrak{su}(n) \arrow[d, "\Phi"] \\ \mathfrak{su}(n)^* \arrow[r, "\mathrm{ad}_X^*"'] & \mathfrak{su}(n)^* \end{tikzcd}$$

[1] In fact, $\langle A, B \rangle = \operatorname{tr}(AB^*)$ defines a Hermitian inner product on the space of complex matrices.

Thus, in this case we have the tangent space to the corresponding adjoint orbit as

$$T_A M = \{\mathrm{ad}_X(A) \mid X \in \mathfrak{su}(n)\}$$

2.2 Diagonalisation and Stabilisers of the coadjoint action

First of all, we note that

$$\mathfrak{su}(n) = \{A \in \mathrm{Mat}(n, \mathbb{C}) \mid A^* + A = 0, \mathrm{tr}(A) = 0\}$$

where A^* is the conjugate transpose of A. In particular, all elements of $\mathfrak{su}(n)$ are skew-hermitian, hence diagonalisable by an element of $\mathrm{SU}(n)$[2]. With this, we can classify the coadjoint orbits based off a diagonal element in the orbit. Consider

$$A = \begin{pmatrix} i\lambda_1 I_{m_1} & & \\ & \ddots & \\ & & i\lambda_k I_{m_k} \end{pmatrix}$$

where I_m is the $m \times m$ identity matrix, $\lambda_j \in \mathbb{R}$, with $\lambda_1 > \lambda_2 > \cdots > \lambda_k$, $m_1 + \cdots + m_k = n$ and $m_1\lambda_1 + \cdots + m_k\lambda_k = 0$. In this case, we have that the orbit is

$$\mathrm{Orb}(A) \cong \mathrm{SU}(n)/\,\mathrm{Stab}(A)$$

where $\mathrm{Stab}(A)$ is the stabiliser of A under the adjoint action. In this case, we have that the stabiliser is the block diagonal subgroup

$$\mathrm{Stab}(A) = \mathrm{S}\left(\mathrm{U}(\mathrm{m}_1) \times \cdots \times \mathrm{U}(\mathrm{m}_\mathrm{k})\right)$$

where we consider $\mathrm{U}(\mathrm{m}_1) \times \cdots \times \mathrm{U}(\mathrm{m}_\mathrm{k}) \leq \mathrm{SU}(\mathrm{n})$ as the block diagonal subgroup, and

$$\mathrm{S}\left(\mathrm{U}(\mathrm{m}_1) \times \cdots \times \mathrm{U}(\mathrm{m}_\mathrm{k})\right) = \left(\mathrm{U}(\mathrm{m}_1) \times \cdots \times \mathrm{U}(\mathrm{m}_\mathrm{k})\right) \cap \mathrm{SU}(\mathrm{n})$$

the subgroup with determinant 1.

3 Kirillov-Kostant-Souriau symplectic form

This section is from [2] Chapter 14. The proof is slightly modified, since here we have an explicit isomorphism between the adjoint and coadjoint representation, which simplifies some of the arguments.

Theorem 3.1. [Kirillov-Kostant-Souriau, [2, Theorem 14.4.1]] Let $M \subseteq \mathfrak{su}(n)^*$ be a coadjoint orbit. Define the 2-form ω on M by

$$\omega_\mu(\mathrm{ad}^*_X(\mu), \mathrm{ad}^*_Y(\mu)) = -\langle \mu, [X, Y] \rangle$$

Then ω is a symplectic form on M.

3.1 ω is well defined

First of all, we show that ω is well defined. That is, it is independent of the choice of $X, Y \in \mathfrak{su}(n)$.

Suppose $Z \in \mathfrak{su}(n)$ is such that $\mathrm{ad}^*_Z(\mu) = \mathrm{ad}^*_X(\mu)$. Then we must have that

$$\langle \mu, [X, Y] \rangle = \langle \mu, [Z, Y] \rangle$$

for all $Y \in \mathfrak{su}(n)$.

[2]From standard linear algebra arguments, we know that they are $\mathrm{U}(n)$-diagonalisable. But if PAP^{-1} is diagonal, then so is $(\lambda P)A(\lambda P)^{-1}$, and by choosing λ appropriately, $\lambda \in \mathrm{SU}(n)$.

3.2 ω is non-degenerate

Suppose we have $X \in \mathfrak{su}(n)$ such that

$$\omega_\mu(\mathrm{ad}_X^*(\mu), \mathrm{ad}_Y^*(\mu)) = \langle \mu, [X, Y] \rangle = 0$$

for all Y. But this is the same as $\mathrm{ad}_X^*(\mu) = 0$. Therefore, ω is non-degenerate.

3.3 ω is closed

First of all, we will need some preliminary results.

Lemma 3.2.

$$\mathrm{ad}^*_{\mathrm{Ad}_g X} = \mathrm{Ad}_g^* \circ \mathrm{ad}_X^* \circ \mathrm{Ad}_{g^*}^*$$

Proof. We will prove the corresponding statement for Ad and ad, and the result will follow by conjuation with the isomorphism Φ.

$$\mathrm{Ad}_g \circ \mathrm{ad}_X \circ \mathrm{Ad}_{g^*}(Y) = gXg^*Ygg^* - gg^*YgXg^* = [gXg^*, Y] = \mathrm{ad}_{\mathrm{Ad}_g X}(Y)$$

□

Lemma 3.3.

$$\mathrm{Ad}_g([X, Y]) = [\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)]$$

Proof. Expand using the definition of Ad. □

Lemma 3.4. $\mathrm{Ad}_g^* : M \to M$ preserves ω, that is,

$$(\mathrm{Ad}_g^*)^* \omega = \omega$$

Proof. Evaluating $\mathrm{ad}^*_{\mathrm{Ad}_g X} = \mathrm{Ad}_g^* \circ \mathrm{ad}_X^* \circ \mathrm{Ad}_{g^{-1}}^*$ at $\nu = \mathrm{Ad}_g^*(\mu)$, we get

$$\mathrm{ad}^*_{\mathrm{Ad}_g X}(\nu) = \mathrm{Ad}_g^* \circ \mathrm{ad}_X^*(\mu) = \mathrm{d}_\mu \mathrm{Ad}_g^* \circ \mathrm{ad}_X^*(\mu)$$

Therefore,

$$\begin{aligned}
((\mathrm{Ad}_g^*)^*\omega)_\mu(\mathrm{ad}_X^*(\mu), \mathrm{ad}_Y^*(\mu)) &= \omega_\nu(\mathrm{d}_\mu \mathrm{Ad}_g^* \cdot \mathrm{ad}_X^*(\mu), \mathrm{d}_\mu \mathrm{Ad}_g^* \cdot \mathrm{ad}_Y^*(\mu)) \\
&= \omega_\nu(\mathrm{ad}^*_{\mathrm{Ad}_g X}(\nu), \mathrm{ad}^*_{\mathrm{Ad}_g Y}(\nu)) \\
&= -\left\langle \nu, [\mathrm{Ad}_g X, \mathrm{Ad}_g Y] \right\rangle \\
&= -\left\langle \nu, \mathrm{Ad}_g([X, Y]) \right\rangle \\
&= -\left\langle \mathrm{Ad}^*_{g^{-1}}(\nu), [X, Y] \right\rangle \\
&= -\langle \mu, [X, Y] \rangle \\
&= \omega_\mu(\mathrm{ad}_X^*(\mu), \mathrm{ad}_Y^*(\mu))
\end{aligned}$$

□

For $\nu \in \mathfrak{su}(n)^*$, define the left-invariant one-form

$$\nu_\ell(g) = (\mathrm{d}_g \ell_{g^{-1}})^*(\nu)$$

for $g \in \mathrm{SU}(n)$. Similarly, for $X \in \mathfrak{su}(n)$, let X_ℓ be the corresponding left invariant vector field on G. Then $\nu_\ell(X_\ell) = \langle \nu, X \rangle$ at all $g \in \mathrm{SU}(n)$.

Fix $v \in M$, and consider the map $\pi : \mathrm{SU}(n) \to M$, defined by

$$\pi(g) = \mathrm{Ad}_g^*(v)$$

We can use this to pullback $\sigma = \pi^*\omega$ to a two form on $\mathrm{SU}(n)$.

Lemma 3.5. σ is left invariant. That is, $\ell_g^*\sigma = \sigma$ for all $g \in \mathrm{SU}(n)$.

Proof. First, notice that $\pi \circ \ell_g = \mathrm{Ad}_g^* \circ \pi$, since

$$\pi(\ell_g(h)) = \mathrm{Ad}_{gh}^*(v) = \mathrm{Ad}_g^* \circ \mathrm{Ad}_h^*(v) = \mathrm{Ad}_g^*(\pi(h))$$

With this,

$$\ell_g^*\sigma = \ell_g^*\pi^*\omega = (\pi \circ \ell_g)^*\omega = (\mathrm{Ad}_g^* \circ \pi)^*\omega = \pi^*(\mathrm{Ad}_g^*)^*\omega = \pi^*\omega = \sigma$$

$\square$

Lemma 3.6. $\sigma(X_\ell, Y_\ell) = -\langle v_\ell, [X_\ell, Y_\ell]\rangle$.

Proof. By left invariance of both sides, suffices to show that the result holds at e. First notice that

$$\mathrm{d}_I\pi(Y) = -\,\mathrm{ad}_Y^*(v)$$

Therefore, π is a submersion at e. By definition of the pullback,

$$\begin{aligned}
\sigma_I(X, Y) &= (\pi^*\omega)_I(X, Y) \\
&= \omega_{\pi(I)}(\mathrm{d}_e\pi \cdot X, \mathrm{d}_e\pi \cdot Y) \\
&= \omega_v(\mathrm{ad}_X^*(v), \mathrm{ad}_Y^*(v)) \\
&= -\langle v, [X, Y]\rangle
\end{aligned}$$

Hence

$$\sigma(X_\ell, Y_\ell)_I = \sigma_I(X, Y) = -\langle v, [X, Y]\rangle = -\langle v_\ell, [X_\ell, Y_\ell]\rangle_I$$

$\square$

Now for a one form α, we have that

$$\mathrm{d}\alpha(X, Y) = X[\alpha(Y)] - Y[\alpha(X)] - \alpha([X, Y])$$

where for a smooth function $f : M \to \mathbb{R}$, and a vector field X on M, $X[f] := \mathrm{d}f(X)$ is a smooth function $M \to \mathbb{R}$.

Since $v_\ell(X_\ell)$ is constant, $Y_\ell[v_\ell(X_\ell)] = 0$. Similarly, $X_\ell[v_\ell(Y_\ell)] = 0$. Therefore, we have that

$$\mathrm{d}v_\ell(X_\ell, Y_\ell) = -v_\ell([X_\ell, Y_\ell]) = \sigma(X_\ell, Y_\ell)$$

Now suppose X, Y are vector fields on G. We want to show that $\sigma(X, Y) = \mathrm{d}v_\ell(X, Y)$. As σ is left invariant,

$$
\begin{aligned}
\sigma(X, Y)_g &= (\ell_{g^{-1}}^* \sigma)_g(X(g), Y(g)) \\
&= \sigma_I(\underbrace{\mathrm{d}\ell_{g^{-1}} \cdot X(g)}_{=:X}, \underbrace{\mathrm{d}\ell_{g^{-1}} \cdot Y(g)}_{=:Y}) \\
&= \sigma_I(X, Y) \\
&= \mathrm{d}\nu_\ell(X_\ell, Y_\ell)_I \\
&= (\ell_g^* \mathrm{d}\nu_\ell)(X_\ell, Y_\ell)_I \\
&= (\mathrm{d}\nu_\ell)_g(\mathrm{d}\ell_g(X_\ell)_I, \mathrm{d}\ell_g(Y_\ell)_I) \\
&= (\mathrm{d}\nu_\ell)_g(\mathrm{d}\ell_g X, \mathrm{d}\ell_g Y) \\
&= (\mathrm{d}\nu_\ell)_g(X_g, Y_g) \\
&= \mathrm{d}\nu_\ell(X, Y)_g
\end{aligned}
$$

With this, $\mathrm{d}\sigma = \mathrm{d}^2\nu_\ell = 0$. Hence $\pi^*\mathrm{d}\omega = \mathrm{d}(\pi^*\omega) = \mathrm{d}\sigma = 0$. Since $\pi \circ \ell_g = \mathrm{Ad}_g^* \circ \ell_g$, and π is a submersion at I, it is in fact a submersion everywhere. Moreover, π is surjective, by definition.

For $\mu \in M$, and $X, Y \in T_\mu M$, we have that

$$\mathrm{d}\omega(\mu)(X, Y) = \mathrm{d}\omega_{\pi(g)}(\mathrm{d}\pi(X), \mathrm{d}\pi(Y)) = (\pi^*\mathrm{d}\omega)(g)(X, Y) = 0$$

where $g \in \mathrm{SU}(n)$ is such that $\pi(g) = \mu$, which exists by surjectivity, and $X, Y \in T_g\,\mathrm{SU}(n)$ such that $\mathrm{d}\pi(X) = X$ and $\mathrm{d}\pi(Y) = Y$, which exists as π is a submersion. Thus, as $\mu \in M$ is arbitrary, ω is closed.

3.4 ω on adjoint orbits

Using the isomorphism Φ, theorem 3.1 and the computation for ad_X^*, we get the following result.

Theorem 3.7. Let $M \subseteq \mathfrak{su}(n)$ be an adjoint orbit. Define the 2-form ω on M by

$$\omega_A([A, B], [A, C]) = -\langle A, [B, C]\rangle = \mathrm{tr}(A[B, C])$$

Then ω is a symplectic form on M.

This will be convenient for us since we can compute the right hand side directly from the matrices.

4 Root decomposition

Consider the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ of trace free $n \times n$ complex matrices. Then we have the Cartan subalgebra $\mathfrak{t}$ of diagonal matrices. Let E_{ij} be the standard basis matrices for $\mathrm{Mat}(n, \mathbb{C})$, $B \in \mathfrak{t}$. Say

$$B = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix}$$

Then $[B, E_{ij}] = (b_i - b_j)E_{ij}$. This means that we have the eigendecomposition

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{t} \oplus \bigoplus_{1 \le i,j \le n, i \ne j} \mathbb{C}E_{ij} \tag{1}$$

In particular, if we restrict this to the subalgebra $\mathfrak{su}(n)$, we get the decomposition

$$\mathfrak{su}(n) = \widetilde{\mathfrak{t}} \oplus \bigoplus_{1 \le i < j \le n} \big(\mathbb{R}(E_{ij} - E_{ji}) \oplus i\mathbb{R}(E_{ij} + E_{ji})\big) \tag{2}$$

where $\widetilde{\mathfrak{t}} = \mathfrak{t} \cap \mathfrak{su}(n)$ is the subalgebra of $\mathfrak{su}(n)$ of diagonal matrices.

5 Kähler structure

In this section, we construct the Kähler structure on adjoint orbits of SU(n). In principle, we only need two of (ω, g, J) as we can recover the third. See Cannas da Silva [1, §13.2] for a table which summarises the relations and the required conditions if we only have two of the three.

We have already constructed the Kirillov-Kostant-Souriau symplectic form ω. In the following, we will construct the Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$[3] and the almost complex structure J. Once we have shown that these form a compatible triple, since ω is closed and J comes from a diffeomorphism between the adjoint orbit and a complex manifold, we will have that ω is in fact a Kähler form.

5.1 At a diagonal element

Let M be an adjoint orbit. We will now focus on the generic case, where elements of M have distinct eigenvalues. Let

$$A = \begin{pmatrix} i\lambda_1 & & \\ & \ddots & \\ & & i\lambda_n \end{pmatrix}$$

be the element in M with $\lambda_i \in \mathbb{R}$, $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. In this case, we have that the stabiliser of A is the torus $T \cong T^{n-1}$ of diagonal matrices in SU(n).

$$\mathrm{T}_A M = \bigoplus_{1 \leq i < j \leq n} \left(\mathbb{R}(E_{ij} - E_{ji}) \oplus i\mathbb{R}(E_{ij} + E_{ji}) \right) \cong \frac{\mathfrak{su}(n)}{\tilde{t}} \cong \mathrm{T}_{[1]} \left(\frac{\mathrm{SU}(n)}{T^{n-1}} \right)$$

where the isomorphism is induced by the quotient map

$$\begin{aligned} \pi : \mathrm{SU}(n) &\to M \\ g &\mapsto \mathrm{Ad}_g(A) = gAg^* \end{aligned}$$

Let $e_{ij} = E_{ij} - E_{ji}$ and $f_{ij} = i(E_{ij} + E_{ji})$.

Lemma 5.1.
1. For any $X, Y, Z \in \mathfrak{su}(n)$, $\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle$,
2. $[A, E_{ij}] = i(\lambda_i - \lambda_j) E_{ij}$
3. $E_{ij} E_{kl} = \delta_{jk} E_{il}$
4. $\langle \cdot, \cdot \rangle$ is $\mathbb{C}$-bilinear.

Proof. Expand. ☐

Using these, we have that

$$\begin{aligned} \left\langle A, [E_{ij}, E_{kl}] \right\rangle &= \left\langle [A, E_{ij}], E_{kl} \right\rangle \\ &= i(\lambda_i - \lambda_j) \left\langle E_{ij}, E_{kl} \right\rangle \\ &= -i(\lambda_i - \lambda_j) \delta_{jk} \operatorname{tr}(E_{il}) \\ &= -i(\lambda_i - \lambda_j) \delta_{jk} \delta_{il} \\ &= i(\lambda_j - \lambda_i) \delta_{il} \delta_{jk} \end{aligned}$$

and so,

$$\begin{aligned} \left\langle A, [e_{ij}, e_{kl}] \right\rangle &= \left\langle A, [E_{ij}, E_{kl}] \right\rangle - \left\langle A, [E_{ji}, E_{kl}] \right\rangle - \left\langle A, [E_{ij}, E_{lk}] \right\rangle + \left\langle A, [E_{ji}, E_{lk}] \right\rangle \\ &= i(\lambda_j - \lambda_i)\delta_{il}\delta_{jk} - i(\lambda_i - \lambda_j)\delta_{jl}\delta_{ik} - i(\lambda_j - \lambda_i)\delta_{ik}\delta_{jl} + i(\lambda_i - \lambda_j)\delta_{jk}\delta_{il} \\ &= 0 \end{aligned}$$

[3] Quite often g will be used for an element of SU(n), and so we will use $\langle\langle \cdot, \cdot \rangle\rangle$ to denote the inner product.

and

$$
\begin{aligned}
-\left\langle A,[f_{ij},f_{kl}]\right\rangle &= \left\langle A,[E_{ij},E_{kl}]\right\rangle + \left\langle A,[E_{ji},E_{kl}]\right\rangle + \left\langle A,[E_{ij},E_{lk}]\right\rangle + \left\langle A,[E_{ji},E_{lk}]\right\rangle \\
&= i(\lambda_j-\lambda_i)\delta_{il}\delta_{jk} + i(\lambda_i-\lambda_j)\delta_{jl}\delta_{ik} + i(\lambda_j-\lambda_i)\delta_{ik}\delta_{jl} + i(\lambda_i-\lambda_j)\delta_{jk}\delta_{il} \\
&= 0
\end{aligned}
$$

Finally, we have that

$$
\begin{aligned}
-i\left\langle A,[e_{ij},f_{kl}]\right\rangle &= \left\langle A,[E_{ij},E_{kl}]\right\rangle - \left\langle A,[E_{ji},E_{kl}]\right\rangle + \left\langle A,[E_{ij},E_{lk}]\right\rangle - \left\langle A,[E_{ji},E_{lk}]\right\rangle \\
&= i(\lambda_j-\lambda_i)\delta_{il}\delta_{jk} - i(\lambda_i-\lambda_j)\delta_{jl}\delta_{ik} + i(\lambda_j-\lambda_i)\delta_{ik}\delta_{jl} - i(\lambda_i-\lambda_j)\delta_{jk}\delta_{il} \\
&= 2i(\lambda_j-\lambda_i)(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}) \\
&= 2i(\lambda_j-\lambda_i)\delta_{ik}\delta_{jl}
\end{aligned}
$$

where the last line is because we require $i<j$ and $k<l$. Hence we have that

$$
\begin{aligned}
\omega_A(e_{ij},e_{kl}) &= 0 \\
\omega_A(f_{ij},f_{kl}) &= 0 \\
\omega_A(e_{ij},f_{ij}) &= 2(\lambda_i-\lambda_j) \\
\omega_A(e_{ij},f_{kl}) &= 0 \text{ for } (i,j)\neq(k,l)
\end{aligned}
$$

We can then define an an almost complex structure on $\mathrm{T}_A M$ by

$$
J_A(e_{ij}) = f_{ij} \quad J_A(f_{ij}) = -e_{ij} \tag{3}
$$

Moreover, we can define an inner product on $\mathrm{T}_A M$ by

$$
\left\langle\!\left\langle e_{ij},e_{ij}\right\rangle\!\right\rangle_A = \left\langle\!\left\langle f_{ij},f_{ij}\right\rangle\!\right\rangle_A = 2(\lambda_i-\lambda_j)
$$

and requiring e_{ij}, f_{ij} to form an orthogonal basis. This is positive definite since we required $\lambda_i > \lambda_j$ for $i<j$. Using this, we find that

$$
\omega_A(e_{ij},f_{ij}) = 2(\lambda_i-\lambda_j) = \left\langle\!\left\langle f_{ij},f_{ij}\right\rangle\!\right\rangle_A = \left\langle\!\left\langle J_A(e_{ij}),f_{ij}\right\rangle\!\right\rangle_A
$$

and J defines an isometry. Hence $(\omega_A, \langle\!\langle\cdot,\cdot\rangle\!\rangle_A, J_A)$ define a compatible triple on the vector space $\mathrm{T}_A M$.

5.2 Complex quotient

Let P be the subgroup of lower triangular matrices in $\mathrm{SL}(n,\mathbb{C})$. Consider the composition $\varphi : \mathrm{SU}(n) \to \mathrm{SL}(n,\mathbb{C})/P$ given by the composition

$$
\mathrm{SU}(n) \hookrightarrow \mathrm{SL}(n) \twoheadrightarrow \mathrm{SL}(n,\mathbb{C})/P
$$

Suppose $\varphi(g)=\varphi(h)$. That is, $gP=hP$. This is true if and only if there exists $p\in P$, such that $h=gp$. In this case, $p=g^{-1}h\in\mathrm{SU}(n)$, therefore, $p\in\mathrm{SU}(n)\cap P=T$, since $p^*=p^{-1}$ is also lower triangular. This means that φ induces a homeomorphism $\mathrm{SU}(n)/T\cong\mathrm{SL}(n,\mathbb{C})/P$. The right hand side is a complex manifold ($\mathrm{SL}(n,\mathbb{C})$) quotiented by a complex Lie group P, so it is a complex manifold. Using the above, we can get a complex structure on $\mathrm{SU}(n)/T\cong M$.

Using the root decompositions eq. (1) for $\mathfrak{sl}(n,\mathbb{C})$ and eq. (2) for $\mathfrak{su}(n)$, we can see that the almost complex structure we defined in eq. (3) is the same as the action of multiplication by i.

5.3 At a general $B\in M$

Fix $g\in\mathrm{SU}(n)$, and let $B=\mathrm{Ad}_g(A)$. Let $\varphi:\mathrm{SU}(n)/T\to\mathrm{SL}(n,\mathbb{C})/P$ be the diffeomorphism from above, which is given by $\varphi([g])=[\![g]\!]$, where $[g]=gT\in\mathrm{SU}(n)/T$ and $[\![g]\!]=gP\in\mathrm{SL}(n,\mathbb{C})/P$.

First of all, $\mathrm{SL}(n,\mathbb{C})$ is a complex Lie group, so left multiplication is holomorphic. That is, $\mathrm{d}\ell_g(iv)=i\mathrm{d}\ell_g(v)$. If $\widetilde{J}$ denotes the complex structure on $\mathrm{SL}(n,\mathbb{C})$, then we have that

$$\widetilde{J}_g = \mathrm{d}\ell_g \circ \widetilde{J}_I \circ \mathrm{d}\ell_{g^{-1}}$$

ℓ_g descends to a biholomorphism on $\mathrm{SL}(n, \mathbb{C})/P \cong \mathrm{SU}(n)/T$, and so the corresponding almost complex structure $\overline{J}$ on $\mathrm{SU}(n)/T$ is given by

$$\overline{J}_{[g]} = \mathrm{d}\ell_g \circ \overline{J}_{[I]} \circ \mathrm{d}\ell_{g^*} \tag{4}$$

Since the diffeomorphism $\mathrm{SU}(n)/T$ is induced by the map $\pi(g) = \mathrm{Ad}_g(A)$, we have that

$$\begin{aligned} J_B &= \mathrm{d}\pi \circ \overline{J}_{[g]} \circ \mathrm{d}\pi^{-1} \\ &= \mathrm{d}\pi \circ \mathrm{d}\ell_g \circ \overline{J}_{[I]} \circ \mathrm{d}\ell_{g^*} \circ \mathrm{d}\pi^{-1} \\ &= \mathrm{d}\left(\pi \circ \ell_g \circ \pi^{-1}\right) \circ J_A \circ \mathrm{d}\left(\pi \circ \ell_{g^*} \circ \pi^{-1}\right) \end{aligned}$$

From the proof of lemma 3.5, we have that $\pi \circ \ell_g = \mathrm{Ad}_g \circ \pi$. Moreover, since Ad_g is a linear map, $\mathrm{d}\,\mathrm{Ad}_g = \mathrm{Ad}_g$. Therefore, we have that the almost complex structure is given by

$$J_B = \mathrm{Ad}_g \circ J_A \circ \mathrm{Ad}_{g^*}$$

We want to show that this is compatible with the Kirillov-Kostant-Souriau symplectic form.

From the proof of lemma 3.2, we have that

$$\mathrm{ad}_{\mathrm{Ad}_g(A)} = \mathrm{Ad}_g \circ \mathrm{ad}_A \circ \mathrm{Ad}_{g^*}$$

In particular, we have that

$$\begin{aligned} \mathrm{T}_B M &= \{\mathrm{ad}_B(X) \mid X \in \mathfrak{su}(n)\} \\ &= \left\{\mathrm{Ad}_g(\mathrm{ad}_A(X)) \mid X \in \mathfrak{su}(n)\right\} \\ &= \mathrm{Ad}_g(\mathrm{T}_A M) \end{aligned}$$

Then we have that

$$\begin{aligned} \omega_B([B, X], [B, Y]) &= -\langle B, [X, Y]\rangle \\ &= -\left\langle \mathrm{Ad}_g(A), \mathrm{Ad}_g([\mathrm{Ad}_{g^*}(X), \mathrm{Ad}_{g^*}(Y)])\right\rangle \\ &= -\left\langle A, [\mathrm{Ad}_{g^*}(X), \mathrm{Ad}_{g^*}(Y)]\right\rangle \\ &= \omega_A([A, \mathrm{Ad}_{g^*}(X)], [A, \mathrm{Ad}_{g^*}(X)]) \\ &= \omega_A(\mathrm{Ad}_{g^*}([B, X]), \mathrm{Ad}_{g^*}([B, Y])) \end{aligned}$$

Note this also follows from lemma 3.5 where we showed $\pi^*\omega$ is left invariant. Therefore, the Riemannian metric is given by

$$\langle\langle X, Y\rangle\rangle_B = \left\langle\left\langle \mathrm{Ad}_{g^*}(X), \mathrm{Ad}_{g^*}(Y)\right\rangle\right\rangle_A$$

References

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