Kähler reduction with momentum

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In this note, we will generalise the Kähler quotient construction to more general $\mu^{-1}(\xi)$, for $\xi \in \mathfrak{su}(n)$. We will assume the Kähler reduction construction, and that (co)adjoint orbits of $\mathfrak{su}(n)$ are Kähler manifolds.

Let X be a Kähler manifold, where $\mathrm{SU}(n)$ acts on X preserving the Kähler structure, and with a moment map $\mu_X : X \to \mathfrak{su}(n)$. Note that we use the minus the Killing form

$$\langle \xi, \eta \rangle = -\kappa(\xi, \eta) = -\operatorname{tr}(\xi\eta)$$

to define an inner product on $\mathfrak{su}(n)$, and hence an isomorphism $\mathfrak{su}(n) \cong \mathfrak{su}(n)^*$.

Let H be a subgroup of $\mathrm{SU}(n)$ preserving $\mu_X^{-1}(\xi)$. Since μ_X is equivariant with respect to the $\mathrm{SU}(n)$ action and the adjoint action, preserving $\mu_X^{-1}(\xi)$ is equivalent to

$$\mathrm{Ad}_h(\xi) = \xi \text{ for all } h \in H$$

Assuming H, acting via the adjoint action, fixes ξ, we would like to make $\mu_X^{-1}(\xi)/H$ into a Kähler manifold.

Lemma 0.1. Suppose X, Y are Kähler manifolds, $\mathrm{SU}(n)$ acts on X and Y preserving the Kähler structure, and we have moment maps $\mu_X : X \to \mathfrak{su}(n)$ and $\mu_Y : Y \to \mathfrak{su}(n)$. Then the map

$$\mu(x, y) = \mu_X(x) + \mu_Y(y)$$

defines a moment map for the action

$$g \cdot (x, y) = (g \cdot x, g \cdot y)$$

of $\mathrm{SU}(n)$ on $X \times Y$.

Note that the product of complex manifolds is a complex manifold, with the natural choice of coordinate charts. If we consider the decomposition

$$\mathrm{T}_{(x,y)}(X \times Y) \cong \mathrm{T}_x X \oplus \mathrm{T}_y Y$$

induced by projection maps, then the Riemannian metric is defined by

$$g((x_1, y_1), (x_2, y_2)) = g_X(x_1, x_2) + g_Y(y_1, y_2)$$

where g_X, g_Y are the Riemannian metrics on X, Y respectively. Similarly, the symplectic form is defined by

$$\omega((x_1, y_1), (x_2, y_2)) = \omega_X(x_1, x_2) + \omega_Y(y_1, y_2)$$

Proof. **Equivariance** follows immediately from the equivariance of μ_X and μ_Y.

$$\mu(g \cdot (x, y)) = \mu(g \cdot x, g \cdot y) = \mu_X(g \cdot x) + \mu_Y(g \cdot y) = \mathrm{Ad}_g\, \mu_X(x) + \mathrm{Ad}_g\, \mu_Y(y) = \mathrm{Ad}_g(\mu_X(x) + \mu_Y(y)) = \mathrm{Ad}_g(\mu(x, y))$$

Hamiltonian function

Next, fix $\eta \in \mathfrak{su}(n)$. Let U^η be the vector field on X generated by η, and Y^η the vector field on V generated by η. Define $\mu_X^\eta(p) = \langle \mu_X(p), \eta \rangle$. Then we have that for any $W \in \mathrm{T}_p X$,

$$(\mathrm{d}\mu_X^\eta)_p(W) = \omega_X(U_p^\eta, W)$$

and a similar statement holds for μ_Y. If we now define

$$\mu^\eta(x, y) = \langle \mu_X(x) + \mu_Y(y), \eta \rangle = \mu_X^\eta(x) + \mu_Y^\eta(y)$$

Then

$$\begin{aligned}
(\mathrm{d}\mu^\eta)_{(x,y)}(u, v) &= (\mathrm{d}\mu_X^\eta)_x(u) + (\mathrm{d}\mu_Y^\eta)_y(v) \\
&= \omega_X(U_x^\eta, u) + \omega_Y(V_y^\eta, v) \\
&= \omega((U^\eta, V^\eta)_{(x,y)}, (u, v))
\end{aligned}$$

But the vector field on $X \times Y$ generated by η is precisely (U^η, V^η). $\square$

Lemma 0.2. Let $\xi \in \mathfrak{su}(n)$, and M be its adjoint orbit, with the symplectic structure given by the Kirillov-Kostant-Souriau form

$$\omega_\xi([\xi, \eta], [\xi, \zeta]) = -\langle \xi, [\eta, \zeta] \rangle$$

Then $\mathrm{SU}(n)$ acts on M via the adjoint action preserving the Kähler structure, and with moment map $\mu(\xi) = -\xi$.

Proof. We will omit the proof that the adjoint action preserves the Kähler structure. It is clear that $\mu(\xi) = -\xi$ is equivariant, since both the actions are the adjoint action.

Fix $\eta \in \mathfrak{su}(n)$. The vector field on M generated by η is precisely $U_\xi^\eta = [\xi, \eta]$. Set $\mu^\eta(\xi) = \langle \mu(\xi), \eta \rangle = -\langle \xi, \eta \rangle$. This extends to a linear map on $\mathfrak{su}(n)$, and so we have that

$$(\mathrm{d}\mu^\eta)_\xi([\xi, \zeta]) = -\langle [\xi, \zeta], \eta \rangle = -\langle \xi, [\zeta, \eta] \rangle = \omega_\xi([\xi, \zeta], [\xi, \eta])$$

as required. $\square$

Theorem 0.3 (Kähler reduction with momentum)**.** Suppose X is a Kähler manifold, where $\mathrm{SU}(n)$ acts on X preserving the Kähler structure. Let $\xi \in \mathfrak{su}(n)$, and

$$H = \{h \in G \mid \mathrm{Ad}_h(\xi) = \xi\}$$

is the stabiliser for the adjoint action of $\mathrm{SU}(n)$. Let $\mu_X : X \to \mathfrak{su}(n)$ be a moment map for the $\mathrm{SU}(n)$ action on X, and suppose $\mathrm{SU}(n)$ acts freely on $\mu^{-1}(\xi)$. Then $\mu_X^{-1}(\xi)/H$ is a Kähler manifold.

Proof. Let M be the adjoint orbit of ξ. Combining the previous lemmas, we have that $\mathrm{SU}(n)$ acts on $X \times M$ preserving the Kähler structure, with moment map

$$\mu(p, \eta) = \mu_X(p) - \eta$$

Step 1: $\mathrm{SU}(n)$ **acts freely on** $\mu^{-1}(0)$**.** Let $(p, \eta) \in \mu^{-1}(0)$, and $g \in \mathrm{SU}(n)$ fixing (p, η). That is,

$$\begin{aligned}
g \cdot p &= p \\
\mathrm{Ad}_g(\eta) &= \eta
\end{aligned}$$

Say $\eta = \mu_X(p) = \mathrm{Ad}_h(\xi)$. Then

$$h^{-1}gh \cdot (h^{-1} \cdot p) = h^{-1} \cdot p$$

But

$$\mu_X(h^{-1} \cdot p) = \mathrm{Ad}_{h^{-1}} \mu_X(p) = \mathrm{Ad}_{h^{-1}} \mathrm{Ad}_h(\xi) = \xi$$

and as $\mathrm{SU}(n)$ acts freely on $\mu^{-1}(\xi)$, we must have that $h^{-1}gh = 1$. So $g = 1$. Using this, we can take the symplectic quotient $\mu^{-1}(0)/\mathrm{SU}(n)$.

Step 2: Bijection $\mu^{-1}(0)/\operatorname{SU}(n) \cong \mu_X^{-1}(\xi)/H$. Define

$$\begin{aligned} F : \mu_X^{-1}(\xi) &\to \mu^{-1}(0) \\ F(p) &= (p, \xi) \end{aligned}$$

Then for $h \in H$,

$$F(h \cdot p) = (h \cdot p, \xi) = (h \cdot p, \operatorname{Ad}_h(\xi)) = h \cdot F(p)$$

Therefore, we have a smooth map Φ making the diagram

$$\begin{tikzcd} \mu_X^{-1}(\xi) \arrow[rr, "F"] \arrow[d, two heads, "\pi_X"'] & & \mu^{-1}(0) \arrow[d, "\pi"] \\ \mu_X^{-1}(\xi)/H \arrow[rr, "\Phi"'] & & \mu^{-1}(0)/\operatorname{SU}(n) \end{tikzcd}$$

commute, where π_X, π are the quotient maps. We would like to show that Φ is a bijection.

Injectivity. If $\Phi([p]) = \Phi([q])$, then $\pi(p, \xi) = \pi(q, \xi)$. Therefore, there exists $g \in \operatorname{SU}(n)$ such that

$$(p, \xi) = (g \cdot q, \operatorname{Ad}_g \xi)$$

Since $\operatorname{Ad}_g(\xi) = \xi$, $g \in H$. Therefore, p and q are in the same H-orbit.

Surjectivity. Let $[(q, \eta)] \in \mu^{-1}(0)/\operatorname{SU}(n)$. Then η is in the adjoint orbit M, so there exists $g \in \operatorname{SU}(n)$ such that $\eta = \operatorname{Ad}_g(\xi)$. In this case,

$$\mu_X(g^{-1} \cdot q) = \operatorname{Ad}_{g^{-1}} \mu(q) = \operatorname{Ad}_{g^{-1}} \operatorname{Ad}_g(\xi) = \xi$$

and so $g^{-1} \cdot q \in \mu_X^{-1}(\xi)$. In this case,

$$\Phi([g^{-1} \cdot q]) = [(g^{-1} \cdot q, \xi)] = [(q, \eta)]$$

and so Φ is a bijection.

Step 3: Φ is a diffeomorphism. Since Φ is a smooth bijection, suffices to show that it is a submersion. As π_X is a surjective submersion, suffices to show that $\pi \circ F$ is a submersion. The map

$$\begin{aligned} \widehat{\operatorname{Ad}} : \operatorname{SU}(n) &\to M \\ g &\mapsto \operatorname{Ad}_g(\xi) \end{aligned}$$

is a submersion, therefore there exists a local right inverse σ, with $\sigma(\xi) = 1$. So $\operatorname{Ad}_{\sigma(\eta)}(\xi) = \eta$. Then for η sufficiently close to ξ, with $\mu_X(q) = \eta$,

$$\mu_X(\sigma(\eta)^{-1} \cdot q) = \operatorname{Ad}_{\sigma(\eta)^{-1}} \mu_X(q) = \xi$$

Define the map

$$\psi(\eta) = \sigma(\eta)^{-1} \cdot q$$

Then we have that

$$\pi(F(\psi(\eta))) = [(\sigma(\eta)^{-1} \cdot q, \xi)] = [q, \operatorname{Ad}_{\sigma(\eta)} \xi] = [(q, \eta)]$$

Therefore, if α is a local right inverse for π, with $\alpha([(q, \eta)]) = \eta$, then

$$(\pi \circ F) \circ (\psi \circ \alpha)([(q, \eta)]) = \pi(F(\psi(\eta))) = [(q, \eta)]$$

Hence $\psi \circ \alpha$ is a local right inverse, and so $\pi \circ F$ is a submersion. $\square$