# Invariance of the hyperKähler metric

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In this note, we will investigate the invariance of the hyperKähler metric from [1]. We will use the same notation as in section 2.

Let $g$ denote the Riemannian metric on $M$. That is,

$$g((\alpha_j, \beta_j), (\tilde{\alpha}_j, \tilde{\beta}_j)) = \sum_{j=0}^{k-1} \operatorname{Re} \operatorname{tr} \left( \alpha_j^* \tilde{\alpha}_j + \beta_j \tilde{\beta}_j^* \right)$$

We have a $\mathrm{SL}(n, \mathbb{C})$ acton on $M$, by

$$\psi_\gamma(\alpha_j, \beta_j) = (\alpha_0, \ldots, \alpha_{k-2}, \gamma\alpha_{k-1}, \beta_0, \ldots, \beta_{k-2}, \beta_{k-1}\gamma^{-1})$$

Since $\psi_\gamma : M \to M$ is linear, $\mathrm{d}\psi_\gamma = \psi_\gamma$. In particular, for

$$\begin{aligned} \omega_J((\alpha_j, \beta_j), (\tilde{\alpha}_j, \tilde{\beta}_j)) &= g(J(\alpha_j, \beta_j), (\tilde{\alpha}_j, \tilde{\beta}_j)) \\ &= \sum_{j=0}^{k-1} \operatorname{Re} \operatorname{tr} \left( -(\beta_j^*)^* \tilde{\alpha}_j + \alpha_j^* \tilde{\beta}_j^* \right) \\ &= \sum_{j=0}^{k-1} \operatorname{Re} \operatorname{tr} \left( -\beta_j \tilde{\alpha}_j + \alpha_j^* \tilde{\beta}_j^* \right) \end{aligned}$$

We will need to show that

But trace is conjugation invariant. Hence $\psi_g^* \omega_J = \omega_J$. Similarly, we have that $\psi_g^* \omega_K = \omega_K$. Therefore, the complex symplectic form $\omega_c = \omega_J + i\omega_K$ is $\psi_g$-invariant. Now notice that if $\tilde{\Phi} = \Phi^c \circ \pi$, then $\tilde{\Phi}$ is a surjective submersion, and

$$\begin{tikzcd} \mu_c^{-1}(0) \arrow[r, "\psi_\gamma"] \arrow[d, "\Phi"'] & \mu_c^{-1}(0) \arrow[d, "\Phi"] \\ \mathrm{Orb}(A) \arrow[r, "\mathrm{Ad}_\gamma"'] & \mathrm{Orb}(A) \end{tikzcd}$$

commutes. This means that the complex-symplectic form $\tilde{\omega}_c$ is $\mathrm{Ad}_\gamma$-invariant. To see this, first note that $\Phi^* \tilde{\omega}_c$ is the restriction of $\omega_c$ to $\mu_c^{-1}(0)$. In this case,

$$\begin{aligned} \Phi^* \operatorname{Ad}_\gamma^* \tilde{\omega}_c &= \psi_\gamma^* \Phi^* \tilde{\omega}_c \\ &= \psi_\gamma^* \iota^* \omega_c \\ &= \iota^* \omega_c \\ &= \Phi^* \tilde{\omega}_c \end{aligned}$$

As $\Phi^*$ is injective, $\operatorname{Ad}_\gamma^* \tilde{\omega}_c = \tilde{\omega}_c$.

Note however that

$$\psi_\gamma^* g((\alpha_j, \beta_j), (\tilde{\alpha}_j, \tilde{\beta}_j)) = g(\psi_\gamma(\alpha_j, \beta_j), \psi_\gamma(\tilde{\alpha}_j, \tilde{\beta}_j))$$