Computation of moment map

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1 HyperKähler moment maps

On $\mathbb{H}^N$, we can construct a hyperKähler structure, using the standard metric given by

$$\langle u, v \rangle = \overline{u}^{\mathsf{T}} v$$

where $\overline{u}$ denotes the (elementwise) quaternionic conjugate of u. The complex structures are given by right multiplication by $-i, -j, -k$ respectively. Let $\omega_I, \omega_J, \omega_K$ be the corresponding Kähler forms, and $\eta = \omega_I i + \omega_J j + \omega_K k$.

Let H be a subgroup of $\mathrm{Sp}(N)$. Then H acts on $\mathbb{H}^N$ preserving the hyperKähler structure. In this case, a hyperKähler moment map is a map $\mu : \mathbb{H}^N \to \mathfrak{h}^* \otimes \mathrm{Im}(\mathbb{H})$, which is equivariant with respect to the H, and with $\mathrm{d}\left(\mu^X\right) = X \lrcorner \eta$

In particular, in [1], we make the choice

$$\mu^X(q) = -\overline{q}^{\mathsf{T}} X q = -\langle q, Xq \rangle$$

where we define $\langle u, v \rangle = \overline{u}^{\mathsf{T}} v$ for elements of $\mathbb{H}^N$.

2 U(n) action

Choose a sequence $(V_0, \dots, V_k)$ of Hermitian vector spaces, with $\dim_{\mathbb{C}}(V_i) = n_i$, $n_0 = 0, n_k = n$. Let

$$M = \bigoplus_{i=0}^{k-1} (\mathrm{Hom}(V_i, V_{i+1}) \oplus \mathrm{Hom}(V_{i+1}, V_i))$$

and we write a point of M as (α_i, β_i), where

$$V_0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} V_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \cdots \underset{\beta_{k-1}}{\overset{\alpha_{k-1}}{\rightleftarrows}} V_k$$

Note that $\langle \alpha, \beta \rangle = \mathrm{tr}(\alpha\beta^*)$ defines a Hermitian metric on $\mathrm{Hom}(V, W)$, and hence on M, we have the metric

$$\left\langle\!\left\langle (\alpha_i, \beta_i), (\tilde{\alpha}_i, \tilde{\beta}_i) \right\rangle\!\right\rangle = \sum_{i=0}^{k-1} \left(\langle \alpha_i, \tilde{\alpha}_i \rangle + \left\langle \beta_i, \tilde{\beta}_i \right\rangle \right)$$

The complex structures are

$$I(\alpha_i, \beta_i) = (i\alpha_i, i\beta_i) \quad J(\alpha_i, \beta_i) = (-\beta_i^*, \alpha_i^*)$$

The Lie group $G = \mathrm{U(n_1)} \times \mathrm{U(n_{k-1})}$ acts on M via

$$\begin{aligned} \alpha_i &\mapsto g_{i+1}\alpha_i g_i^{-1} = g_{i+1}\alpha_i g_i^* \\ \beta_i &\mapsto g_i \beta_i g_{i+1}^{-1} = g_i \beta_i g_{i+1}^* \end{aligned}$$

Now notice that $\langle \alpha, \beta \rangle$ as above induces an isomorphism $\mathfrak{u}(m) \cong \mathfrak{u}(m)^*$, via $X \mapsto \langle X, \cdot \rangle$.

3 Moment map

Now let $X_i \in \mathfrak{u}(n_i)$, and let $X = (0, \dots, X_i, \dots, 0) \in \mathfrak{u}(n_1) \oplus \cdots \oplus \mathfrak{u}(n_{k-1})$. Let $q = (\alpha_i, \beta_i) \in M$. Then the action of X is

$$Xq = (0, \dots, X_i\alpha_{i-1}, -\alpha_i X_i, \dots, 0, 0, \dots, -\beta_{i-1}X_i, X_i\beta_i, \dots, 0)$$

In particular, we have that

$$\begin{aligned}
\langle\langle q, Xq \rangle\rangle &= \langle \alpha_{i-1}, X_i\alpha_{i-1} \rangle - \langle \alpha_i, \alpha_i X_i \rangle - \langle \beta_{i-1}, \beta_{i-1}X_i \rangle + \langle \beta_i, X_i\beta_i \rangle \\
&= \operatorname{tr}\big(\alpha_{i-1}\alpha_{i-1}^* X^* - \alpha_i X_i^* \alpha_i^* - \beta_{i-1}X_i^*\beta_{i-1}^* + \beta_i\beta_i^* X^*\big) \\
&= \operatorname{tr}\big((\alpha_i^*\alpha_i - \beta_i\beta_i^* + \beta_{i-1}^*\beta_{i-1} - \alpha_{i-1}\alpha_{i-1}^*)X_i\big)
\end{aligned}$$

which gives us

$$\mu_r = (\alpha_{i-1}\alpha_{i-1}^* - \beta_{i-1}^*\beta_{i-1} + \beta_i\beta_i^* - \alpha_i^*\alpha_i)$$

Next, we can take

$$\begin{aligned}
\langle\langle q, X \cdot (-J)(q) \rangle\rangle &= -\langle \alpha_{i-1}, -X_i\beta_{i-1}^* \rangle + \langle \alpha_i, -\beta_i X_i^* \rangle + \langle \beta_{i-1}, \alpha_{i-1}^* X_i \rangle - \langle \beta_i, \alpha_i^* X_i \rangle \\
&= \operatorname{tr}(\alpha_{i-1}\beta_{i-1}X_i^* - \alpha_i X_i^*\beta_i + \beta_{i-1}X_i^*\alpha_{i-1} - \beta_i X_i^*\alpha_i) \\
&= -2\operatorname{tr}((\alpha_{i-1}\beta_{i-1} - \beta_i\alpha_i)X_i)
\end{aligned}$$

which gives us

$$\mu_c = (\alpha_{i-1}\beta_{i-1} - \beta_i\alpha_i)$$

Note also that $IX = XI$ and $JX = XJ$, and that $I, J, K = IJ$ define isometries with respect to $\langle\langle \cdot, \cdot \rangle\rangle$, so we also have that

$$\langle\langle q, X \cdot (-J)q \rangle\rangle = \langle\langle Jq, Xq \rangle\rangle$$

4 How does this work?

Now for simplicity, consider the case of $\mathbb{H}$. In this case, we can write $p \in \mathbb{H}$ as $p = z_0 + jz_1$, where $z_0, z_1 \in \mathbb{C}$. Then the hermitian inner product is given by

$$\langle z_0 + jz_1, w_0 + jw_1 \rangle = \overline{z}_0 w_0 + \overline{z}_1 w_1$$

Writing $z_0 = a + bi, z_1 = c + di$, $p = a + bi + cj - dk$. Then the quaternionic conjugate is $\overline{p} = a - bi - cj + dk = \overline{z}_0 - jz_1$. This means that

$$\begin{aligned}
\overline{p}q &= (\overline{z}_0 - jz_1)(w_0 + jw_1) \\
&= \overline{z}_0 w_0 + \overline{z}_0 jw_1 - jz_1 w_0 - jz_1 jw_1
\end{aligned}$$

Next, if $z_0 = a + bi$, then $z_0 j = aj + bk = j(a - bi) = j\overline{z}_0$. Using this, we get that

$$\overline{p}q = (\overline{z}_0 w_0 + \overline{z}_1 w_1) + j(z_0 w_1 - z_1 w_0)$$

Next, notice that $pj = -\overline{z}_1 + j\overline{z}_0$, and so

$$\langle pj, q \rangle = -z_1 w_0 + z_0 w_1$$

References

[1] Piotr Z. Kobak and Andrew Swann. “Classical nilpotent orbits as hyperkähler quotients”. In: *Int. J. Math.* 07.02 (Apr. 1996). Publisher: World Scientific Publishing Co., pp. 193–210. ISSN: 0129-167X. DOI: `10.1142/S0129167X96000116`. URL: `https://www.worldscientific.com/doi/10.1142/S0129167X96000116` (visited on 07/27/2023).