Complex structure on $\mathfrak{sl}(n, \mathbb{C})$ and coadjoint orbits of $\mathrm{SU}(n)$

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First of all, notice that

$$\mathfrak{sl}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \mathrm{tr}(A) = 0\}$$

is naturally a complex vector space, and that we have the decomposition

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{su}(n) \oplus i\mathfrak{su}(n)$$

With respect to this decomposition, the complex structure J on $\mathfrak{sl}(n, \mathbb{C})$ is given by the matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

Next, let $M = \mathrm{SU}(n) \times \mathfrak{su}(n)$. The tangent space of M at $p = (g, X)$ is

$$\mathrm{T}_p M = \mathrm{T}_g \mathrm{SU}(n) \times \mathrm{T}_X \mathfrak{su}(n) = g\mathfrak{su}(n) \times \mathfrak{su}(n)$$

Therefore, we have a natural isomorphism $\mathrm{T}_p M \cong \mathfrak{su}(n) \oplus \mathfrak{su}(n)$, given by

$$(u, v) \mapsto (g^{-1}u, v)$$

This then gives us a natural complex structure on $\mathrm{T}_p M$, by

$$J(u, v) = (-gv, g^{-1}u) \in \mathrm{T}_g \mathrm{SU}(n) \times \mathfrak{su}(n)$$

But we also have the symplectic form, coming from the isomorphism $\mathrm{T}^* \mathrm{SU}(n) \cong \mathrm{SU}(n) \times \mathfrak{su}(n)^*$ as vector bundles, given at $q = (g, \xi)$ by

$$\omega_q((v, \phi), (w, \psi)) = -\phi(g^{-1}w) + \psi(g^{-1}v) + \xi([g^{-1}v, g^{-1}w])$$

where $v, w \in \mathrm{T}_g \mathrm{SU}(n) = g\mathfrak{su}(n)$, $\phi, \psi \in \mathfrak{su}(n)^*$. Next, as $\mathrm{SU}(n)$ is compact, fix a bi-invariant metric $\langle \cdot, \cdot \rangle$ on $\mathrm{SU}(n)$. This gives us an isomorphism $R : \mathfrak{su}(n) \to \mathfrak{su}(n)^*$ by

$$R(X)(Y) = \langle X, Y \rangle$$

For concreteness, we can take $\langle A, B \rangle = -\mathrm{tr}(AB)$. Say $\xi = R(X)$, $\phi = R(y)$, $\psi = R(z)$. Then we have that

$$\omega_p((v, y), (w, z)) = -\left\langle y, g^{-1}w \right\rangle + \left\langle z, g^{-1}v \right\rangle + \left\langle X, [g^{-1}v, g^{-1}w] \right\rangle$$

and this gives the Riemannian metric

$$\begin{aligned}
\langle\!\langle (v, y), (w, z) \rangle\!\rangle &= \omega_p((v, y), J(w, z)) \\
&= \omega_p((v, y), (-gz, g^{-1}w)) \\
&= \langle y, z \rangle + \left\langle g^{-1}w, g^{-1}v \right\rangle - \left\langle X, [g^{-1}v, z] \right\rangle \\
&= \langle y, z \rangle + \langle w, v \rangle - \left\langle X, [g^{-1}v, z] \right\rangle
\end{aligned}$$