Riemannian metric

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In this note, we will compute the Riemannian metric on the adjoint orbit, following [1].

Choose a sequence $n_0, \dots, n_k$, with $n_0 = 0$ and $n_k = n$. Then define the space

$$M = \bigoplus_{j=0}^{k-1} \left(\mathrm{Hom}(\mathbb{C}^{n_j}, \mathbb{C}^{n_{j+1}}) \oplus \mathrm{Hom}(\mathbb{C}^{n_{j+1}}, \mathbb{C}^{n_j})\right)$$

We will identify M with the matrix space

$$M = \bigoplus_{j=1}^{k-1} \left(\mathrm{Mat}_{n_{j+1} \times n_j}(\mathbb{C}) \oplus \mathrm{Mat}_{n_j \times n_{j+1}}(\mathbb{C})\right)$$

and write a general point as (α_j, β_j), where $\alpha_j : \mathbb{C}^{n_j} \to \mathbb{C}^{n_{j+1}}$ and $\beta_j : \mathbb{C}^{n_{j+1}} \to \mathbb{C}^{n_j}$. Define the moment maps

$$\begin{aligned}
\mu_r(\alpha_j, \beta_j) &= (\alpha_{j-1}\alpha_{j-1}^* - \beta_{j-1}^*\beta_{j-1} + \beta_j\beta_j^* - \alpha_j^*\alpha_j)_{j=1}^{k-1} \\
\mu_c(\alpha_j, \beta_j) &= (\alpha_{j-1}\beta_{j-1} - \beta_j\alpha_j)_{j=1}^{k-1}
\end{aligned}$$

Computing the derivatives at the fixed point $p = (\alpha_j, \beta_j)$,

$$\begin{aligned}
\mathrm{d}\mu_r(\delta_j, \varepsilon_j) &= \left(\alpha_{j-1}\delta_{j-1}^* + \delta_{j-1}\alpha_{j-1}^* - \varepsilon_{j-1}^*\beta_{j-1} - \beta_{j-1}^*\varepsilon_{j-1} + \varepsilon_j\beta_j^* + \beta_j\varepsilon_j^* - \alpha_j^*\delta_j - \delta_j^*\alpha_j\right)_{j=1}^{k-1} \\
\mathrm{d}\mu_c(\delta_j, \varepsilon_j) &= \left(\delta_{j-1}\beta_{j-1} + \alpha_{j-1}\varepsilon_{j-1} - \beta_j\delta_j - \varepsilon_j\alpha_j\right)_{j=1}^{k-1}
\end{aligned}$$

and the tangent space of $\mu^{-1}(0) = \mu_c^{-1}(0) \cap \mu_r^{-1}(0)$ is

$$\mathrm{T}_p\mu^{-1}(0) = \ker(\mathrm{d}\mu_r) \cap \ker(\mathrm{d}\mu_c)$$

Next, note that if we differentiate the $G = \mathrm{U(n_1)} \times \dots \times \mathrm{U(n_{k-1})}$ action, we will get the kernel of the quotient map, which is

$$V = \left\{(X_{j+1}\alpha_j - \alpha_j X_j, X_j\beta_j - \beta_j X_{j+1}) \mid X_j \in \mathfrak{u}(n_j)\right\}$$

The Riemannian metric on the quotient space $\mu^{-1}(0)/G$ is induced by the linear isomorphism

$$\mathrm{T}_{[p]}(\mu^{-1}(0)/G) \cong H := V^\perp$$

given by the quotient map. Next, note that the map $\tilde{\Phi}^c(\alpha_j, \beta_j) = \alpha_{k-1}\beta_{k-1}$ defines a diffeomorphism $\Phi^c : \mu^{-1}(0)/G \to N$, where N is a nilpotent orbit (when restricted to an open subset of $\mu^{-1}(0)$).

$$\begin{tikzcd}
\mu^{-1}(0) \arrow[d, two heads, "\pi"'] \arrow[dr, "\tilde{\Phi}^c"] & \\
\mu^{-1}(0)/G \arrow[r, "\Phi^c"] & N
\end{tikzcd}$$

Therefore, we can compute the Riemannian metric on $\mu^{-1}(0)/G$ by considering its pullback to $\mu^{-1}(0)$. Moreover, since N is a complex submanifold of $\mathfrak{sl}(n, \mathbb{C})$, it also has a natural Riemannian metric g from its Kähler structure. We can then consider the pullback $(\tilde{\Phi}^c)^*g$ as an inner product on H.

Suppose $(\delta_j, \varepsilon_j) \in H$. Then for all $(u_j, v_j) \in V$, say $u_j = X_{j+1}\alpha_j - \alpha_j X_j$, $v_j = X_j\beta_j - \beta_j X_{j+1}$,

$$\begin{aligned}\sum_{j=1}^{k-1} \operatorname{Re}\left(\operatorname{tr}\left(\delta_j u_j^*\right) + \operatorname{tr}\left(\varepsilon_j v_j^*\right)\right) &= \sum_{j=1}^{k-1} \operatorname{Re}\left(\operatorname{tr}\left((\alpha_j^*\delta_j - \varepsilon_j\beta_j^*)X_j\right) + \operatorname{tr}\left((\beta_j^*\varepsilon_j - \delta_j\alpha_j^*)X_{j+1}\right)\right) \\ &= \operatorname{Re}\sum_{j=1}^{k-1} \operatorname{tr}\left((\alpha_j^*\delta_j - \varepsilon_j\beta_j^* + \beta_{j-1}^*\varepsilon_{j-1} - \delta_{j-1}\alpha_{j-1}^*)X_j\right)\end{aligned}$$

Therefore, a sufficient condition is

$$\alpha_j^*\delta_j - \varepsilon_j\beta_j^* + \beta_{j-1}^*\varepsilon_{j-1} - \delta_{j-1}\alpha_{j-1}^* = 0$$

Finally, we compute

$$\mathrm{d}\Phi^c(\delta, \varepsilon) = \delta_{k-1}\beta_{k-1} + \alpha_{k-1}\varepsilon_{k-1}$$

Combining all of the above, we have the following conditions

1. $\ker(\mathrm{d}\mu_r)$
$$\alpha_{j-1}\delta_{j-1}^* + \delta_{j-1}\alpha_{j-1}^* - \varepsilon_{j-1}^*\beta_{j-1} - \beta_{j-1}^*\varepsilon_{j-1} + \varepsilon_j\beta_j^* + \beta_j\varepsilon_j^* - \alpha_j^*\delta_j - \delta_j^*\alpha_j = 0$$
2. $\ker(\mathrm{d}\mu_c)$
$$\delta_{j-1}\beta_{j-1} + \alpha_{j-1}\varepsilon_{j-1} - \beta_j\delta_j - \varepsilon_j\alpha_j = 0$$
3. $\mu_c = 0$
$$\alpha_{j-1}\beta_{j-1} - \beta_j\alpha_j = 0$$
4. $\mu_r = 0$
$$\alpha_{j-1}\alpha_{j-1}^* - \beta_{j-1}^*\beta_{j-1} + \beta_j\beta_j^* - \alpha_j^*\alpha_j = 0$$
5. Orthogonality
$$\alpha_j^*\delta_j - \varepsilon_j\beta_j^* + \beta_{j-1}^*\varepsilon_{j-1} - \delta_{j-1}\alpha_{j-1}^* = 0$$

Note also that 5. implies 1.
With all of these, it is then clear that

$$J(\delta_j, \varepsilon_j) = (-\varepsilon_j^*, \delta_j^*)$$

defines a linear map $H \to H$. Hence we have the complex structure J on N, given by

$$J(\delta_{k-1}\beta_{k-1} + \alpha_{k-1}\varepsilon_{k-1}) = \alpha_{k-1}\delta_{k-1}^* - \varepsilon_{k-1}^*\beta_{k-1}$$

References

[1] Piotr Z. Kobak and Andrew Swann. “Classical nilpotent orbits as hyperkähler quotients”. In: *Int. J. Math.* 07.02 (Apr. 1996). Publisher: World Scientific Publishing Co., pp. 193–210. ISSN: 0129-167X. DOI: `10.1142/S0129167X96000116`. URL: `https://www.worldscientific.com/doi/10.1142/S0129167X96000116` (visited on 07/27/2023).