Semisimple Lie Algebras

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This document sketches the definition of semisimple Lie algebras and the Killing form. In addition, we will look at representations, and the root space decomposition. The main reference is Humphreys' *Introduction to Lie Algebras and Representation Theory*, where all of the skipped proofs can be found.

Throughout, let F be an algebraically closed field, with $\mathrm{char}(F) = 0$, $\mathfrak{g}$ is a Lie algebra over F.

1 Definitions

Definition 1.1 (Lie Algebra)
Let $\mathfrak{g}$ be a vector space over F. Suppose $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear form such that

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,
2. *(Jacobi identity)* $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

Then $(\mathfrak{g}, [\cdot, \cdot])$ is called a *Lie algebra* over F.

Definition 1.2 (abelian)
$\mathfrak{g}$ is called *abelian* if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

Definition 1.3 (subalgebra)
A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a (Lie) *subalgebra* if for all $x, y \in \mathfrak{h}$, $[x, y] \in \mathfrak{h}$.

Definition 1.4 (Ideal)
A subspace I of $\mathfrak{g}$ is called an *ideal* if for all $x \in \mathfrak{g}$ and $y \in I$, $[x, y] \in I$.

Proposition 1.5. Every ideal is a subalgebra.

Definition 1.6 (centre)
The *centre* of $\mathfrak{g}$ is the ideal

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, z] = 0 \text{ for all } z \in \mathfrak{g}\}$$

Definition 1.7 (derived algebra)

The *derived algebra* of $\mathfrak{g}$ is the ideal $[\mathfrak{g}, \mathfrak{g}]$.

Proposition 1.8. The following are equivalent:

1. $\mathfrak{g}$ is abelian,
2. $Z(\mathfrak{g}) = \mathfrak{g}$,
3. $[\mathfrak{g}, \mathfrak{g}] = 0$.

Definition 1.9 (simple)
Suppose $\mathfrak{g}$ is not simple, and the only ideals are 0 and $\mathfrak{g}$. Then we say $\mathfrak{g}$ is *simple*.

Definition 1.10 (homomorphism)
Suppose $\mathfrak{g}, \mathfrak{h}$ are Lie algebras over F. Then a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is called a *homomorphism* if for all $x, y \in \mathfrak{g}$, $\phi([x, y]) = [\phi(x), \phi(y)]$.

Proposition 1.11. Suppose $\phi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism. Then $\ker(\phi)$ is an ideal, and $\operatorname{im}(\phi)$ is a subalgebra. Moreover, $\mathfrak{g}/\ker(\phi) \cong \operatorname{im}(\phi)$.

2 Solvable Lie algebras

Definition 2.1 (Derived series, solvable)
Define the sequence of ideals by

$$\mathfrak{g}^{(0)} = \mathfrak{g} \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$$

The sequence $\mathfrak{g}^{(i)}$ is called the *derived series* of $\mathfrak{g}$. We say that $\mathfrak{g}$ is solvable if $\mathfrak{g}^{(n)} = 0$ for some n.

Proposition 2.2. $\mathfrak{g}$ has a unique maximal solvable ideal.

Proof. Existence follows by Zorn's lemma. Suppose S is a maximal solvable ideal for $\mathfrak{g}$, I is any solvable ideal. Then $S + I$ is solvable, and $S + I \supseteq S$, so $S + I = S$. Thus $I \subseteq S$, so S is unique. $\square$

Definition 2.3 (radical, semisimple)
The unique maximal solvable ideal of $\mathfrak{g}$ is called the *radical* of $\mathfrak{g}$, denoted $\operatorname{rad}(\mathfrak{g})$. We say that $\mathfrak{g}$ is *semisimple* if $\operatorname{rad}(\mathfrak{g}) = 0$.

Lemma 2.4. $\mathfrak{g}/\operatorname{rad}(\mathfrak{g})$ is semisimple.

Proposition 2.5. $\mathfrak{g}$ is semisimple if and only if it has no nonzero abelian ideals.

Proof. Any nonzero abelian ideal would be contained in $\mathrm{rad}(\mathfrak{g})$, as any abelian Lie algebra is solvable. Conversely, suppose $\mathrm{rad}(\mathfrak{g})$ is nonzero. Then the last nonzero term $\mathrm{rad}(\mathfrak{g})^{(n-1)}$ of the derived series satisfies

$$[\mathrm{rad}(\mathfrak{g})^{(n-1)}, \mathrm{rad}(\mathfrak{g})^{(n-1)}] = \mathrm{rad}(\mathfrak{g})^{(n)} = 0$$

Moreover, $\mathrm{rad}(\mathfrak{g})^{(n-1)}$ is an ideal of $\mathfrak{g}$. □

3 Killing form

Suppose in addition that $\mathfrak{g}$ is finite dimensional.

Definition 3.1 (adjoint representation)
The *adjoint representation* is the homomorphism $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $\mathrm{ad}(x)(y) = [x, y]$.

Definition 3.2 (Killing form)
The *Killing form* of $\mathfrak{g}$ is a symmetric bilinear form on $\mathfrak{g}$, defined by

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}(x) \cdot \mathrm{ad}(y))$$

Proposition 3.3. κ is associative, that is,

$$\kappa([x, y], z) = \kappa(x, [y, z])$$

Lemma 3.4. Let I be an ideal of $\mathfrak{g}$. Then I is a Lie algebra, with Killing form κ_I. Then $\kappa_I = \kappa|_{I \times I}$.

Definition 3.5 (radical, nondegenerate)
Suppose β is a symmetric bilinear form on $\mathfrak{g}$. Define the *radical* of β to be

$$\mathrm{rad}(\beta) = \{x \in \mathfrak{g} \mid \beta(x, y) = 0 \text{ for all } y \in \mathfrak{g}\}$$

Then $\mathrm{rad}(\beta)$ is a subspace of $\mathfrak{g}$. We say that β is *nondegenerate* if $\mathrm{rad}(\beta) = 0$.

Proposition 3.6. $\mathrm{rad}(\kappa)$ is an ideal of $\mathfrak{g}$.

Lemma 3.7. Let $x_1, \dots, x_n$ be a basis of $\mathfrak{g}$. Then κ is nondegenerate if and only if the matrix $(\kappa(x_i, x_j))$ is invertible.

Theorem 3.8. Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is semisimple if and only if κ is nondegenerate.

Theorem 3.9. Let $\mathfrak{g}$ be a semisimple Lie algebra. Then there exists ideals $I_1, \ldots, I_r$ of $\mathfrak{g}$, such that

$$\mathfrak{g} = I_1 \oplus \cdot \oplus I_r$$

as vector spaces. Every simple ideal of $\mathfrak{g}$ is one of the I_j, and the Killing form of I_j is $\kappa|_{I_j \times I_j}$.

Corollary 3.10. If $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, and all ideals and quotients of $\mathfrak{g}$ are semisimple. Moreover, each ideal of $\mathfrak{g}$ is a direct sum of simple ideals in $\mathfrak{g}$.

4 Representations

Definition 4.1 (representation)
A *representation* of $\mathfrak{g}$ is a homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$, where V is a vector space over F.

Definition 4.2 ($\mathfrak{g}$-module)
Let V be a vector space, then V is a $\mathfrak{g}$-*module* if there exists a map $\mathfrak{g} \times V \to V$, $(x, v) \mapsto x \cdot v$, such that

1. $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$
2. $x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w)$
3. $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$

Proposition 4.3. Suppose (ϕ, V) is a representation of $\mathfrak{g}$. Then

$$x \cdot v = \phi(x)(v) \tag{*}$$

makes V into a $\mathfrak{g}$-module. Conversely, if V is a $\mathfrak{g}$-module, then $(*)$ defines a representation of $\mathfrak{g}$ on V.

Definition 4.4 (homomorphism)
A *homomorphism* of $\mathfrak{g}$-modules is a linear map $\phi : V \to W$ such that $x \cdot \phi(v) = \phi(x \cdot v)$.

Definition 4.5 (irreducible)
A $\mathfrak{g}$-module V is *irreducible* if it has precisely two $\mathfrak{g}$-submodules, 0 and V. Note in particular 0 is not irreducible.

Definition 4.6 (completely reducible)
A $\mathfrak{g}$-module V is *completely reducible* if it is a direct sum of irreducible $\mathfrak{g}$-modules. Equivalently, for each $\mathfrak{g}$-submodule W of V, there exists a $\mathfrak{g}$-submodule W' of V such that $V = W \oplus W'$.

Lemma 4.7 (Schur)**.** Suppose $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is an irreducible representation, then the only endomorphisms of V commuting with all $\phi(x)$ are scalar multiples of the identity.

Definition 4.8 (dual module)
Let V be a $\mathfrak{g}$-module. Then the dual vector space V^* is an $\mathfrak{g}$-module, with action defined by

$$(x \cdot f)(v) = -f(x \cdot v)$$

Definition 4.9 (tensor module)
Let V, W be $\mathfrak{g}$-modules. Then $V \otimes W$ is a $\mathfrak{g}$-module, with action defined by

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w)$$

Definition 4.10 (Hom module)
Let V, W be $\mathfrak{g}$-modules. Then $\mathrm{Hom}(V, W) \simeq V^* \otimes W$ is a $\mathfrak{g}$-module, with $\mathfrak{g}$ action defined by

$$(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v)$$

4.1 Casimir element

Suppose $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a faithful representation. Define a symmetric bilinear form

$$\beta(x, y) = \mathrm{tr}(\phi(x) \cdot \phi(y))$$

Then β is associative (as for the Killing form), and nondegenerate.

Now suppose that $\mathfrak{g}$ is semisimple, β any nongenerate symmetric associative bilinear form on $\mathfrak{g}$. Let $x_1, \ldots, x_n$ be a basis of $\mathfrak{g}$, and $x^1, \ldots, x^n$ the β-dual basis of $\mathfrak{g}$[1].

[1] β defines an inner product, therefore we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$, using β. More preceisely, $\beta(x_i, x^j) = \delta_{ij}$.