Low dimensional examples

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In this note, we study low dimensional and short examples of the construction in [1]. We will use the notation as in the paper. In addition, we will write

- $[a_1^{n_1}, \dots, a_\ell^{n_\ell}]$ for the Jordan type with n_i Jordan blocks of size a_i.
- $\mathcal{O}\left(a_1^{n_1}, \dots, a_\ell^{n_\ell}\right)$ for the corresponding orbit.
- $\mathcal{M}\left(n_1, \dots, n_{k-1}\right)$ for the space of diagrams of the form

$$\begin{tikzcd} 0 \arrow[r, shift left] & \mathbb{C}^{n_1} \arrow[l, shift left] \arrow[r, shift left] & \mathbb{C}^{n_2} \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & \mathbb{C}^{n_{k-1}} \arrow[l, shift left] \arrow[r, shift left] & \mathbb{C}^{n} \arrow[l, shift left] \end{tikzcd}$$

  and we will write $\mathbf{n} = (n_1, \dots, n_{k-1})$.
- $\mathcal{N}(\mathbf{n}) = \mu_c^{-1}(0)/G_\mathbb{C}$ is the quotient space.

Case $k = 1$

In this case, the diagram is

$$\begin{tikzcd} 0 \arrow[r, shift left, "\alpha_0"] & V_1 \arrow[l, shift left, "\beta_0"] \end{tikzcd}$$

So there is only one point in M, which is $(0, 0)$. In this case, the image of Φ^c is the zero orbit.

Case $k = 2$

In this case, we have the diagram

$$\begin{tikzcd} \mathbb{C}^m \arrow[r, shift left] & \mathbb{C}^n \arrow[l, shift left] \end{tikzcd}$$

which means that X has $X^2 = 0$, and $\operatorname{rank}(X) \leq m$. Equality can be achieved if $n \geq 2k$, and X has Jordan type $[2^m, 1^{n-2m}]$. In this case, the quotient space is

$$\bigcup_{\ell=0}^{m} \mathcal{O}\left(2^\ell, 1^{n-2\ell}\right)$$

Orderings

For the general case, we have the following statement:

Theorem. Let X, Y be nilpotent $n \times n$ matrices. Then Y is in the closure of the orbit of X if and only if $\operatorname{rank}(Y^i) \leq \operatorname{rank}(X^i)$ for all i.

We can see that the $k = 2$ case above is a special case of this, and the theorem by Kobak and Swann.

Moreover, we can see that if $n_i \leq m_i$ for all i, then we have a natural embedding $\mathcal{M}(\mathbf{n}) \leq \mathcal{M}(\mathbf{m})$. In addition, since

Theorem. If $\operatorname{rank}(X^i) = n_{k-i}$ for all i, then the quotient of $\mathcal{M}(\mathbf{n})$ is the closure of the orbit of X.

and the result is sharp, in the sense that if $n_{k-i} < \operatorname{rank}(X^i)$, then X can't be in $\mathcal{N}(\mathbf{n})$, as X^i factors through $\mathbb{C}^{n_{k-i}}$. Therefore, we can use this to explicitly write down $\mathcal{N}(\mathbf{n})$ as a union of orbits.

A Mathematica notebook to compute the Hasse diagram for a fixed n, and the ranks of the Jordan blocks is at `code/Hasse.nb`.

Examples

We can use the above notebook to compute examples for small n. For $n \leq 5$, we get a linear order. Since the maximal element is always the Jordan block, which comes from the diagram

$$0 \rightleftarrows \mathbb{C} \rightleftarrows \mathbb{C}^2 \rightleftarrows \cdots \rightleftarrows \mathbb{C}^{n-1} \rightleftarrows \mathbb{C}^n$$

Therefore, all the other orbits can be constructed from "sub-diagrams" of the above.

For $n = 6$, we no longer get a linear order. $\operatorname{rank}([4, 1^2]) = [3, 2, 1]$ and $\operatorname{rank}([3, 3]) = [4, 2]$.

References

[1] Piotr Z. Kobak and Andrew Swann. "Classical nilpotent orbits as hyperkähler quotients". In: *Int. J. Math.* 07.02 (Apr. 1996). Publisher: World Scientific Publishing Co., pp. 193–210. ISSN: 0129-167X. DOI: `10.1142/S0129167X96000116`. URL: `https://www.worldscientific.com/doi/10.1142/S0129167X96000116` (visited on 07/27/2023).