HyperKähler quotients

Shing Tak Lam

July 31, 2023

Let M be a hyperKähler manifold, that is, we have:

1. a $4n$-manifold M,
2. a Riemannian metric g on M,
3. almost complex structures I, J, K on M, with $(M, g, I), (M, g, J), (M, g, K)$ all Kähler manifolds, and satisfying the quaternionic relations $I^2 = J^2 = K^2 = IJK = -1$.

Let G be a Lie group, acting on M preserving the Kähler structures, say with moment maps μ_I, μ_J, μ_K respectively. Then we can define the *hyperKähler moment map*

$$\mu = \mu_I i + \mu_J j + \mu_K k : M \to \mathfrak{g}^* \otimes \mathrm{Im}(\mathbb{H})$$

We want to show that $\mu^{-1}(0)/G$ is a hyperKähler manifold. Fix the complex structure I, and we can define the *real* and *complex moment maps*

$$\begin{aligned} \mu_r &= \mu_I & &: M \to \mathfrak{g}^* \\ \mu_c &= \mu_J + \mu_K i & &: M \to \mathfrak{g}^*_{\mathbb{C}} \end{aligned}$$

With this in mind, we have the following diagram for how to obtain the hyperKähler structure on $\mu^{-1}(0)/G$:

$$\begin{tikzcd} & M \arrow[dl, two heads, "k"'] & \\ \mu_r^{-1}(0)/G & & \mu_c^{-1}(0) \arrow[ul, hook'] \arrow[dl, two heads, "k"] \\ & \mu^{-1}(0)/G \arrow[ul, hook'] & \end{tikzcd} \tag{1}$$

where

1. $A \overset{k}{\twoheadrightarrow} B$ indicates that B is the Kähler quotient of A,
2. $A \hookleftarrow B$ indicates that B is a submanifold of A,
3. each horizontal level has the same dimension.

$\mu_c^{-1}(0)$ is a complex submanifold.

Let $\omega_I, \omega_J, \omega_K$ be the Kähler forms on M. Fix $X \in \mathfrak{g}$ and let $X^\#$ be the corresponding vector field on M generated by X. Then for any vector field Y on M, we have that

$$\mathrm{d}\mu_c^X(Y) = \mathrm{d}\mu_J^X(Y) + i\mathrm{d}\mu_K^X(Y) = \omega_J(X^\#, Y) + i\omega_K(X^\#, Y) = g(J \cdot X^\#, Y) + ig(K \cdot X^\#, Y)$$

Therefore,

$$d\mu_c^X(I \cdot Y) = g(J \cdot X^\#, I \cdot Y) + ig(K \cdot X^\#, I \cdot Y) = -g(K \cdot X^\#, Y) + ig(J \cdot X^\#, Y) = i d\mu_c^X(Y)$$

and so μ_c^X is holomorphic. Thus, $\mu_c^{-1}(0) = \mu_J^{-1}(0) \cap \mu_K^{-1}(0)$ is a complex submanifold of M. In particular, $\mu_c^{-1}(0)$ is a Kähler manifold.

Moreover, G acts on $\mu_c^{-1}(0)$ by equivariance, and it preserves the Kähler structure, and the moment map is $\mu_r|_{\mu_c^{-1}(0)}$. Therefore, if we take the Kähler quotient, we get $\mu^{-1}(0)/G$.

Compatibility of the diagram

Let $\pi : \mu_r^{-1}(0) \to \mu_r^{-1}(0)/G$ be the quotient map, $V_p = \ker(d\pi_p)$ is a vector bundle on $\mu_r^{-1}(0)$. Let H_p be the orthogonal complement of V_p in $\mathrm{T}_p\mu^{-1}(0)$, given by (the restriction of) the Riemannian metric g. Using this, one can show that the complex structure $I_p : T_pM \to T_pM$ restricts to a map $H_p \to H_p$. Moreover, $\pi_p : H_p \to \mathrm{T}_{\pi(p)}\left(\mu_r^{-1}(0)/G\right)$ is a linear isomorphism.

We will use the notation

$$\begin{aligned} H_p &\cong \mathrm{T}_{\pi(p)}\left(\mu_r^{-1}(0)/G\right) \\ X &\mapsto X_* \\ Y^* &\leftarrow\!\shortmid Y \end{aligned} \tag{2}$$

for this isomorphism. Using this, we have the formulae

$$\begin{aligned} \widetilde{\omega}(X, Y) &= \omega(X^*, Y^*) \\ \widetilde{g}(X, Y) &= g(X^*, Y^*) \\ \widetilde{I}(X) &= I(X^*)_* \end{aligned} \tag{3}$$

for the Kähler structure $(\widetilde{\omega}, \widetilde{g}, \widetilde{I})$ on $\mu_r^{-1}(0)/G$.

Similarly, let $\psi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ be the quotient map. Then we have a similar isomorphism to the above, which we will denote by

$$\begin{aligned} \overline{H}_p &\cong \mathrm{T}_{\psi(p)}\left(\mu^{-1}(0)/G\right) \\ X &\mapsto X_\bigstar \\ Y^\bigstar &\leftarrow\!\shortmid Y \end{aligned} \tag{4}$$

With this, we then get the formulae

$$\begin{aligned} \overline{\omega}(X, Y) &= \omega(X^\bigstar, Y^\bigstar) \\ \overline{g}(X, Y) &= g(X^\bigstar, Y^\bigstar) \\ \overline{I}(X) &= I(X^\bigstar)_\bigstar \end{aligned} \tag{5}$$

for the Kähler structure $(\overline{\omega}, \overline{g}, \overline{I})$ on $\mu^{-1}(0)/G$.

However, ψ is just the restriction of π to $\mu^{-1}(0) \subseteq \mu_r^{-1}(0)$, $\overline{H}_p = H_p \cap \mathrm{T}_p\mu^{-1}(0)$. With this in mind, we can see that $X_* = X_\bigstar$ and $Y^* = Y^\bigstar$. What this also implies is that eq. (5) is just the restriction of eq. (3) to $\mu^{-1}(0)/G$, $\mu^{-1}(0)/G$ is a complex submanifold of $\mu_r^{-1}(0)/G$, and that whether we take the Kähler quotient first or the complex submanifold first, we get the same result.

HyperKähler triple

Repeating the above, we get three Kähler structures on $\mu^{-1}(0)/G$. We want to show that they are compatible. That is, we need to show that the Riemannian metrics are the same, and that the complex structures satisfy the quaternionic relations.

We first consider the Riemannian metric. We have the following diagram of inclusions and quotients:

$$
\begin{tikzcd}
 & M & \\
\mu_r^{-1}(0) \arrow[ur, hook] \arrow[d, two heads] & & \\
\mu_r^{-1}(0)/G & & \\
 & & \mu^{-1}(0) \arrow[uull, hook] \arrow[uuul, hook] \arrow[dl, two heads] \\
 & \mu^{-1}(0)/G \arrow[uul, hook] &
\end{tikzcd}
$$

From the fact that eq. (5) and eq. (3) give the same result, we can see that the Riemannian metric is the one given by first restricting to $\mu^{-1}(0)$ and then taking the Riemannian quotient. Therefore, in all three cases the Riemannian metric is the same.

Moreover, ψ is the same in all three cases, therefore, we can compute using eq. (5) that

$$
\bar{I}\,\bar{J}(X) = \bar{I}(J(X^*)_*) = (IJ(X^*))_* = K(X^*)_* = \overline{K}(X)
$$

So $\bar{I}\,\bar{J} = \overline{K}$.

Complex quotients

The starting point is the following statement

Theorem. Let X be a Kähler manifold, G acting on X preserving the Kähler structure. Suppose μ is a moment map for this action, and $G_{\mathbb{C}}$ acts holomorphically on X. Then the natural map

$$
\mu^{-1}(0)/G \to X/G^{\mathbb{C}}
$$

is a biholomorphism.

The above statement is false. We need extra assumptions on X, G and μ, and the right hand side should be $\widetilde{X}/G^{\mathbb{C}}$ for some open subset $\widetilde{X}$ of X. However, it is useful to have the above statement in mind.

We would like to apply this to the hyperKähler quotient. From eq. (1), we can see that there are two possible ways to do this. The difference in these two cases is that the Kähler manifold X which we apply the theorem to (and hence have to check the conclusions for) are different.

Applying to G acting on (M, I)

In this case, we have the following diagram of inclusions and quotients:

$$
\begin{tikzcd}
 & M \arrow[dd, two heads] & \\
\mu_r^{-1}(0) \arrow[ur, hook] \arrow[d, two heads] & & \\
\mu_r^{-1}(0)/G \arrow[r, no head, "\cong"] & M/G_{\mathbb{C}} & \mu_c^{-1}(0) \arrow[uul, hook] \arrow[dl, two heads] \\
\mu^{-1}(0)/G \arrow[u, hook] \arrow[r, dashed, no head] & \mu_c^{-1}(0)/G_{\mathbb{C}} \arrow[u, hook] &
\end{tikzcd}
$$

Applying the theorem gives us that the natural map $\phi : \mu_r^{-1}(0)/G \to M/G_{\mathbb{C}}$ is a biholomorphism. We want to show that ϕ maps $\mu^{-1}(0)/G$ to $\mu_c^{-1}(0)/G_{\mathbb{C}}$. Denote orbits in $\mu_r^{-1}(0)/G$ by $[x]$ and orbits in $M/G_{\mathbb{C}}$ by $[\![x]\!]$. Then $\phi([x]) = [\![x]\!]$. Moreover, as $G_{\mathbb{C}}$ acts on $\mu_c^{-1}(0)$, we get that

$$\phi^{-1}(\mu_c^{-1}(0)/G_{\mathbb{C}}) = \left\{[x] \mid x \in \mu_r^{-1}(0) \cap \mu_c^{-1}(0) = \mu^{-1}(0)\right\} = \mu^{-1}(0)/G$$

Therefore, we have a biholomorphism $\mu^{-1}(0)/G \cong \mu_c^{-1}(0)/G_{\mathbb{C}}$.

Applying to G acting on $\mu_c^{-1}(0)$

In this case, we have

$$\begin{tikzcd}
& M & \\
& & \mu_c^{-1}(0) \arrow[ul, hook'] \arrow[dd, two heads] \\
\mu^{-1}(0) \arrow[uur, hook] \arrow[urr, hook] \arrow[d, two heads] & & \\
\mu^{-1}(0)/G \arrow[rr, dash, "\cong"] & & \mu_c^{-1}(0)/G_{\mathbb{C}}
\end{tikzcd}$$

and the theorem directly gives us the fact that $\mu^{-1}(0)/G \cong \mu_c^{-1}(0)/G_{\mathbb{C}}$.