Coadjoint orbits of $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SU}(n)$

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In this note, we study the coadjoint orbits of $\mathrm{SU}(n)$, and of its complexification $\mathrm{SL}(n, \mathbb{C})$.

1 Killing form

First of all, note that the bilinear form β on $\mathfrak{sl}(n, \mathbb{C})$

$$\beta(A, B) = -\operatorname{tr}(AB)$$

is non-degenerate. Therefore, we have an isomorphism $R : \mathfrak{sl}(n, \mathbb{C}) \to \mathfrak{sl}(n, \mathbb{C})^*$, given by

$$R(A)(B) = \beta(A, B)$$

With this, the coadjoint action of $\mathrm{SL}(n, \mathbb{C})$ on $\mathfrak{sl}(n, \mathbb{C})^*$ is given by

$$\mathrm{Ad}_g^*(R(A))(B) = R(A)(\mathrm{Ad}_{g^{-1}}(B)) = -\operatorname{tr}\left(Ag^{-1}Bg\right) = -\operatorname{tr}\left(gAg^{-1}B\right) = R(gAg^{-1})(B)$$

Therefore, up to identification by R, the adjoint and coadjoint orbits are the same. The same result holds for the restriction of β to $\mathfrak{su}(n)$. Therefore, in what follows, we will consider the adjoint orbits instead.

2 Adjoint orbits of $\mathrm{SU}(n)$

First of all, we note that

$$\mathfrak{su}(n) = \left\{A \in \mathrm{Mat}(n, \mathbb{C}) \mid A^\dagger + A = 0, \operatorname{tr}(A) = 0\right\}$$

where $A^\dagger$ is the conjugate transpose of A. In particular, all elements of $\mathfrak{su}(n)$ are skew-hermitian, hence diagonalisable by an element of $\mathrm{SU}(n)$[1]. With this, we can classify the coadjoint orbits based off a diagonal element in the orbit. Consider

$$A = \begin{pmatrix} i\lambda_1 I_{m_1} & & \\ & \ddots & \\ & & i\lambda_k I_{m_k} \end{pmatrix}$$

where $\lambda_j \in \mathbb{R}$, with $\lambda_1 > \lambda_2 > \cdots > \lambda_k$, $m_1 + \cdots + m_k = n$ and $m_1\lambda_1 + \cdots + m_k\lambda_k = 0$. In this case, we have that the orbit is

$$\mathrm{Orb}(A) \cong \mathrm{SU}(n)/\,\mathrm{Stab}(A)$$

where $\mathrm{Stab}(A)$ is the stabiliser of A under the adjoint action. In this case, we have that the stabiliser is the block diagonal subgroup

$$\mathrm{Stab}(A) = \mathrm{S}\left(\mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_k)\right)$$

where we consider $\mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_k) \leq \mathrm{U}(n)$ as the block diagonal subgroup, and

$$\mathrm{S}\left(\mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_k)\right) = \left(\mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_k)\right) \cap \mathrm{SU}(n)$$

[1] From standard linear algebra arguments, we know that they are $\mathrm{U}(n)$-diagonalisable. But if PAP^{-1} is diagonal, then so is $(\lambda P)A(\lambda P)^{-1}$, and by choosing λ appropriately, $\lambda \in \mathrm{SU}(n)$.

the subgroup with determinant 1. Therefore, the coadjoint orbit is diffeomorphic to the flag manifold

$$\mathcal{F}(m_1, \dots, m_k) = \frac{\mathrm{SU}(n)}{\mathrm{S}\left(\mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_k)\right)}$$

In particular, note that $\mathcal{F}(p, n-p)$ is diffeomorphic to the Grassmannian $\mathrm{Gr}(p, n)$ of p-dimensional subspaces of $\mathbb{C}^n$. More generally, a generic element of $\mathcal{F}(m_1, \dots, m_k)$ can be represented as $V_1, \dots, V_k$, where V_j is a dimension m_j subspace of $\mathbb{C}^n$, with $V_j \perp V_k$ for all $j \neq k$. This is because we can set V_j to be the λ_j eigenspace and vice versa.

3 Adjoint orbits of $\mathrm{SL}(n, \mathbb{C})$

In this case, we have that the Lie algebra is

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{su}(n)_{\mathbb{C}} = \{A \in \mathrm{Mat}(n, \mathbb{C}) \mid \mathrm{tr}(A) = 0\}$$

Define the Jordan block

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} = \lambda I + J_n \in \mathrm{Mat}(n, \mathbb{C})$$

where $J_n = J_n(0)$. Then we know that every element of $\mathfrak{sl}(n, \mathbb{C})$ is $\mathrm{SL}(n, \mathbb{C})$-conjugate to a matrix in Jordan normal form, say

$$A = \begin{pmatrix} J_{m_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{m_k}(\lambda_k) \end{pmatrix}$$

where $m_1 + \cdots + m_k = n$ and $m_1\lambda_1 + \cdots + m_k\lambda_k = 0$. As above, we would like to compute the stabiliser of A under the conjugation action.

3.1 Diagonalisable case

Suppose A was diagonalisable. By conjugating by a permutation matrix, we can assume that A is of the form

$$A = \begin{pmatrix} \lambda_1 I_{\ell_1} & & \\ & \ddots & \\ & & \lambda_p I_{\ell_p} \end{pmatrix}$$

with $\lambda_1, \dots, \lambda_p$ distinct. In this case, we have that the stabiliser is

$$\mathrm{Stab}(A) = \mathrm{S}(\mathrm{GL}(\ell_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(\ell_p, \mathbb{C}))$$

where we consider $\mathrm{GL}(\ell_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(\ell_p, \mathbb{C}) \leq \mathrm{GL}(n, \mathbb{C})$ as the block diagonal subgroup, and

$$\mathrm{S}(\mathrm{GL}(\ell_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(\ell_p, \mathbb{C})) = \big(\mathrm{GL}(\ell_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(\ell_p, \mathbb{C})\big) \cap \mathrm{SL}(n, \mathbb{C})$$

3.2 Jordan block

Now suppose $A = J_n(\lambda)$. Since $\mathrm{tr}(A) = 0$, we must have $\lambda = 0$. In this case, it is easy to show that the stabiliser is the subgroup

$$P = \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & \cdots & a_n \\ & a_1 & a_2 & & \ddots & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & a_1 & a_2 \\ & & & & a_1 \end{pmatrix} \;\middle|\; a_1^n = 1, a_j \in \mathbb{C} \right\}$$

of upper triangular Toeplitz matrices. In fact, we prove something more general in the next subsection.

3.3 General case

In general, for $B \in \mathrm{Mat}(n, \mathbb{C})$, the equation $BA = AB$ becomes equations of the form

$$J_\ell(\lambda)X = XJ_m(\mu)$$

for some $X \in \mathrm{Mat}(\ell \times m, \mathbb{C})$. Assume without loss of generality that $\ell \geq m$. Then note that

$$J_\ell(\lambda - \mu)X = J_\ell(\lambda)X - \mu X = XJ_m(\mu) - \mu X = XJ_m(0) = XJ_m$$

Lemma. Let X be a $m \times n$ matrix, with $J_m X = XJ_n$. Then X is strictly upper triangular and Toeplitz, and if $m < n$, then the first $n - m$ columns of X are zero.

Proof. Say the entries of X are (x_{ij}). Then

$$x_{i+1,j+1} = e_i^\mathsf{T} J_m X e_{j+1} = e_i^\mathsf{T} X J_n e_{j+1} = x_{i,j}$$

and so X is Toeplitz. Moreover,

$$J_m X e_1 = XJ_n e_1 = 0$$

So $x_{i,1} = 0$ for all $i > 1$. Similarly, we have that

$$J_m^\ell X e_\ell = XJ_n^\ell e_\ell = 0$$

hence $x_{i,j} = 0$ for $i > j$. Therefore, X is upper triangular. Now suppose $m \leq n$. We will show that the last row is zero, since this will show the first $n - m$ columns are zero, and that the Toeplitz part is strictly upper triangular. This follows from the fact that

$$x_{m,n-\ell} = e_m^\mathsf{T} X e_{n-\ell} = e_m^\mathsf{T} X J_n^\ell e_n = e_m^\mathsf{T} J_m^\ell X e_n = 0$$

□

This means that if $\lambda = \mu$, then X must satisfy the conclusions of the lemma. On the other hand, if $\lambda \neq \mu$, then we have that

$$J_\ell(\lambda - \mu)^n X = XJ_m{}^m = 0$$

and so $X = 0$.

Therefore, we can write a generic matrix B with $BA = AB$ blockwise as B_{ij}, where $B_{ij} \in \mathrm{Mat}(m_i \times m_j, \mathbb{C})$, corresponding to $J_{m_i}(\lambda_i)$ and $J_{m_j}(\lambda_j)$. In particular, we have that

$$B_{ij} = \begin{cases} 0 & \text{if } \lambda_j \neq \lambda_i \\ \begin{pmatrix} T_{ij} \\ 0 \end{pmatrix} & \text{if } \lambda_j = \lambda_i \text{ and } m_i > m_j \\ \begin{pmatrix} 0 & T_{ij} \end{pmatrix} & \text{if } \lambda_j = \lambda_i \text{ and } m_i < m_j \\ T_{ij} & \text{if } \lambda_j = \lambda_i \text{ and } m_i = m_j \end{cases}$$

where T_{ij} is a strictly upper triangular Toeplitz matrix, of size $\min(m_i, m_j)$.