Bi-invariant metric on compact Lie groups

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1 Metrics on Lie groups

Let G be a Lie group, g be a metric on G.

Definition 1.1 ({left, right, bi-} invariant)
g is left invariant if $\ell_a^* g = g$ for all $a \in G$. That is, for all $a, b \in G$, $u, v \in T_b G$,

$$g_b(u, v) = g_{ab}(\mathrm{d}_b \ell_a(u), \mathrm{d}_b \ell_a(v))$$

We can define right invariant similarly. g is bi-invariant if it is both left and right invariant.

Proposition 1.2. There is a 1-1 correspondence

$$\{\text{left invariant metrics on } G\} \leftrightarrow \{\text{inner products on } \mathfrak{g}\}$$

Proof. Any left invariant metric must satisfy

$$g_a(u, v) = g_e(\mathrm{d}_a \ell_{a^{-1}}(u), \mathrm{d}_a \ell_{a^{-1}}(v))$$

Conversely, given an inner product g_e on $\mathfrak{g}$, the above formula defines a left invariant metric on G. □

2 Bi-invariant metrics

Recall the adjoint representation of a Lie group is $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, defined by

$$\mathrm{Ad}_a = \mathrm{d}_e(r_{a^{-1}} \circ \ell_a) = \mathrm{d}_a r_{a^{-1}} \circ \mathrm{d}_e \ell_a$$

Definition 2.1 (Ad-invariant)
An inner product on $\mathfrak{g}$ is Ad-invariant if Ad_a is an isometry for all $a \in G$. That is,

$$\langle \mathrm{Ad}_a(u), \mathrm{Ad}_a(v) \rangle = \langle u, v \rangle$$

Proposition 2.2. There is a 1-1 correspondence

$$\{\text{bi-invariant metrics on } G\} \leftrightarrow \{\text{Ad-invariant inner products on } \mathfrak{g}\}$$

Proof. It is clear that any bi-invarant metric g will give an Ad-invariant inner product g_e on $\mathfrak{g}$. Conversely, suppose g_e is an Ad-invariant inner product on $\mathfrak{g}$. Define g as in the left invariant case. Then it is easy to check that g is also right invariant. □

Corollary 2.3. Suppose G has a bi-invariant metric. Then $\mathrm{Ad}(G) \subseteq \mathrm{O}(\mathfrak{g})$. In particular, $\overline{\mathrm{Ad}(G)}$ is compact.

3 Haar measure and Weyl's unitary trick

This section is all from Part II Representation Theory.

Theorem 3.1 (Haar measure)**.** Let G be a compact group, then there exists a unqiue regular Borel measure μ which is

(i) translation invariant, i.e. $\mu(gX) = \mu(X) = \mu(Xg)$ for any measurable set X,

(ii) regular, i.e.

$$\mu(X) = \inf\{\mu(U) \mid X \subseteq U, U \text{ open}\} = \sup\{\mu(K) \mid K \subseteq X, K \text{ compact}\}$$

(iii) normalised, i.e. $\mu(G) = 1$.

In the remainder of this section, G is a compact Lie group, μ is the Haar measure on G.

Corollary 3.2. In particular, for $\gamma \in G$, $f \in L^1(\mu)$,

$$\int_G f(\gamma x)\mathrm{d}\mu(x) = \int_G f(x)\mathrm{d}\mu(x) = \int_G f(x\gamma)\mathrm{d}\mu(x)$$

Recall that if $\rho : G \to \mathrm{GL}(V)$ is a representation, an inner product on V is G-invariant if $\langle \rho(\gamma)x, \rho(\gamma)y\rangle = \langle x, y\rangle$ for all $x, y \in V$, $\gamma \in G$.

Theorem 3.3 (Weyl's unitary trick)**.** Let G be a compact Lie group. Then for every representation $\rho : G \to \mathrm{GL}(V)$, there exists a G-invariant inner product on V.

Proof. First, fix any inner product $(\cdot, \cdot)$ on V. Now define

$$\langle u, v\rangle = \int_G (\rho(\gamma)u, \rho(\gamma)v)\mathrm{d}\mu(\gamma)$$

Using translation invariance of the integral, it follows that $\langle \cdot, \cdot\rangle$ is G-invariant. ☐

4 Bi-invariant metrics on compact Lie groups

Theorem 4.1. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. Then there exists a G-invariant inner product on V if and only if $\overline{\rho(G)} \subseteq \mathrm{GL}(V)$ is compact.

Proof. If there exists a G-invarant inner product, then each $\rho(\gamma)$ is an isometry. Hence we have that $\rho(G) \subseteq \mathrm{O}(V, \langle \cdot, \cdot\rangle)$. As $O(V, \langle \cdot, \cdot\rangle)$ is compact, we have that $\overline{\rho(G)}$ is compact.

Conversely, if $H = \overline{\rho(G)}$ is compact, then it is a compact subgroup of $\mathrm{GL}(V)$. Consider the inclusion representation $H \hookrightarrow \mathrm{GL}(V)$. Therefore, we have an H-invariant inner product on V. But if $f = \rho(\gamma) \in H$, then

$$\langle \rho(\gamma)u, \rho(\gamma)v\rangle = \langle f(u), f(v)\rangle = \langle u, v\rangle$$

So $\langle \cdot, \cdot\rangle$ is G-invariant. ☐

Corollary 4.2. Let G be a Lie group. Then an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ induces a bi-invariant metric on G if and only if $\overline{\mathrm{Ad}(G)}$ is compact. In particular, every compact Lie group admits a bi-invariant metric.

Finally, recall the adjoint representation of $\mathfrak{g}$ is $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $\mathrm{ad} = \mathrm{d}_e \mathrm{Ad}$. More explicitly, $\mathrm{ad}_x(y) = [x, y]$. Then we have the following:

Proposition 4.3. An inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ induces a bi-invariant metric on G if and only if for all $u, v, w \in \mathfrak{g}$,

$$\langle \mathrm{ad}_u(v), w \rangle = - \langle v, \mathrm{ad}_u(w) \rangle$$

if and only if

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle$$

for all $x, y, z \in \mathfrak{g}$.

Lemma 4.4. A Lie group is simple if and only if the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is irreducible.

Proposition 4.5. Suppose G is a simple Lie group, then the bi-invariant metric on G is unique up to scaling, if it exists.

Proof. By Schur's lemma? □