Kähler structure on reduction via complex manifolds

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Throughout, let M be a compact Kähler manifold, G a compact Lie group acting on X preserving the Kähler structure. Let $\mu : M \to \mathfrak{g}^*$ be the moment map for the action. Suppose in addition that G acts freely on $\mu^{-1}(0)$. Then we have that

$$M /\!\!/ G := \mu^{-1}(0)/G$$

has a Kähler structure. In this note, we show another way of thinking about this, which is that

$$\mu^{-1}(0)/G \cong M^{\min}/G_{\mathbb{C}}$$

where M^s is an open submanifold of M, and $G_{\mathbb{C}}$ is the complexification of G. In this case, the fact that $M /\!\!/ G$ has a complex structure becomes clear.

Fix a G-invariant Riemannian metric on M, and define $f : M \to \mathbb{R}$ by

$$f(x) = \left\|\mu(x)\right\|^2$$

We want to consider f as a Morse function on M. For $x \in M$, define the curve $(x_t)_{t\geq 0}$ by

$$\begin{aligned} x_0 &= x \\ \frac{\mathrm{d}x_t}{\mathrm{d}t} &= -\operatorname{grad} f(x_t) \end{aligned}$$

The set of limit points of the trajectory is

$$\omega(x) = \{y \in M \mid \text{every neighbourhood of } y \text{ contains } x_t \text{ for } t \text{ arbitrarily large}\}$$

With this, we define

$$M^{\min} = \left\{x \in X \mid \omega(x) \subseteq \mu^{-1}(0)\right\}$$

Then $M^{\min}$ is a $G_{\mathbb{C}}$ invariant open subset of M, with $\mu^{-1}(0) \subseteq M^{\min}$. The inclusion map induces a natural continuous map

$$\mu^{-1}(0)/G \to M^{\min}/G_{\mathbb{C}}$$

which is a homeomorphism. The proof of this relies on the fact that $G \cdot \mu^{-1}(0) = M^{\min}$, hence the map is surjective. Another argument shows the map is bijective. Therefore it is a continuous bijection from a compact space to a Hausdorff space, hence a homeomorphism.