Ideas from “Classical nilpotent orbits as hyperkähler quotients”

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In this note, we provide a high level overview for the paper [1] by Kobak and Swann. We will use the same notation and numbering as in the paper, but we will omit technical details from the proofs.

1 Introduction

Here, we define the hyperKähler structure on $\mathbb{H}^n$, where the complex structures are given by right multiplication by $-i, -j, -k$ respectively, and the Riemannian metric is the Euclidean metric from the isomorphism $\mathbb{H}^n \cong \mathbb{R}^{4n}$ of vector spaces. For a subgroup H of $\mathrm{Sp}(N)$, H acts (on the left) on $\mathbb{H}^n$, and the *hyperKähler moment map* for this action is an equivariant map

$$\mu : \mathbb{H}^n \to \mathfrak{h}^* \otimes \mathrm{Im}(\mathbb{H}) \cong \mathfrak{h} \otimes \mathrm{Im}(\mathbb{H})$$

where $\mathrm{d}\big(\mu^X\big) = X \lrcorner \eta$, $\eta = \omega_I i + \omega_J j + \omega_K k$ the quaternion values form given by the symplectic forms. What this means is that if we write

$$\mu = \mu_I i + \mu_J j + \mu_K k$$

then μ_I is a moment map for the action of H on $\mathbb{H}^n$ with respect to the complex structure I, and so on.

Throughout, we will have the moment map

$$\mu^X(q) = -\overline{q}^\top X q$$

for $X \in \mathfrak{h}$, which we consider to be an $N \times N$ quaternionic matrix.

2 The Constructions

First of all, we reduce to the case where $\mathcal{O}$ is an adjoint orbit of a classical simple Lie algebra over $\mathbb{C}$, since in the general case we can take product/sums.

In each case, we first specify a hyperKähler vector space M and the group G, then we prove that the complex symplectic quotient by $G^{\mathbb{C}}$ is what we want, then we show that the complex quotient is the same as the hyperKähler quotient.

2.1 The Special Linear Group

In this case, let $V_0, \dots, V_k$ be Hermitian vector spaces, $\dim(V_i) = n_i$, $n_0 = 0$ and $n_k = n$. Then we can define the vector space

$$M = \bigoplus_{i=0}^{k-1} (\mathrm{Hom}(V_i, V_{i+1}) \oplus \mathrm{Hom}(V_{i+1}, V_i))$$

and we write each point $p = (\alpha_i, \beta_i)$ as the diagram

$$0 = V_0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} V_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} V_2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{k-1}}{\overset{\alpha_{k-1}}{\rightleftarrows}} V_k = \mathbb{C}^n$$

We have a left $\mathbb{H}$ action on M, given by

$$i(\alpha_i, \beta_i) = (i\alpha_i, i\beta_i) \quad j(\alpha_i, \beta_i) = (-\beta_i^*, \alpha_i^*)$$

which makes M into a quaternionic vector space. In this case, the Lie group action of $G = \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1})$ on M is

$$\begin{aligned} \alpha_i &\mapsto g_{i+1}\alpha_i g_i^{-1} \\ \beta_i &\mapsto g_i \beta_i g_{i+1}^{-1} \end{aligned} \tag{1}$$

where $g_i \in U_{n_i}$, $g_0 = g_k = 1$. The moment map in this case is $\mu = i\mu_r + 2k\mu_c : M \to \mathfrak{g}^* \otimes \mathrm{Im}(\mathbb{H})$, where (up to identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ via the Killing form),

$$\begin{aligned} \mu_r &= (\alpha_{i-1}\alpha_{i-1}^* - \beta_{i-1}^*\beta_{i-1} + \beta_i\beta_i^* - \alpha_i^*\alpha_i)_{i=1}^{k-1} && \in \mathfrak{g} \otimes i\mathbb{R} = i\mathfrak{g} = i\mathfrak{u}(n_1, \mathbb{C}) \oplus \cdots \oplus i\mathfrak{u}(n_{k-1}, \mathbb{C}) \\ \mu_c &= (\alpha_{i-1}\beta_{i-1} - \beta_i\alpha_i)_{i=1}^{k-1} && \in \mathfrak{g} \otimes \mathbb{C} = \mathfrak{gl}(n_1, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(n_{k-1}, \mathbb{C}) \end{aligned}$$

2.1.1 The Complex Quotient

Fix a point $p = (\alpha_i, \beta_i) \in \mu_c^{-1}(0)$. In this case, we can define $X = \alpha_{k-1}\beta_{k-1} \in \mathrm{End}(\mathbb{C}^n)$. Then $X^k = 0$, as $p \in \mu_c^{-1}(0)$. Moreover, the action of $G^{\mathbb{C}} = \mathrm{GL}(n_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(n_{k-1}, \mathbb{C})$ preserves X, so we have a well defined map

$$\begin{aligned} \Phi^c : \mu_c^{-1}(0)/G^{\mathbb{C}} &\to \mathcal{N} \\ (\alpha_i, \beta_i) &\mapsto \alpha_{k-1}\beta_{k-1} \end{aligned}$$

where $\mathcal{N}$ is the nilpotent variety of $\mathfrak{sl}(n, \mathbb{C})$[1].

Theorem 2.1. The map Φ^c, restricted to the set of closed $G^{\mathbb{C}}$ orbits, is injective. Furthermore, its image consists of a union of closures of nilpotent orbits in $\mathfrak{sl}(n, \mathbb{C})$. If there exists $X \in \mathfrak{sl}(n, \mathbb{C})$ such that $\mathrm{rank}(X^i) = n_{k-i}$ for all i, then the image is precisely the closure of the nilpotent orbit containing X.

Proof sketch. First of all, notice that eq. (1) defines a $\mathrm{GL}(n, \mathbb{C})$ action on M, by taking $g_k = g$, and $g_i = 1$ for $i < k$. This action preserves $\mu_c^{-1}(0)$, and Φ^c is equivariant with respect to this action[2]. Therefore, the image of Φ^c is a union of nilpotent orbits.

To show the injectivity statement, we use the $\mathrm{GL}(n, \mathbb{C})$ action to assume without loss of generality that X is in Jordan normal form. In fact, we can assume that X is a Jordan block, each β_i is surjective and each α_i is injective using the $G^{\mathbb{C}}$ action and the fact that the orbits are closed. In this case, by an appropriate choice of basis, using the $G^{\mathbb{C}} \times \mathrm{GL}(n, \mathbb{C})$-action, we can assume β has matrix

$$\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix}$$

With this, α_1 is upper triangular, and by induction all α_i are upper triangular, and we get uniqueness.

The rest of the statement follows from the facts that

1. If X is nilpotent, the numbers $\mathrm{rank}(X^i)$ determines the Jordan normal form of X, and so the nilpotent orbit of X,
2. If X, Y are nilpotent, then Y is in the closure of the nilpotent orbit containing X if and only if $\mathrm{rank}(Y^i) \leq \mathrm{rank}(X^i)$ for all i.

□

2.1.2 Equivalence of Kähler and Complex Quotients

We have the following result by Kirwan (paraphrased):

[1] Any nilpotent endomorphism necessarily has all eigenvalues being zero, and so it is trace free.
[2] In fact, Φ^c is a complex symplectic moment map for a $\mathrm{GL}(n, \mathbb{C})$ action on the quotient.

Theorem 2.2. Let X be a Kähler manifold, G a compact Lie group acting on X preserving the Kähler structure, such that $G^{\mathbb{C}}$ also acts holomorphically on X. Let μ be the Kähler moment map for the action of G, satisfying condition $(\star)$. Let

$$X^{\min} = \left\{ y \;\middle|\; \text{limit under steepest decent of } \left\|\mu\right\|^2 \text{ lies in } \mu^{-1}(0) \right\}$$

Then $x \in G^{\mathbb{C}}\mu^{-1}(0)$ if and only if $x \in X^{\min}$ and the orbit $G^{\mathbb{C}}x$ is closed in $X^{\min}$. In this case, the map

$$\mu^{-1}(0)/G \to G^{\mathbb{C}}\mu^{-1}(0) /\!\!/ G^{\mathbb{C}}$$

is a homeomorphism, where $G^{\mathbb{C}}\mu^{-1}(0) /\!\!/ G^{\mathbb{C}}$ is the set of closed $G^{\mathbb{C}}$ orbits in $G^{\mathbb{C}}\mu^{-1}(0)$.

We will return to the condition $(\star)$ later, but for now, we will first assume that $(\star)$ holds in the cases which we want, and then prove that it holds later on.

First, since $(M, \omega_I, \omega_J, \omega_K)$ is hyperKähler, (M, ω_I) is Kähler. Moreover, the group G acts on (M, ω_I) preserving the Kähler structure, with moment map μ_r. Therefore, applying theorem 2.2, we get that

$$\mu_r^{-1}(0)/G \cong G^{\mathbb{C}}\mu_r^{-1}(0) /\!\!/ G^{\mathbb{C}}$$

Next, we assume $M^{\min} = M$, so $G^{\mathbb{C}}\mu_r^{-1}(0)$ is just the set of points for which the orbit $G^{\mathbb{C}}x$ is closed. In this case, we have a natural inclusion

$$X = \mu_c^{-1}(0) \cap G^{\mathbb{C}}\mu_r^{-1}(0) \subseteq G^{\mathbb{C}}\mu_r^{-1}(0)$$

Since X is $G^{\mathbb{C}}$ invariant, we have an induced map

$$X/G^{\mathbb{C}} \hookrightarrow G^{\mathbb{C}}\mu_r^{-1}(0) /\!\!/ G^{\mathbb{C}} \cong \mu_r^{-1}(0)/G$$

Finally, we want to find the image of this map. But this is just $\mu^{-1}(0)/G$, which is the hyperKähler quotient. Therefore, all that remains is to show that $M^{\min} = M$, and that $(\star)$ holds.

$\underline{M^{\min} = M}$: For this, it suffices to show that the critical points of $\left\|\mu_r\right\|^2$ are global minima. Since $\mu_r^* = \mu_r$, we have that $\operatorname{grad}\left(\left\|\mu_r\right\|^2\right) = 2(\mathrm{d}\mu_r)\mu_r$, which vanishes if and only if $\mu_r = 0$ and so $\left\|\mu_r\right\|^2 = 0$.

The condition $(\star)$ is that $\underline{\text{the trajectories of the gradient flow of } \left\|\mu_r\right\|^2 \text{ are bounded}}$. In this case, we have that

$$\left\|\mu_r(x)\right\|^2 \leq \|x\|^4$$

for all $x \in M$. **The paper then claims that this implies each trajectory is bounded, but I don't see why this is true.**

Theorem 2.7. The hyperKähler quotient of M by G is a union of nilpotent orbits in $\mathfrak{sl}(n, \mathbb{C})$. If there is a nilpotent element $X \in \mathfrak{sl}(n, \mathbb{C})$, with $\operatorname{rank}(X^i) = n_{k-i}$ for all i, then the quotient is isomorphic to the closure of the nilpotent orbit containing X.

2.2 Orthogonal and Symplectic Lie Algebras

For $\mathfrak{o}(n, \mathbb{C})$ and $\mathfrak{sp}(n, \mathbb{C})$, we write

$$\mathfrak{o}(n, \mathbb{C}) = \mathfrak{c}_0^{\mathbb{C}}$$
$$\mathfrak{sp}(n, \mathbb{C}) = \mathfrak{c}_1^{\mathbb{C}}$$

and the corresponding Lie groups $C_\delta^{\mathbb{C}}$ are Lie groups acting on $V = \mathbb{C}^{(1+\delta)n}$ preserving a non-degenerate bilinear form B such that $B(u, v) = (-1)^\delta B(v, u)$.

If $X \in \mathfrak{c}_\delta^{\mathbb{C}}$ is nilpotent, with $X^k = 0$, then $X^{k-i}V$ has a bilinear form B_i, with

$$B_i(u, v) = (-1)^{k-i+\delta} B_i(v, u) \tag{2}$$

As a matrix, $B_{i-1} = XB_i$ restricted to $X^{k-i}V$. Therefore, we consider the same construction as in the previous subsection, except we require each V_i to have a bilinear form B_i satisfying eq. (2). Moreover, let $A^\dagger$ be the adjoint of A with respect to B_i, then we require that

$$(A^*)^\dagger = (A^\dagger)^*$$

Let M_δ be the vector subspace of M, given by

$$\beta_i = \alpha_i^\dagger$$

Note $(\alpha_i^\dagger)^\dagger = -\alpha_i$. Then M_δ is also a flat hyperKähler manifold. Let $H = C_1 \times \cdots \times C_n$ be the subgroup of G, where C_i preserves B_i. In particular, C_i is $\mathrm{O}(n_i)$ or $\mathrm{Sp}(n_i/2)$. H acts on M_δ, and the moment map is the same as above, just with $\beta_i = \alpha_i^\dagger$.

Theorem 2.8. The hyperKähler quotient of M_δ by H is a union of closures of nilpotent orbits of $C_\delta^\mathbb{C}$, where $C_0^\mathbb{C} = \mathrm{O}(n, \mathbb{C})$ and $C_1^\mathbb{C} = \mathrm{Sp}(n/2, \mathbb{C})$. Moreover, this quotient agrees with the algebraic quotient $\mu_c^{-1}(0)/\!\!/H^\mathbb{C}$. If there is an $X \in \mathfrak{c}_\delta^\mathbb{C}$ with $\mathrm{rank}(X^i) = n_{k-i}$ for all i, then the hyperKähler quotient is the closure of the nilpotent orbit containing X.

Proof sketch. First, we note that nilpotent orbits in $\mathfrak{o}(n, \mathbb{C})$ and $\mathfrak{sp}(n/2, \mathbb{C})$ are the intersections of the $\mathfrak{sl}(n, \mathbb{C})$ orbits with $\mathfrak{c}_\delta^\mathbb{C}$. Therefore, suffices to show that for any $X \in \mathfrak{c}_\delta^\mathbb{C}$, there exists $(\alpha_1, \ldots, \alpha_{k-1})$ in a closed $H^\mathbb{C}$-orbit of $\mu_c^{-1}(0) \cap M_\delta$, with $\alpha_{k-1}\alpha_{k-1}^\dagger = X$. This is because we have natural maps

$$\frac{\mu_c^{-1}(0) \cap M_\delta}{H^\mathbb{C}} \to \frac{\mu_c^{-1}(0)}{G^\mathbb{C}} \xrightarrow{\Phi^c} \mathcal{N}$$

for which the composition is $[(\alpha_i)] \mapsto \alpha_{k-1}\alpha_{k-1}^\dagger$.

To prove the claim, we again consider Jordan blocks, and define the B_is and α_is appropriately. □

3 Consequences and Examples

Lemma 3.1. Let H be a Lie group acting on $\mathbb{H}^N$ preserving the complex structures. Let a non-zero quaternion $a \in \mathbb{H}^*$ act on $q \in \mathbb{H}^N$ on the right, that is, $q \mapsto qa^{-1}$, and on $p \in \mathrm{Im}(\mathbb{H})$ by conjugation. That is, $p \mapsto apa^{-1}$. Then the map $\mu : \mathbb{H}^N \to \mathfrak{h}^* \otimes \mathrm{Im}(\mathbb{H})$ defined as above is the unique moment map for the action of H on $\mathbb{H}^N$ which is equivariant with respect to the action of $\mathbb{H}^*$.

Using this, we get that the set $\mu^{-1}(0)$ is $\mathbb{H}^*$-invariant, and if K is another Lie group so that $H \times K$ also acts on $\mathbb{H}^N$ preserving the complex structures, then the moment map for the action of $H \times K$ is the direct sum of the moment maps for H and for K, and the hyperKähler quotient by $H \times K$ is the hyperKähler quotient by H followed by the hyperKähler quotient by K.

3.1 Quaternionic Kähler metrics

Omitted.

3.2 Finite Quotients

3.3 HyperKähler Quotients

References

[1] Piotr Z. Kobak and Andrew Swann. "Classical nilpotent orbits as hyperkähler quotients". In: *Int. J. Math.* 07.02 (Apr. 1996). Publisher: World Scientific Publishing Co., pp. 193–210. ISSN: 0129-167X. DOI: 10.1142/S0129167X96000116. URL: https://www.worldscientific.com/doi/10.1142/S0129167X96000116 (visited on 07/27/2023).