The complex symplectic forms

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In this note, we will study the complex symplectic form from the construction in [1].

1 Tangent spaces

Let M be as in [1], and write a generic point as (α_j, β_j). Then we have the real and complex moment maps, which are

$$\begin{aligned} \mu_r(\alpha_j, \beta_j) &= (\alpha_{j-1}\alpha_{j-1}^* - \beta_{j-1}^*\beta_{j-1} + \beta_j\beta_j^* - \alpha_j^*\alpha_j)_{j=1}^{k-1} \\ \mu_c(\alpha_j, \beta_j) &= (\alpha_{j-1}\beta_{j-1} - \beta_j\alpha_j)_{j=1}^{k-1} \end{aligned}$$

First of all, we want to compute the tangent space to $\mu_c^{-1}(0)$. By standard arguments, we have that

$$\mathrm{T}_{(\alpha_j,\beta_j)}\mu_c^{-1}(0) = \ker\left((\mathrm{d}\mu_c)_{(\alpha_j,\beta_j)}\right)$$

We can compute the derivative, since

$$\begin{aligned} \mu_c(\alpha_j + \delta_j, \beta_j + \varepsilon_j) &= (\alpha_{j-1} + \delta_{j-1})(\beta_{j-1} + \varepsilon_{j-1}) - (\beta_j + \varepsilon_j)(\alpha_j + \delta_j) \\ &= \alpha_{j-1}\beta_{j-1} - \beta_j\alpha_j + \delta_{j-1}\beta_{j-1} + \alpha_{j-1}\varepsilon_{j-1} - \beta_j\delta_j - \varepsilon_j\alpha_j + \text{ higher order terms} \\ &= \mu_c(\alpha_j, \beta_j) + \delta_{j-1}\beta_{j-1} + \alpha_{j-1}\varepsilon_{j-1} - \beta_j\delta_j - \varepsilon_j\alpha_j + \text{ higher order terms} \end{aligned}$$

Hence we have that

$$\mathrm{T}_{(\alpha,\beta)}\mu_c^{-1}(0) = \left\{(\delta_j, \varepsilon_j) \mid \delta_{j-1}\beta_{j-1} + \alpha_{j-1}\varepsilon_{j-1} - \beta_j\delta_j - \varepsilon_j\alpha_j = 0\right\}$$

Next, we have the map $\Phi^c : \mu_c^{-1}(0) \to \mathcal{N}$, given by $\Phi^c(\alpha, \beta) = \alpha_{k-1}\beta_{k-1}$. The derivative of this map is given by

$$\Phi^c(\alpha + \delta, \beta + \varepsilon) = (\alpha_{k-1} + \delta_{k-1})(\beta_{k-1} + \varepsilon_{k-1}) = \Phi^c(\alpha, \beta) + \delta_{k-1}\beta_{k-1} + \alpha_{k-1}\varepsilon_{k-1} + \text{ higher order terms}$$

Therefore, the map $\mathrm{d}\Phi^c$ is given by

$$\mathrm{d}\Phi^c(\delta, \varepsilon) = \delta_{k-1}\beta_{k-1} + \alpha_{k-1}\varepsilon_{k-1}$$

Restricting to an open subset giving us the top nilpotent orbit N given by M, Φ^c is a submersion.
Fix the point (α, β) and let $X = \alpha_{k-1}\beta_{k-1}$. In this case, we must have that

$$\mathrm{d}\Phi^c(\delta, \varepsilon) \in \mathrm{T}_X N = \{[X, Y] \mid Y \in \mathfrak{sl}(n, \mathbb{C})\}$$

Now given $Y \in \mathfrak{sl}(n, \mathbb{C})$, setting

$$\delta_j^0 = \begin{cases} 0 & j < k-1 \\ -Y\alpha_{k-1} & j = k-1 \end{cases}$$

$$\varepsilon_j^0 = \begin{cases} 0 & j < k-1 \\ \beta_{k-1}Y & j = k-1 \end{cases}$$

gives us an element of $T_{(\alpha,\beta)}\mu_c^{-1}(0)$, with $\mathrm{d}\Phi^c(\delta^0, \varepsilon^0) = [X, Y]$. More generally, since we have a diffeomorphism

$$\mu_c^{-1}(0)/G^{\mathbb{C}} \cong N$$

induced by Φ^c, we can compute the (affine) space of possible choices of δ, ε. The $G^{\mathbb{C}} = \mathrm{GL}(n_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(n_{k-1}, \mathbb{C})$ action is given by

$$(\alpha, \beta) \mapsto (g_{j+1}\alpha_j g_j^{-1}, g_j\beta_j g_{j+1}^{-1})$$

where $g_0, g_k = 1$. Therefore, for a generic element $(X_1, \ldots, X_{k-1}) \in \mathfrak{g}^{\mathbb{C}} = \mathfrak{gl}(n_1, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(n_{k-1}, \mathbb{C})$, the infinitesimal action is given by

$$(X_{j+1}\alpha_j - \alpha_j X_j, X_j\beta_j - \beta_j X_{j+1})$$

Let

$$V^{\mathbb{C}} = \left\{(X_{j+1}\alpha_j - \alpha_j X_j, X_j\beta_j - \beta_j X_{j+1}) \mid X_j \in \mathfrak{gl}(n_j, \mathbb{C})\right\}$$

be the subspace given by the $G^{\mathbb{C}}$ action. This is precisely the kernel of $\mathrm{d}\Phi^c$, as we have

$$(\alpha_{k-1} - \alpha_{k-1}X_{k-1})\beta_{k-1} + \alpha_{k-1}(X_{k-1}\beta_{k-1} - \beta_{k-1}) = 0$$

Moreover,

$$\begin{aligned}
&(X_j\alpha_{j-1} - \alpha_{j-1}X_{j-1})\beta_{j-1} + \alpha_{j-1}(X_{j-1}\beta_{j-1} - \beta_{j-1}X_j) - \beta_j(X_{j+1}\alpha_j - \alpha_j X_j) - (X_j\beta_j - \beta_j X_{j+1})\alpha_j \\
&= [X_j, \alpha_{j-1}\beta_{j-1} - \beta_j\alpha_j] \\
&= 0
\end{aligned}$$

So it is a subspace. On $\mu_c^{-1}(0)$, we have a Riemannian metric induced from M, which is

$$g((\alpha, \beta), (\gamma, \delta)) = \sum_{j=0}^{k-1} \mathrm{Re}\,\mathrm{tr}\left(\alpha_j\gamma_j^* + \beta_j^*\delta_j\right)$$

We would like to find $(u, v) \in V^{\mathbb{C}}$, so that $(\delta^0, \varepsilon^0) + (u, v) \in (V^{\mathbb{C}})^{\perp}$.
Say

$$\begin{aligned}
u_j &= X_{j+1}\alpha_j - \alpha_j X_j \\
v_j &= X_j\beta_j - \beta_j X_{j+1} \\
x_j &= Z_{j+1}\alpha_j - \alpha_j Z_j \\
y_j &= Z_j\beta_j - \beta_j Z_{j+1}
\end{aligned}$$

Then for $j = k - 1$, we have

2 Complex Kirillov-Konstant-Souriau form

Define the form

$$\omega_c([X, Y], [X, Z]) = \langle X, [Y, Z]\rangle = \mathrm{tr}(X[Y, Z]) = \mathrm{tr}(XYZ - XZY)$$

We can then compute the pullback $(\Phi^c)^*\omega_c$

3 Complex symplectic form on quotient

References

[1] Piotr Z. Kobak and Andrew Swann. "Classical nilpotent orbits as hyperkähler quotients". In: *Int. J. Math.* 07.02 (Apr. 1996). Publisher: World Scientific Publishing Co., pp. 193–210. ISSN: 0129-167X. DOI: 10.1142/S0129167X96000116. URL: https://www.worldscientific.com/doi/10.1142/S0129167X96000116 (visited on 07/27/2023).