Invariance of the hyperKähler metric

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In this note, we will investigate the invariance of the hyperKähler metric from [1]. We will use the same notation as in section 2.

Let g denote the Riemannian metric, that is,

$$g((\alpha_j, \beta_j), (\widetilde{\alpha}_j, \widetilde{\beta}_j)) = \sum_{j=0}^{k-1} \left(\operatorname{Re} \operatorname{tr} \left(\alpha_j^* \widetilde{\alpha}_j \right) + \operatorname{Re} \operatorname{tr} \left(\beta_j^* \widetilde{\beta}_j \right) \right)$$

Therefore, we have that

$$\begin{aligned} \omega_J((\alpha_j, \beta_j), (\widetilde{\alpha}_j, \widetilde{\beta}_j)) &= g(J(\alpha_j, \beta_j), (\widetilde{\alpha}_j, \widetilde{\beta}_j)) \\ &= g((-\beta_j^*, \alpha_j^*), (\widetilde{\alpha}_j, \widetilde{\beta}_j)) \\ &= \sum_{j=0}^{k-1} \left(\operatorname{Re} \operatorname{tr} \left(-\beta_j \widetilde{\alpha}_j \right) + \operatorname{Re} \operatorname{tr} \left(\alpha_j \widetilde{\beta}_j \right) \right) \end{aligned}$$

We also have a $\mathrm{SL}(n, \mathbb{C})$ action on M, given by

$$\psi_\gamma(\alpha_j, \beta_j) = (\alpha_0, \dots, \alpha_{k-2}, \gamma \alpha_{k-1}, \beta_0, \dots, \beta_{k-2}, \beta_{k-1} \gamma^{-1})$$

$\psi_\gamma : M \to M$ is linear, therefore the derivative is itself. In particular,

$$\begin{aligned} \psi_\gamma^* \omega_J((\alpha_j, \beta_j), (\widetilde{\alpha}_j, \widetilde{\beta}_j)) &= \sum_{j=0}^{k-2} \left(\operatorname{Re} \operatorname{tr} \left(-\beta_j \widetilde{\alpha}_j \right) + \operatorname{Re} \operatorname{tr} \left(\alpha_j \widetilde{\beta}_j \right) \right) + \operatorname{Re} \operatorname{tr} \left(-\beta_{k-1} \gamma^{-1} \gamma \widetilde{\alpha}_{k-1} \right) + \operatorname{Re} \operatorname{tr} \left(\gamma \alpha_{k-1} \widetilde{\beta}_{k-1} \gamma^{-1} \right) \\ &= \omega_J((\alpha_j, \beta_j), (\widetilde{\alpha}_j, \widetilde{\beta}_j)) \end{aligned}$$

using the fact that trace is conjugation invariant. Similarly, $\psi_\gamma^* \omega_K = \omega_K$ as trace is $\mathbb{C}$-linear, and so the Re becomes $-\operatorname{Im}$.

Define $\omega_c = \omega_J + i\omega_K$ for the complex symplectic form on M. Let $\Phi = \Phi^c \circ \pi : \mu_c^{-1}(0) \to \mathcal{N}$ be defined by

$$\Phi(\alpha_j, \beta_j) = \alpha_{k-1} \beta_{k-1}$$

Let N be the image of Φ. Using the results from [1], N has a hyperKähler metric $\widetilde{\omega}_c$, satisfying

$$\Phi^* \widetilde{\omega}_c = i^* \omega_c$$

where $i : \mu_c^{-1}(0) \to M$ is the inclusion. Since $\Phi : \mu_c^{-1}(0) \to N$ is a surjective submersion, this completely determines $\widetilde{\omega}_c$. We would like to show $(\mathrm{Ad}_\gamma)^* \widetilde{\omega}_c = \widetilde{\omega}_c$ for all $\gamma \in \mathrm{SL}(n, \mathbb{C})$. As

$$\begin{tikzcd} \mu_c^{-1}(0) \arrow[r, "\psi_\gamma"] \arrow[d, "\Phi"] & \mu_c^{-1}(0) \arrow[d, "\Phi"] \\ \mathrm{Orb}(A) \arrow[r, "\mathrm{Ad}_\gamma"'] & \mathrm{Orb}(A) \end{tikzcd}$$

commutes,

$$\Phi^*(\mathrm{Ad}_\gamma)^*\widetilde{\omega}_c = \psi_\gamma^*\Phi^*\widetilde{\omega}_c = \psi_\gamma^* \iota^*\omega_c = \iota^*\omega_c = \Phi^*\widetilde{\omega}_c$$

Therefore, we have that $\widetilde{\omega}_c$ is invariant under the $\mathrm{SL}(n, \mathbb{C})$ action.

Moreover, the Riemannian metric is invariant under the action of the compact subgroup $\mathrm{SU}(n)$. To see this,

$$\begin{aligned}\psi_\gamma^* g((\alpha_j, \beta_j), (\widetilde{\alpha}_j, \widetilde{\beta}_j)) &= \sum_{j=0}^{k-2} \left(\mathrm{Re}\,\mathrm{tr}\left(\alpha_j^*\widetilde{\alpha}_j\right) + \mathrm{Re}\,\mathrm{tr}\left(\beta_j^*\widetilde{\beta}_j\right)\right) + \mathrm{Re}\,\mathrm{tr}\left(\alpha_j\gamma^*\gamma\widetilde{\alpha}_j\right) + \mathrm{Re}\,\mathrm{tr}\left(\gamma\beta_j^*\widetilde{\beta}_j\gamma^*\right) \\ &= g((\alpha_j, \beta_j), (\widetilde{\alpha}_j, \widetilde{\beta}_j))\end{aligned}$$

References

[1] Piotr Z. Kobak and Andrew Swann. "Classical nilpotent orbits as hyperkähler quotients". In: *Int. J. Math.* 07.02 (Apr. 1996). Publisher: World Scientific Publishing Co., pp. 193–210. ISSN: 0129-167X. DOI: 10.1142/S0129167X96000116. URL: https://www.worldscientific.com/doi/10.1142/S0129167X96000116 (visited on 07/27/2023).