Kähler structure on coadjoint orbits of SU(2)

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First, notice as every element in $\mathfrak{su}(2)$ can be diagonalised by an element of SU(2), each coadjoint orbit $\mathcal{O}$ contains precisely one element of the form

$$A = \begin{pmatrix} i\xi & \\ & -i\xi \end{pmatrix}$$

with $\xi \geq 0$. We will use the following basis of $\mathfrak{su}(2)$:

$$\mathbf{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{j} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad \mathbf{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Note that we have $[\mathbf{i}, \mathbf{j}] = 2\mathbf{k}$ and cyclic permutations of this.

Fix $\xi > 0$, and consider the coadjoint orbit

$$\mathcal{O} = \{a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \mid a^2 + b^2 + c^2 = \xi^2\}$$

Fix $B \in \mathcal{O}$, say $B = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Let

$$\begin{aligned} u &= [B, \mathbf{i}] = 2c\mathbf{j} - 2b\mathbf{k} \\ v &= [B, \mathbf{j}] = -2c\mathbf{i} + 2a\mathbf{k} \\ w &= [B, \mathbf{k}] = 2b\mathbf{i} - 2a\mathbf{j} \end{aligned}$$

Note that u, v, w span $\mathrm{T}_B\mathcal{O}$, and $au + bv + cw = 0$. Assume without loss of generality that $c \neq 0$. Then u, v form a basis of $\mathrm{T}_B\mathcal{O}$. The KKS symplectic form is

$$\omega_B([B, X], [B, Y]) = -\langle B, [X, Y]\rangle = \mathrm{tr}(B[X, Y])$$

In particular, this gives us that

$$\omega_B(u, u) = \omega_B(v, v) = 0 \quad \text{and} \quad \omega_B(u, v) = 2\,\mathrm{tr}(B\mathbf{k}) = -4c$$

Hence ω is represented by the matrix

$$\omega = \begin{pmatrix} 0 & -4c \\ 4c & 0 \end{pmatrix}$$

with respect to the basis u, v.

Now suppose $B = \gamma A \gamma^\dagger$, where $\gamma = p\mathbf{1} + q\mathbf{i} + r\mathbf{j} + s\mathbf{k} \in \mathrm{SU}(2)$. Computing, we find that

$$\begin{aligned} a &= 2\xi(pr + qs) \\ b &= 2\xi(rs - pq) \\ c &= \xi(p^2 - q^2 - r^2 + s^2) \end{aligned}$$

In this specific example, we have that the Hermitian metric is

$$h(u, u) = 4\xi(\gamma\mathbf{i}\gamma^\dagger)_{21}\overline{(\gamma\mathbf{i}\gamma^\dagger)_{21}}$$

and so on. This is independent of the choice of γ, since any other choice would be $\lambda\gamma$, where $\lambda \in S^1 \subseteq \mathbb{C}$, and

$$(\lambda\gamma)\xi(\lambda\gamma)^\dagger = \lambda\overline{\lambda}\gamma\xi\gamma^\dagger = \gamma\xi\gamma^\dagger$$

Computing, we have that

$$\begin{aligned}
h(u,u) &= 4\xi\left(\frac{b^2+c^2}{\xi^2} + 16pqrs\right)\\
h(v,v) &= 4\xi\left(\frac{a^2+c^2}{\xi^2} - 16pqrs\right)\\
h(u,v) &= 4\xi\left(\frac{ab}{\xi^2} + 8(p^2-s^2)qr + i(\cdots)\right)
\end{aligned}$$

where we use the fact that the Riemannian metric g is the real part of h, and so we omit the imaginary part of $h(u,v)$. Moreover, using the fact that

$$\mathrm{SU}(2) = \left\{\begin{pmatrix}\alpha & -\overline{\beta}\\ \beta & \overline{\alpha}\end{pmatrix}\,\middle|\, |\alpha|^2+|\beta|^2 = 1\right\}$$

it is easy to see that for an appropriate choice of $\lambda = e^{i\theta}$, we can set β pure imaginary, and so $q = 0$. With this, we get that the Riemannian metric is

$$g = \frac{4}{\xi}\begin{pmatrix}b^2+c^2 & ab\\ ab & a^2+c^2\end{pmatrix}$$

We can see that $\det(g) = 16c^2 > 0$ by assumption that $c \neq 0$.

Finally, we wish to compute the almost complex structure. We have that

$$J = \tilde{g}^{-1}\tilde{\omega}$$

where $\tilde{g}, \tilde{\omega} : \mathrm{T}_B\mathcal{O} \to \mathrm{T}_B^*\mathcal{O}$ are the left maps, i.e.

$$\tilde{g}(u)(v) = g(u,v) \quad \tilde{\omega}(u)(v) = \omega(u,v)$$

If u^*, v^* is the dual basis of u, v, then we have that

$$[\tilde{g}] = [g]^\mathsf{T} = [g] \quad [\tilde{\omega}] = [\omega]^\mathsf{T} = -[\omega]$$

Therefore, we have that

$$J = \frac{1}{4\xi c}\begin{pmatrix}a^2+c^2 & -ab\\ -ab & b^2+c^2\end{pmatrix}\begin{pmatrix}0 & 4c\\ -4c & 0\end{pmatrix} = \frac{1}{c\xi}\begin{pmatrix}ab & a^2+c^2\\ -(b^2+c^2) & -ab\end{pmatrix}$$

and

$$J^2 = \frac{1}{\xi^2c^2}\begin{pmatrix}-c^2(a^2+b^2+c^2) & 0\\ 0 & -c^2(a^2+b^2+c^2)\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$