

Coadjoint Orbits of $SU(n)$

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In this note, we will consider the coadjoint orbits of $SU(n)$, and show that they are Kähler manifolds.

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Notation

Throughout this document, we will use the following notation:

- $SU(n)$ is the group of $n \times n$ unitary matrices with determinant 1.
- $\mathfrak{su}(n)$ is its Lie algebra.
- $\text{Ad} : SU(n) \rightarrow \text{GL}(\mathfrak{su}(n))$ is the adjoint representation.
- $\text{Ad}^* : SU(n) \rightarrow \text{GL}(\mathfrak{su}(n)^*)$ the coadjoint representaton.
- $\text{ad} : \mathfrak{su}(n) \rightarrow \mathfrak{gl}(\mathfrak{su}(n))$ the adjoint representation, given by $\text{ad}_X(Y) = [X, Y]$.
- M will be a (co)adjoint orbit, $A \in M$ a diagonal element.
- $\langle \cdot, \cdot \rangle$ will denote both the pairing $\mathfrak{su}(n)^* \times \mathfrak{su}(n) \rightarrow \mathbb{R}$ and the inner product on $\mathfrak{su}(n)$,
- $\Phi : \mathfrak{su}(n) \rightarrow \mathfrak{su}(n)^*$ is the isomorphism induced by the inner product,
- $\ell_g(h) = gh$ is the left multiplication by g map,
- $\langle\langle \cdot, \cdot \rangle\rangle$ the Riemannian metric on the (co)adjoint orbit.

1 Adjoint and Coadjoint Orbits

Define the Lie algebra

$$\mathfrak{su}(n) = \{X \in \text{Mat}(n, \mathbb{C}) \mid X^* + X = 0, \text{tr}(X) = 0\} \quad (1)$$

where X^* is the conjugate transpose of X , and with the Lie bracket being the matrix commutator. We can define the adjoint representation of $\text{SU}(n)$ as

$$\begin{aligned} \text{Ad} : \text{SU}(n) &\rightarrow \text{GL}(\mathfrak{su}(n)) \\ \text{Ad}_g(X) &= gXg^* \end{aligned}$$

Taking the dual representation, we get the coadjoint representation, which is

$$\begin{aligned} \text{Ad}^* : \text{SU}(n) &\rightarrow \text{GL}(\mathfrak{su}(n)^*) \\ \text{Ad}_g^*(\beta)(X) &= \langle \beta, \text{Ad}_{g^{-1}}(X) \rangle \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is used here to denote the pairing $\mathfrak{su}(n)^* \times \mathfrak{su}(n) \rightarrow \mathbb{R}$. We will use the same notation for the inner product on $\mathfrak{su}(n)$, which should not be an issue as the inner product defines a natural isomorphism. Now note that $-\kappa$, where κ is the Killing form, defines an inner product

$$\langle X, Y \rangle = -\text{tr}(XY) = \text{tr}(XY^*)$$

defines an inner product on $\mathfrak{su}(n)^1$, which means that we have a natural isomorphism

$$\begin{aligned} \Phi : \mathfrak{su}(n) &\rightarrow \mathfrak{su}(n)^* \\ X &\mapsto \langle X, \cdot \rangle \end{aligned}$$

With this, suppose $\beta = \Phi(B)$, then

$$\text{Ad}_g^*(\beta)(X) = \langle B, \text{Ad}_{g^{-1}}(X) \rangle = -\text{tr}(Bg^{-1}Xg) = -\text{tr}(BAg^{-1}X) = \Phi(\text{Ad}_g(B))(X)$$

Therefore, the following diagram commutes

$$\begin{array}{ccc} \mathfrak{su}(n) & \xrightarrow{\text{Ad}_g} & \mathfrak{su}(n) \\ \downarrow \Phi & & \downarrow \Phi \\ \mathfrak{su}(n)^* & \xrightarrow{\text{Ad}_g^*} & \mathfrak{su}(n)^* \end{array}$$

or equivalently, Φ defines an isomorphism of representations between Ad and Ad^* .

2 Root decomposition

Consider the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ of trace free $n \times n$ complex matrices. Then we have the Cartan subalgebra \mathfrak{t} of diagonal matrices. Let E_{ij} be the standard basis matrices for $\text{Mat}(n, \mathbb{C})$, $B \in \mathfrak{t}$. Say

$$B = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix}$$

Then $[B, E_{ij}] = (b_i - b_j)E_{ij}$. This means that we have the eigendecomposition

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{t} \oplus \bigoplus_{1 \leq i, j \leq n, i \neq j} \mathbb{C}E_{ij} \quad (2)$$

¹In fact, $\langle A, B \rangle = \text{tr}(AB^*)$ defines a Hermitian inner product on the space of complex matrices.

In particular, if we restrict this to the subalgebra $\mathfrak{su}(n)$, we get the decomposition

$$\mathfrak{su}(n) = \tilde{\mathfrak{t}} \oplus \bigoplus_{1 \leq i < j \leq n} (\mathbb{R}(E_{ij} - E_{ji}) \oplus i\mathbb{R}(E_{ij} + E_{ji})) \quad (3)$$

where $\tilde{\mathfrak{t}} = \mathfrak{t} \cap \mathfrak{su}(n)$ is the subalgebra of $\mathfrak{su}(n)$ of diagonal matrices.

3 Tangent space and Diagonalisation

3.1 Diagonalisation and Stabilisers of the coadjoint action

First of all, we note that elements of $\mathfrak{su}(n)$ are skew-hermitian, hence diagonalisable by an element of $SU(n)$ ². With this, we can classify the coadjoint orbits based off a diagonal element in the orbit. Consider

$$A = \begin{pmatrix} i\lambda_1 I_{m_1} & & \\ & \ddots & \\ & & i\lambda_k I_{m_k} \end{pmatrix}$$

where I_m is the $m \times m$ identity matrix, $\lambda_j \in \mathbb{R}$, with $\lambda_1 < \lambda_2 < \dots < \lambda_k$, $m_1 + \dots + m_k = n$ and $m_1\lambda_1 + \dots + m_k\lambda_k = 0$. In this case, we have that the orbit is

$$\text{Orb}(A) \cong SU(n)/\text{Stab}(A)$$

where $\text{Stab}(A)$ is the stabiliser of A under the adjoint action. In this case, we have that the stabiliser is the block diagonal subgroup

$$\text{Stab}(A) = S(U(m_1) \times \dots \times U(m_k))$$

where we consider $U(m_1) \times \dots \times U(m_k) \leq SU(n)$ as the block diagonal subgroup, and

$$S(U(m_1) \times \dots \times U(m_k)) = (U(m_1) \times \dots \times U(m_k)) \cap SU(n)$$

the subgroup with determinant 1.

3.2 Tangent space

Let M be an adjoint orbit. We will now focus on the generic case, that is, the eigenvalues of A are distinct. In this case, we have that

$$\text{Stab}(A) = T^{n-1}$$

is the torus of diagonal matrices in $SU(n)$, and we have a diffeomorphism $M \simeq SU(n)/T$. Therefore, the tangent space at A is

$$T_A M = \frac{\mathfrak{su}(n)}{\tilde{\mathfrak{t}}} = \bigoplus_{1 \leq i < j \leq n} (\mathbb{R}(E_{ij} - E_{ji}) \oplus i\mathbb{R}(E_{ij} + E_{ji})) = \{\text{ad}_A(X) \mid X \in \mathfrak{su}(n)\}$$

where the last equality follows from the fact that

$$\text{ad}_A(E_{ij}) = i(\lambda_i - \lambda_j)E_{ij}$$

and that diagonal matrices commute. Next, for $g \in SU(n)$, $\text{Ad}_g : \mathfrak{su}(n) \rightarrow \mathfrak{su}(n)$ is an invertible linear map, hence a diffeomorphism. In particular, this means that if $B = \text{Ad}_g(A)$, then

$$T_B M = \text{Ad}_g(T_A M) = \{\text{Ad}_g(\text{ad}_A(X)) \mid X \in \mathfrak{su}(n)\} = \{\text{ad}_B(X) \mid X \in \mathfrak{su}(n)\} \quad (4)$$

using the fact that

$$\text{ad}_{\text{Ad}_g(A)} = \text{Ad}_g \circ \text{ad}_A \circ \text{Ad}_{g^*}$$

²From standard linear algebra arguments, we know that they are $U(n)$ -diagonalisable. But if PBP^{-1} is diagonal, then so is $(\lambda P)B(\lambda P)^{-1}$, and by choosing λ appropriately, $\lambda \in SU(n)$.

which we will show in the proof of lemma 4.2, and that Ad_{g^*} is a bijection. Moreover, if $\beta = \Phi(B)$, then

$$\begin{aligned}
\langle (\text{ad}_X)^*(\beta), Y \rangle &= \langle \beta, \text{ad}_X(Y) \rangle \\
&= \langle B, [X, Y] \rangle \\
&= -\text{tr}(BXY - BYX) \\
&= -\text{tr}(BXY - XBY) \\
&= \langle [B, X], Y \rangle \\
&= \langle -\text{ad}_X(B), Y \rangle
\end{aligned}$$

Therefore, we have that the following diagram commutes.

$$\begin{array}{ccc}
\mathfrak{su}(n) & \xrightarrow{-\text{ad}_X} & \mathfrak{su}(n) \\
\downarrow \Phi & & \downarrow \Phi \\
\mathfrak{su}(n)^* & \xrightarrow{\text{ad}_X^*} & \mathfrak{su}(n)^*
\end{array}$$

Thus, we have the tangent space to the corresponding coadjoint orbit is

$$T_B \tilde{M} = \{\text{ad}_X^*(B) \mid X \in \mathfrak{su}(n)\}$$

4 Kirillov-Kostant-Souriau symplectic form

This section is from [2] Chapter 14. The proof is slightly modified, since here we have an explicit isomorphism between the adjoint and coadjoint representation, which simplifies some of the arguments.

Theorem 4.1. [Kirillov-Kostant-Souriau, [2, Theorem 14.4.1]] Let $M \subseteq \mathfrak{su}(n)^*$ be a coadjoint orbit. Define the 2-form ω on M by

$$\omega_\mu(\text{ad}_X^*(\mu), \text{ad}_Y^*(\mu)) = -\langle \mu, [X, Y] \rangle$$

Then ω is a symplectic form on M .

4.1 ω is well defined

First of all, we show that ω is well defined. That is, it is independent of the choice of $X, Y \in \mathfrak{su}(n)$.

Suppose $Z \in \mathfrak{su}(n)$ is such that $\text{ad}_Z^*(\mu) = \text{ad}_X^*(\mu)$. Then we must have that

$$\langle \mu, [X, Y] \rangle = \langle \mu, [Z, Y] \rangle$$

for all $Y \in \mathfrak{su}(n)$.

4.2 ω is non-degenerate

Suppose we have $X \in \mathfrak{su}(n)$ such that

$$\omega_\mu(\text{ad}_X^*(\mu), \text{ad}_Y^*(\mu)) = \langle \mu, [X, Y] \rangle = 0$$

for all Y . But this is the same as $\text{ad}_X^*(\mu) = 0$. Therefore, ω is non-degenerate.

4.3 ω is closed

First of all, we will need some preliminary results.

Lemma 4.2.

$$\mathrm{ad}_{\mathrm{Ad}_g X}^* = \mathrm{Ad}_g^* \circ \mathrm{ad}_X^* \circ \mathrm{Ad}_g^{*-1}$$

Proof. We will prove the corresponding statement for Ad and ad , and the result will follow by conjugation with the isomorphism Φ .

$$\mathrm{Ad}_g \circ \mathrm{ad}_X \circ \mathrm{Ad}_g^{-1}(Y) = gXg^*Ygg^* - gg^*YgXg^* = [gXg^*, Y] = \mathrm{ad}_{\mathrm{Ad}_g X}(Y)$$

□

Lemma 4.3.

$$\mathrm{Ad}_g([X, Y]) = [\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)]$$

Proof. Expand using the definition of Ad .

□

Lemma 4.4. $\mathrm{Ad}_g^* : M \rightarrow M$ preserves ω , that is,

$$(\mathrm{Ad}_g^*)^* \omega = \omega$$

Proof. Evaluating $\mathrm{ad}_{\mathrm{Ad}_g X}^* = \mathrm{Ad}_g^* \circ \mathrm{ad}_X^* \circ \mathrm{Ad}_g^{*-1}$ at $v = \mathrm{Ad}_g^*(\mu)$, we get

$$\mathrm{ad}_{\mathrm{Ad}_g X}^*(v) = \mathrm{Ad}_g^* \circ \mathrm{ad}_X^*(\mu) = d_\mu \mathrm{Ad}_g^* \circ \mathrm{ad}_X^*(\mu)$$

Therefore,

$$\begin{aligned} ((\mathrm{Ad}_g^*)^* \omega)_\mu(\mathrm{ad}_X^*(\mu), \mathrm{ad}_Y^*(\mu)) &= \omega_v(d_\mu \mathrm{Ad}_g^* \cdot \mathrm{ad}_X^*(\mu), d_\mu \mathrm{Ad}_g^* \cdot \mathrm{ad}_Y^*(\mu)) \\ &= \omega_v(\mathrm{ad}_{\mathrm{Ad}_g X}^*(v), \mathrm{ad}_{\mathrm{Ad}_g Y}^*(v)) \\ &= -\langle v, [\mathrm{Ad}_g X, \mathrm{Ad}_g Y] \rangle \\ &= -\langle v, \mathrm{Ad}_g([X, Y]) \rangle \\ &= -\langle \mathrm{Ad}_{g^{-1}}^*(v), [X, Y] \rangle \\ &= -\langle \mu, [X, Y] \rangle \\ &= \omega_\mu(\mathrm{ad}_X^*(\mu), \mathrm{ad}_Y^*(\mu)) \end{aligned}$$

□

For $v \in \mathfrak{su}(n)^*$, define the left-invariant one-form

$$v_\ell(g) = (d_g \ell_{g^{-1}})^*(v)$$

for $g \in \mathrm{SU}(n)$. Similarly, for $X \in \mathfrak{su}(n)$, let X_ℓ be the corresponding left invariant vector field on G . Then $v_\ell(X_\ell) = \langle v, X \rangle$ at all $g \in \mathrm{SU}(n)$.

Fix $v \in M$, and consider the map $\pi : \mathrm{SU}(n) \rightarrow M$, defined by

$$\pi(g) = \mathrm{Ad}_g^*(v)$$

We can use this to pullback $\sigma = \pi^* \omega$ to a two form on $\mathrm{SU}(n)$.

Lemma 4.5. σ is left invariant. That is, $\ell_g^* \sigma = \sigma$ for all $g \in \mathrm{SU}(n)$.

Proof. First, notice that $\pi \circ \ell_g = \text{Ad}_g^* \circ \pi$, since

$$\pi(\ell_g(h)) = \text{Ad}_{gh}^*(v) = \text{Ad}_g^* \circ \text{Ad}_h^*(v) = \text{Ad}_g^*(\pi(h))$$

With this,

$$\ell_g^* \sigma = \ell_g^* \pi^* \omega = (\pi \circ \ell_g)^* \omega = (\text{Ad}_g^* \circ \pi)^* \omega = \pi^* (\text{Ad}_g^*)^* \omega = \pi^* \omega = \sigma$$

□

Lemma 4.6. $\sigma(X_\ell, Y_\ell) = -\langle v_\ell, [X_\ell, Y_\ell] \rangle$.

Proof. By left invariance of both sides, suffices to show that the result holds at e . First notice that

$$d_I \pi(Y) = -\text{ad}_Y^*(v)$$

Therefore, π is a submersion at e . By definition of the pullback,

$$\begin{aligned} \sigma_I(X, Y) &= (\pi^* \omega)_I(X, Y) \\ &= \omega_{\pi(I)}(d_e \pi \cdot X, d_e \pi \cdot Y) \\ &= \omega_v(\text{ad}_X^*(v), \text{ad}_Y^*(v)) \\ &= -\langle v, [X, Y] \rangle \end{aligned}$$

Hence

$$\sigma(X_\ell, Y_\ell)_I = \sigma_I(X, Y) = -\langle v, [X, Y] \rangle = -\langle v_\ell, [X_\ell, Y_\ell] \rangle_I$$

□

Now for a one form α , we have that

$$d\alpha(X, Y) = X[\alpha(Y)] - Y[\alpha(X)] - \alpha([X, Y])$$

where for a smooth function $f : M \rightarrow \mathbb{R}$, and a vector field X on M , $X[f] := df(X)$ is a smooth function $M \rightarrow \mathbb{R}$.

Since $v_\ell(X_\ell)$ is constant, $Y_\ell[v_\ell(X_\ell)] = 0$. Similarly, $X_\ell[v_\ell(Y_\ell)] = 0$. Therefore, we have that

$$dv_\ell(X_\ell, Y_\ell) = -v_\ell([X_\ell, Y_\ell]) = \sigma(X_\ell, Y_\ell)$$

Now suppose U, V are vector fields on $\text{SU}(n)$. We want to show that $\sigma(U, V) = dv_\ell(U, V)$. As σ is left invariant,

$$\begin{aligned} \sigma(U, V)_g &= (\ell_{g^{-1}}^* \sigma)_g(U_g, V_g) \\ &= \sigma_I(\underbrace{d\ell_{g^{-1}} \cdot U_g}_{=: X}, \underbrace{d\ell_{g^{-1}} \cdot V_g}_{=: Y}) \\ &= \sigma_I(X, Y) \\ &= dv_\ell(X_\ell, Y_\ell)_I \\ &= (\ell_g^* dv_\ell)(X_\ell, Y_\ell)_I \\ &= (dv_\ell)_g(d\ell_g(X_\ell)_I, d\ell_g(Y_\ell)_I) \\ &= (dv_\ell)_g(d\ell_g X, d\ell_g Y) \\ &= (dv_\ell)_g(U_g, V_g) \\ &= dv_\ell(U, V)_g \end{aligned}$$

With this, $d\sigma = d^2 v_\ell = 0$. Hence $\pi^* d\omega = d(\pi^* \omega) = d\sigma = 0$. Since $\pi \circ \ell_g = \text{Ad}_g^* \circ \ell_g$, and π is a submersion at I , it is in fact a submersion everywhere. Moreover, π is surjective, by definition.

For $\mu \in M$, and $X, Y \in T_\mu M$, we have that

$$d\omega_\mu(X, Y, Z) = d\omega_{\pi(g)}(d\pi(U), d\pi(V), d\pi(W)) = (\pi^*d\omega)_g(U, V, W) = 0$$

where $g \in \mathrm{SU}(n)$ is such that $\pi(g) = \mu$, which exists by surjectivity, and $U, V, W \in T_g \mathrm{SU}(n)$ such that $d\pi(U) = X, d\pi(V) = Y$ and $d\pi(W) = Z$, which exists as π is a submersion. Thus, as $\mu \in M$ is arbitrary, ω is closed.

4.4 ω on adjoint orbits

Using the isomorphism Φ , theorem 4.1 and the computation for ad_X^* , we get the following result.

Theorem 4.7. Let $M \subseteq \mathfrak{su}(n)$ be an adjoint orbit. Define the 2-form ω on M by

$$\omega_A([A, B], [A, C]) = -\langle A, [B, C] \rangle = \mathrm{tr}(A[B, C])$$

Then ω is a symplectic form on M .

This will be convenient for us since we can compute the right hand side directly from the matrices.

5 Kähler structure

In this section, we construct the Kähler structure on adjoint orbits of $\mathrm{SU}(n)$. In principle, we only need two of (ω, g, J) as we can recover the third. See Cannas da Silva [1, §13.2] for a table which summarises the relations and the required conditions if we only have two of the three.

We have already constructed the Kirillov-Kostant-Souriau symplectic form ω . In the following, we will construct the Riemannian metric $\langle \cdot, \cdot \rangle^3$ and the almost complex structure J . Once we have shown that these form a compatible triple, since ω is closed and J comes from a diffeomorphism between the adjoint orbit and a complex manifold, we will have that ω is in fact a Kähler form.

5.1 At a diagonal element

Let M be an adjoint orbit. Recall from section 3 that the stabiliser of A is the torus $T \cong T^{n-1}$ of diagonal matrices in $\mathrm{SU}(n)$.

$$T_A M = \bigoplus_{1 \leq i < j \leq n} (\mathbb{R}(E_{ij} - E_{ji}) \oplus i\mathbb{R}(E_{ij} + E_{ji})) \cong \frac{\mathfrak{su}(n)}{\mathfrak{t}} \cong T_{[1]} \left(\frac{\mathrm{SU}(n)}{T^{n-1}} \right)$$

where the isomorphism is induced by the quotient map

$$\begin{aligned} \pi : \mathrm{SU}(n) &\rightarrow M \\ g &\mapsto \mathrm{Ad}_g(A) = gAg^* \end{aligned}$$

Let $e_{ij} = E_{ij} - E_{ji}$ and $f_{ij} = i(E_{ij} + E_{ji})$.

Lemma 5.1. 1. For any $X, Y, Z \in \mathfrak{su}(n)$, $\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle$,

$$2. [A, E_{ij}] = i(\lambda_i - \lambda_j)E_{ij}$$

$$3. E_{ij}E_{kl} = \delta_{jk}E_{il}$$

$$4. \langle \cdot, \cdot \rangle \text{ is } \mathbb{C}\text{-bilinear.}$$

Proof. Expand. □

³Quite often g will be used for an element of $\mathrm{SU}(n)$, and so we will use $\langle \cdot, \cdot \rangle$ to denote the inner product.

Using these, we have that

$$\begin{aligned}
\langle A, [E_{ij}, E_{kl}] \rangle &= \langle [A, E_{ij}], E_{kl} \rangle \\
&= i(\lambda_i - \lambda_j) \langle E_{ij}, E_{kl} \rangle \\
&= -i(\lambda_i - \lambda_j) \delta_{jk} \operatorname{tr}(E_{il}) \\
&= -i(\lambda_i - \lambda_j) \delta_{jk} \delta_{il} \\
&= i(\lambda_j - \lambda_i) \delta_{il} \delta_{jk}
\end{aligned}$$

and so,

$$\begin{aligned}
\langle A, [e_{ij}, e_{kl}] \rangle &= \langle A, [E_{ij}, E_{kl}] \rangle - \langle A, [E_{ji}, E_{kl}] \rangle - \langle A, [E_{ij}, E_{lk}] \rangle + \langle A, [E_{ji}, E_{lk}] \rangle \\
&= i(\lambda_j - \lambda_i) \delta_{il} \delta_{jk} - i(\lambda_i - \lambda_j) \delta_{jl} \delta_{ik} - i(\lambda_j - \lambda_i) \delta_{ik} \delta_{jl} + i(\lambda_i - \lambda_j) \delta_{jk} \delta_{il} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
-\langle A, [f_{ij}, f_{kl}] \rangle &= \langle A, [E_{ij}, E_{kl}] \rangle + \langle A, [E_{ji}, E_{kl}] \rangle + \langle A, [E_{ij}, E_{lk}] \rangle + \langle A, [E_{ji}, E_{lk}] \rangle \\
&= i(\lambda_j - \lambda_i) \delta_{il} \delta_{jk} + i(\lambda_i - \lambda_j) \delta_{jl} \delta_{ik} + i(\lambda_j - \lambda_i) \delta_{ik} \delta_{jl} + i(\lambda_i - \lambda_j) \delta_{jk} \delta_{il} \\
&= 0
\end{aligned}$$

Finally, we have that

$$\begin{aligned}
-i \langle A, [e_{ij}, f_{kl}] \rangle &= \langle A, [E_{ij}, E_{kl}] \rangle - \langle A, [E_{ji}, E_{kl}] \rangle + \langle A, [E_{ij}, E_{lk}] \rangle - \langle A, [E_{ji}, E_{lk}] \rangle \\
&= i(\lambda_j - \lambda_i) \delta_{il} \delta_{jk} - i(\lambda_i - \lambda_j) \delta_{jl} \delta_{ik} + i(\lambda_j - \lambda_i) \delta_{ik} \delta_{jl} - i(\lambda_i - \lambda_j) \delta_{jk} \delta_{il} \\
&= 2i(\lambda_j - \lambda_i) (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl}) \\
&= 2i(\lambda_j - \lambda_i) \delta_{ik} \delta_{jl}
\end{aligned}$$

where the last line is because we require $i < j$ and $k < l$. Hence we have that

$$\begin{aligned}
\omega_A(e_{ij}, e_{kl}) &= 0 \\
\omega_A(f_{ij}, f_{kl}) &= 0 \\
\omega_A(e_{ij}, f_{ij}) &= 2(\lambda_j - \lambda_i) \\
\omega_A(e_{ij}, f_{kl}) &= 0 \text{ for } (i, j) \neq (k, l)
\end{aligned}$$

We can then define an almost complex structure on $T_A M$ by

$$J_A(e_{ij}) = f_{ij} \quad J_A(f_{ij}) = -e_{ij} \quad (5)$$

Moreover, we can define an inner product on $T_A M$ by

$$\langle\langle e_{ij}, e_{ij} \rangle\rangle_A = \langle\langle f_{ij}, f_{ij} \rangle\rangle_A = 2(\lambda_j - \lambda_i)$$

and requiring e_{ij}, f_{ij} to form an orthogonal basis. This is positive definite since we required $\lambda_i < \lambda_j$ for $i < j$. Using this, we find that

$$\omega_A(e_{ij}, f_{ij}) = 2(\lambda_j - \lambda_i) = \langle\langle f_{ij}, f_{ij} \rangle\rangle_A = \langle\langle J_A(e_{ij}), f_{ij} \rangle\rangle_A$$

and that J defines an isometry. Hence $(\omega_A, \langle\langle \cdot, \cdot \rangle\rangle_A, J_A)$ define a compatible triple on the vector space $T_A M$.

5.2 Complex quotient

Let P be the subgroup of lower triangular matrices in $\mathrm{SL}(n, \mathbb{C})$. Consider the composition $\varphi : \mathrm{SU}(n) \rightarrow \mathrm{SL}(n, \mathbb{C})/P$ given by the composition

$$\mathrm{SU}(n) \hookrightarrow \mathrm{SL}(n) \twoheadrightarrow \mathrm{SL}(n, \mathbb{C})/P$$

Suppose $\varphi(g) = \varphi(h)$. That is, $gP = hP$. This is true if and only if there exists $p \in P$, such that $h = gp$. In this case, $p = g^{-1}h \in \mathrm{SU}(n)$, therefore, $p \in \mathrm{SU}(n) \cap P = T$, since $p^* = p^{-1}$ is also lower triangular. This means that φ induces a homeomorphism $\mathrm{SU}(n)/T \cong \mathrm{SL}(n, \mathbb{C})/P$. The right hand side is a complex manifold $\mathrm{SL}(n, \mathbb{C})$ quotiented by a complex Lie group P , so it is a complex manifold. Using the above, we can get a complex structure on $\mathrm{SU}(n)/T \cong M$.

Using the root decompositions eq. (2) for $\mathfrak{sl}(n, \mathbb{C})$ and eq. (3) for $\mathfrak{su}(n)$, we can see that the almost complex structure we defined in eq. (5) is the same as the action of multiplication by i .

5.3 At a general $B \in M$

Fix $g \in \mathrm{SU}(n)$, and let $B = \mathrm{Ad}_g(A)$. Let $\varphi : \mathrm{SU}(n)/T \rightarrow \mathrm{SL}(n, \mathbb{C})/P$ be the diffeomorphism from above, which is given by $\varphi([g]) = \llbracket g \rrbracket$, where $[g] = gT \in \mathrm{SU}(n)/T$ and $\llbracket g \rrbracket = gP \in \mathrm{SL}(n, \mathbb{C})/P$.

First of all, $\mathrm{SL}(n, \mathbb{C})$ is a complex Lie group, so left multiplication is holomorphic. That is, $d\ell_g(iv) = id\ell_g(v)$. If \tilde{J} denotes the complex structure on $\mathrm{SL}(n, \mathbb{C})$, then we have that

$$\tilde{J}_g = d\ell_g \circ \tilde{J}_I \circ d\ell_{g^{-1}}$$

ℓ_g descends to a biholomorphism on $\mathrm{SL}(n, \mathbb{C})/P \cong \mathrm{SU}(n)/T$, and so the corresponding almost complex structure \bar{J} on $\mathrm{SU}(n)/T$ is given by

$$\bar{J}_{[g]} = d\ell_g \circ \bar{J}_{[I]} \circ d\ell_{g^*} \quad (6)$$

Since the diffeomorphism $\mathrm{SU}(n)/T$ is induced by the map $\pi(g) = \mathrm{Ad}_g(A)$, we have that

$$\begin{aligned} J_B &= d\pi \circ \bar{J}_{[g]} \circ d\pi^{-1} \\ &= d\pi \circ d\ell_g \circ \bar{J}_{[I]} \circ d\ell_{g^*} \circ d\pi^{-1} \\ &= d(\pi \circ \ell_g \circ \pi^{-1}) \circ J_A \circ d(\pi \circ \ell_{g^*} \circ \pi^{-1}) \end{aligned}$$

From the proof of lemma 4.5, we have that $\pi \circ \ell_g = \mathrm{Ad}_g \circ \pi$. Moreover, since Ad_g is a linear map, $d\mathrm{Ad}_g = \mathrm{Ad}_g$. Therefore, we have that the almost complex structure is given by

$$J_B = \mathrm{Ad}_g \circ J_A \circ \mathrm{Ad}_{g^*}$$

We want to show that this is compatible with the Kirillov-Kostant-Souriau symplectic form.

Recall from eq. (4) that

$$T_B M = \mathrm{Ad}_g(T_A M)$$

Then we have that

$$\begin{aligned} \omega_B([B, X], [B, Y]) &= -\langle B, [X, Y] \rangle \\ &= -\langle \mathrm{Ad}_g(A), \mathrm{Ad}_g([\mathrm{Ad}_{g^*}(X), \mathrm{Ad}_{g^*}(Y)]) \rangle \\ &= -\langle A, [\mathrm{Ad}_{g^*}(X), \mathrm{Ad}_{g^*}(Y)] \rangle \\ &= \omega_A([A, \mathrm{Ad}_{g^*}(X)], [A, \mathrm{Ad}_{g^*}(Y)]) \\ &= \omega_A(\mathrm{Ad}_{g^*}([B, X]), \mathrm{Ad}_{g^*}([B, Y])) \end{aligned}$$

Note this also follows from lemma 4.5 where we showed $\pi^* \omega$ is left invariant. Therefore, the Riemannian metric is given by

$$\langle\langle X, Y \rangle\rangle_B = \langle\langle \mathrm{Ad}_{g^*}(X), \mathrm{Ad}_{g^*}(Y) \rangle\rangle_A$$

References

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