Kirillov-Kostant-Souriau symplectic form

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Throughout, let G be a Lie group, with Lie algebra $\mathrm{T}_e G = \mathfrak{g}$.

Let G act on itself by conjugation, that is, $C_g(h) = ghg^{-1}$. This is a smooth map, with $C_g(e) = e$. Taking the derivative, we have

$$\mathrm{Ad}_g = \mathrm{d}\big(C_g\big) : \mathfrak{g} \to \mathfrak{g}$$

The map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, $\mathrm{Ad}(g) = \mathrm{Ad}_g$ is called the *adjoint representation* of G. Dualising, we get the *coadjoint representation*, that is, $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$, given by

$$\mathrm{Ad}^*_g = (\mathrm{Ad}_{g^{-1}})^*$$

On the other hand, if we differentiate Ad, we get $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$.

1 Infinitesimal action

Let M be a manifold, $\Phi : G \times M \to M$ a smooth action.

Definition 1.1 (proper action)
The action Φ is proper if the map

$$(g, x) \mapsto (\Phi(g, x), x)$$

is proper, i.e. the preimage of a compact set is compact.

Proposition 1.2. Suppose Φ is a free and proper action. Then the quotient M/G is a smooth manifold, and $\pi : M \to M/G$ is a smooth submersion.

Definition 1.3 (infinitesimal action)
Suppose $\Phi : G \times M \to M$ is an action. For $\xi \in \mathfrak{g}$, define $\Phi^\xi : \mathbb{R} \times M \to M$

$$\Phi^\xi(t, m) = \Phi(\exp(t\xi), x)$$

Then Φ^ξ defines an $\mathbb{R}$-action, i.e. $\Phi^\xi(t, \cdot)$ is a flow on M. The corresponding vector field

$$\xi_M(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi^\xi(t, x)$$

is called the infinitesimal action of ξ.

In particular, for the (co)adjoint representation, we find that

$$\xi_{\mathfrak{g}}(\eta) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{\exp(t\xi)}(\eta) = \mathrm{ad}_\xi(\eta) = [\xi, \eta]$$

and if $\langle , \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the usual pairing, then

$$
\begin{aligned}
\langle \xi_{\mathfrak{g}^*}(\alpha), \eta \rangle &= \left\langle \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}^*_{\exp(t\xi)}(\alpha), \eta \right\rangle \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left\langle \mathrm{Ad}^*_{\exp(t\xi)}(\alpha), \eta \right\rangle \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left\langle \alpha, \mathrm{Ad}_{\exp(-t\xi)}(\eta) \right\rangle \\
&= \left\langle \alpha, \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{\exp(-t\xi)}(\eta) \right\rangle \\
&= \langle \alpha, -[\xi, \eta] \rangle \\
&= \left\langle \alpha, -\mathrm{ad}_\xi(\eta) \right\rangle \\
&= -\left\langle (\mathrm{ad}_\xi)^*(\alpha), \eta \right\rangle
\end{aligned}
$$

So $\xi_{\mathfrak{g}^*} = -(\mathrm{ad}_\xi)^*$, and $\xi_{\mathfrak{g}^*}(\alpha)(\eta) = -\langle \alpha, [\xi, \eta] \rangle$.

2 Coadjoint orbits

For $\mu \in \mathfrak{g}^*$, we'll write $\mathrm{Orb}(\mu) = \{\mathrm{Ad}^*_g(\mu) \mid g \in G\}$ for the *coadjoint orbit* of μ. We will assume without proof that this is a submanifold of $\mathfrak{g}^*$, which is diffeomorphic to G/G_μ, where $G_\mu = \{g \in G \mid \mathrm{Ad}^*_g(\mu) = \mu\}$ is the *isotropy* or *stabiliser* of μ.

2.1 Tangent space

Let $\mathcal{O}$ be a coadjoint orbit, $\mu \in \mathcal{O}$. For $\xi \in \mathfrak{g}$, let $g(t) = \exp(t\xi)$. Then $g'(0) = \xi$. Now define

$$\mu(t) = \mathrm{Ad}^*_{g(t)} \mu$$

which is a curve in $\mathcal{O}$ (which means it is a curve in the vector space $\mathfrak{g}^*$), with $\mu(0) = \mu$. By definition, for any $\eta \in \mathfrak{g}$,

$$\langle \mu(t), \eta \rangle = \left\langle \mu, \mathrm{Ad}_{g(t)^{-1}} \eta \right\rangle$$

We can differentiate this at $t = 0$, to get

$$\left\langle \mu'(0), \eta \right\rangle = -\left\langle \mu, \mathrm{ad}_\xi(\eta) \right\rangle = -\left\langle (\mathrm{ad}_\xi)^* \mu, \eta \right\rangle$$

where as usual we use the isomorphism $\mu'(0) \in \mathrm{Hom}(\mathrm{T}_0\mathbb{R}, \mathrm{T}_\mu \mathfrak{g}^*) \simeq \mathfrak{g}^*$. This then gives us that

$$\mathrm{T}_\mu \mathcal{O} = \left\{ (\mathrm{ad}_\xi)^*(\mu) \mid \xi \in \mathfrak{g} \right\}$$

Moreover, this also gives us that the infinitesimal generator is

$$\xi_{\mathfrak{g}^*}(\mu) = -(\mathrm{ad}_\xi)^*(\mu)$$

3 Kirillov-Kostant-Souriau symplectic form

Theorem 3.1. Let G be a Lie group, $\mathcal{O} \subseteq \mathfrak{g}^*$ be a coadjoint orbit. Define the 2-form ω on $\mathcal{O}$ by

$$\omega(\mu)(\xi_{\mathfrak{g}^*}(\mu), \eta_{\mathfrak{g}^*}(\mu)) = -\langle \mu, [\xi, \eta] \rangle$$

Then ω and $-\omega$ are symplectic forms on $\mathcal{O}$.

We will only prove the result for ω. The proof for $-\omega$ is similar.

3.1 ω is well defined

First of all, we show that ω is well defined. That is, it is independent of the choice of $\xi, \eta \in \mathfrak{g}$.

Suppose $\zeta \in \mathfrak{g}$ is such that $\zeta_{\mathfrak{g}^*}(\mu) = \xi_{\mathfrak{g}^*}(\mu)$. Then as $\xi_{\mathfrak{g}^*} = (\mathrm{ad}_\xi)^*$, we must have that

$$\langle \mu, [\xi, \eta] \rangle = \langle \mu, [\zeta, \eta] \rangle$$

for all $\eta \in \mathfrak{g}$.

3.2 ω is non-degenerate

Since the pairing $\langle , \rangle$ is non-degenerate, $\omega(\mu)(\xi_{\mathfrak{g}^*}(\mu), \eta_{\mathfrak{g}^*}(\mu))$ for all $\eta_{\mathfrak{g}^*}(\mu)$ implies that $\langle \mu, [\xi, \eta] \rangle = 0$, for all η. But this then means that $\xi_{\mathfrak{g}^*}(\mu) = 0$, so ω is non-degenerate.

3.3 ω is closed

First of all, we will need some preliminary results.

Lemma 3.2.

$$(\mathrm{Ad}_\xi)_{\mathfrak{g}^*} = \mathrm{Ad}_g^* \circ \xi_{\mathfrak{g}^*} \circ \mathrm{Ad}_{g^{-1}}^*$$

Proof. Let $h(t) = \exp(t\xi)$. Then

$$\begin{aligned}
(\mathrm{Ad}_g\, \xi)_{\mathfrak{g}^*}(\mu) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{gh(t)g^{-1}}^*(\mu) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_g^* \,\mathrm{Ad}_{h(t)}^* \,\mathrm{Ad}_{g^{-1}}^*(\mu) \\
&= \mathrm{Ad}_h^* \circ \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{h(t)}^* \right) \circ \mathrm{Ad}_{g^{-1}}^*(\mu) \\
&= \mathrm{Ad}_h^* \circ \xi_{\mathfrak{g}^*} \circ \mathrm{Ad}_{g^{-1}}^*(\mu)
\end{aligned}$$

where we used the fact that Ad_g^* is a linear map, so we can exchange it with the derivative operator. □

Lemma 3.3.

$$\mathrm{Ad}_g([\xi, \eta]) = [\mathrm{Ad}_g(\xi), \mathrm{Ad}_g(\eta)]$$

Proof. First, notice that

$$C_g(C_h(k)) = ghkh^{-1}g^{-1} = C_g(h)C_g(k)C_g(h^{-1})$$

Differentiating this at $h = e$ and $k = e$ gives the result. □

Lemma 3.4. $\mathrm{Ad}_g^* : \mathcal{O} \to \mathcal{O}$ preserves ω, that is,

$$(\mathrm{Ad}_g^*)^* \omega = \omega$$

Proof. Evaluating $(\mathrm{Ad}_\xi)_{\mathfrak{g}^*} = \mathrm{Ad}_g^* \circ \xi_{\mathfrak{g}^*} \circ \mathrm{Ad}_{g^{-1}}^*$ at $\nu = \mathrm{Ad}_g^*(\mu)$, we get

$$(\mathrm{Ad}_g\, \xi)_{\mathfrak{g}^*}(\nu) = \mathrm{Ad}_g^* \circ \xi_{\mathfrak{g}^*}(\mu) = \mathrm{d}_\mu \mathrm{Ad}_g^* \circ \xi_{\mathfrak{g}^*}(\mu)$$

Therefore,

$$
\begin{aligned}
((\mathrm{Ad}_g^*)^*\omega)(\mu)(\xi_{\mathfrak{g}^*}(\mu),\eta_{\mathfrak{g}^*}(\mu)) &= \omega(\nu)(\mathrm{d}_\mu \mathrm{Ad}_g^* \cdot \xi_{\mathfrak{g}^*}(\mu), \mathrm{d}_\mu \mathrm{Ad}_g^* \cdot \eta_{\mathfrak{g}^*}(\mu)) \\
&= \omega(\nu)((\mathrm{Ad}_g \xi)_{\mathfrak{g}^*}(\nu), (\mathrm{Ad}_g \eta)_{\mathfrak{g}^*}(\nu)) \\
&= -\left\langle \nu, [\mathrm{Ad}_g \xi, \mathrm{Ad}_g \eta]\right\rangle \\
&= -\left\langle \nu, \mathrm{Ad}_g([\xi,\eta])\right\rangle \\
&= -\left\langle \mathrm{Ad}_{g^{-1}}^*(\nu), [\xi,\eta]\right\rangle \\
&= -\langle \mu, [\xi,\eta]\rangle \\
&= \omega(\mu)(\xi_{\mathfrak{g}^*}(\mu), \eta_{\mathfrak{g}^*}(\mu))
\end{aligned}
$$

□

For $\nu \in \mathfrak{g}^*$, define the left-invariant one-form

$$\nu_\ell(g) = (\mathrm{d}_g \ell_{g^{-1}})^*(\nu)$$

for $g \in G$. Similarly, for $\xi \in \mathfrak{g}$, let ξ_ℓ be the corresponding left invariant vector field on G. Then $\nu_\ell(\xi_\ell) = \langle \nu, \xi \rangle$ at all $g \in G$.

Fix $\nu \in \mathcal{O}$, and consider the map $\varphi_\nu : G \to \mathcal{O}$, defined by

$$\varphi_\nu(g) = \mathrm{Ad}_g^*(\nu)$$

We can use this to pullback $\sigma = (\varphi_\nu)^*\omega$ to a two form on G.

Lemma 3.5. σ is left invariant. That is, $\ell_g^*\sigma = \sigma$ for all $g \in G$.

Proof. First, notice that $\varphi_\nu \circ \ell_g = \mathrm{Ad}_g^* \circ \varphi_\nu$, since

$$\varphi_\nu(\ell_g(h)) = \mathrm{Ad}_{gh}^*(\nu) = \mathrm{Ad}_g^* \circ \mathrm{Ad}_h^*(\nu) = \mathrm{Ad}_g^*(\varphi_\nu(h))$$

With this,

$$\ell_g^*\sigma = \ell_g^*\varphi^*\omega = (\varphi \circ \ell_g)^*\omega = (\mathrm{Ad}_g^* \circ \varphi_\nu)^*\omega = (\varphi_\nu)^*(\mathrm{Ad}_g^*)^*\omega = (\varphi_\nu)^*\omega = \sigma$$

□

Lemma 3.6. $\sigma(\xi_\ell, \eta_\ell) = -\langle \nu_\ell, [\xi_\ell, \eta_\ell]\rangle$.

Proof. By left invariance of both sides, suffices to show that the result holds at e. First notice that

$$\mathrm{d}_e\varphi_\nu(\eta) = \eta_{\mathfrak{g}_*}(\nu)$$

Therefore, φ_ν is a submersion at e. By definition of the pullback,

$$
\begin{aligned}
\sigma(e)(\xi,\eta) &= (\varphi_\nu)^*\omega(e)(\xi,\eta) \\
&= \omega(\varphi_\nu(e))(\mathrm{d}_e\varphi_\nu \cdot \xi, \mathrm{d}_e\varphi_\nu \cdot \eta) \\
&= \omega(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) \\
&= -\langle \nu, [\xi,\eta]\rangle
\end{aligned}
$$

Hence

$$\sigma(\xi_\ell,\eta_\ell)(e) = \sigma(e)(\xi,\eta) = -\langle \nu, [\xi,\eta]\rangle = -\langle \nu_\ell, [\xi_\ell,\eta_\ell]\rangle (e)$$

□

Now for a one form α, we have that

$$\mathrm{d}\alpha(X, Y) = X[\alpha(Y)] - Y[\alpha(X)] - \alpha([X, Y])$$

where for a smooth function $f : M \to \mathbb{R}$, and a vector field X on M, $X[f] := \mathrm{d}f(X)$ is a smooth function $M \to \mathbb{R}$.

Since $\nu_\ell(\xi_\ell)$ is constant, $\eta_\ell[\nu_\ell(\xi_\ell)] = 0$. Similarly, $\xi_\ell[\nu_\ell(\eta_\ell)] = 0$. Therefore, we have that

$$\mathrm{d}\nu_\ell(\xi_\ell, \eta_\ell) = -\nu_\ell([\xi_\ell, \eta_\ell]) = \sigma(\xi_\ell, \eta_\ell)$$

Now suppose X, Y are vector fields on G. We want to show that $\sigma(X, Y) = \mathrm{d}\nu_\ell(X, Y)$. As σ is left invariant,

$$\begin{aligned}
\sigma(X, Y)(g) &= (\ell_{g^{-1}}^* \sigma)(g)(X(g), Y(g)) \\
&= \sigma(e)(\underbrace{\mathrm{d}\ell_{g^{-1}} \cdot X(g)}_{=\xi}, \underbrace{\mathrm{d}\ell_{g^{-1}} \cdot Y(g)}_{=\eta}) \\
&= \sigma(e)(\xi, \eta) \\
&= \mathrm{d}\nu_\ell(\xi_\ell, \eta_\ell)(e) \\
&= (\ell_g^* \mathrm{d}\nu_\ell)(\xi_\ell, \eta_\ell)(e) \\
&= (\mathrm{d}\nu_\ell)(g)(\mathrm{d}\ell_g \cdot \xi_\ell(e), \mathrm{d}\ell_g \cdot \eta_\ell(e)) \\
&= (\mathrm{d}\nu_\ell)(g)(\mathrm{d}\ell_g \cdot \xi, \mathrm{d}\ell_g \cdot \eta) \\
&= (\mathrm{d}\nu_\ell)(g)(X(g), Y(g)) \\
&= \mathrm{d}\nu_\ell(X, Y)(g)
\end{aligned}$$

With this, $\mathrm{d}\sigma = \mathrm{d}^2\nu_\ell = 0$. Hence $(\varphi_\nu)^*\mathrm{d}\omega = \mathrm{d}((\varphi_\nu)^*\omega) = \mathrm{d}\sigma = 0$. Since $\varphi_\nu \circ \ell_g = \mathrm{Ad}_g^* \circ \ell_g$, and φ_ν is a submersion at e, it is infact a submersion everywhere. Moreover, φ_ν is surjective, by definition.

For $\mu \in \mathcal{O}$, and $X, Y \in T_\mu\mathcal{O}$, we have that

$$\mathrm{d}\omega(\mu)(X, Y) = \mathrm{d}\omega_{\varphi_\nu(g)}(\mathrm{d}\varphi_\nu(\xi), \mathrm{d}\varphi_\nu(\eta)) = ((\varphi_\nu)^*\mathrm{d}\omega)(g)(\xi, \eta) = 0$$

where $g \in G$ is such that $\varphi_\nu(g) = \mu$, which exists by surjectivity, and $\xi, \eta \in T_gG$ such that $\mathrm{d}\varphi_\nu(\xi) = X$ and $\mathrm{d}\varphi_\nu(\eta) = Y$, which exists as φ_ν is a submersion. Thus, as $\mu \in \mathcal{O}$ is arbitrary, ω is closed.