Computation of hyperKähler metric on nilpotent $\mathrm{SL}(2, \mathbb{C})$ orbits

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In this note, we will compute explicitly the hyperKähler metric on the nilpotent $\mathrm{SL}(2, \mathbb{C})$ orbit using [1].

In this case, the only nonzero nilpotent orbit is the orbit of

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which corresponds to the diagram

$$0 \rightleftarrows \mathbb{C} \rightleftarrows \mathbb{C}^2$$

Choosing a basis for $\mathbb{C}$ and $\mathbb{C}^2$, we can identify $M = \mathbb{C}^4$, with

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \qquad \beta = \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}$$

The complex moment map in this case is

$$\mu_c(\alpha, \beta) = \beta\alpha = \alpha_1\beta_1 + \alpha_2\beta_2$$

The action of the group $\mathrm{U}(1) \cong \mathrm{S}^1$ on M is given by

$$g \cdot (\alpha, \beta) = (\alpha g^{-1}, g\beta)$$

In terms of the components, this is given by

$$\lambda \cdot (\alpha_1, \alpha_2, \beta_1, \beta_2) = (\lambda^{-1}\alpha_1, \lambda^{-1}\alpha_2, \lambda\beta_1, \lambda\beta_2)$$

Define the map $\Phi : M \to \mathcal{N}$ by

$$\Phi(\alpha, \beta) = \alpha\beta = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 \\ \alpha_2\beta_1 & \alpha_2\beta_2 \end{pmatrix}$$

where $\mathcal{N}$ is the nilpotent variety of $\mathfrak{sl}(2, \mathbb{C})$. In particular, as we are interested in the nonzero orbit, we will want to consider the restriction of Φ to $M \setminus \{0\}$. In particular, Φ factors as

$$\begin{tikzcd} \mu_c^{-1}(0) \arrow[d, two heads, "\pi"'] \arrow[dr, "\Phi"] & \\ \mu_c^{-1}(0)/G^{\mathbb{C}} \arrow[r, "\overline{\Phi}"'] & \mathcal{N} \end{tikzcd}$$

From [1, Theorem 2.7], $\overline{\Phi}$ defines a diffeomorphism onto its image. In particular, this means that we can compute the hyperKähler metric on the adjoint orbit of A.

Let $\mathrm{Orb}(A)$ denote the adjoint orbit of A. Then we have that

$$\mathrm{T}_A \mathrm{Orb}(A) = \mathrm{span}_{\mathbb{C}} \{e, f\}$$

where

$$e = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

If $\omega_c = \omega_J + i\omega_K$ is the complex-symplectic form on M, and $\hat{\omega}_c$ is the complex-symplectic form on $\mu_c^{-1}(0)$, then we have that

$$\pi^* \hat{\omega}_c = i^* \omega_c$$

where $i : \mu_c^{-1}(0) \to M$ is the inclusion map. In particular, if we transport across the diffeomorphism $\overline{\Phi}$, and let $\tilde{\omega}_c$ be the complex-symplectic form on $\mathrm{Orb}(A)$, then we have that

$$\Phi^* \tilde{\omega}_c = i^* \omega_c$$

We would like to compute $\tilde{\omega}_c$ at A. One preimage is $(1, 0, 0, 1) \in M$. Computing $\mathrm{d}\Phi$ at $(1, 0, 0, 1)$, we find that

$$\mathrm{d}\Phi_{(1,0,0,1)}(h_1, h_2, k_1, k_2) = \begin{pmatrix} k_1 & h_1 + k_2 \\ 0 & h_2 \end{pmatrix}$$

Next, using standard arguments, we find that

$$\mathrm{T}_{(1,0,0,1)}\mu_c^{-1}(0) = \{k_1 + h_2 = 0\}$$

In particular, setting

$$\hat{e} = (-2, 0, 0, 0) \qquad \hat{f} = (0, 1, -1, 0)$$

we have that $\mathrm{d}\Phi(\hat{e}) = e$ and $\mathrm{d}\Phi(\hat{f}) = f$. Therefore, to compute $\tilde{\omega}_c$, all we need to do is compute $\omega_c(\hat{e}, \hat{f})$.

In terms of the coordinates, the complex structure J is given by

$$J(\alpha_1, \alpha_2, \beta_1, \beta_2) = (-\overline{\beta}_1, -\overline{\beta}_2, \overline{\alpha}_1, \overline{\alpha}_2)$$

and so

$$\omega_J(\hat{e}, \hat{f}) = g(J(-2, 0, 0, 0), (0, 1, -1, 0)) = g((0, 0, -2, 0), (0, 1, -1, 0)) = 2$$

where g is the standard Riemannian metric on $\mathbb{C}^4 \cong \mathbb{R}^8$. Next, we have that

$$\omega_K(\hat{e}, \hat{f}) = g(IJ(-2, 0, 0, 0), (0, 1, -1, 0)) = g((0, 0, -2i, 0), (0, 1, -1, 0)) = 0$$

Therefore, $\tilde{\omega}_c(e, f) = 2$. $\mathrm{SL}(2, \mathbb{C})$ acts on $\mu_c^{-1}(0)$ via

$$\gamma \cdot (\alpha, \beta) = (\gamma\alpha, \beta\gamma^{-1})$$

which means that

$$\Phi(\gamma \cdot (\alpha, \beta)) = \gamma\Phi(\alpha, \beta)\gamma^{-1}$$

i.e. if $\psi_\gamma(\alpha, \beta) = (\gamma\alpha, \beta\gamma^{-1})$, then

$$\begin{tikzcd} \mu_c^{-1}(0) \arrow[r, "\psi_\gamma"] \arrow[d, "\Phi"'] & \mu_c^{-1}(0) \arrow[d, "\Phi"] \\ \mathrm{Orb}(A) \arrow[r, "\mathrm{Ad}_\gamma"'] & \mathrm{Orb}(A) \end{tikzcd}$$

commutes. We would like to show that $\mathrm{Ad}_\gamma^* \tilde{\omega}_c = \tilde{\omega}_c$. Since Φ is a surjective submersion, it suffices to show that

$$\Phi^* \tilde{\omega}_c = \Phi^* \mathrm{Ad}_g^* \tilde{\omega}_c$$

i.e.

$$\psi_\gamma^* i^* \omega_c = i^* \omega_c$$

That is,

$$\omega_c(\mathrm{d}\psi_\gamma(u), \mathrm{d}\psi_\gamma(v)) = \omega_c(u, v)$$

for all $u, v \in \mathrm{T}_{(1,0,0,1)}\mu_c^{-1}(0)$. In particular, we need to check this on the subspace spanned by $\hat{e}, \hat{f}$. This means that all we need to compute is

$$\omega_c(\mathrm{d}\psi_\gamma(\hat{e}), \mathrm{d}\psi_\gamma(\hat{f}))$$

Since ψ_γ gives a linear map $M \to M$, $\mathrm{d}\psi_\gamma = \psi_\gamma$. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then it is easy to show that

$$\mathrm{d}\psi_\gamma(\hat{e}) = (-2a, -2c, 0, 0) \qquad \mathrm{d}\psi_\gamma(\hat{f}) = (b, d, -d, b)$$

This then shows that

$$\begin{aligned} \omega_J(\mathrm{d}\psi_\gamma(\hat{e}), \mathrm{d}\psi_\gamma(\hat{f})) &= g(J(-2a, -2c, 0, 0), (b, d, -d, b)) \\ &= g((0, 0, -2a, -2c), (b, d, -d, b)) \\ &= \mathrm{Re}(2ad - 2bc) \\ &= 2 \end{aligned}$$

and

$$\begin{aligned} \omega_K(\mathrm{d}\psi_\gamma(\hat{e}), \mathrm{d}\psi_\gamma(\hat{f})) &= g(IJ(-2a, -2c, 0, 0), (b, d, -d, b)) \\ &= g((0, 0, -2ai, -2ci), (b, d, -d, b)) \\ &= \mathrm{Re}((2ad - 2bc)i) \\ &= 0 \end{aligned}$$

Therefore, we have that $\tilde{\omega}_c$ is invariant under the adjoint action of $\mathrm{SL}(2, \mathbb{C})$.

References

[1] Piotr Z. Kobak and Andrew Swann. "Classical nilpotent orbits as hyperkähler quotients". In: *Int. J. Math.* 07.02 (Apr. 1996). Publisher: World Scientific Publishing Co., pp. 193–210. ISSN: 0129-167X. DOI: 10.1142/S0129167X96000116. URL: https://www.worldscientific.com/doi/10.1142/S0129167X96000116 (visited on 07/27/2023).