HyperKähler structure on the nilpotent adjoint orbits of $\mathrm{SL}(n, \mathbb{C})$

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August 28, 2023

In this document, we will outline the construction of hyperKähler metrics on the nilpotent adjoint orbits of $\mathrm{SL}(n, \mathbb{C})$. In chapter 1, we will follow [KS96], and define the hyperKähler metric using a finite dimensional quotient. In chapter 2, we will follow [Kro90b], and consider a space of solutions to Nahm's equations, which gives us a diffeomorphism between the space of solutions and a nilpotent orbit. Finally, in chapter 3 we will show how to modify the arguments from [Kro90a] to work in the nilpotent case.

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Chapter 1

Kobak-Swann construction

In this chapter, we follow [KS96] and construct a hyperKähler metric on the adjoint orbits of $\mathrm{SL}(n, \mathbb{C})$ using a finite dimensional hyperKähler quotient.

1.1 Flat HyperKähler space

First, choose a sequence $(V_0, \ldots, V_k)$ of Hermitian vector spaces, with $\dim_{\mathbb{C}}(V_i) = n_i$, $n_0 = 0$ and $n_k = n$. Define

$$M = \bigoplus_{j=0}^{k-1} \big(\mathrm{Hom}_{\mathbb{C}}(V_j, V_{j+1}) \oplus \mathrm{Hom}_{\mathbb{C}}(V_{j+1}, V_j)\big)$$

We will write a point in M in terms of linear maps (α_j, β_j), where $\alpha_j : V_j \to V_{j+1}$ and $\beta_j : V_{j+1} \to V_j$. The quaternionic structure on M is given by

$$I(\alpha_j, \beta_j) = (i\alpha_j, i\beta_j) \qquad J(\alpha_j, \beta_j) = (-\beta_j^*, \alpha_j^*)$$

where * denotes the Hermitian adjoint. That is, if we choose a basis for each V_j, the conjugate transpose. The metric on M is given by the (real) inner product,

$$\big\|(\alpha_j, \beta_j)\big\|^2 = \sum_{j=0}^{k-1} \mathrm{Re}\,\mathrm{tr}\big(\alpha_j^*\alpha_j + \beta_j\beta_j^*\big)$$

This metric, along with the complex structures $I, J, K = IJ$, makes M into a flat hyperKähler manifold, as I, J, K satisfy the quaternionic relations, and are isometries with respect to the above norm.

1.2 HyperKähler quotient

Now define the Lie group

$$G = \mathrm{U(n_1)} \times \cdots \times \mathrm{U(n_{k-1})}$$

This acts on M by

$$g \cdot (\alpha_j, \beta_j) = (g_{j+1}\alpha_j g_j^{-1}, g_j\beta_j g_{j+1}^{-1}) \tag{1.1}$$

where $g = (g_0, \ldots, g_k)$, $g_0 = g_k = 1$, $g_j \in \mathrm{U(n_j)}$ for $1 \le j \le k-1$. The corresponding hyperKähler moment map for this action is

$$\mu = i\mu_r + 2k\mu_c : M \to \mathfrak{g}^* \otimes \mathrm{Im}(\mathbb{H})$$

where $\mathfrak{g} \cong \mathfrak{g}^* \cong \mathfrak{u}(n_1) \oplus \cdots \oplus \mathfrak{u}(n_{k-1})$. The real moment map is

$$\mu_r(\alpha_j, \beta_j) = (\alpha_{j-1}\alpha_{j-1}^* - \beta_{j-1}^*\beta_{j-1} + \beta_j\beta_j^* - \alpha_j^*\alpha_j) \in \mathfrak{g} \otimes i\mathbb{R} \cong i\mathfrak{g}$$

and the complex moment map is

$$\mu_c(\alpha_j, \beta_j) = (\alpha_{j-1}\beta_{j-1} - \beta_j\alpha_j) \in \mathfrak{g} \otimes \mathbb{C}$$

Using this, we can consider various (hyper)Kähler quotients. In particular, we would like to show that the Kähler quotient $\mu_r^{-1}(0)/G$ is homeomorphic to the complex quotient $G^{\mathbb{C}}\mu_r^{-1}(0)/\!\!/G^{\mathbb{C}}$, where $G^{\mathbb{C}} = \mathrm{GL}(n_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(n_{k-1}, \mathbb{C})$ is the complexification of G, and $G^{\mathbb{C}}\mu_r^{-1}(0)/\!\!/G^{\mathbb{C}}$ is the set of closed $G^{\mathbb{C}}$ orbits in $G^{\mathbb{C}}\mu_r^{-1}(0)$.

Define $f : M \to \mathbb{R}$, $f(x) = \left\|\mu_r\right\|^2$. For a fixed point $x_0 \in M$, the path of steepest descent of x under f is a function $x : [0, \infty) \to M$, such that

$$x(0) = x_0$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\nabla f(x(t))$$

From the following results of Kirwan,

Theorem 1.2.1 ([KS96, Theorem 2.2], [Kir84, p. 101])**.** Let X be a compact Kähler manifold and let G be a compact Lie group acting on X holomorphically and isometrically, such that the complexification $G^{\mathbb{C}}$ also acts holomorphically on X. Let μ be a Kähler moment map for the action of G. Then $x \in X$ lies in $G^{\mathbb{C}}\mu^{-1}(0)$ if and only if

$$x \in X^{\min} := \left\{y \mid \text{ limit under the steepest descent of } f \text{ lies in } \mu^{-1}(0)\right\}$$

and the orbit $G^{\mathbb{C}}x$ is closed in $X^{\min}$. Moreover, the map $\mu^{-1}(0)/G \to G^{\mathbb{C}}\mu^{-1}(0)/\!\!/G^{\mathbb{C}}$, sending the G-orbit of $x \in \mu^{-1}(0)$ to the $G^{\mathbb{C}}$ orbit of x, is a homeomorphism.

Proposition 1.2.2 ([KS96, Condition 2.4], [Kir84, p. 9.1])**.** The above theorem can be applied to non-compact manifolds, provided that every path of steepest descent for f is contained in a compact subset of X.

it suffices to show that for all $x_0 \in M$, the path of steepest descent of x under f is bounded, and the limit points lie in $\mu_r^{-1}(0)$. Noting that $f(x) \leq ||x||^4$, we have thet the paths of steepest descent for f are bounded. To show that the limit points lie in $\mu_r^{-1}(0)$, we note that $\mu_r^* = \mu_r$, and so

$$\nabla f = 2(\mathrm{d}\mu_r)\mu_r$$

and hence the critical points of μ_r are when $\mu_r = 0$. With all of this in mind, the natural map

$$\mu_r^{-1}(0)/G \to G^{\mathbb{C}}\mu_r^{-1}(0)/\!\!/G^{\mathbb{C}}$$
$$[x] \mapsto [x]$$

is a homeomorphism. Now consider $\mu^{-1}(0) = \mu_r^{-1}(0) \cap \mu_c^{-1}(0)$, then the hyperKähler quotient $\mu^{-1}(0)/G$ is a submanifold of $\mu_r^{-1}(0)/G$, which using the above identification, corresponding to closed $G^{\mathbb{C}}$ orbits of $\mu_c^{-1}(0)$.

Therefore, in the next section, we will consider closed $G^{\mathbb{C}}$ orbits in the complex quotient $\mu_c^{-1}(0)/G^{\mathbb{C}}$.

1.3 Complex quotient

Define a map

$$X : M \to \mathrm{End}(\mathbb{C}^n)$$
$$X(\alpha_j, \beta_j) = \alpha_{k-1}\beta_{k-1}$$

If $\mu_c(\alpha_j, \beta_j) = 0$, then

$$X^2 = \alpha_{k-1}\beta_{k-1}\alpha_{k-1}\beta_{k-1} = \alpha_{k-1}\alpha_{k-2}\beta_{k-2}\beta_{k-1}$$

and in general,

$$X^k = \alpha_{k-1} \cdots \alpha_0 \beta_0 \cdots \beta_{k-1} = 0$$

Therefore, X is a nilpotent matrix. Moreover, let $G^{\mathbb{C}}$ act via eq. (1.1). Then this action preserves X, and also preserves $\mu_c^{-1}(0)$ setwise. Therefore, we have a well defined map

$$\begin{aligned} \Phi^c : \mu_c^{-1}(0)/G^{\mathbb{C}} &\to \mathcal{N} \\ \Phi^c([\alpha_j, \beta_j]) &= \alpha_{k-1}\beta_{k-1} \end{aligned}$$

where $\mathcal{N} \subseteq \mathfrak{sl}(n, \mathbb{C})$ is the variety of nilpotent matrices. The main theorem is

Theorem 1.3.1 ([KS96, Theorem 2.1])**.** The map Φ^c, restricted to the set of closed $G^{\mathbb{C}}$ orbits, is injective. Furthermore, its image consistes of a union of closures of nilpotent orbits in $\mathfrak{sl}(n, \mathbb{C})$. If there exists $X \in \mathfrak{sl}(n, \mathbb{C})$ such that $\mathrm{rank}(X^i) = n_{k-i}$ for $i = 0, \ldots, k$, then the image is precisely the closure of the nilpotent orbit containing X.

Proof. First of all, notice that we have a $\mathrm{GL}(n, \mathbb{C})$ action, using eq. (1.1), setting $g_0 = \cdots = g_{k-1} = 1$ and $g_k \in \mathrm{GL}(n, \mathbb{C})$. In this case, the action preserves the set $\mu_c^{-1}(0)$, and we have that

$$X(g \cdot (\alpha_j, \beta_j)) = gX(\alpha_j, \beta_j)g^{-1}$$

Therefore, the image of Φ^c is a union of nilpotent orbits.

1.3.1 Injectivity

Let X be a point in the image, (α_j, β_j) be a point in a closed $G^{\mathbb{C}}$ orbit, with $X(\alpha_j, \beta_j) = X$. We will show that this orbit is unique.

Jordan Normal Form. Using the $\mathrm{GL}(n, \mathbb{C})$ action, we may assume without loss of generality that X is in Jordan Normal Form, and we can write

$$V_k = V_k^1 \oplus \cdots \oplus V_k^r$$

where each V_k^j is a cyclic subspace for X. Set $V_{k-1}^i = \beta_{k-1}(V_k^i) \subseteq V_{k-1}$. Since $X = \alpha_{k-1}\beta_{k-1}$ preserves V_k^i, we must have that $\alpha(V_{k-1}^i) \subseteq V_k^i$, and by considering the action of a nilpotent Jordan block, we must also have that

$$\dim(V_{k-1}^i) \geq \dim(V_k^i) - 1$$

More generally, if we set

$$V_j^i = \beta_j(V_{j+1}^i)$$

and assume inductively that $\alpha_{j+1}(V_{j+1}^i) \subseteq V_{j+2}^i$, then

$$\alpha_j(V_j^i) = \alpha_j\beta_j(V_{j+1}^i) = \beta_{j+1}\alpha_{j+1}(V_{j+1}^i) \subseteq \beta_{j+1}(V_{j+2}^i) = V_{j+1}^i$$

Therefore, we can now assume without loss of generality that $r = 1$, i.e. X is a single nilpotent Jordan block.

β_j **surjective.** We will now show that we can assume that each β_j is surjective. For each j, ket V_i^0 be such that

$$V_j = V_j^0 \oplus \mathrm{Im}(\beta_j)$$

We will use the $G^{\mathbb{C}}$ action to modify (α_j, β_j) such that $\alpha_{k-1}|_{V_{k-1}^0} = 0$. Let $g_j \in \mathrm{GL}(n_j, \mathbb{C})$ be multiplication by λ on V_j^0, and id on $\mathrm{Im}(\beta_j)$. Then with respect to the splitting, $g = (g_1, \ldots, g_{k-1})$ acts, via eq. (1.1) as

$$\alpha_{k-1} = \begin{pmatrix} A_{11} & A_{12} \end{pmatrix} \mapsto \begin{pmatrix} \lambda^{-1}A_{11} & A_{12} \end{pmatrix}$$
$$\beta_{k-1} = \begin{pmatrix} 0 \\ B_{21} \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ B_{21} \end{pmatrix}$$
$$\alpha_j = \begin{pmatrix} A_{j11} & 0 \\ A_{j21} & A_{j22} \end{pmatrix} \mapsto \begin{pmatrix} A_{j11} & 0 \\ \lambda^{-1}A_{j21} & A_{j22} \end{pmatrix}$$
$$\beta_j = \begin{pmatrix} 0 & 0 \\ B_{j21} & B_{j22} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ \lambda^{-1}B_{j21} & B_{j22} \end{pmatrix}$$

Note that the top left entry of α_j is zero, since

$$\alpha_j(\beta_j(v)) = \beta_{j+1}\alpha_{j+1}(v) \in \operatorname{Im}(\beta_{j+1})$$

Since we assumed the $G^{\mathbb{C}}$ orbit is closed, letting $|\lambda| \to \infty$ gives us a new point with $A_{11} = 0$. By repeating the process above, using $g = (g_1, \dots, g_{k-2}, 1), \dots, g = (g_1, 1, \dots, 1)$, we can get

$$\alpha_j|_{V_j^0} = 0$$

as required.

α_j **injective.** A very similar argument to the above shows that we can assume that each α_j is injective. This then means that $\dim(V_i) = \dim(V_{i-1}) + 1$ for all i, by considering the rank of X^i.

Standard form for β_j Using the $G^{\mathbb{C}}$ action, we can consider a change of basis, so that each β_j is of the form

$$\beta_j = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

which follows by induction.

Each α_j **is upper triangular.** We will prove this by induction. The complex moment map equations state that

$$\beta_j\alpha_j = \alpha_{j-1}\beta_{j-1}$$

In particular, we have that

$$0 = \beta_1\alpha_1 = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{111} \\ \alpha_{121} \end{pmatrix} = \begin{pmatrix} \alpha_{121} \end{pmatrix}$$

Therefore, α_1 is upper triangular. Now suppose α_{j-1} is upper triangular, i.e. $\alpha_{j-1,a,b} = 0$ for $a > b$. Then

$$(\beta_{j-1}\alpha_{j-1})_{a,b} = \alpha_{j-1,a+1,b}$$

which is zero if $a \geq b$, and $(\alpha_j\beta_j)_{a,b} = \alpha_{j,a,b-1}$. Therefore, $\alpha_{j,a,b} = 0$ for $a > b$, and the nonzero entries of α_{j-1} are determined by the nonzero entries of α_j.

Uniqueness. Now $X = \alpha_{k-1}\beta_{k-1}$ has (a, b) entry $\alpha_{k-1,a,b-1}$. Therefore, α_{k-1}, and thus all the other α_i, are uniquely determined by X. Thus, the orbit is unique.

1.3.2 Closures of nilpotent orbits

We will first need the following lemmas.

Lemma 1.3.2. Let X be a nilpotent matrix. Then the numbers $\operatorname{rank}(X^i)$ determine the nilpotent orbit of X in $\mathfrak{sl}(n, \mathbb{C})$.

Proof. We may put X into Jordan normal form. In this case, the nilpotent orbits are determined by the sizes of the Jordan blocks. Using some combinatorics, we can recover the sizes of the Jordan blocks from the numbers $\operatorname{rank}(X^i)$. □

Lemma 1.3.3. Suppose X, Y are nilpotent $n \times n$ matrices. Then Y lies in the closure of the nilpotent orbit containing X if and only if $\operatorname{rank}(Y^i) \leq \operatorname{rank}(X^i)$ for all i.

Proof. Again using Jordan normal form, we can reduce to the case where X is a Jordan block. Noting that for all $\lambda \neq 0$,

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ \vdots & & & \ddots & \lambda \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

has Jordan normal form

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

Taking the limit as $\lambda \to 0$ gives a Jordan block of size one smaller. Repeating this process, for any of the nonzero entries, we can show that any $n \times n$ nilpotent matrix is in the closure of the orbit of X. The rank condition just gives us the sizes of the Jordan blocks, and shows that this process works for any such X, Y. □

Note that if $X = \alpha_{k-1}\beta_{k-1}$, then we have an upper bound

$$\operatorname{rank}(X^i) \leq n_{k-i} \tag{1.2}$$

and in fact (apart from the fact that X has to be nilpotent), this is the only constraint. Therefore, the image of Φ^c is a union of closures of nilpotent orbits, or equivalently, the union of orbits which satisfy eq. (1.2). Moreover, if there exists X such that $\operatorname{rank}(X^i) = n_{k-i}$, then the image must be the closure of its nilpotent orbit. □

Remark 1.3.4. From the proof, we can see that in fact, we could have assumed $n_0 < n_1 < \cdots < n_k$ without loss of generality.

Combining this result with the discussion in section 1.2, we have the following:

Theorem 1.3.5 ([KS96, Theorem 2.7])**.** The hyperKähler quotient of M by G is a union of closures of nilpotent orbits in $\mathfrak{sl}(n, \mathbb{C})$. If there is a nilpotent element $X \in \mathfrak{sl}(n, \mathbb{C})$ with $\operatorname{rank}(X^i) = n_{k-i}$ for all i, then the quotient is isomorphic to the closure of the nilpotent orbit containing X.

1.4 Complex-symplectic form

In this section, we compute the complex-symplectic form $\omega_c = \omega_J + i\omega_K$ given by the hyperKähler quotient.

Throughout, we will restrict to an open subset of M, corresponding to the "top" nilpotent orbit in the image. Moreover, we will need to consider points in a closed $G^{\mathbb{C}}$ orbit. For simplicity of notation, we won't write the restrictions.

Fix a point $p = (\alpha_j, \beta_j) \in \mu_c^{-1}(0)$. Then

$$(\mathrm{d}\mu_c)_p(\delta_j, \varepsilon_j) = \left(\delta_{j-1}\beta_{j-1} + \alpha_{j-1}\varepsilon_{j-1} - \beta_j\delta_j - \varepsilon_j\alpha_j\right)_{j=1}^{k-1}$$

where we write a generic tangent vector as $(\delta_j, \varepsilon_j)$, $\delta_j : V_j \to V_{j+1}$ and $\varepsilon_j : V_{j+1} \to V_j$. Therefore, the tangent space to $\mu_c^{-1}(0)$ at p is

$$\mathrm{T}_p\mu_c^{-1}(0) = \ker((\mathrm{d}\mu_c)_p) = \left\{(\delta_j, \varepsilon_j) \mid \delta_{j-1}\beta_{j-1} + \alpha_{j-1}\varepsilon_{j-1} - \beta_j\delta_j - \varepsilon_j\alpha_j\right\}$$

We have maps

$$\begin{tikzcd} \mu_c^{-1}(0) \arrow[d, two heads, "\pi"'] \arrow[dr, "\tilde{\Phi}^c"] & \\ \mu_c^{-1}(0)/G^{\mathbb{C}} \arrow[r, "\Phi^c"'] & N \end{tikzcd}$$

where N is the "top" nilpotent orbit, π is the quotient map. The complex-symplectic form $\tilde{\omega}_c$ on $\mu_c^{-1}(0)/G^c$ satisfies

$$\pi^*\tilde{\omega}_c = i^*\omega_c$$

where ω_c is the complex-symplectic form on M, and $i : \mu_c^{-1}(0) \hookrightarrow G^{\mathbb{C}}$ is the inclusion map. Therefore, it is determined by its pullback to $\mu_c^{-1}(0)$. Let $\hat{\omega}_c$ be the complex-symplectic form on N, defined by the "complex Kirillov-Kostant-Souriau" formula, i.e.

$$(\hat{\omega}_c)_X([X, Y], [X, Z]) = \mathrm{tr}(X[Y, Z])$$

Then

$$(\tilde{\Phi}^c)^*\hat{\omega}_c = \pi^*((\Phi^c)^*\hat{\omega}_c)$$

Therefore, to show that the complex-symplectic form on N given by the quotient is the complex KKS form, it suffices to show that

$$(\tilde{\Phi}^c)^*\hat{\omega}_c = i^*\omega_c$$

The map $\tilde{\Phi}^c$ is defined by

$$\tilde{\Phi}^c(\alpha_j, \beta_j) = \alpha_{k-1}\beta_{k-1}$$

Therefore, its derivative is given by

$$\mathrm{d}\tilde{\Phi}^c(\delta, \varepsilon) = \delta_{k-1}\beta_{k-1} + \alpha_{k-1}\varepsilon_{k-1}$$

Let $X = \alpha_{k-1}\beta_{k-1}$. We would like to relate the above to a Lie bracket with X. Fix $Y \in \mathfrak{sl}(n, \mathbb{C})$, and define

$$\delta_j^Y = \begin{cases} 0 & j < k-1 \\ -Y\alpha_{k-1} & j = k-1 \end{cases}$$
$$\varepsilon_j^Y = \begin{cases} 0 & j < k-1 \\ \beta_{k-1}Y & j = k-1 \end{cases}$$

Then $\mathrm{d}\tilde{\Phi}^c(\delta_j^Y, \varepsilon_j^Y) = [X, Y]$. Moreover, by substituting into the definition of $\mathrm{T}_p\mu_c^{-1}(0)$, we find that $(\delta_j^Y, \varepsilon_j^Y) \in \mathrm{T}_p\mu_c^{-1}(0)$. Therefore,

$$(\tilde{\Phi}^c)^*((\delta^Y, \varepsilon^Y), (\delta^Z, \varepsilon^Z)) = (\hat{\omega}_c)_X([X, Y], [X, Z]) = \mathrm{tr}(X[Y, Z])$$

Noting that

$$J(\delta^Y, \varepsilon^Y) = \begin{cases} (0, 0) & j < k-1 \\ (-Y^*\beta_{k-1}^*, -\alpha_{k-1}^*Y^*) & j = k-1 \end{cases}$$

We then get that

$$
\begin{aligned}
\omega_J((\delta^Y, \varepsilon^Y), (\delta^Z, \varepsilon^Z)) &= g(J(\delta^Y, \varepsilon^Y), (\delta^Z, \varepsilon^Z)) \\
&= \mathrm{Re}\left(\mathrm{tr}\left(-(-Y^*\beta_{k-1}^*)^* Z\alpha_{k-1}\right) + \mathrm{tr}\left((-\alpha_{k-1}^* Y^*)^*\beta_{k-1} Z\right)\right) \\
&= \mathrm{Re}\left(\mathrm{tr}(\alpha_{k-1}\beta_{k-1} YZ) - \mathrm{tr}(\alpha_{k-1}\beta_{k-1} ZY)\right) \\
&= \mathrm{Re}\,\mathrm{tr}(X[Y, Z])
\end{aligned}
$$

Next, for K, we have that

$$
K(\delta^Y, \varepsilon^Y) = \begin{cases} (0, 0) & j < k-1 \\ (-iY^*\beta_{k-1}^*, -i\alpha_{k-1}^* Y^*) & j = k-1 \end{cases}
$$

and

$$
\begin{aligned}
\omega_K((\delta^Y, \varepsilon^Y), (\delta^Z, \varepsilon^Z)) &= g(K(\delta^Y, \varepsilon^Y), (\delta^Z, \varepsilon^Z)) \\
&= \mathrm{Re}\left(\mathrm{tr}\left(-(-iY^*\beta_{k-1}^*)^* Z\alpha_{k-1}\right) + \mathrm{tr}\left((-i\alpha_{k-1}^* Y^*)^*\beta_{k-1} Z\right)\right) \\
&= -\,\mathrm{Im}\left(\mathrm{tr}(\alpha_{k-1}\beta_{k-1} ZY) - \mathrm{tr}(\alpha_{k-1}\beta_{k-1} YZ)\right) \\
&= \mathrm{Im}\,\mathrm{tr}(X[Y, Z])
\end{aligned}
$$

Combining these two, we then have that

$$
\omega_c((\delta^Y, \varepsilon^Y), (\delta^Z, \varepsilon^Z)) = \mathrm{tr}(X[Y, Z])
$$

as required.

Chapter 2

Kronheimer's construction

In this chapter, we will follow [Kro90b] and consider a space of solutions to Nahm's equations. In particular, we will fill in the details in some of the proofs.

2.1 Introduction

The inner product on $\mathfrak{su}(n)$ is given by $-\kappa$, where κ is the Killing form. That is,

$$\langle A, B \rangle = -\operatorname{tr}(AB)$$

Define

$$\begin{aligned} \varphi : \mathfrak{su}(n) \times \mathfrak{su}(n) \times \mathfrak{su}(n) &\to \mathbb{R} \\ \varphi(A_1, A_2, A_3) &= \sum_{j=1}^{3} \langle A_j, A_j \rangle + \langle A_1, [A_2, A_3] \rangle \end{aligned}$$

We are interested in studying the gradient flow of φ. That is, $A_1, A_2, A_3 : I \to \mathfrak{su}(n)$ such that

$$(\dot{A}_1, \dot{A}_2, \dot{A}_3) = -\nabla\varphi(A_1, A_2, A_3) \tag{2.1}$$

First of all, notice that

$$\varphi(A_1 + H_1, A_2, A_3) = \varphi(A_1, A_2, A_3) + 2\langle H_1, A_1 \rangle + \langle H_1, [A_2, A_3] \rangle$$

and that $\langle A_1, [A_2, A_3] \rangle = \langle A_2, [A_3, A_1] \rangle = \langle A_3, [A_1, A_2] \rangle$. Therefore, eq. (2.1) becomes

$$\begin{aligned} \dot{A}_1 &= -2A_1 - [A_2, A_3] \\ \dot{A}_2 &= -2A_2 - [A_3, A_1] \\ \dot{A}_3 &= -2A_3 - [A_1, A_2] \end{aligned} \tag{2.2}$$

The critical points of eq. (2.2) are triples (A_1, A_2, A_3) satisfying

$$[A_1, A_2] = -2A_3 \quad [A_2, A_3] = -A_1 \quad [A_3, A_1] = -2A_2$$

Recall that the Lie algebra $\mathfrak{su}(2)$ has basis

$$e_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

satifying the above relations. Therefore, critical points of eq. (2.2) correspond to Lie algebra homomorphisms $\rho : \mathfrak{su}(2) \to \mathfrak{su}(n)$. From this, we see that at all critical points of eq. (2.2), φ is nonnegative, and it is zero only at $(0, 0, 0)$.

Next, we will identify $\mathfrak{su}(n) \times \mathfrak{su}(n) \times \mathfrak{su}(n) \cong \mathrm{L}(\mathfrak{su}(2), \mathfrak{su}(\mathrm{n}))$, the space of linear maps $\mathfrak{su}(2) \to \mathfrak{su}(n)$, sending (A_1, A_2, A_3) to the linear map A given by $e_i \mapsto A_i$.

The adjoint action of $\mathrm{SU}(n)$ on $\mathfrak{su}(n)$ is given by

$$\mathrm{Ad}_g(A) = gAg^{-1}$$

and this induces an action on $\mathrm{L}(\mathfrak{su}(2), \mathfrak{su}(\mathrm{n}))$ by

$$g \cdot A : e_i \mapsto gA_ig^{-1}$$

For any Lie algebra homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{su}(n)$, define

$$C(\rho) = \{g \cdot \rho \mid g \in \mathrm{SU}(n)\}$$

for the critical manifold of all homomorphisms which are conjugate to ρ via the adjoint action. For Lie algebra homomorphisms $\rho_-, \rho_+ : \mathfrak{su}(2) \to \mathfrak{su}(n)$, define $M(\rho_-, \rho_+)$ for the space of solutions $A(t)$ to eq. (2.2), with boundary conditions

$$\begin{aligned} &\lim_{t\to-\infty} A(t) \in C(\rho_-) \\ &\lim_{t\to\infty} A(t) = \rho_+ \end{aligned} \tag{2.3}$$

Note that we are considering parametrised trajectories, therefore there is a natural $\mathbb{R}$-action sending $A(t)$ to $A(t+c)$.

For a Lie algebra homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{su}(n)$, we can extend it to a Lie algebra homomorphism $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{sl}(n, \mathbb{C})$, and define

$$H = \rho\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad X = \rho\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \qquad Y = \rho\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

We will then define $\mathcal{N}(\rho)$ for the nilpotent orbit of Y in $\mathfrak{sl}(n, \mathbb{C})$, and the affine subspace

$$S(\rho) = Y + Z(X)$$

where $Z(X) = \{A \in \mathfrak{sl}(n, \mathbb{C}) \mid [A, X] = 0\}$. Using this, we have

Theorem 2.1.1 ([Kro90b, Theorem 1])**.** For any pair of homomorphisms ρ_-, ρ_+, there is a diffeomorphism

$$M(\rho_-, \rho_+) \cong \mathcal{N}(\rho_-) \cap S(\rho_+)$$

If $\rho_+ = 0$, then $S(\rho_+) = \mathfrak{sl}(n, \mathbb{C})$, and in this case, we have a diffeomorphism

$$M(\rho_-, 0) \cong \mathcal{N}(\rho_-)$$

Moreover, every nilpotent orbit is $\mathcal{N}(\rho)$ for some homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{su}(n)$, which means that we have a description of all nilpotent orbits in $\mathfrak{sl}(n, \mathbb{C})$.

We will defer the proof of theorem 2.1.1 to section 2.5, but below, we will provide a sketch proof, which will also function as an outline for the note.

First, we will extend eq. (2.2) to a system of equations eq. (2.4). In this case, we have an action of a gauge group. Writing

$$\alpha = \frac{1}{2}(A_0 + iA_1) \qquad \beta = \frac{1}{2}(A_2 + iA_3)$$

We can split Nahm's equations into a real equation eq. (2.6) and a complex equation eq. (2.7). In lemma 2.2.4 and lemma 2.2.5, we show that using the group action, we can assume that the solution to the complex equation takes a given form. In particular, in proposition 2.2.6, we prove that the solutions are parametrised by an element of $S(\rho_+) \cap \mathcal{N}(\rho_-)$.

Thus, we have a bijection between the space of (equivalence classes of) solutions of the complex equation and $S(\rho_+) \cap \mathcal{N}(\rho_-)$. Since each solution to eq. (2.2) gives us a solution to the real and complex equations, this gives us a map $\mathcal{M}(\rho_-, \rho_+) \to S(\rho_+) \cap \mathcal{N}(\rho_-)$.

Working now with the real equation, using proposition 2.4.3, we can show that the map is injective. On the other hand, in proposition 2.4.7, we show that within each equivalence class of complex trajectories, there exists a trajectory which satisfies the real equation. Decomposing into hermitian and anti-hermitian parts, we can use this to recover a solution to the extended equations eq. (2.4). Finally, we use the group action to show that we can take $A_0 = 0$, and recover a solution to the original equations eq. (2.2). Thus, the map is also surjective.

2.2 Complex trajectories

2.2.1 Gauge group

First of all, we will extend eq. (2.2) by considering $A_0, \ldots, A_3 : \mathbb{R} \to \mathfrak{su}(n)$, satisfying the equations

$$\begin{aligned} \dot{A}_1 &= -2A_1 - [A_0, A_1] - [A_2, A_3] \\ \dot{A}_2 &= -2A_2 - [A_0, A_2] + [A_1, A_3] \\ \dot{A}_3 &= -2A_3 - [A_0, A_3] - [A_1, A_2] \end{aligned} \tag{2.4}$$

Define the group

$$\mathcal{G} = \{g : \mathbb{R} \to \mathrm{SU}(n)\}$$

with pointwise operations. Then $\mathcal{G}$ acts $A = (A_0, \ldots, A_3)$ by

$$(g \cdot A)(t) = \left(g(t)A_0(t)g(t)^{-1} - \frac{\mathrm{d}g}{\mathrm{d}t} \cdot g(t)^{-1}, g(t)A_1(t)g(t)^{-1}, g(t)A_2(t)g(t)^{-1}, g(t)A_3(t)g(t)^{-1} \right) \tag{2.5}$$

For brevity, when clear, we will write this as

$$g \cdot A = (gA_0g^{-1} - \dot{g}g^{-1}, gA_1g^{-1}, gA_2g^{-2}, gA_3g^{-1})$$

Note that $\dot{g}(t) \in \mathrm{T}_{g(t)}\,\mathrm{SU}(n) = g(t)\mathfrak{su}(n)$, and so $\dot{g}(t)g(t)^{-1} \in g(t)\mathfrak{su}(n)g(t)^{-1} = \mathfrak{su}(n)$. First, we will show that eq. (2.4) is invariant under the action eq. (2.5). To see this, the transformed right hand side (for the first equation) is

$$\begin{aligned} -2gA_1g^{-1} - [gA_0g^{-1} - \dot{g}g^{-1}, gA_1g^{-1}] - [gA_2g^{-1}, gA_3g^{-1}] &= g(-2A_1 - [A_0, A_1] - [A_2, A_3])g^{-1} + [\dot{g}g^{-1}, gA_1g^{-1}] \\ &= g\dot{A}_1g^{-1} + \dot{g}A_1g^{-1} - gA_1g^{-1}\dot{g}g^{-1} \end{aligned}$$

which is precisely $\frac{\mathrm{d}}{\mathrm{d}t}\left(gA_1g^{-1}\right)$. Moreover, in eq. (2.5), we can always choose g to make $A_0 = 0$, by considering the linear ODE

$$\dot{g} = gA_0$$

Therefore, we don't change the problem much by considering eq. (2.4).

2.2.2 Complex equations

Next, we will break the symmetry in the equations, by choosing A_1 to be 'special'. More precisely, we will consider $\alpha, \beta : \mathbb{R} \to \mathfrak{sl}(n, \mathbb{C})$, defined by

$$\alpha = \frac{1}{2}(A_0 + iA_1) \qquad \beta = \frac{1}{2}(A_2 + iA_3)$$

In this case, we have the followiing expressions:

$$\begin{aligned} \alpha^* &= \frac{1}{2}(-A_0 + iA_1) \\ \alpha + \alpha^* &= iA_1 \\ [\alpha, \alpha^*] &= \frac{1}{2}i[A_0, A_1] \\ [\beta, \beta^*] &= \frac{1}{2}i[A_2, A_3] \end{aligned}$$

and so the first equation in eq. (2.4) can be written as the *real equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha + \alpha^*) + 2(\alpha + \alpha^*) + 2([\alpha, \alpha^*] + [\beta, \beta^*]) = 0 \tag{2.6}$$

and using

$$[\alpha, \beta] = \frac{1}{4}([A_0, A_2] + [A_3, A_1]) + \frac{1}{4}i([A_0, A_3] + [A_1, A_2])$$

the second equation in eq. (2.4) becomes the *complex equation*

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} + 2\beta + 2[\alpha, \beta] = 0 \tag{2.7}$$

As above, the real equation is invariant under the action of $\mathcal{G}$. But in this case, the complex equation is invariant under the action of the complex gauge group

$$\mathcal{G}^c = \{\mathbb{R} \to \mathrm{SL}(n, \mathbb{C})\}$$

via eq. (2.5). In particular, the action is given by

$$g \cdot (\alpha, \beta) = \left(g\alpha g^{-1} - \frac{1}{2}\dot{g}g^{-1}, g\beta g^{-1}\right)$$

and so substituting into eq. (2.7), we get

$$\dot{g}\beta g^{-1} + g\dot{\beta}g^{-1} - g\beta g^{-1}\dot{g}g^{-1} + 2g\beta g^{-1} + 2g[\alpha, \beta]g^{-1} - [\dot{g}g^{-1}, g\beta g^{-1}] = g\left(\dot{\beta} + 2\beta + 2[\alpha, \beta]\right)g^{-1}$$

2.2.3 Complex trajectories

Let $\rho_+, \rho_- : \mathfrak{su}(2) \to \mathfrak{su}(n)$ be Lie algebra homomorphisms. Extend them to Lie algebra homomorphisms $\mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{sl}(n, \mathbb{C})$, and define

$$H_\pm = \rho_\pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X_\pm = \rho_\pm \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y_\pm = \rho_\pm \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Definition 2.2.1 ([Kro90b, Definition 5])

A *complex trajectory* associated to ρ_+, ρ_- is a pair of smooth functions $\alpha, \beta : \mathbb{R} \to \mathfrak{sl}(n, \mathbb{C})$, which satisfy the complex equation eq. (2.7), and the boundary conditions

$$\begin{aligned} \lim_{t\to\infty} 2\alpha(t) &= H_+ \\ \lim_{t\to-\infty} 2\alpha(t) &= gH_-g^{-1} \\ \lim_{t\to\infty} \beta(t) &= Y_+ \\ \lim_{t\to-\infty} \beta(t) &= gY_-g^{-1} \end{aligned} \tag{2.8}$$

for some $g \in \mathrm{SU}(n)$. Moreover, we require that the convergence in eq. (2.8) is exponential, that is,

$$\left\|2\alpha(t) - H_+\right\| < Ke^{-\eta t}$$

for some $\eta, K > 0$ and so on. Note the choice of norm here does not matter, as all norms on $\mathfrak{sl}(n, \mathbb{C})$ are equivalent.

Now define the subgroup $\mathcal{G}_0^c$ of $\mathcal{G}^c$ by

$$\mathcal{G}_0^c = \left\{g \in \mathcal{G}^c \mid g \text{ bounded}, \lim_{t\to\infty} g(t) = 1\right\}$$

Using the operator norm, which satisfies $\left\|gh\right\| \leq \left\|g\right\|\left\|h\right\|$, it is clear that $\mathcal{G}_0^c$ is closed under multiplication. Therefore, all we need to show is that it is closed under inverses. One proof is as follows:

By Cayley-Hamilton, we have coefficients $c_1(t), \ldots, c_{n-1}(t)$ such that

$$g(t)^n + c_{n-1}g(t)^{n-1} + \cdots + c_1(t)g(t) + 1 = 0$$

Multiplying by $g(t)^{-1}$, we get

$$g(t)^{-1} = -\left(g(t)^{n-1} + c_{n-1}g(t)^{n-2} + \cdots + c_1(t)\right)$$

The $c_i(t)$ are the elementary symmetric functions in the eigenvalues of $g(t)$, and the eigenvalues of $g(t)$ are bounded, since any eigenvalue λ of $g(t)$ necessarily satisfies $|\lambda| \leq \left\|g(t)\right\|$. Therefore, the coefficients on the right hand side are bounded. Hence by the triangle inequality, we have a bound on $\left\|g(t)^{-1}\right\|$.

Definition 2.2.2 ([Kro90b, Definition 6])
We say that two complex trajectories (α, β) and (α', β') are *equivalent* if there exists $g \in \mathcal{G}_0^c$ such that

$$(\alpha', \beta') = g \cdot (\alpha, \beta)$$

i.e. they are in the same $\mathcal{G}_0^c$ orbit.

2.2.4 Classification of complex trajectories

First of all, note that under the $\mathcal{G}^c$ action, we can always make $\alpha = 0$. In particular, we need

$$\dot{g} = 2g\alpha$$

Assuming this, the complex equation eq. (2.7) becomes

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} + 2\beta = 0$$

which has solution

$$\beta(t) = e^{-2t}\beta_0$$

for some β_0. Therefore, the only local invariant under the $\mathcal{G}^c$ (and $\mathcal{G}_0^c$) action is the conjugacy class of β_0. Reversing the $\mathcal{G}^c$ action, we find that a generic local solution is

$$\alpha = \frac{1}{2}g^{-1}\dot{g}$$
$$\beta = e^{-2t}g^{-1}\beta_0 g$$

As a consequence of this, we have

Lemma 2.2.3 ([Kro90b, Lemma 9]). If (α, β) and (α', β') are complex trajectories which are equal outside of some compact set $K \subseteq \mathbb{R}$, then (α, β) and (α', β') are equivalent.

Proof. Without loss of generality, we may assume $K = [-M, M]$ for some $M > 0$. Using the $\mathcal{G}^c$ action, we may assume that

$$\alpha(t) = 0 \qquad \beta(t) = e^{-2t}\beta_0$$

Now let $g \in \mathcal{G}^c$ be such that

$$g \cdot (\alpha', \beta') = (0, e^{-2t}\beta_0')$$

In particular, as

$$\dot{g} = 2g\alpha'$$

$\dot{g} = 0$ for $t \notin [-M, M]$, and so g is constant outside of $[-M, M]$. Say $g = g_-$ for $t < -M$ and $g = g_+$ for $t > M$. By left multiplication by g_+^{-1}, we can assume $g_+ = 1$. This means that for $t > M$, $e^{-2t}\beta'(t) = e^{-2t}\beta_0'$. But in this case $\beta = \beta'$, so $\beta_0 = \beta_0'$. Hence $g \cdot (\alpha', \beta') = (\alpha, \beta)$, and so they are equivalent. □

Lemma 2.2.4 ([Kro90b, Lemma 10]). Let (α, β) be a solution of the complex equation eq. (2.7), satisfying the boundary equations eq. (2.8) at $t \to -\infty$. That is,

$$\lim_{t\to-\infty} 2\alpha(t) = gH_- g^{-1} \qquad \lim_{t\to-\infty} \beta(t) = gY_- y^{-1}$$

with exponential convergence. Then there exists a gauge transformation $g_- : \mathbb{R} \to \mathrm{SL}(n, \mathbb{C})$ such that $(\alpha', \beta') = g_- \cdot (\alpha, \beta)$ is the constant solution

$$2\alpha' = H_- \qquad \beta' = Y_-$$

and $g_-(t)$ converges as $t \to -\infty$.

Proof. By conjugation, without loss of generality $g = 1$. Considering the ODE

$$\dot{g}_0 = 2g_0\alpha - H_- g_0$$

We can find g_0 such that

$$H_- = 2g_0\alpha g_0^{-1} - \dot{g}_0 g_0^{-1}$$

with the boundary condition $g_0(t) \to 1$, as $t \to -\infty$, since $2\alpha(t) \to H_-$ exponentially.
Using this, we get a transformed solution $(\alpha'', \beta'') = g_0 \cdot (\alpha, \beta)$, with $2\alpha'' = H_-$. In this case, the complex equation becomes

$$\frac{\mathrm{d}\beta''}{\mathrm{d}t} + 2\beta'' + [H_-, \beta''] = 0$$

Trying the ansatz

$$\begin{aligned} \beta''(t) &= e^{-2t}\operatorname{Ad}_{f(t)}(\omega) = e^{-2t} f\omega f^{-1} \\ f(t) &= \exp(Xt) \end{aligned}$$

We have that

$$\dot{f} = Xf$$

and so

$$\begin{aligned} \dot{\beta}'' &= -2e^{-2t} f\omega f^{-1} + e^{-2t}\dot{f}\omega f^{-1} - e^{-2t} f\omega f^{-1}\dot{f} f^{-1} \\ &= -2\beta'' + X\beta'' - \beta'' X \end{aligned}$$

Therefore, the complex equation becomes

$$\dot{\beta}'' + 2\beta'' + H_-\beta'' - \beta'' H_- = -2\beta'' + X\beta'' - \beta'' X + 2\beta'' + H_-\beta'' - \beta'' H_- = [X + H_-, \beta'']$$

Hence setting $X = -H_-$, we get a solution. By dimensionality arguments, this is the general solution.
Using the composition

$$\mathfrak{sl}(2, \mathbb{C}) \xrightarrow{\rho_-} \mathfrak{sl}(n, \mathbb{C}) \xrightarrow{\mathrm{ad}} \mathfrak{gl}(\mathfrak{sl}(n, \mathbb{C}))$$

We get a representation of $\mathfrak{sl}(2, \mathbb{C})$ on $\mathfrak{sl}(n, \mathbb{C})$. Therefore, we have a decomposition[1]

$$\mathfrak{sl}(n, \mathbb{C}) = \bigoplus_{k\in\mathbb{Z}} V_k$$

where V_λ is the λ-eigenspace of $\operatorname{ad}(H_-)$. Since we want $\beta'' \to Y_-$ as $t \to -\infty$, we will try the ansatz $\omega = Y_- + \delta$. By linearity, we can first compute the case of $\omega = Y_-$.
First of all, notice that we also have that $\dot{f} = fX = -fH_-$, and so in this case

[1] See section 2.A for more details on this, as well as some more details on the representation theory of $\mathfrak{sl}(2, \mathbb{C})$.

$$\begin{aligned}\dot{\beta}'' &= -2e^{-2t} f Y_- f^{-1} - e^{-2t} f H_- Y_- f^{-1} + f Y_- f^{-1} f H_- f^{-1} \\ &= -2\beta'' - f[H_-, Y_-]f^{-1} \\ &= 0\end{aligned}$$

as $[H_-, Y_-] = \rho_-([H, Y]) = \rho_-(-2Y) = -2Y_-$. Therefore, as $\beta''(0) = Y_-$ in this case, it is constant. Now by linearity, say $\delta = \sum_k \delta_k$, where $\delta_k \in V_k$. Then for $\omega = \delta_k$,

$$\dot{\beta}'' = -2\beta'' - f[H_-, \delta_k]f^{-1} = -(2 + k)\beta''$$

This gives the solution

$$\beta''(t) = e^{-(2+k)t}\beta''(0) = e^{-(2+k)t}\delta_k$$

Since we require $\beta''(t) \to 0$ as $t \to -\infty$ in this case, we need $-(2 + k) > 0$, i.e. $k < -2$. Hence the general solution in this case is

$$\beta''(t) = Y_- + e^{-2t} \exp(-H_- t)\delta \exp(H_- t)$$

where $\delta \in \bigoplus_{k<-2} V_k$. Now notice that g_0 from earlier was not uniquely determined. We can still act on the solution by a gauge transformation g_1, which preserves $2\alpha'' = H_-$, and approaches 1 at $t \to -\infty$. That is, we have the equation

$$H_- = g_1 H_- g_1^{-1} - \dot{g}_1 g_1^{-1}$$

which we can rearrange to

$$\dot{g}_1 = g_1 H_- - H_- g_1$$

Trying the ansatz

$$\begin{aligned} g_1(t) &= f(t)\sigma f(t)^{-1} \\ f(t) &= \exp(-H_- t)\end{aligned}$$

for $\sigma \in \mathrm{SL}(n, \mathbb{C})$, we find that this gives the general solution for the equation. For the boundary condition, suppose further that $\sigma = \exp(\gamma)$, for some $\gamma \in \mathfrak{sl}(n, \mathbb{C})$. Define

$$h_t(s) = f \exp(s\gamma) f^{-1}$$

and note that $g_1(t) = h_t(1)$. Then

$$\begin{aligned}\frac{\mathrm{d}h_t}{\mathrm{d}s} &= f \exp(s\gamma)\gamma f^{-1} \\ &= h_t(s) \cdot f\gamma f^{-1}\end{aligned}$$

Set $\varphi(t) = f\gamma f^{-1}$, then we have that

$$\dot{\varphi} = -f[H, \gamma]f^{-1}$$

This equation is linear in γ, and so for simplicity, we will assume $\gamma \in V_k$. In this case, $\dot{\varphi} = -k\varphi$, and so $\varphi(t) = e^{-kt}\gamma$. Substituting this in, we get that

$$\frac{\mathrm{d}h_t}{\mathrm{d}s} = e^{-kt} h_t \cdot \gamma$$

and so, integrating this equation, we find that

$$h_t(s) = \exp\left(se^{-kt}\gamma\right) \implies g_1(t) = \exp\left(e^{-kt}\gamma\right)$$

Thus, for $g_1 \to 1$ as $t \to -\infty$, we must have $k < 0$. Therefore, the general solution is

$$g_1(t) = \exp(-H_- t)\exp(\gamma)\exp(H_- t)$$

where $\gamma \in \bigoplus_{k<0} V_k$. Therefore, if we consider $(\alpha', \beta') = g_1 \cdot (\alpha'', \beta'')$, we would get that $2\alpha' = H_-$, and

$$\beta'(t) = Y_- + e^{-2t}\exp(-H_- t)(\exp(\gamma)(Y_- + \delta)\exp(-\gamma) - Y_-)\exp(H_- t)$$

Therefore, all that remains to show is that for all $\delta \in \bigoplus_{k<-2} V_k$, there exists $\gamma \in \bigoplus_{k<0} V_k$ such that

$$\exp(\gamma)(Y_- + \delta)\exp(-\gamma) - Y_- = 0$$

We will use the implicit function theorem for this. Expand the left hand side near $\gamma = \delta = 0$, the terms linear in γ, δ are

$$f(\gamma, \delta) = \delta + \gamma Y_- - Y_- \gamma = \delta - [Y_-, \gamma]$$

From the representation theory of $\mathfrak{sl}(2, \mathbb{C})$, we have a linear map

$$[Y_-, \cdot] : \bigoplus_{k<0} V_k \to \bigoplus_{k<-2} V_k$$

and so we have a map

$$f : \bigoplus_{k<0} V_k \oplus \bigoplus_{k<-2} V_k \to \bigoplus_{k<-2} V_k$$

The map $\gamma \mapsto f(\gamma, 0)$ is surjective, for example by decomposing $\mathfrak{sl}(n, \mathbb{C})$ as a direct sum of $\mathfrak{sl}(2, \mathbb{C})$ representations. Therefore if we have a decomposition

$$\bigoplus_{k<0} V_k = K \oplus W$$

where $K = \ker(f(\cdot, 0))$, then the map $\hat{f} : W \to \bigoplus_{k<-2} V_k$, given by $\hat{f}(\gamma) = f(\gamma, 0)$, is an isomorphism. We can then apply the implicit function theorem to

$$F : \left(\bigoplus_{k<-2} V_k \oplus K\right) \oplus W \to \bigoplus_{k<-2} V_k$$
$$F((\delta, k), \gamma') = \exp\big((\gamma', k)\big)(Y_- + \delta)\exp\big(-(\gamma', k)\big) - Y_-$$

which then gives us a neighbourhood U of 0 in $\bigoplus_{k<-2} V_k$, and a neighbourhood V of 0 in W, and a map $g : U \times V \to W$ such that

$$F(x, g(x)) = 0$$

for all $x \in U \times V$. Therefore, for $\delta \in U$, setting $\gamma = g(\delta, 0)$ gives the required result. Finally, we will use homogeneity to extend the result to all of $\bigoplus_{k<-2} V_k$. First of all, we note that the condition is invariant under the substitution

$$\gamma = f\hat{\gamma}f^{-1}$$
$$\delta = e^{-2t} f\hat{\delta}f^{-1}$$

where $f(t) = \exp(-H_- t)$, since we have that $Y_- = e^{-2t} f Y_- f^{-1}$, and that $\exp\big(f\hat{\gamma}f^{-1}\big) = f\exp(\hat{\gamma})f^{-1}$. Now suppose $[H, v] = mv$, and let $\varphi = fvf^{-1}$. Then

$$\begin{aligned}\dot{\varphi} &= f\dot{v}f^{-1} - fvf^{-1}\dot{f}f^{-1} \\ &= -fHvf^{-1} + fvHf^{-1} \\ &= -mfvf^{-1} \\ &= -m\varphi\end{aligned}$$

Hence $\varphi(t) = e^{-mt}v$. Therefore in the limit $t \to -\infty$ (as $m < 0$), we have that $\gamma \to 0$, and so we can apply the result for small δ. $\square$

There is a very similar result for the limit at $t \to \infty$.

Lemma 2.2.5 ([Kro90b, Lemma 11]). Let (α, β) be a solution of the complex equation eq. (2.7) satisfying the boundary equations eq. (2.3) at $t \to \infty$. That is,

$$\lim_{t\to\infty} 2\alpha(t) = H_+ \qquad \lim_{t\to\infty} \beta(t) = Y_+$$

with exponential convergence. Then there exists a unique gauge transformation $g_+ : \mathbb{R} \to SL(n, \mathbb{C})$, with $g_+(t) \to 1$ as $t \to \infty$, such that the transformed solution $(\alpha', \beta') = g_+ \cdot (\alpha, \beta)$ satisfies

$$2\alpha' = H_+ \qquad \beta'(0) \in S(\rho_+)$$

Proof. The proof is very similar to the previous lemma. We find a gauge transformation g_0, approaching 1 as $t \to \infty$, such that $(\alpha'', \beta'') = g \cdot (\alpha, \beta)$ satisfies

$$\begin{aligned} 2\alpha'' &= H_+ \\ \beta''(t) &= Y_+ + e^{-2t} \exp(-H_+t)\varepsilon \exp(H_+t) \end{aligned}$$

with

$$\varepsilon \in \bigoplus_{k>-2} V_k$$

where in this case, V_k is the k-eigenspace of $\mathrm{ad}(H_+)$. As above, we have a further choice of gauge transformation g_1 of the form

$$g_1(t) = \exp(-H_+t)\exp(\gamma)\exp(H_+t)$$

where $\gamma \in \bigoplus_{i>0} V_k$. Using this, the solution becomes

$$\beta''(t) = Y_+ + e^{-2t} \exp(-H_+t)(\exp(\gamma)(Y_+ + \varepsilon)\exp(-\gamma) - Y_+)\exp(H_+t)$$

Recall that $S(\rho_+) = Y_+ + Z(X_+)$. Therefore, we need to show that for each $\varepsilon \in \bigoplus_{k>-2} V_k$, there exists $\gamma \in \bigoplus_{k>0} V_k$ such that

$$\exp(\gamma)(Y_+ + \varepsilon)\exp(-\gamma) - Y_+ \in Z(X_+)$$

Expanding the left hand side near $\gamma = \varepsilon = 0$, to first order we have

$$f(\gamma, \varepsilon) = \varepsilon - [Y_+, \gamma]$$

In this case, we have a linear map

$$[Y_+, \cdot] : \bigoplus_{k>0} V_k \to \bigoplus_{k>-2} V_k$$

which is injective, and its image satisfies

$$\bigoplus_{k>-2} V_k = \mathrm{Im}([Y_+, \cdot]) \oplus Z(X_+)$$

Therefore, for each ε, there exists a unique γ such that $f(\gamma, \varepsilon) \in Z(X_+)$. Hence the linearisation has a unique solution, and so by the implicit function theorem, for ε sufficiently small, there exists γ such that $\exp(\gamma)(Y_+ + \varepsilon)\exp(-\gamma) - Y_+ \in Z(X_+)$. Finally, we can use homogeneity to extend the result to all of $\bigoplus_{k>-2} V_k$ as above. $\square$

Now let (α', β') be a solution of the complex equation eq. (2.7) satisfying the boundary conditions eq. (2.3). Define a gauge transformation $g : \mathbb{R} \to \mathrm{SL}(n, \mathbb{C})$ via

$$g(t) = \begin{cases} g_-(t) & t \leq 0 \\ g_+(t) & t \geq 1 \end{cases} \tag{2.9}$$

and smooth on all of $\mathbb{R}$. Then $g(t)$ is bounded, since g_- and g_+ are, as they converge in the limit $t \to \pm\infty$. Therefore, $g \in \mathcal{G}_0^c$, and $(\alpha, \beta) = g \cdot (\alpha', \beta')$ is given by

$$\begin{aligned} \alpha(t) &= \begin{cases} \frac{1}{2} H_- & t \leq 0 \\ \frac{1}{2} H_+ & t \geq 1 \end{cases} \\ \beta(t) &= \begin{cases} Y_- & t \leq 0 \\ Y_+ + e^{-2t} \exp(-H_+ t) \varepsilon \exp(H_+ t) & t \geq 1 \end{cases} \end{aligned} \tag{2.10}$$

and hence every complex trajectory is equivalent to one of this form. Moreover, we can choose ε such that $Y_+ + \varepsilon \in S(\rho_+)$, and in this case, ε is uniquely determined.

Since (α, β) is locally equivalent to the constant solution $(-\frac{1}{2}H_-, Y_-)$, the element $Y_+ + \varepsilon$ must be conjugate to Y_- in $\mathfrak{sl}(n, \mathbb{C})$. That is, $Y_+ + \varepsilon \in \mathcal{N}(\rho_-)$. Conversely, given $Y_+ + \varepsilon \in S(\rho_+) \cap \mathcal{N}(\rho_-)$, we can always find a solution satisfying eq. (2.10).

Proposition 2.2.6 ([Kro90b, Proposition 7])**.** The equivalence classes of complex trajectories associated to ρ_+, ρ_- are parametrised by $S(\rho_+) \cap \mathcal{N}(\rho_-)$.

Proof. We have already seen that each trajectory is equivalent to one in the form eq. (2.10), which is parametrised by the element $Y_+ + \varepsilon \in S(\rho_+) \cap \mathcal{N}(\rho_-)$. Using lemma 2.2.3, we see that two trajectories which are equal outside of $[0, 1]$ are equivalent. Therefore, the equivalence classes are parametrised by $Y_+ + \varepsilon \in S(\rho_+) \cap \mathcal{N}(\rho_-)$. ☐

2.3 Nahm's equations

Consider the change of variables

$$T_i = e^{2t} A_i \qquad s = -\frac{1}{2} e^{-2t}$$

Using this, eq. (2.2) becomes

$$\begin{aligned} \frac{\mathrm{d}T_1}{\mathrm{d}s} &= -[T_2, T_3] \\ \frac{\mathrm{d}T_2}{\mathrm{d}s} &= -[T_3, T_1] \\ \frac{\mathrm{d}T_3}{\mathrm{d}s} &= -[T_1, T_2] \end{aligned}$$

which are Nahm's equations. The same change of variables also transforms eq. (2.4) into

$$\begin{aligned} \frac{\mathrm{d}T_1}{\mathrm{d}s} + [T_0, T_1] + [T_2, T_3] = 0 \\ \frac{\mathrm{d}T_2}{\mathrm{d}s} + [T_0, T_2] + [T_3, T_1] = 0 \\ \frac{\mathrm{d}T_3}{\mathrm{d}s} + [T_0, T_3] + [T_1, T_2] = 0 \end{aligned}$$

Using this, we can also consider the action of the gauge group on this system. Recall that the action is given by eq. (2.5), which is:

$$g \cdot A = (gA_0g^{-1} - \dot{g}g^{-1}, gA_1g^{-1}, gA_2g^{-1}, gA_3g^{-1})$$

Note that

$$\dot{g} = \frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\mathrm{d}g}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = e^{-2t}\frac{\mathrm{d}g}{\mathrm{d}s}$$

In this case, the gauge group action becomes

$$\begin{aligned} g \cdot T &= g \cdot (e^{-2t}T_0, e^{-2t}T_1, e^{-2t}T_2, e^{-2t}T_3) \\ &= \left(e^{-2t}gT_0g^{-1} - e^{-2t}\frac{\mathrm{d}g}{\mathrm{d}s}g^{-1}, e^{-2t}gT_1g^{-1}, e^{-2t}gT_2g^{-1}, e^{-2t}gT_3g^{-1}\right) \\ &= \left(gT_0g^{-1} - \frac{\mathrm{d}g}{\mathrm{d}s}g^{-1}, gT_1g^{-1}, gT_2g^{-1}, gT_3g^{-1}\right) \end{aligned}$$

This is the same as the action as in [Don84, Equation 1.6]. Finally, we can consider the $\mathrm{SL}(n, \mathbb{C})$ valued paths

$$\tilde{\alpha} = e^{2t}\alpha = \frac{1}{2}(T_0 + iT_1) \qquad \tilde{\beta} = e^{2t}\beta = \frac{1}{2}(T_2 + iT_3)$$

In this case the real and complex equations become

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}(\tilde{\alpha} + \tilde{\alpha}^*) + 2([\tilde{\alpha}, \tilde{\alpha}^*] + [\tilde{\beta}, \tilde{\beta}^*]) = 0 \\ \frac{\mathrm{d}\tilde{\beta}}{\mathrm{d}s} + 2[\tilde{\alpha}, \tilde{\beta}] = 0 \end{aligned} \tag{2.11}$$

With all of this in mind, this allows us to use the results from [Don84].

2.4 Real equation

Recall the real equation eq. (2.6),

$$\hat{F}(\alpha, \beta) = \frac{\mathrm{d}}{\mathrm{d}t}(\alpha + \alpha^*) + 2(\alpha + \alpha^*) + 2([\alpha, \alpha^*] + [\beta, \beta^*]) = 0$$

Write $(\alpha', \beta') = g \cdot (\alpha, \beta)$, and we will regard $\hat{F}(\alpha', \beta') = 0$ as an equation for g. First of all, notice that the real equation is invariant under the action of $\mathcal{G}$, and so the action of g only depends on the corresponding path

$$\tilde{g} : \mathbb{R} \to \mathrm{SL}(n, \mathbb{C}) / \mathrm{SU}(n) = \mathcal{H}$$

From the polar decomposition of $\mathrm{SL}(n, \mathbb{C})$, we can write any $A \in \mathrm{SL}(n, \mathbb{C})$ uniquely as $A = UP$, where $U \in \mathrm{SU}(n)$ and P is hermitian, with positive eigenvalues and $\det(P) = 1$. Hence we can choose

$$\mathcal{H} = \{A \in \mathrm{SL}(n, \mathbb{C}) \mid A \text{ hermitian, with positive eigenvalues}\}$$

For each g, we define $h = h(g) = g^*g$, which gives us a path $h : \mathbb{R} \to \mathcal{H}$.

2.4.1 Uniqueness

Lemma 2.4.1 ([Kro90b, Lemma 12]). Suppose (α, β) satisfies the complex equation on an interval $[-N, N]$. Then for any $h_-, h_+ \in \mathcal{H}$, there exists $g : [-N, N] \to \mathrm{SL}(n, \mathbb{C})$ continuous and smooth on the interior, with $h = h(g)$ satisfying

$$h(-N) = h_- \qquad h(N) = h_+$$

and such that $(\alpha', \beta') = g \cdot (\alpha, \beta)$ satisfies the real equation $\hat{F}(\alpha', \beta') = 0$ on $[-N, N]$.

Proof. See [Don84, Proposition 2.8]. The main idea is that the real equation (for Nahm's equations) is the Euler-Lagrange equations for a functional, and so the result follows by the direct method of the calculus of variations. To get the result, we apply [Don84, Proposition 2.8] with

$$`\alpha' := \tilde{\alpha} = e^{2t}\alpha \qquad `\beta' = \tilde{\beta} = e^{2t}\beta$$

and modify the interval $[\varepsilon, 2-\varepsilon]$ to $[-N, N]$. The work in section 2.3 shows that g has the required properties. ☐

Now for $h \in \mathcal{H}$, with eigenvalues $\lambda_1, \ldots, \lambda_k$, define

$$\Psi(h) = \log \max(\lambda_i)$$

Since $\det(h) = 1$, $\Psi(h) = 0$ if and only if $h = 1$. Moreover, if $h(t)$ is continuous, then $\Psi(h(t))$ is as well.

Lemma 2.4.2 ([Kro90b, Lemma 13])**.** If $(\alpha', \beta') = g \cdot (\alpha, \beta)$ over some interval in $\mathbb{R}$, then with $h = g^*g$,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Psi(h) + 2\frac{\mathrm{d}}{\mathrm{d}t}\Psi(h) \geq -2\left(\left|\hat{F}(\alpha,\beta)\right| + \left|\hat{F}(\alpha',\beta')\right|\right)$$

weakly[a]. Note the norm on the right hand side is defined using the Killing form.

[a]See section 2.B for the definition.

Proof. We want to use [Don84, Lemma 2.10]. First, we will write the left hand side in terms of s. In this case, we have

$$\begin{aligned}
\frac{\mathrm{d}\Psi}{\mathrm{d}s} &= \frac{\mathrm{d}\Psi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} \\
\frac{\mathrm{d}^2\Psi}{\mathrm{d}s^2} &= \frac{\mathrm{d}^2\Psi}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + \frac{\mathrm{d}\Psi}{\mathrm{d}t}\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} \\
&= e^{4t}\left(\frac{\mathrm{d}^2\Psi}{\mathrm{d}t^2} + 2\frac{\mathrm{d}\Psi}{\mathrm{d}t}\right)
\end{aligned}$$

Next, note that the real equation for Nahm's equations, eq. (2.11), is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}(\tilde{\alpha}+\tilde{\alpha}^*) + 2([\tilde{\alpha},\tilde{\alpha}^*] + [\tilde{\beta},\tilde{\beta}^*]) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{2t}(\alpha+\alpha^*)\right)\frac{\mathrm{d}t}{\mathrm{d}s} + 2e^{4t}([\alpha,\alpha^*] + [\beta,\beta^*]) \\
&= e^{4t}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\alpha+\alpha^*) + 2(\alpha+\alpha^*) + 2([\alpha,\alpha^*] + [\beta,\beta^*])\right)
\end{aligned}$$

Therefore, compared to [Don84, Lemma 2.10], we have a factor of e^{4t} on both sides, which is a positive function. Therefore, the result follows. ☐

Proposition 2.4.3 ([Kro90b, Proposition 8(b)])**.** Suppose (α', β') and (α'', β'') are equivalent complex trajectories, satisfying the real equation eq. (2.6), then $(\alpha'', \beta'') = g \cdot (\alpha', \beta')$ for some $g \in \mathcal{G}$, i.e. $g : \mathbb{R} \to \mathrm{SU}(n)$, with $g(t) \to 1$ as $t \to \infty$.

Proof. Suppose (α', β') and $(\alpha'', \beta'') = g \cdot (\alpha', \beta')$ both satisfy the real equation. Setting $h = h(g)$ and $\Psi = \Psi(h)$, we find that

$$\ddot{\Psi} + 2\dot{\Psi} \geq 0$$

Using the same computation as in the previous lemma, this implies that

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}s^2} \geq 0$$

and so $\Psi(s)$ is convex. The other conditions transform to $\Psi : (-\infty, 0) \to \mathbb{R}$ as: $\Psi(s) \to 0$ as $s \to 0$, $\Psi(s)$ bounded and nonnegative. This then implies that Ψ must be identically zero. Hence $h = 1$, and so $g^*g = 1$. That is, g takes values in $\mathrm{SU}(n)$. ☐

2.4.2 Existence

Let (α, β) be a solution to the complex equations. We can assume without loss of generality that (α, β) is in the form eq. (2.10).

Lemma 2.4.4 ([Kro90b, Lemma 14]). If (α, β) are in the form as in eq. (2.10), and $\varepsilon \in Z(X_+)$, then

$$\begin{cases} \hat{F}(\alpha, \beta) = 0 & \text{on } (-\infty, 0] \\ \left|\hat{F}(\alpha, \beta)\right| \leq Ce^{-4t} & \text{on } [0, \infty) \end{cases}$$

Proof. In both cases, since $\rho_\pm$ are representations of $\mathfrak{su}(2)$, we have that

$$\begin{aligned} H_\pm^* &= H_\pm \\ X_\pm^* &= Y_\pm \\ Y_\pm^* &= X_\pm \end{aligned}$$

Thus, in the first case, we have

$$2H_- + 2[Y_-, X_-] = 0$$

which is true as ρ_- is a representation of $\mathfrak{sl}(2, \mathbb{C})$. For the second case, let

$$\varepsilon(t) = e^{-2t} \exp(-H_+ t)\varepsilon \exp(H_+ t)$$

and we have that

$$\alpha = \frac{1}{2} H_+ \qquad \beta(t) = Y_+ + \varepsilon(t)$$

Computing each part, we have

$$\begin{aligned} \alpha + \alpha^* &= H_+ \\ [\alpha, \alpha^*] &= 0 \\ [\beta, \beta^*] &= [Y_+ + \varepsilon(t), Y_+^* + \varepsilon(t)^*] \\ &= -H_+ + [\varepsilon(t), X_+] + [Y_+, \varepsilon(t)^*] + [\varepsilon(t), \varepsilon(t)^*] \\ &= -H_+ + 2[\varepsilon(t), X_+] + [\varepsilon(t), \varepsilon(t)^*] \end{aligned}$$

We want to show that $[\varepsilon(t), X_+] = 0$. Set $f = \exp(-H_+ t)$, then this is equivalent to showing $\varphi = 0$, where $\varphi(t) = [\varepsilon, e^{2t} f^{-1} X_+ f]$. Since $\varphi(0) = 0$, as $\varepsilon \in Z(X_+)$, suffices to show $\dot{\varphi} = 0$. Computing,

$$\begin{aligned} \dot{\varphi} &= [\varepsilon, 2e^{-2t} f^{-1} X_+ f - e^{2t} f^{-1} H_+ X_+ f + e^{2t} f^{-1} X_+ H_+ f] \\ &= 0 \end{aligned}$$

as $[H, X] = 2X$. Therefore, we have that $\hat{F}(\alpha, \beta) = 2[\varepsilon(t), \varepsilon(t)^*]$. In this case, we have that $|\varepsilon(t)| = e^{-2t}|\varepsilon|$, and so using the fact that the norm is (up to a constant) submultiplicative, we have that

$$\left|\hat{F}(\alpha, \beta)\right| \leq Ce^{-4t}$$

Since $\hat{F}$ is bounded on $[0, 1]$, making C larger if necessary, we have that $\left|\hat{F}(\alpha, \beta)\right| \leq Ce^{-4t}$ on $[0, \infty)$. $\square$

Using lemma 2.4.1, for each $N \in \mathbb{N}$, we can find a complex gauge transformation $g_N : [-N, N] \to \mathrm{SL}(n, \mathbb{C})$, such that $g_N \cdot (\alpha, \beta)$ satisfies the real equation, and $h_N = g_N^* g_N$ satisfies the Dirichlet boundary condition $h_N(\pm N) = 1$. We will now show that the h_N have a smooth limit as $N \to \infty$.

Lemma 2.4.5 ([Kro90b, Lemma 15]). Let C be the constant from lemma 2.4.4. Define the C^1 function $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(t) = \begin{cases} C/4 & t \leq 0 \\ Ce^{-2t}/2 - Ce^{-4t}/4 & t \geq 0 \end{cases}$$

Then for all N, we have $\Psi(h_N) < \psi$ on $[-N, N]$.

Proof. We have that

$$\ddot{\psi} + 2\dot{\psi} = \begin{cases} 0 & t < 0 \\ -2Ce^{-4t} & t > 0 \end{cases}$$

and lemma 2.4.2 and lemma 2.4.4 gives us that

$$\ddot{\Psi} + 2\dot{\Psi} \geq -2\left|\hat{F}(\alpha, \beta)\right| \geq \begin{cases} 0 & t \leq 0 \\ -2Ce^{-4t} & t \geq 0 \end{cases}$$

Therefore, we have that $(\ddot{\Psi} - \ddot{\psi}) + 2(\dot{\Psi} - \dot{\psi}) \geq 0$. Using the change of variables $s = -\frac{1}{2}e^{-2t}$ as before, we find that

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}(\Psi - \psi) \geq 0$$

That is, $\Psi - \psi$, as a function of s, is convex. Therefore, by the maximum principle for convex functions, the maximum value of $\Psi - \psi$ is attained at one of the end points. By assumption, $h_N(\pm N) = 1$, and so $\Psi(\pm N) = 0$. Thus, $\Psi(-N) - \psi(-N) = -C/4 < 0$. Similarly, $\Psi(N) - \psi(N) = Ce^{-2N}/2 - Ce^{-4N}/4 < 0$. Therefore, $\Psi - \psi$ is negative on $[-N, N]$. In fact, it is bounded away from zero. □

Lemma 2.4.6 ([Kro90b, Corollary 16, Lemma 17]). The h_N converges in the C^∞ topology on compact subsets, to a smooth path $h : \mathbb{R} \to \mathcal{H}$, such that

(i) h is bounded, and for large t,
$$|h(t) - 1| \leq C'e^{-2t}$$
for some $C' > 0$,

(ii) if $g = h^{1/2}$, $(\alpha', \beta') = g \cdot (\alpha, \beta)$, then $\hat{F}(\alpha', \beta') = 0$,

(iii) the derivative $\mathrm{d}h/\mathrm{d}t$ is bounded, and for large t,
$$\left|\frac{\mathrm{d}h}{\mathrm{d}t}\right| < C''e^{-2t}$$
for some $C'' > 0$

Proof. Omitted. □

Using this, we can prove:

Proposition 2.4.7. For every complex trajectory (α, β), there is an equivalent trajectory $(\alpha', \beta') = g \cdot (\alpha, \beta)$ which satisfies the real equation $\hat{F}(\alpha', \beta') = 0$.

Proof. Using g from lemma 2.4.6, we all we need to show is that (α', β') satisfy the boundary conditions eq. (2.8). First of all, using lemma 2.4.6 (iii), we find that $(\alpha', \beta') - (\alpha, \beta)$ decays exponentially as $t \to \infty$. In particular, this means that the boundary conditions at $t \to \infty$ are satisfied.

For the boundary conditions at $t \to -\infty$, split (α', β') into hermitian and skew-hermitian parts, to get a solution (A_0, A_1, A_2, A_3) of the extended gradient flow equations eq. (2.4). Using a real gauge transformation

$g \in \mathcal{G}$, we can make $A_0 = 0$, which gives us a solution (A'_1, A'_2, A'_3) of the gradient flow equations eq. (2.2). By lemma 2.4.6 (iii), this is a bounded trajectory. Therefore, it approaches a critical point. Hence the boundary conditions at $t \to -\infty$ are satisfied, although it might be for a different representation ρ_-. However, by lemma 2.2.4, the conjugacy class of the representation ρ_- given in the limit, is uniquely determined by the orbit in which β' lies, which is the same orbit as β. □

2.5 Proof of theorem 2.1.1

Suppose $A(t)$ is a solution to the gradient flow equations eq. (2.2) satisfying the boundary conditions eq. (2.3). Setting $A_0 = 0$, we obtain a complex trajectory (α, β). The only thing we need to check that the convergence at $t \to \pm\infty$ is exponential.

Now sure how this is true. In the paper (p. 482), Kronheimer claims that it is a standard property of the gradient flow equations eq. (2.2)

Therefore, we have a map from $M(\rho_-, \rho_+)$ to the space of equivalence classes of complex trajectories.

Proposition 2.4.3 shows that this map is injective. To see this, suppose $A, A' \in M(\rho_-, \rho_+)$ give equivalent complex trajectories. Then there exists $g : \mathbb{R} \to \mathrm{SU}(n)$, with $g \cdot A = A'$, and $g(t) \to 1$ as $t \to \infty$. But in this case, $A_0 = A'_0 = 0$, which means that

$$-\dot{g}g^{-1} = 0 \implies \dot{g} = 0$$

and so $g(t) = 1$ for all t. That is, $A = A'$.

By proposition 2.4.7, in each equivalence class there is a complex trajectory (α', β') satisfying the real equation. Decomposing (α', β') into hermitian and skew-hermitian parts, we get a solution (A_0, A_1, A_2, A_3) of the extended equations eq. (2.4). Moreover, A_0 decays exponenitally, so there exists a real gauge transformation $g : \mathbb{R} \to \mathrm{SU}(n)$, with $g(t) \to 1$ as $t \to 1$, such that

$$gA_0g^{-1} - \dot{g}g^{-1} = 0$$

Therefore, from this, we obtain a solution to the original equations. Thus, the map from $M(\rho_-, \rho_+)$ to the space of equivalence classes of complex trajectories is surjective.

2.6 Nilpotent orbit

As we are predominantly interested in the nilpotent orbits, we will consider the case where $\rho_+ = 0$. Define $M(\rho) = M(\rho, 0)$ to be the space of solutions to eq. (2.2), satisfying the boundary conditions

$$\lim_{t\to-\infty} A(t) \in C(\rho) \qquad \lim_{t\to\infty} A(t) = 0$$

In this case, theorem 2.1.1 becomes

$$M(\rho) \cong \mathcal{N}(\rho)$$

First of all, given $A = (A_1, A_2, A_3) \in M(\rho)$, we send it to the equivalence class of the complex trajectory

$$\alpha = iA_1 \qquad \beta = A_2 + iA_3$$

Putting (α, β) into the form of eq. (2.10), we have that

$$\begin{aligned} \alpha(t) &= \begin{cases} \frac{1}{2}H & t \le 0 \\ 0 & t \ge 1 \end{cases} \\ \beta(t) &= \begin{cases} Y & t \le 0 \\ e^{-2t}\varepsilon & t \ge 1 \end{cases} \end{aligned} \tag{2.12}$$

where ε is conjugate to Y, i.e. $\varepsilon \in \mathcal{N}(\rho)$. Note however in this case, we don't need to use lemma 2.2.4 for $t \le 0$, we could just leave it as is, and just use lemma 2.2.5. Therefore, to compute ε, we can just solve the ODE

$$\dot{g} = 2g\alpha = 2igA_1$$
$$\lim_{t\to\infty} g(t) = 1$$

and in this case, g will transform β to $e^{-2t}\varepsilon$ for some $\varepsilon \in \mathfrak{sl}(n, \mathbb{C})$. More precisely, $\varepsilon = g(0)\beta(0)g(0)^{-1}$.

2.6.1 Nahm's equations

Consider the complex equation for Nahm's equations, that is, using the change of variables

$$s = -\frac{1}{2}e^{-2t} \qquad \tilde{\alpha} = e^{2t}\alpha \qquad \tilde{\beta} = e^{2t}\beta$$

we have the complex equation

$$\frac{\mathrm{d}\tilde{\beta}}{\mathrm{d}s} + 2[\tilde{\alpha}, \tilde{\beta}] = 0$$

The tangent space to the adjoint orbit M of $\tilde{\beta}$ at $\tilde{\beta}$ is

$$\left\{[X, \tilde{\beta}] \mid X \in \mathfrak{sl}(n, \mathbb{C})\right\}$$

which means that $\frac{\mathrm{d}\tilde{\beta}}{\mathrm{d}s} \in \mathrm{T}_{\tilde{\beta}}M$. Therefore $\tilde{\beta}$ stays in the same adjoint orbit of $\mathfrak{sl}(n, \mathbb{C})$. We can transfer this back to the original equations, since for a *nilpotent* matrix A, A and λA are conjugate, for all $\lambda \in \mathbb{C}$. That is, β stays within the same nilpotent orbit.

2.6.2 Boundary conditions to Nahm's equations

In this case, we would like to translate the boundary conditions from $A = (A_1, A_2, A_3)$ to boundary conditions on $T = (T_1, T_2, T_3)$, where $T = e^{2t}A$.

The boundary condition $A \to 0$ at $t \to \infty$ becomes

$$e^{-2t}T \to 0 \implies sT \to 0$$

as $s \to 0$. The boundary condition $\lim\limits_{t\to-\infty} A(t) \in C(\rho)$ becomes

$$\lim_{s\to-\infty} e^{-2t}T \in C(\rho) \implies \lim_{s\to-\infty} sT \in -\frac{1}{2}C(\rho)$$

That is, we have the boundary conditions

$$\begin{aligned} \lim_{s\to 0} sT &= 0 \\ \lim_{s\to-\infty} sT &\in C(\sigma) \end{aligned} \tag{2.13}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3) = -\frac{1}{2}\rho$ satisfies

$$[\sigma_1, \sigma_2] = \sigma_3$$
$$[\sigma_2, \sigma_3] = \sigma_1$$
$$[\sigma_3, \sigma_1] = \sigma_2$$

In particular, given such a triple, define

$$T_i = \frac{\sigma_i}{s-1} \tag{2.14}$$

Then this is a solution to Nahm's equations, satisfying the boundary conditions eq. (2.3). In fact, all solutions will be asymptotic to (a conjugate of) this one as $s \to -\infty$.

2.6.3 Map for ε using Nahm's equations

Setting $\tilde{\alpha} = e^{2t}\alpha$ and $\tilde{\beta} = e^{2t}\beta$, then we have the complex equation coming from Nahm's equations, i.e.

$$\frac{\mathrm{d}\tilde{\beta}}{\mathrm{d}s} + [\tilde{\alpha}, \tilde{\beta}] = 0$$

Therefore, if we instead solve for $g \cdot (\tilde{\alpha}, \tilde{\beta}) = (0, \tilde{\beta}')$, i.e.

$$\frac{\mathrm{d}g}{\mathrm{d}s} = 2g\tilde{\alpha}$$

with the boundary condition that $g \to 1$ as $s \to 0$, then $\tilde{\beta}'$ is constant, and this constant is exactly ε. In fact, this is the same equation as in lemma 2.2.5, just with a change of variables. Therefore, in this case,

$$\tilde{\beta}(t) = g(t)^{-1}\varepsilon g(t)$$

and since $g(t) \to 1$ as $s \to 0$, we have that

$$\varepsilon = \lim_{s\to 0} \tilde{\beta}(s)$$

2.A Representation theory of $\mathfrak{sl}(2, \mathbb{C})$

In this section, we will sketch the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ that is required for the analysis in this chapter. For more details, see [Hum73, Section 7].

Let V be a complex vector space. Then a representation of $\mathfrak{sl}(2, \mathbb{C})$ is a Lie algebra homomorphism $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(V)$. When clear, we will write $X \cdot v := \rho(X)(v)$. Choose the basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for $\mathfrak{sl}(2, \mathbb{C})$. The commutators are $[H, X] = 2X, [H, Y] = -2Y, [X, Y] = H$

We first note that $\rho(H)$ is diagonalisable, and so we have a Jordan decomposition

$$V = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda$$

where

$$V_\lambda = \{v \in V \mid H \cdot v = \lambda v\}$$

is the λ-eigenspace of H. In fact, we have:

1. $$V = \bigoplus_{\lambda\in\mathbb{Z}} V_\lambda$$
2. if $v \in V_\mu$, then $X \cdot v \in V_{\mu+2}$ and $Y \cdot v \in V_{\mu-2}$.

2.B Weak inequalities

Let $\Omega \subseteq \mathbb{R}^n$ be open and connected.

Definition 2.B.1 (positive)
We say that $u \in \mathcal{D}'(\Omega)$ is *positive* if for all $\phi \in C_c^\infty(\Omega)$, with $\phi \geq 0$, $u[\phi] \geq 0$. We write this as $u \geq 0$.

Definition 2.B.2 (derivative)
The derivative of a distribution $u \in \mathcal{D}'(\Omega)$ is the distribution Du given by

$$Du[\phi] = -u[D\phi]$$

Finally, recall that we have an embedding $T : L^1_{\text{loc.}}(\Omega) \to \mathcal{D}'(\Omega)$, given by

$$T_f(\phi) = \int_\Omega f\phi \mathrm{d}x$$

We will abuse notation and write $Df = DT_f$. With this, let

$$L = \sum_{k=0}^{d} a_k D^k$$

be a linear differential operator, $a_k : \Omega \to \mathbb{R}$ smooth. Suppose $Lf \geq 0$. Then we say that the differential inequality $Lf \geq 0$ holds *weakly*.

Chapter 3

HyperKähler manifold structure

In this chapter, we will construct the hyperKähler manifold structure on $M(\rho)$, following [Kro90a]. Note however that [Kro90a] is for the *regular semisimple* orbit, therefore to transfer it to the nilpotent orbit, we will need to make some changes. This chapter also contains a few things which overlap with the previous chapter, but for ease of reading, we will include them as well.

On the other hand, the emphasis on this section is not to fill in details, but rather to show how to modify the arguments. Therefore, we will omit some details, and refer the reader to [Kro90a] for more details.

3.1 Nahm's equations

Recall that if we set $B = e^{2t}A$ and $s = -\frac{1}{2}e^{-t}$, then the equations eq. (2.2) becomes Nahm's equations

$$\begin{aligned}\frac{\mathrm{d}B_1}{\mathrm{d}s} + [B_2, B_3] = 0\\ \frac{\mathrm{d}B_2}{\mathrm{d}s} + [B_3, B_1] = 0\\ \frac{\mathrm{d}B_3}{\mathrm{d}s} + [B_1, B_2] = 0\end{aligned} \tag{3.1}$$

and from section 2.6.2, we have the boundary conditions

$$\lim_{s\to 0} sB = 0$$
$$\lim_{s\to -\infty} sB \in C(\sigma)$$

where $\sigma = -\frac{1}{2}\rho$, satisfies

$$[\sigma_1, \sigma_2] = \sigma_3$$
$$[\sigma_2, \sigma_3] = \sigma_1$$
$$[\sigma_3, \sigma_1] = \sigma_2$$

Moreover, since we assume the limits $s \to 0$ exists, we will consider the equations on the half interval $(-\infty, 0]$.

Adding in a fourth function $B_0 : (-\infty, 0] \to \mathfrak{su}(n)$, we can write the extended Nahm's equations as

$$\begin{aligned}\frac{\mathrm{d}B_1}{\mathrm{d}s} + [B_0, B_1] + [B_2, B_3] = 0\\ \frac{\mathrm{d}B_2}{\mathrm{d}s} + [B_0, B_2] + [B_3, B_1] = 0\\ \frac{\mathrm{d}B_3}{\mathrm{d}s} + [B_0, B_3] + [B_1, B_2] = 0\end{aligned} \tag{3.2}$$

We have three equations for four variables, so the system is underdetermined. We can introduce the gauge group as before, with

$$\mathcal{G} = \{g : (-\infty, 0] \to \mathfrak{su}(n)\}$$
$$\mathcal{G}_0 = \{g : (-\infty, 0] \to \mathfrak{su}(n), g(0) = 1\}$$

with the action given by

$$g \cdot (B_0, B_1, B_2, B_3) := \left(gB_0g^{-1} - \frac{\mathrm{d}g}{\mathrm{d}s}g^{-1}, gB_1g^{-1}, gB_2g^{-1}, gB_3g^{-1}\right)$$

Using this, if we ignore the boundary conditions, then the space of solutions to eq. (3.1) are the same as the space of solutions to eq. (3.2) modulo the action of $\mathcal{G}_0$, since we can always use $\mathcal{G}_0$ to make $B_0 = 0$. Moreover, by an element of $\mathcal{G}_0$, we can assume that $\lim_{s\to-\infty} sB = \sigma$.

We would like a version of [Kro90a, Lemma 3.4], since this would give us the space which we are interested in. However, since the boundary conditions are different, we won't have an exponential rate of convergence. Let

$$B^0 = \left(0, \frac{\sigma_1}{s-1}, \frac{\sigma_2}{s-1}, \frac{\sigma_3}{s-1}\right)$$

be the "asymptotic" solution to Nahm's equations. We are interested in solutions which are asymptotic to this one.

Lemma 3.1.1. Let B be a solution to eq. (3.2), with $sB \to \sigma$ as $s \to -\infty$. Then there exists $C, \delta > 0$ such that

$$\left|B - B^0\right| \le \frac{Ke^{\eta t}}{|s|} = \frac{K}{|s|^{1+\delta}} \le \frac{C}{(1-s)^{1+\delta}}$$

Proof. In [Kro90b, Proof of Theorem 1, p. 482] (also see section 2.5), we have exponential convergence of A, i.e.

$$|A - \rho| \le Ke^{\eta t}$$

for some $K, \eta > 0$, as $t \to -\infty$. This then gives us that in the limit as $s \to -\infty$,

$$\left|B - B^0\right| \le \frac{Ke^{\eta t}}{|s|} = \frac{K}{|s|^{1+\zeta}} \le \frac{C}{(1-s)^{1+\zeta}}$$

□

Let Ω_1 denote the space of all C^1 maps

$$b = (b_0, b_1, b_2, b_3) : (-\infty, 0] \to \mathfrak{su}(n) \otimes \mathbb{R}^4$$

The boundary conditions on $M(\rho)$ gives us a norm condition, which is

$$\left\|b\right\|_1 = \sup_{s\le 0}((1-s)^{1+\delta}\left|b_j\right|) + \sup_{s\le 0}((1-s)^{2+\delta}\left|\dot{b}_j\right|) < \infty$$

for some $\delta > 0$, and define the affine space

$$\mathscr{A} = B^0 + \Omega_1 = \left\{B^0 + b \mid b \in \Omega_1\right\}$$

By lemma 3.1.1, all of the solutions which we are interested in belong to $\mathscr{A}$. For any path $u : (-\infty, 0] \to \mathfrak{su}(n)$, define

$$\nabla_B u = \left(\frac{\mathrm{d}u}{\mathrm{d}s} + [B_0, u], [B_1, u], [B_2, u], [B_3, u]\right)$$

and we have that [Kro90a, Proposition 3.7]

$$M(\rho) \cong \{B \in \mathscr{A} \text{ satisfying eq. (3.2)}\} /\mathscr{G}$$

where $\mathscr{G}$ is the group

$$\mathscr{G} = \left\{ g : (-\infty, 0] \to \mathrm{SU}(n) \mid g(0) = 1, g^{-1}\nabla_B g \in \Omega_1 \right\}$$

acting via eq. (2.5). The condition $g^{-1}\nabla_B g \in \Omega_1$ just means that g carries B to another element of $\mathscr{A}$.

3.2 HyperKähler structure on Ω_1

Ω_1 inherits a natural quaternionic structure from $\mathbb{R}^4 \cong \mathbb{H}$, and we have a norm defined by $\Omega_1 \subseteq L^2$. That is, we have the L^2 inner product

$$\langle\!\langle b, c \rangle\!\rangle = \sum_{j=0}^{3} \int_{-\infty}^{0} \left\langle b_j(s), c_j(s) \right\rangle \mathrm{d}s$$

and the complex structures are given by

$$\begin{aligned} I(b_0, b_1, b_2, b_3) &= (-b_1, b_0, -b_3, b_2) \\ J(b_0, b_1, b_2, b_3) &= (-b_2, b_3, b_0, -b_1) \\ K(b_0, b_1, b_2, b_3) &= (-b_3, -b_2, b_1, b_0) \end{aligned}$$

Since $\mathscr{A}$ is an affine space modelled on Ω, this means that $\mathscr{A}$ inherits a natural hyperKähler structure.

3.2.1 Integrability

A priori, it is not clear that Ω_1 is a subspace of L^2. However, from the decay condition, we have that

$$|b(s)|^2 \leq K|s|^{-2}$$

for some $K > 0$. The integral

$$\int_{-\infty}^{-1} \frac{1}{x^2} \mathrm{d}x$$

is finite, and elements of Ω_1 are bounded on $[-1, 0]$. Therefore, they are in L^2. In fact, the decay condition shows that the elements are also in L^1.

3.3 Tangent space

Let ∇_B^* be the L^2 adjoint of ∇_B, i.e.

$$\nabla_B^* u = -\frac{\mathrm{d}u_0}{\mathrm{d}s} - \sum_{j=0}^{3} [B_j, u_j]$$

Using this, we have

Proposition 3.3.1 ([Kro90a, Proposition 3.9]). $M(\rho)$ is a smooth manifold, and the tangent space to M at a (the equivalence class) of a solution

$$B = (B_0(s), B_1(s), B_2(s), B_3(s))$$

to eq. (3.2) can be identified with the set of solutions in Ω of the linear equations

$$
\begin{aligned}
\frac{\mathrm{d}b_0}{\mathrm{d}s} + [B_0, b_0] + [B_1, b_1] + [B_2, b_2] + [B_3, b_3] = 0 \\
\frac{\mathrm{d}b_1}{\mathrm{d}s} + [B_0, b_1] - [B_1, b_0] + [B_2, b_3] - [B_3, b_2] = 0 \\
\frac{\mathrm{d}b_2}{\mathrm{d}s} + [B_0, b_2] - [B_1, b_3] - [B_2, b_0] + [B_3, b_1] = 0 \\
\frac{\mathrm{d}b_3}{\mathrm{d}s} + [B_0, b_3] + [B_1, b_2] - [B_2, b_1] - [B_3, b_0] = 0
\end{aligned}
\tag{3.3}
$$

Equivalently, it is given by the equation

$$
\nabla_B^*(b) = \nabla_B^*(Ib) = \nabla_B^*(Jb) = \nabla_B^*(Kb) = 0
$$

Using this, the tangent space to M at B is a subspace of Ω, which is invariant under I, J, K. Therefore, M inherits three almost complex structures satisfying the quaternionic relations. In fact, this and the L^2 metric on Ω makes M into a hyperKähler manifold.

Below, we will sketch how to modify the proof of [Kro90a, Lemma 3.8] to take into account the different boundary conditions in the nilpotent case. With these modifications, we will prove that the operator $\nabla_B^*\nabla_B$ on a given space is invertible, and so the same proof as in [Kro90a, Proposition 3.9] will work.

The space Ω_0 can be defined as in the paper, i.e.

$$
\Omega_0 = \{u \mid u(0) = 0, \nabla_B u \in \Omega_1 \text{ for some } B \in \mathscr{A}\}
$$

with norm $\|u\|_0 = \left\|\nabla_B u\right\|_1$ and we can define

$$
\Omega_0' = \left\{ v \;\middle|\; \|v\|_0' = \sup_s((1-s)^{2+\delta}|v(s)|) < \infty \right\}
$$

We are interested in the operator

$$
\begin{aligned}
\nabla_B^*\nabla_B(u) &= \nabla_B^*\left(\frac{\mathrm{d}u}{\mathrm{d}s} + [B_0, u], [B_1, u], [B_2, u], [B_3, u]\right) \\
&= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}u}{\mathrm{d}s} + [B_0, u]\right) - \left[B_0, \frac{\mathrm{d}u}{\mathrm{d}s}\right] - \sum_{j=0}^{3}[B_j, [B_j, u]]
\end{aligned}
$$

Considering the case $B = B^0$, we get the equation

$$
\frac{\mathrm{d}^2u}{\mathrm{d}s^2} - \frac{1}{(s-1)^2}\Lambda u = -v
$$

where Λ is the nonnegative self-adjoint operator

$$
\Lambda u = \sum_{j=0}^{3}[\sigma_j^*, [\sigma_j, u]]
$$

Λ is diagonalisable, and so we have the equations

$$
\ddot{u} - \frac{\lambda^2}{(s-1)^2}u = v
$$

3.3.1 The norm on Ω_0

Here, we will compute the norm on Ω_0, using $B = B^0$. In particular, we have that

$$
\nabla_B u = \left(\frac{\mathrm{d}u}{\mathrm{d}s}, \frac{1}{s-1}[\sigma_1, u], \frac{1}{s-1}[\sigma_2, u], \frac{1}{s-1}[\sigma_3, u]\right)
$$

The first term gives us

$$(1-s)^{1+\delta}\left|\frac{\mathrm{d}u}{\mathrm{d}s}\right| \quad \text{and} \quad (1-s)^{2+\delta}\left|\frac{\mathrm{d}^2u}{\mathrm{d}s^2}\right|$$

The rest of the terms are bounded by

$$\frac{(1-s)^{1+\delta}}{s-1}K|u| \leq K'(1-s)^{\delta}|u|$$

for some constant K' independent of u. For the derivative, we have the terms

$$\frac{(1-s)^{2+\delta}}{(1-s)^2}[\sigma_1, u] \leq C(1-s)^{\delta}|u|$$
$$\frac{(1-s)^{2+\delta}}{1-s}\left[\sigma_1, \frac{\mathrm{d}u}{\mathrm{d}s}\right] \leq C'(1-s)^{1+\delta}\left|\frac{\mathrm{d}u}{\mathrm{d}s}\right|$$

Therefore, to bound the norm on Ω_0, all we need is a bound on

$$(1-s)^{\delta}|u(s)|$$
$$(1-s)^{1+\delta}\left|\frac{\mathrm{d}u}{\mathrm{d}s}\right|$$
$$(1-s)^{2+\delta}\left|\frac{\mathrm{d}^2u}{\mathrm{d}s^2}\right|$$

3.3.2 Case $\lambda > 0$

As in [Kro90a], we now consider the case where v is compactly supported. Define $f(s) = (1-s)^{\delta}u(s)$. Consider a maxima of f, so s_0 such that $f'(s_0) = 0$, $f''(s_0) \leq 0$. Computing, we find that

$$u''(s_0) - \frac{\delta(\delta+1)}{(1-s_0)^2}u(s_0) \leq 0$$

Using the equation for u'', we then get the equation that

$$\frac{\lambda^2 - \delta(\delta+1)}{(1-s_0)^2}u(s_0) \leq v(s_0)$$

Since $\lambda > 0$, for δ sufficiently small, $\lambda^2 - \delta(\delta+1) > 0$. In this case, we find that

$$f(s_0) \leq K(1-s_0)^{\delta+2}v(s_0)$$

and at a minima, we have the reverse inequality. Therefore, we must have that

$$\sup_s \left((1-s)^{\delta}|u(s)|\right) \leq K \sup_s \left((1-s)^{2+\delta}|v(s)|\right)$$

for some constant K which is independent of v.

The bound on $(1-s)^{2+\delta}\left|\mathrm{d}^2u/\mathrm{d}s^2\right|$ follows from the ODE and the Ω_0' bound on v. Therefore, all we need is a bound on $(1-s_0)^{1+\delta}|\mathrm{d}u/\mathrm{d}s|$. But this follows from a similar argument to the above. Set $g(s) = (1-s)^{1+\delta}u'(s)$. Then at a maxima/minima of g,

$$g'(s_0) = (1-s_0)^{1+\delta}u''(s_0) - (1+\delta)(1-s_0)^{\delta}u'(s_0) = 0$$

and so

$$u'(s_0) = \frac{1-s_0}{1+\delta}u''(s_0)$$

which means that

$$(1-s_0)^{1+\delta}u'(s_0) = \frac{(1-s_0)^{2+\delta}}{1+\delta}u''(s_0)$$

Hence the bound on the second derivative gives us a bound on the first derivative.

3.3.3 Case $\lambda = 0$

In this case, we have the equation

$$\ddot{u} = -v$$

This immediately gives us the bound on the second derivative. The same argument as in the previous subsection shows that this gives us a bound on $(1-s)^{1+\delta}\dot{u}$. Therefore, all we need is a bound on $(1-s)^{2+\delta}u$. Defining $h(s) = (1-s)^{2+\delta}$, and using the same argument again, we get a bound on h.

3.3.4 Bounded below

We have shown that the operator $\nabla_B^*\nabla_B : \Omega_0 \to \Omega_0'$ is bounded below. In particular, this means that we can use density to extend our arguments above to all of Ω_0', since the image of an operator which is bounded below is closed. Moreover, since it is a bijection (injectivity follows from the same reason as in [Kro90a]), the inverse is then a bounded linear map.

The bound that δ^2 is smaller than the least positive eigenvalue is replaced by the requirement that $\delta(\delta+1) < \lambda^2$, where λ^2 is the least positive eigenvalue.

3.4 Adjoint orbit

Define a map $\phi : M(\rho) \to \mathcal{N}$ by

$$\phi(B) = B_2(0) + iB_3(0)$$

and from the previous section, we have that $\phi(B)$ is in the same adjoint orbit as Y. In terms of the complex coordinates

$$\alpha(s) = \frac{1}{2}(B_0(s) + iB_1(s)) \qquad \beta(s) = \frac{1}{2}(B_2(s) + iB_3(s))$$

and a tangent vector $(\delta\alpha, \delta\beta) = (b_0, b_1, b_2, b_3)$, the complex structure I is just

$$I(\delta\alpha, \delta\beta) = (i\delta\alpha, i\delta\beta)$$

In this case, $\phi(\alpha, \beta) = 2\beta(0)$, and ϕ can easily be extended to $\mathscr{A}$, as

$$\phi(B_0, B_1, B_2, B_3) = B_2(0) + iB_3(0)$$

Hence

$$\begin{aligned} \phi(B+b) &= \phi(B_0 + b_0, B_1 + b_1, B_2 + b_2, B_3 + b_3) \\ &= B_2(0) + b_2(0) + iB_3(0) + ib_3(0) \\ &= \phi(B) + b_2(0) + ib_3(0) \end{aligned}$$

and so, $\mathrm{d}\phi_B(b) = b_2(0) + ib_3(0)$. Therefore, in this case we have that ϕ is holomorphic with respect to the complex structures I on $M(\rho)$ and i on the adjoint orbit (which as a complex submanifold of $\mathfrak{sl}(n,\mathbb{C})$ is naturally Kähler).

Bibliography

[Don84] S. K. Donaldson. "Nahm's equations and the classification of monopoles". en. In: *Commun.Math. Phys.* 96.3 (Sept. 1984), pp. 387–407. ISSN: 1432-0916. DOI: 10.1007/BF01214583. URL: https://doi.org/10.1007/BF01214583 (visited on 03/15/2023).

[Hum73] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Publisher: Springer New York. Jan. 1973. DOI: 10.1007/978-1-4612-6398-2.

[Kir84] Frances Clare Kirwan. *Cohomology of Quotients in Symplectic and Algebraic Geometry. (MN-31), Volume 31*. Vol. 104. Princeton University Press, 1984. ISBN: 9780691083704. URL: http://www.jstor.org/stable/j.ctv10vm2m8 (visited on 08/23/2023).

[Kro90a] P. B. Kronheimer. "A Hyper-KÄHlerian Structure on Coadjoint Orbits of a Semisimple Complex Group". en. In: *Journal of the London Mathematical Society* s2-42.2 (1990). _eprint: https://onlinelibrary.wiley.com/doi/pc 42.2.193, pp. 193–208. ISSN: 1469-7750. DOI: 10.1112/jlms/s2-42.2.193. URL: https://onlinelibrary.wiley.com/doi/abs/10.1112/jlms/s2-42.2.193 (visited on 03/15/2023).

[Kro90b] P. B. Kronheimer. "Instantons and the geometry of the nilpotent variety". In: *Journal of Differential Geometry* 32.2 (Jan. 1990). Publisher: Lehigh University, pp. 473–490. ISSN: 0022-040X. DOI: 10.4310/jdg/1214445316. URL: https://projecteuclid.org/journals/journal-of-differential-geometry/volume-32/issue-2/Instantons-and-the-geometry-of-the-nilpotent-variety/10.4310/jdg/1214445316.full (visited on 03/15/2023).

[KS96] Piotr Z. Kobak and Andrew Swann. "Classical nilpotent orbits as hyperkähler quotients". In: *Int. J. Math.* 07.02 (Apr. 1996). Publisher: World Scientific Publishing Co., pp. 193–210. ISSN: 0129-167X. DOI: 10.1142/S0129167X96000116. URL: https://www.worldscientific.com/doi/10.1142/S0129167X96000116 (visited on 07/27/2023).