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Contents

Part I

Pre-requisites

Chapter 1

Differential geometry

1.1 Smooth manifolds

1.1.1 Smooth manifolds

Definition 1.1.1 (topological manifold)

A topological n-manifold is a topological space X, such that for every $p \in X$, there exists an open neighbourhood U of p in X, and an open set V in $\mathbb{R}^n$, and a homeomorphism $\varphi : U \to V$.

Moreover, we require X to be Hausdorff and second countable.

1. φ as above is called a chart,
2. a collection of charts where the domains form an (open) cover of X is called an atlas,
3. U is a coordinate patch,
4. if $x_1, \ldots, x_n$ the standard coordinate functions on $\mathbb{R}^n$, then $x_1 \circ \varphi, \ldots, x_n \circ \varphi$ are local coordinates on U. We will usually abuse notation and denote them by $x_1, \ldots, x_n$.
5. if we have charts $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$, the map
$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$
is called the transition map.

Definition 1.1.2 (smooth function)

Given an atlas $\mathcal{A}$ and a an open subsets $W \subseteq X$, a function $f : W \to \mathbb{R}$ is smooth with respect to $\mathcal{A}$ if $f \circ \varphi^{-1}$ is smooth for all $\varphi \in \mathcal{A}$. That is, if all local coordinate expressions $f(x_1, \ldots, x_n)$ are smooth.

Definition 1.1.3 (smooth atlas)

An atlas $\mathcal{A}$ on X is smooth if all the transition functions $\varphi_b eta \circ \varphi_\alpha^{-1}$ are smooth.

Definition 1.1.4 (smoothly equivalent, smooth structure)

Two smooth atlases $\mathcal{A}$ and $\mathcal{B}$ are smoothly equivalent if $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas. This defines an equivalence relation, and an equivalence class is called a smooth structure.

Definition 1.1.5 (smooth manifold)

A smooth n-manifold X is a topological n-manifold with a smooth structure.

1.1.2 Smooth maps

Throughout, fix smooth manifolds X, Y, with atlases $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$ and $\{\psi_\beta : S_\beta \to T_\beta\}$ respectively.

Definition 1.1.6 (smooth map)

A continuous map $F : X \to Y$ is smooth if for all α, β,
$$\psi_\beta \circ F \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap F^{-1}(S_\beta)) \to T_\beta$$
is smooth.

Lemma 1.1.7. Smoothness is local. That is, if $F : X \to Y$ is smooth, and $U \subseteq X$ is open, then $F|_U$ is also smooth.

Lemma 1.1.8. The composition of smooth maps is smooth.

Definition 1.1.9 (diffeomorphism)

A diffeomorphism is a smooth map $F : X \to Y$ with a smooth inverse.

1.1.3 Tangent spaces

Throughout, let X be a smooth n-manifold.

Definition 1.1.10 (curve based at a point)

Let X be a manifold, $p \in X$, then a curve based at p is a smooth map $\gamma : I \to X$, where $I \subseteq \mathbb{R}$ is an open interval containing 0, and $\gamma(0) = p$.

Definition 1.1.11 (agree to first order)

Given curves γ_1, γ_2 at p, we say that they agree to first order if there exists a chart φ near p, such that $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$ in $\mathbb{R}^n$.

Write π_p^φ for the map $\gamma \to (\varphi \circ \gamma)'(0)$.

Lemma 1.1.12. Agreement to first order is independent of the choice of charts. Moreover, it is an equivalence relation.

Definition 1.1.13 (tangent space)

The tangent space of X at p is

$$\mathrm{T}_pX = \frac{\{\text{curves based at } p\}}{\text{agreement to first order}}$$

Elements of T_pX are called tangent vectors at p, and we write $[\gamma]$ for the equivalence class of γ.

Proposition 1.1.14. T_pX is an n-dimensional vector space.

Proof. Given a chart φ at p, π_p^φ induces an injective map $\pi_p^\varphi : \mathrm{T}_pX \to \mathbb{R}^n$. We want to show that this is surjective.

Given $v \in \mathbb{R}^n$, let $\gamma(t) = \varphi^{-1}(\varphi(p) + tv)$. Then $\pi_p^\varphi([\gamma]) = v$, so π_p^φ is surjective.

Therefore, we can transport the $\mathbb{R}$-vector space structure using π_p^φ. □

Definition 1.1.15 (basis of the tangent space)

Let φ be a chart at p, with corresponding local coordinates $x_1, \ldots, x_n$, define

$$\frac{\partial}{\partial x_i} = (\pi_p^\varphi)^{-1}(e_i) \in \mathrm{T}_pX$$

where e_i is the i-th standard basis vector in $\mathbb{R}^n$.

Remark 1.1.16. $\frac{\partial}{\partial x_i}$ depends on the whole set of coordinates $x_1, \ldots, x_n$.

Lemma 1.1.17. On overlaps of charts,

$$\frac{\partial}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j}$$

1.1.4 Derivatives of smooth maps

Fix manifolds X, Y, with a smooth map $F : X \to Y$.

Definition 1.1.18 (derivative)
The derivative of F at $p \in X$ is the linear map

$$\mathrm{D}_p F : \mathrm{T}_p X \to \mathrm{T}_{F(p)} Y$$

given by $\mathrm{D}_p F([\gamma]) = [F \circ \gamma]$.

Lemma 1.1.19. $\mathrm{D}_p F$ is well defined and linear.

Lemma 1.1.20 (chain rule)**.** Suppose we have smooth maps $F : X \to Y, G : Y \to Z$, then $G \circ F$ is smooth, with

$$\mathrm{D}_p(G \circ F) = \mathrm{D}_{F(p)} G \circ \mathrm{D}_p F$$

Definition 1.1.21 (immersion, submersion, local diffeomorphism)
A smooth map $X \to Y$ is an immersion (submersion, local diffeomorphism) (at a point p) if $\mathrm{D}_p F$ is injective (surjective, bijective) (at p).

Definition 1.1.22 (reguular point, regular value)
$p \in X$ is a regular point for $F : X \to Y$ if F is a submersion at p. $q \in X$ is a regular value for $F : X \to Y$ if for all $p \in F^{-1}(q)$, F is a submersion at p.

If $p \in X$ is not a regular point, then it is a critical point. Similarly, if $q \in Y$ is not a regular value, then it is a critical value.

Lemma 1.1.23. Suppose $F : X \to Y$ is a local diffeomorphism at p. Then there exists open sets U, V of $p, F(p)$ respectively, such that $F : U \to V$ is a diffeomorphism.

Lemma 1.1.24 (local immersion)**.** Suppose $F : X \to Y$ is an immersion at p. Given local coordinates $x_1, \dots, x_n$ on X, there exists local coordinates $y_1, \dots, y_m$ on Y, such that locally, F looks like the inclusion

$$\mathbb{R}^n = \mathbb{R}^n \oplus 0 \hookrightarrow \mathbb{R}^m$$

Lemma 1.1.25 (local submersion)**.** Suppose $F : X \to Y$ is a submersion at p. Given local coordinates $y_1, \dots, y_m$ on Y, there exists local coordinates $x_1, \dots, x_n$ on X, such that locally, F looks like the projection

$$\mathbb{R}^n \to \mathbb{R}^m = \mathbb{R}^m \oplus 0$$

1.1.5 Submanifolds

Throughout, let X be an n-manifold.

Definition 1.1.26 (submanifold)

A subset $Z \subseteq X$ is a submanifold of codimension k if for all $p \in Z$, there exists local coordinates $x_1, \dots, x_n$ on X about p, such that Z is locally given by

$$\{x_1 = \dots = x_k = 0\}$$

We say that Z is properly embedded if the above holds for all $p \in X$.

Lemma 1.1.27. Let $x_1, \dots, x_n$ be as above. Then $x_{k+1}, \dots, x_n$ define local coordinates on Z, making it into a smooth $(n-k)$-manifold. Moreover, the inclusion map $\iota : Z \hookrightarrow X$ is an immersion, and a homeomorphism onto its image.

Definition 1.1.28 (embedding)

A smooth map $F : X \to Y$ is an embedding if it is an immersion and a homeomorphism onto its image.

Lemma 1.1.29. The image of an embedding $F : X \to Y$ is a submanifold of Y, which is diffeomorphic to X.

Proposition 1.1.30. Suppose $F : X \to Y$ is a smooth map, $q \in Y$ a regular value of F, then $F^{-1}(q)$ is a submanifold of X, of codimension $\dim(Y)$.

Theorem 1.1.31 (Sard)**.** The set of critical values of $F : X \to Y$ has measure zero in Y.

Corollary 1.1.32. The set of regular values is dense in Y.

Remark 1.1.33. Note on the other hand that regular points need not exist.

Definition 1.1.34 (transverse)
Submanifolds Y, Z of X are transverse if for all $p \in Y \cap Z$,

$$\mathsf{T}_p X = \mathsf{T}_p Y + \mathsf{T}_p Z$$

Proposition 1.1.35. If Y, Z are submanifolds of codimension k, l respectively, intersecting transversally, then $Y \cap Z$ is a submanifold of codimension $k + l$.

1.2 Vector bundles and tensors

1.2.1 Vector bundles

Definition 1.2.1 (vector bundle)
A vector bundle of rank k over a manifold B is

(i) a manifold E,

(ii) a smooth map $\pi : E \to B$,

(iii) an open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of B,

(iv) for each α, a diffeomorphism $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$,

such that

1. $\mathrm{pr}_1 \circ \Phi_\alpha = \pi$ on $\pi^{-1}(U_\alpha)$,
2. for all α, β, the map
$$\Phi_\beta \circ \Phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$
is of the form
$$\Phi_\beta \circ \Phi_\alpha^{-1}(b, v) = (b, g_{\beta\alpha}(b)(v))$$
where $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{R})$ is smooth.

We call

- E the total space,
- π the projection,
- the Φ_α local trivialisations,
- $g_{\beta\alpha}$ the transition functions,

and we write E_b for the fibre $\pi^{-1}(b)$.

Remark 1.2.2. Replacing $\mathbb{R}$ with $\mathbb{C}$ we get complex vector bundles.

Definition 1.2.3 (trivial bundle)
The trivial bundle of rank k over B is $B \times \mathbb{R}^k$ with the obvious trivialisation.

Notation 1.2.4. For a vector space V, write $\underline{V}$ for the trivial bundle $B \times V$.

Definition 1.2.5 (tangent bundle)
The tangent bundle of an n-manfold X is a rank n vector bundle, given by

(i) $\mathrm{T}X = \bigsqcup_{p \in X} \mathrm{T}_p X = \{(p, v) \mid p \in X, v \in \mathrm{T}_p X\}$. On any coordinate neighbourhood U of X, with coordinates $x_1, \dots, x_n$, and chart φ, then we have a chart on $\mathrm{T}X$ given by
$$\psi\left(p, \sum_i a_i \partial_i\right) = (\varphi(p), (a_1, \dots, a_n)) \subseteq \mathbb{R}^{2n}$$

(ii) and $\pi(p, v) = p$

Definition 1.2.6 (section)
A section s of a vector bundle $\pi : E \to B$ is a smooth map $s : B \to E$, such that $\pi \circ s = \mathrm{id}$.

Definition 1.2.7 (vector field)
A section of TX is called a vector field.

Definition 1.2.8 (morphism of vector bundles)
Given vector bundles $\pi_1 : E_1 \to B_1$ and $\pi_2 : E_2 \to B_2$, and a smooth map $F : B_1 \to B_2$, a morphism of vector bundles covering F is a smooth map $G : E_1 \to E_2$, such that

1. $\pi_2 \circ G = F \circ \pi_1$,
2. for all $p \in B$, the map $G_p : (E_1)_p \to (E_2)_{F(p)}$ is linear.

Definition 1.2.9 (isomorphism of vector bundles)
An isomorphism of vector bundles over B is a morphism covering id_B, with a two sided inverse.

Definition 1.2.10 (subbundle, quotient bundles)
Given a vector bundle $\pi : E \to B$ of rank k, a subbundle of rank l is a submanifold $F \subseteq E$, such that B is covered by the local trivialisations

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$$

under which F is given by $U_\alpha \times (\mathbb{R}^l \oplus 0)$.

1.2.2 Gluing

Suppose we have the following data:

(i) A manifold B,

(ii) and open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of B,

(iii) for each α, β, a smooth map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{R})$,

such that

1. $g_{\alpha\alpha}(x) = \mathrm{id}$,
2. $g_{\gamma\alpha} = g_{\gamma\beta} \circ g_{\beta\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

Then, define

$$E = \frac{\bigsqcup_{\alpha \in \mathcal{A}} (U_\alpha \times \mathbb{R}^k)}{(b, v) \sim (b, g_{\beta\alpha}(b)(v))}$$

and defining π by projecting onto the first factor, we get a vector bundle $\pi : E \to B$.

Lemma 1.2.11. Suppose $E \to B$ is a vector bundle. Then the transition functions satisfy 1. and 2., and E is isomorphic to the vector bundle constructed above.

1.2.3 Constructions on vector bundles

Definition 1.2.12 (pullback)

Given a vector bundle $\pi : E \to B$, and a smooth map $F : B' \to B$, the pullback bundle F^*E over B' is given by:

(i) The total space is still E, with fibres $E_{F(p)}$.

(ii) Suppose $E \to B$ is trivialised over $\{U_\alpha\}_{\alpha\in\mathcal{A}}$, with transition functions $g_{\beta\alpha}$. Then F^*E is trivialised over $\{F^{-1}(U_\alpha)\}$, with transition functions $F^*g_{\beta\alpha} = g_{\beta\alpha} \circ F$.

Definition 1.2.13 (dual bundle)

Suppose $E \to B$ is a vector bundle. Then the dual bundle $E^\vee \to B$ has total space

$$E^\vee = \bigsqcup_{p\in V} (E_p)^\vee$$

trivialised over the same open cover, with transition functions $(g^\vee_{\beta\alpha})^{-1}$.

1.2.4 Cotangent bundle

Definition 1.2.14 (cotangent bundle)

The cotangent bundle of X is $\mathrm{T}^*X = (\mathrm{T}X)^\vee$. We write T^*_pX for the fibre at p, the cotangent space of X at p.

Proposition 1.2.15. Given function elements $(f, U), (g, V)$ about $p \in X$, we say that $f, g : U \cap V \to \mathbb{R}$ agree to first order if $\mathrm{D}_pf = \mathrm{D}_pg$. Then we have a natural isomorphism

$$\frac{\text{function elements about } p \in X}{\text{agreement to first order}} \simeq \mathrm{T}^*_pX$$

Proof. Define map $e : \{\text{function elements about } p\} \to \mathrm{T}^*_pX$ by

$$e(f)([\gamma]) = (f \circ \gamma)'(0)$$

Then the result follows by the first isomorphism theorem for vector spaces. □

Definition 1.2.16 (differential of function)

For a function $f : X \to \mathbb{R}$, define $\mathrm{d}_pf = e(f) \in \mathrm{T}^*_pX$ as above. Then this defines a smooth section of T^*X, denoted $\mathrm{d}f$, called the differential of f.

Definition 1.2.17 (1-form)

A section of T^*_pX is called a 1-form.

Remark 1.2.18. The 1-forms $\mathrm{d}x_i$ form a basis of T_p^*X. Moreover, $\mathrm{d}x_i$ depends only on x_i, and not the other coordinate functions.

Definition 1.2.19 (pullback)

Given a smooth map, the map

$$(\mathrm{D}_p F)^\vee : \mathrm{T}^*_{F(p)} Y \to \mathrm{T}_p^* X$$

is called the pullback by F, denoted by F^*.

Lemma 1.2.20. Suppose $F : X \to Y$, $g : Y \to \mathbb{R}$ smooth. Then

$$F^*(\mathrm{d}g) = \mathrm{d}(F^* g) = \mathrm{d}(g \circ F)$$

1.2.5 Tensors and forms

Definition 1.2.21 (direct sum)

Suppose E, F are vector bundles over B, trivialised over $\{U_\alpha\}$, and with transition functions $g_{\beta\alpha}, h_{\beta\alpha}$ respectively. Then define the direct sum bundle $E \oplus F$, with fibres $E_p \oplus F_p$, and transition functions $g_{\beta\alpha} \oplus h_{\beta\alpha}$.

Remark 1.2.22. We can define the tensor product of vector bundles in a similar way, and tensor powers and symmetric, exterior powers.

Proposition 1.2.23. For $F : X \to Y$ smooth, $\mathrm{D}F$ is a section of $T^*X \otimes F^*TY$.

Proof. $\mathrm{Hom}(\mathrm{T}_p X, \mathrm{T}_{F(p)} Y) = \mathrm{T}_p^* X \otimes \mathrm{T}_{F(p)} Y = (\mathrm{T}^* X \otimes F^* \mathrm{T} Y)_p$ □

Definition 1.2.24 (tensor)

A tensor of type (p, q) on a manifold X is a section of

$$(\mathrm{T}X)^{\otimes p} \otimes (\mathrm{T}^* X)^{\otimes q}$$

Definition 1.2.25 (differential form)

An r form is a section of

$$\Lambda^r \mathrm{T}^* X$$

The space of all r-forms is denoted $\Omega^r(X)$.

From now on, we will write local coordinates with "up" indices, i.e. $x^1, \ldots, x^n$, and repeated indices, once up and once down are summed over. Up indices correspond to ∂_i factors, and down indices correspond to $\mathrm{d}x^i$ factors.

Notation 1.2.26. For $I = (i_1 < \cdots < i_r)$, write

$$\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_r}$$

Fix smooth manifolds X, Y, and a smooth map $F : X \to Y$. Then we have

Definition 1.2.27 (pushforward at a point)

For $p \in X$, a tensor of type $(r, 0)$ at p, we can push forward this to $(\mathrm{T}_{F(0)}Y)^{\otimes r}$ by applying $(\mathrm{D}_p F)^{\otimes r}$. We call this operation the pushforward, denoted by F_*.

Definition 1.2.28 (pullback at a point)

For $p \in X$, a tensor of type $(0, r)$ at $F(p)$, we can pull back this to $(\mathrm{T}_p^* X)^{\otimes r}$ by applying $\left((\mathrm{D}_p F)^\vee\right)^{\otimes r}$. Similarly, we can pull back an r form. We call this operation the pullback, denoted by F^*.

Definition 1.2.29 (pullback)

Given a tensor T of type $(0, r)$ on Y, we can pull this back to a tensor F^*T on X, by $(F^*T)_p = F^*(T_{F(p)})$. Similarly, we can pull back an r-form.

1.3 Differential forms

1.3.1 Exterior derivative

Definition 1.3.1 (exterior derivative)
Let α be a p-form, say $\alpha = \alpha_I \mathrm{d}x^I$. Then define the exterior derivative by

$$\mathrm{d}\alpha = \mathrm{d}\alpha_I \wedge \mathrm{d}x^I = \frac{\partial \alpha_I}{\partial x^j}\mathrm{d}x^j \wedge \mathrm{d}x^I$$

Proposition 1.3.2. The exterior derivative is well defined. That is, it is independent of the choice of local coordinates. Moreover,

(i) it is $\mathbb{R}$-linear,

(ii) it agrees with the differential on 0-forms,

(iii) $d^2 = 0$,

(iv) if $F : X \to Y$ smooth, α is a p-form on Y, then

$$F^*(\mathrm{d}\alpha) = d(F^*\alpha)$$

(v) given a p-form α and a q-form β,

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta$$

1.3.2 de Rham cohomology

Definition 1.3.3 (closed, exact)
A differential form α is closed if $\mathrm{d}\alpha = 0$, and α is exact if $\alpha = \mathrm{d}\beta$ for some β. We write $Z^r(X), B^r(X) \subseteq \Omega^r(X)$ for the spaces of closed and exact r-forms respectively.

Definition 1.3.4 (de Rham cohomology)
The r-th de Rham cohomology group of X is

$$\mathrm{H}^r_{\mathrm{dR}}(X) = \frac{Z^r(X)}{B^r(X)}$$

which is well defined as $d^2 = 0$.

Remark 1.3.5. $\mathrm{H}^r_{\mathrm{dR}}(X) = 0$ for $r > \dim(X)$, as there are no r forms in that case. Furthermore, $\mathrm{H}^r_{\mathrm{dR}}(X) = 0$ for $r < 0$, by convention.

Proposition 1.3.6 (functoriality)**.** Suppose $F : X \to Y$ is smooth. Then F^* induces a linear map

$$F^* : \mathrm{H}^r_{\mathrm{dR}}(Y) \to \mathrm{H}^r_{\mathrm{dR}}(X)$$

Proposition 1.3.7. The wedge product descends to $\mathrm{H}^*_{\mathrm{dR}}(X)$, making it into a unital graded-commutative associative algebra.

Corollary 1.3.8. The map $F^* : \mathrm{H}^*_{\mathrm{dR}}(Y) \to \mathrm{H}^*_{\mathrm{dR}}(X)$ is a unital algebra homomorphism.

Proposition 1.3.9 (homotopy invariance)**.** Suppose $F_0, F_1 : X \to Y$ are smoothly homotopic. Then the induced maps $F_0^*, F_1^* : \mathrm{H}^*_{\mathrm{dR}}(Y) \to \mathrm{H}^*_{\mathrm{dR}}(X)$ are equal.

Corollary 1.3.10. If $F : X \to Y$ is a smooth homotopy equivalence, then the induced map $F^* : \mathrm{H}^*_{\mathrm{dR}}(Y) \to \mathrm{H}^*_{\mathrm{dR}}(X)$ is an isomorphism.

1.3.3 Orientations

Definition 1.3.11 (orientation of a vector space)

For a vector space V, an orientation of V is a nonzero element of $\Lambda^n V$, up to rescaling by positive scalars, where $n = \dim(V)$.

Definition 1.3.12 (orientation of a vector bundle)

An orientation of a rank k vector bundle $E \to X$ is a nowhere zero section of $\Lambda^r E$, again up to rescaling by positive scalars.

Definition 1.3.13 (orientation of a manifold)

An orientation of a manifold X is an orientation of the tangent bundle $\mathrm{T}X$.

Definition 1.3.14 (volume form)

A volume form on an n-manifold X is a nowhere zero n-form, i.e. a nowhere zero section of $\Lambda^n \mathrm{T}^* X$.

Proposition 1.3.15. Volume forms and orientations are equivalent.

1.3.4 Integration

Definition 1.3.16 (partition of unity)

Given an open cover $\{U_\alpha\}$ of X, a partition of unity subordinate to the cover is a collection of smooth functions $p_\alpha : X \to \mathbb{R}_{\geq 0}$, such that

(i) $\mathrm{supp}(p_\alpha) \subseteq U_\alpha$,

(ii) the collection is locally finite. That is, for all $x \in X$, there exists an open neighbourhood V of x, such that all but finitely many p_α is zero on V,

(iii) $\sum_\alpha p_\alpha = 1$.

Lemma 1.3.17. For any open cover $\{U_\alpha\}$, there exists a partition of unity subordinate to the cover.

Definition 1.3.18 (integral)
Let X be an oriented n-manifold, ω a compactly supported n form on X. Then the integral of ω, $\int_X \omega$ is defined by

1. Cover X by coordinate neighbourhoods $\{U_\alpha\}$, with coordinates $x^1_\alpha, \ldots, x^n_\alpha$. Without loss of generality, suppose the x^i_α are positively oriented, that is, $\partial_{x^1_\alpha} \wedge \cdots \wedge \partial_{x^n_\alpha}$ represents the orientation.
2. Choose a subordinate partition of unity $\{p_\alpha\}$.
3. On each U_α, $p_\alpha \omega = f_\alpha \mathrm{d}x^1_\alpha \wedge \cdots \wedge \mathrm{d}x^n_\alpha$, where f_α is a smooth function.
4. $$\int_X \omega = \sum_\alpha \int_{\mathbb{R}^n} f_\alpha \mathrm{d}x^1_\alpha \cdots \mathrm{d}x^n_\alpha$$

where the integral on the right hand side is the usual integral on $\mathbb{R}^n$.

Lemma 1.3.19. The integral is well defined.

1.3.5 Stokes' theorem

Definition 1.3.20 (smooth manifold with boundary)
A smooth n-manifold with boundary is as defined as a manifold, except the codomain of each chart is an open set in $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$.

Definition 1.3.21 (interior, boundary)
Let X be a manifold with boundary, $p \in X$. Then if we have a chart $\varphi : U \to V$ near p, if $\varphi(p) \in \{0\} \times \mathbb{R}^{n-1}$, then we say that p is a boundary point, $p \in \partial X$. Otherwise, p is an interior point, $p \in \mathrm{Int}(X)$.

Lemma 1.3.22. The boundary and interior are well defined, i.e. independent of choice of charts.

We can define smooth maps and smooth functions as for manifolds.

Definition 1.3.23 (orientation on boundary)
Suppose X is an oriented n-manifold with boundary, then we can orient ∂X as follows.

Given $p \in \partial X$, choose $\mathfrak{o}_X \in \Lambda^n X$ representing the orientation of X. Choose a vector $\mathbf{n} \in \mathrm{T_p X}$ transverse to ∂X and pointing outwards. Then orient ∂X with the orientation $\mathfrak{o}_{\partial X}$ such that

$$\mathfrak{o}_X = \mathbf{n} \wedge \mathfrak{o}_{\partial \mathrm{X}}$$

Theorem 1.3.24 (Stokes' theorem)**.** Given an oriented n-manifold with boundary X, and a compactly supported $(n-1)$-form ω on X, then

$$\int_X \mathrm{d}\omega = \int_{\partial X} \omega$$

Proposition 1.3.25 (integration by parts)**.** Given an oriented n-manifold with boundary X, a $p-1$ form α and an $n-p$ form β on X, at least one of which is compactly supported. Then

$$\int_X (\mathrm{d}\alpha) \wedge \beta = \int_{\partial X} \alpha \wedge \beta + (-1)^p \int_X \alpha \wedge \mathrm{d}\beta$$

Proposition 1.3.26. If X is a compact oriented n-manifold, then integration over X defines a linear map $\int_X : \mathrm{H}^n_{\mathrm{dR}}(X) \to \mathbb{R}$.

Corollary 1.3.27. Suppose X is a compact orientable n-manifold. Then $\mathrm{H}^n_{\mathrm{dR}}(X) \neq 0$.

1.4 Connections on vector bundles

1.4.1 Connections

Definition 1.4.1 (E-valued r-form)
Given a vector bundle E over B, and E-valued r-form is a section of

$$E \otimes \Lambda^r \mathrm{T}^* B$$

Definition 1.4.2 (V-valued r-form)
Given a vector space V, a V-valued r-form is a $\underline{V}$-valued r-form.

Notation 1.4.3. Write $\Omega^r(E)$ for the space E valued r-forms, and $\Gamma(E) = \Omega^0(E)$ for the space of sections of E.

Notation 1.4.4. Let $\mathfrak{gl}(k, \mathbb{R})$ be the vector space of $k \times k$ real matrices.

Definition 1.4.5 (connection)
Let $E \to B$ be a vector bundle. A connection $\mathcal{A}$ on E is

1. for each trivialisation $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$, we have a $\mathfrak{gl}(k, \mathbb{R})$ valued 1-form A_α on U_α,
2. such that on overlaps,
$$A_\alpha = g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} + g_{\beta\alpha}^{-1} \mathrm{d} g_{\beta\alpha}$$

Definition 1.4.6 (covariant derivative)
Given a connection $\mathcal{A}$ on E, the covariant derivative of $s \in \Gamma(E)$ is the E-valued 1-form $\mathrm{d}^{\mathcal{A}} s$, given under Φ_α by

$$\mathrm{d}^{\mathcal{A}} s = \mathrm{d} v_\alpha + A_\alpha v_\alpha$$

where $v_\alpha = \mathrm{pr}_2 \circ \Phi_\alpha \circ s|_{U_\alpha}$ is the $\mathbb{R}^k$-valued function given by s.

Definition 1.4.7 (horizontal, covariantly constant)
A section $s \in \Gamma(E)$ is horizontal, or covariantly constant, if $\mathrm{d}^{\mathcal{A}} s = 0$.

Lemma 1.4.8. Given a connection $\mathcal{A}$ on $E \to B$, the covariant derivative $\mathrm{d}^{\mathcal{A}} : \Gamma(E) \to \Omega^1(E)$ is $\mathbb{R}$-linear, and satisfies the Leibniz rule

$$\mathrm{d}^{\mathcal{A}}(f \cdot s) = f \cdot \mathrm{d}^{\mathcal{A}} s + s \otimes \mathrm{d} f$$

Conversly, any linear map satisfying the Leibniz rule defines a connection.

Lemma 1.4.9. Any vector bundle $E \to B$ admits a connection.

Definition 1.4.10 (End(E))
Let $E \to B$ be a vector bundle. Then define

$$\mathrm{End}(E) = E \otimes E^\vee$$

Proposition 1.4.11. A section M of $\mathrm{End}(E)$ is the same as a smooth map $M_\alpha : U_\alpha \to \mathfrak{gl}(k, \mathbb{R})$ for all α, such that on overlaps,

$$M_\beta = g_{\beta\alpha} M_\alpha g_{\beta\alpha}^{-1}$$

Proposition 1.4.12. If $\mathcal{A}$ is a connection on E, and Δ is an $\mathrm{End}(E)$-valued 1-form, then we can define a connection $\mathcal{A} + \Delta$ in trivialisations by $A_\alpha + \Delta_\alpha$. Moreover, any connection on E is of this form. Therefore, the space of connections on E is an affine space modelled on $\Omega^1(\mathrm{End}(E))$.

1.4.2 Curvature

Fix a vector bundle $E \to B$, with a connection $\mathcal{A}$.

Definition 1.4.13 (exterior covariant derivative)
The exterior covariant derivative $\mathrm{d}^{\mathcal{A}} : \Omega^\bullet(E) \to \Omega^{\bullet+1}(E)$ is the unique $\mathbb{R}$-linear extension of $\mathrm{d}^{\mathcal{A}} : \Gamma(E) \to \Omega^1(E)$ such that

$$\mathrm{d}^{\mathcal{A}}(\sigma \wedge \omega) = (\mathrm{d}^{\mathcal{A}}\sigma) \wedge \omega + (-1)^r \sigma \wedge \mathrm{d}\omega$$

for E-valued r-form σ, and a differential form ω. In trivialisations, σ is an $\mathbb{R}^k$-valued r form σ_α, and

$$\mathrm{d}^{\mathcal{A}}\sigma = \mathrm{d}\sigma_\alpha + A_\alpha \wedge \sigma_\alpha$$

Proposition 1.4.14. There is a unique $\mathrm{End}(E)$-valued 2-form F such that for any E-valued form σ, we have that

$$(\mathrm{d}^{\mathcal{A}})^2\sigma = F \wedge \sigma$$

Definition 1.4.15 (curvature)
F in the above proposition is called the curvature of $\mathcal{A}$. $\mathcal{A}$ is flat if $F = 0$.

1.4.3 Parallel transport

Fix a vector bundle $E \to [0, 1]$ with connection $\mathcal{A}$.

Lemma 1.4.16. For each $s_0 \in E_0$, there exists a unique horizontal section s of E, with $s(0) = s_0$. Moreover, s depends linearly on s_0.

Definition 1.4.17 (parallel transport)
The parallel transport of s_0 from 0 to 1 is the element $s(1) \in E_1$. Since s depends linearly on s_0, parallel

transport defines a linear map $E_0 \to E_1$.

Now suppose $E \to B$ is any vector bundle, $\gamma : [0,1] \to B$ is a curve. Let $\mathcal{A}$ be a connection on $E \to B$.

Definition 1.4.18 (pullback connection)
We can define a connection $\gamma^*\mathcal{A}$ on γ^*E via the $\mathfrak{gl}(k,\mathbb{R})$ valued 1-forms γ^*A_α.

Definition 1.4.19 (horizontal lift, parallel transport, holonomy)
Given $s_0 \in E_{\gamma(0)}$, the horizontal lift of γ with respect of $\mathcal{A}$, at s_0 is the unique horizontal section of γ^*E starting at s_0.

Parallel transport along γ is the linear map $\mathcal{P}_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ given by $\mathcal{P}_\gamma(s_0) = s(1)$. If γ is a loop, then $\mathcal{P}_\gamma$ is the holonomy of $\mathcal{A}$ along γ.

1.5 Flows and the Lie Derivative

1.5.1 Flows

Let v be a vector field on X.

Definition 1.5.1 (integral curve)
An integral curve of v is a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to X$, such that

$$\dot{\gamma}(t) = v(\gamma(t))$$

Definition 1.5.2 (local flow)
A local flow of v is a smooth map $\Phi : U \to X$, where

1. $U \subseteq X \times \mathbb{R}$ is an open neighbourhood of $X \times 0$, and $U \cap \{p\} \times \mathbb{R}$ is connected for all $p \in X$.
2. $\Phi(\cdot, 0) = \mathrm{id}$,
3. $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(p,t) = v(\Phi(p,t))$ for all $(p,t) \in U$.

We will write $\Phi^t = \Phi(\cdot, t)$.

Lemma 1.5.3. Local flows always exist.

Lemma 1.5.4. Any local flow $\Phi : U \to X$ of v satisfies $\Phi^s \circ \Phi^t = \Phi^{s+t}$, whenever this makes sense.

Definition 1.5.5 (complete vector field)
A vector field v is complete if it admits a global flow, i.e. a flow defined on $X \times \mathbb{R}$.

Lemma 1.5.6. Compactly supported vector fields are complete.

1.5.2 Lie derivative

Let v be a vector field, with flow Φ.

Definition 1.5.7 (Lie derivative)
The Lie derivative of a tensor T along v is

$$\mathcal{L}_v T = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Phi^t)^* T$$

Remark 1.5.8. The brackets in the above expression is

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big((\Phi^t)^* T\big)$$

Lemma 1.5.9. For general t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^t\right)^* T = (\Phi^t)^* \mathcal{L}_v T$$

Lemma 1.5.10. If f is a function on X, then $\mathcal{L}_v(f) = \mathrm{d}f(v)$. If $\alpha = \alpha_i \mathrm{d}x^i$ is a 1-form, then

$$\mathcal{L}_v \alpha = \left(v^j \frac{\partial \alpha_i}{\partial x_j} + \alpha_j \frac{\partial v_j}{\partial x^i} \right) \mathrm{d}x^i$$

Lemma 1.5.11. For a vector field w, and a 1-form α, we have

$$\mathcal{L}_v(w^i \alpha_i) = (\mathcal{L}_v w)^i \alpha_i + w^i (\mathcal{L}_v \alpha)_i$$

and if S, T are tensors, then

$$\mathcal{L}_v(S \otimes T) = (\mathcal{L}_v S) \otimes T + S \otimes (\mathcal{L}_v T)$$

Corollary 1.5.12. If v, w are vector fields, then

$$\mathcal{L}_v(w) = \left(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j} \right) \mathrm{d}x^i$$

Definition 1.5.13 (Lie bracket)
The Lie bracket of vector fields v, w is

$$[v, w] = \mathcal{L}_v w = -\mathcal{L}_w v$$

This makes the space of vector fields on X into a Lie algebra.

Lemma 1.5.14. Let $F : X \to Y$ be a diffeomorphism, v a vector field on Y, T a tensor on Y, then

$$F^*(\mathcal{L}_v T) = \mathcal{L}_{F^* v}(F^* T)$$

Definition 1.5.15
Given a vector field v, and an r-form α, $\iota_v \alpha$ or $v \lrcorner \alpha$ is the $(r-1)$-form defined by

$$(\iota_v \alpha)_{i_1 \cdots i_{r-1}} = v^j \alpha_{j i_1 \cdots i_{r-1}}$$

Proposition 1.5.16 (Cartan's magic formula).

$$\mathcal{L}_v \alpha = \mathrm{d}(\iota_v \alpha) + \iota_v(\mathrm{d}\alpha)$$

1.6 More connections

1.6.1 Tangent bundle

Suppose $\mathcal{A}$ is a connection on $\mathrm{T}X \to X$.

Definition 1.6.1 (Coordinate trivialisations)
Given local coordinates $x^1, \ldots, x^n$ on X, we have a corresponding trivialisation $\partial_{x^1}, \ldots, \partial_{x^n}$ of $\mathrm{T}X$, known as the coordinate trivialisation. We write the components of the local trivialisation 1-form as $\Gamma^i{}_{jk}$, where the $_k$ is the 1-form index, and $^i{}_j$ are the $\mathfrak{gl}(n, \mathbb{R})$ indices.

Remark 1.6.2. The $\Gamma^i{}_{jk}$ do *not* give a tensor of type $(1, 2)$.

Definition 1.6.3 (Solder form)
The Solder form is the $\mathrm{T}X$-valued 1-form θ, given by the fibrewise identity map, under the identification

$$\mathrm{T}X \otimes \mathrm{T}^*X = \mathrm{End}(\mathrm{T}X)$$

In coordinate trivialisations, θ is given by $e_i \otimes \mathrm{d}x^i$.

Definition 1.6.4 (torsion)
The torsion T of $\mathcal{A}$ is the E-valued 2-form $\mathrm{d}^{\mathcal{A}}\theta$, given in coordinate trivialisations by

$$\mathrm{d}\left(e_i \otimes \mathrm{d}x^i\right) + A_\alpha \wedge (e_l \otimes \mathrm{d}x^l) = \Gamma^i{}_{lk} e_i \otimes \mathrm{d}x^k \wedge \mathrm{d}x^l$$

$\mathcal{A}$ is torsion free if $T = 0$.

Proposition 1.6.5 (First Bianchi identity).

$$\mathrm{d}^{\mathcal{A}} T = F \wedge \theta$$

Definition 1.6.6 (geodesic)
A curve γ in X is a geodesic if $\dot{\gamma}$ is horizontal as a section of $\gamma^*\mathrm{T}X$, i.e. if and only if the geodesic equation

$$\ddot{\gamma}^i + \Gamma^i{}_{jk}\dot{\gamma}^j\dot{\gamma}^k = 0$$

holds.

1.6.2 Orthogonal vector bundles

Let $E \to B$ be a vector bundle of rank k.

Definition 1.6.7 (inner product)
An inner product on E is a section g of $E^\vee \otimes E^\vee$, which is a fibrewise symmetric positive definite bilinear form.

Lemma 1.6.8. E admits an inner product.

Definition 1.6.9 (orthogonal vector bundle, orthogonal trivialisation)
An orthogonal vector bundle is a vector bundle with an inner product g. An orthogonal trivialisation is a trivialisation where g is the standard inner product on $\mathbb{R}^k$.

Fix an inner product g on E.

Lemma 1.6.10. E can be covered by orthogonal trivialisations.

Definition 1.6.11 (orthogonal connection)
A connection $\mathcal{A}$ on E is orthogonal if g is covariantly constant using the induced connection on $E^\vee \otimes E^\vee$.

Lemma 1.6.12. Orthogonal connections exist, and form an affine space for $\Omega^1(\mathfrak{o}(E))$, where $\mathfrak{o}(E)$ is the bundle of skew-adjoint endomorphisms of the fibres of E.

Lemma 1.6.13. The curvature of an orthogonal connection is an $\mathfrak{o}(E)$ valued 2-form.

1.7 Riemannian geometry

Definition 1.7.1 (Riemannian metric, Riemannian manifold)

A Riemannian metric is an inner product on $TX \to X$. A Riemannian manifold (X, g) is a manifold X with a Riemannian metric g.

Lemma 1.7.2. Every manifold admits a Riemannian metric.

Definition 1.7.3 (dual metric)

Given a metric g_{ij} on $TX \to X$, let g^{ij} denote the corresponding metric on $T^*X \to X$. That is, $g^{ij}g_{jk} = \delta^i{}_k$.

Definition 1.7.4 (raising and lowering indices)

We denote contraction with g_{ij} or g^{ij} by raising and lowering indices. For example, $v_i = g_{ij}v^j$.

Notation 1.7.5 (Symmetric product). Define

$$\mathrm{d}x^i\mathrm{d}x^j = \frac{1}{2}\left(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i\right)$$

So the standard Euclidean inner product on $\mathbb{R}^n$ is $\mathrm{d}x^i\mathrm{d}x^i$.

Theorem 1.7.6 (fundamental theorem of Riemannian geometry)**.** (X, g) admits a unique torsion free orthogonal connection.

Definition 1.7.7 (Levi-Civita connection)

The unique torsion free orthogonal connection on (X, g) is called the Levi-Civita connection. In coordinates, it is given by[a]

$$\Gamma_{ijk} = \frac{1}{2}\left(\partial_j g_{ik} + \partial_k g_{ji} - \partial_i g_{jk}\right)$$

[a] after lowering the ^{i} index

Let (X, g) be a Riemannian manifold, with Levi-Civita connection ∇.

Definition 1.7.8 (Riemann tensor)

The curvature of ∇ is the Riemann tensor $R^i{}_{jkl}$, which is an $\mathfrak{o}(TX)$ valued 2-form, viewed as a tensor of type $(1, 3)$.

1.7.1 Hodge theory

Let (X, g) be an oriented Riemannian n-manifold. Then g induces an inner product on $\Lambda^p T^*X$ for all p. Moreover, if $\alpha^1, \ldots, \alpha^n$ are a fibrewise orthonormal basis of 1-forms, then α^I form a fibrewise orthonormal basis of $\Lambda^p T^*X$.

In addition, from the orientation, we have a volume form ω. Therefore, by the metric, we can assume it is *the* positively oriented unit volume form. Now given a p-form β, there exists a unique $n - p$ form $\star\beta$ such that

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \, \omega$$

for all p-forms α. More concretely, $\star\alpha^I = \pm\alpha^J$, where $J = \{1, \ldots, n\} \setminus I$. Assuming the α^I are positively oriented, then the sign is $+$ if and only if I, J is an even permutation of $\{1, \ldots, n\}$.

Definition 1.7.9 (Hodge star)
The map $\star : \Omega^p(X) \to \Omega^{n-p}(X)$ is called the Hodge star operator.

Proposition 1.7.10. $\star$ is a fibrewise linear isometry, with $\star^2 = (-1)^{p(n-p)} \, \mathrm{id}$.

Definition 1.7.11 (inner product on forms)
Suppose X is compact, then we have an inner product on $\Omega^p(X)$ given by

$$\langle \alpha, \beta \rangle_X = \int_X \langle \alpha, \beta \rangle \, \omega = \int_X \alpha \wedge \star\beta$$

Lemma 1.7.12. For any $p-1$ form α and p-form β, we have that

$$\langle \mathrm{d}\alpha, \beta \rangle_X = (-1)^p \left\langle \alpha, \star^{-1} \mathrm{d} \star \beta \right\rangle_X$$

Definition 1.7.13 (codifferential)
The map $\delta : \Omega^\bullet(X) \to \Omega^{\bullet-1}(X)$ defined by

$$\delta = (-1)^p \star^{-1} \mathrm{d}\star$$

is called the codifferential.

Lemma 1.7.14. $\delta^2 = 0$.

Definition 1.7.15 (Laplace-Beltrami operator, harmonic)
The Laplace-Beltrami operator $\Delta : \Omega^\bullet(X) \to \Omega^\bullet(X)$ is defined by $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$.
A form α is harmonic if $\Delta\alpha = 0$. The space of harmonic p-forms is denoted by $\mathcal{H}^p(X)$.

Lemma 1.7.16. α is harmonic if and only if it is closed and coclosed, i.e. $d\alpha = 0$ and $\delta\alpha = 0$.

Theorem 1.7.17 (Hodge decomposition)**.** For all p, the space $\mathcal{H}^p(X)$ is finite dimensional, and we have orthogonal decompositions

$$
\begin{aligned}
\Omega^p(X) &= \mathcal{H}^p(X) \oplus \Delta\Omega^p(X) \\
&= \mathcal{H}^p(X) \oplus \mathrm{d}\delta\Omega^p(X) \oplus \delta\mathrm{d}\Omega^p(X) \\
&= \mathcal{H}^p(X) \oplus \mathrm{d}\Omega^{p-1}(X) \oplus \delta\Omega^{p+1}(X)
\end{aligned}
$$

Theorem 1.7.18. The map $\mathcal{H}^p(X) \to \mathrm{H}^p_{\mathrm{dR}}(X)$, given by $\alpha \mapsto [\alpha]$ is an isomorphism.

1.8 Lie groups and principal bundles

1.8.1 Lie groups and Lie algebras

Definition 1.8.1 (Lie group)
A Lie group is a manifold G, which is also a group, such that multiplication and inversion are smooth maps.

Definition 1.8.2 (embedded Lie subgroup)
An embedded Lie subgroup H of G is a submanifold which is also a subgroup. The restriction of the group operations to H makes H a Lie group.

Definition 1.8.3 (left, right translation, conjugation)
For $g \in G$, we get diffeomorphisms $L_g, R_g, C_g : G \to G$, given by

$$L_g(x) = gx \quad R_g(x) = xg \quad C_g(x) = gxg^{-1}$$

are called left translation, right translation, and conjugation by g, respectively.

Definition 1.8.4 (left, right, conjugation invariant)
A tensor T is left invariant if $(L_g)_* T = T$ for all $g \in G$. We can define right invariant and conjugation invariant tensors similarly.

T is bi-invariant if it is both left and right invariant.

Lemma 1.8.5. For any $h \in G$, the map $T \mapsto T_h$ is an isomorphism between the set of left invariant tensors of type (p, q) and tensors of type (p, q) at h.

Definition 1.8.6 (Lie algebra)
The Lie algebra $\mathfrak{g}$ of G is

$$\mathfrak{g} = \mathrm{T}_e G$$

Notation 1.8.7. For $\xi \in \mathfrak{g}$, define the left-invariant vector field

$$\ell_\xi(g) = (L_g)_* \xi$$

Lemma 1.8.8. The Lie bracket of left invariant vector fields is left invariant.

Definition 1.8.9 (Lie bracket)
The Lie bracket on $\mathfrak{g}$ is given by

$$[\xi, \eta] = \zeta$$

where $\zeta \in \mathfrak{g}$ is the unique element such that $[\ell_\xi, \ell_\eta] = \ell_\zeta$. This makes $\mathfrak{g}$ into a Lie algebra.

Definition 1.8.10 (smooth group action)
An action of a Lie group G on a manifold X is smooth if the action map $\sigma : G \times X \to X$ is smooth.

Definition 1.8.11 (adjoint representation)
The adjoint representation of G on $\mathfrak{g}$ is given by

$$\mathrm{Ad}_g(\xi) = (C_g)_*\xi$$

Definition 1.8.12 (infitesimal action)
Given a smooth left action of G on X, the infitesimal action of $\xi \in G$ on $x \in X$ is given by

$$\xi \cdot x = \mathrm{D}_{(e,x)}\sigma(\xi, 0) = [\gamma(t)x]$$

where $\gamma(t)$ is any curve representing ξ. We can define $x \cdot \xi$ for the analogous right action.

1.8.2 Principal bundles

Fix a Lie group G.

Definition 1.8.13 (principal G-bundle)
A principal G bundle P over B is defined as in the same way as for a vector bundle, except the trivialisations are

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$$

and on overlaps, $\Phi_\beta \circ \Phi_\alpha^{-1}(b, g) = (b, g_{\beta\alpha}(b)g)$ where $g_{\beta\alpha} : U_\alpha \times U_\beta \to G$ is smooth.

Definition 1.8.14 (frame bundle)
Given a rank k bundle $E \to B$, its frame bundle $F(E) \to B$ is the principal $\mathrm{GL}(k, \mathbb{R})$ bundle, with

$$F(E)_b = \{\text{ordered bases of } E_b\}$$

Moreover, if E has an inner product, the orthonormal frame bundle is the principal $O(k)$ bundle, with

$$F_O(E)_b = \{\text{ordered orthonormal bases of } E_b\}$$

Remark 1.8.15. Most definitions, such as sections, pullbacks, constructions by gluing etc. carry over from vector bundles. On the other hand, there is no zero section.

Lemma 1.8.16. A G-bundle P has a right G-action, defined by right translation on each fibre, i.e.

$$\Phi_\alpha^{-1}(b, x)g = \Phi_\alpha^{-1}(b, xg)$$

Lemma 1.8.17. Sections s of P over an open $U \subseteq B$ correspond to trivialisations Φ of P over U, i.e. given Φ, we can define $s(b) = \Phi^{-1}(b, e)$, and given s, we can define $\Phi(s(b)g) = (b, g)$.

1.8.3 Connections on principal bundles

Fix a principal G-bundle $P \to B$, and write $R_g : P \to P$ for the diffeomorphism arising from the right action of $g \in G$.

Definition 1.8.18 (connection)
A connection on P is a $\mathfrak{g}$-valued 1-form $\mathcal{A}$ on P, such that

1. $\mathcal{A}(p \cdot \xi) = \xi$ for all $p \in P$ and $\xi \in \mathfrak{g}$, where $p \cdot \xi$ is the infitesimal right action of ξ on p.
2. $R_g^* \mathcal{A} = \mathrm{Ad}_{g^{-1}} \mathcal{A}$.

Given a local section s_α, the local connection 1-form is $\mathcal{A}_\alpha = s_\alpha^* \mathcal{A}$.

Lemma 1.8.19. On overlaps, we have

$$\mathcal{A}_\alpha = \mathrm{Ad}_{g_{\beta\alpha}^{-1}} \mathcal{A}_\beta + (L_{g_{\beta\alpha}^{-1}})_* \mathrm{d}g_{\beta\alpha}$$

Remark 1.8.20. If $P = F(E)$ is a frame bundle, then a connection on P is the same as a connection on E.

Definition 1.8.21 (curvature)
The curvature of $\mathcal{A}$ is the $\mathfrak{g}$-valued 2-form $\mathcal{F}$ on P, given by

$$\mathcal{F} = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A} \wedge \mathcal{A}]$$

where

$$\left[\left(\sum_i \xi_i \otimes \alpha_i\right) \wedge \left(\sum_j \eta_j \otimes \beta_j\right)\right] = \sum_{i,j} [\xi_i, \eta_j] \otimes \alpha_i \wedge \beta_j$$

$\mathcal{A}$ is flat if $\mathcal{F} = 0$.

Chapter 2

Symplectic geometry

2.1 Symplectic manifolds

2.1.1 Symplectic linear algebra

Definition 2.1.1 (skew-symmetric bilinear form)
Let V be a real vector space, a blinear map $\Omega : V \times V \to \mathbb{R}$ is skew-symmetric if $\Omega(v, w) = -\Omega(w, v)$ for all $v, w \in V$.

Theorem 2.1.2 (canonical basis). Let Ω be a skew-symmetric bilinear form on V. Then there exists a basis $u_1, \dots, u_k, e_1, \dots, e_n, f_1, \dots, f_n$ of V, such that

1. $\Omega(u_i, v) = 0$ for all $i = 1, \dots, k$ and $v \in V$,
2. $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$ for all $i, j = 1, \dots, n$,
3. $\Omega(e_i, f_j) = \delta_{ij}$

Definition 2.1.3 (left map)
Given a bilinear map $\Omega : V \times V \to \mathbb{R}$, define $\tilde{\Omega} : V \to V^*$ by $\tilde{\Omega}(v)(u) = \Omega(v, u)$.

Lemma 2.1.4. $\ker(\tilde{\Omega}) = U = \operatorname{span}\{u_1, \dots, u_k\}$

Definition 2.1.5 (symplectic)
A skew-symmetric bilinear form Ω is symplectic if $\ker(\tilde{\Omega}) = U = 0$. Then Ω is called a linear symplectic structure on V, and (V, Ω) is called a symplectic vector space.

Definition 2.1.6 (symplectic, isotropic subspace)
A subspace W of V is

1. symplectic if $\Omega|_W$ is symplectic,
2. isotropic if $\Omega|_W = 0$.

Definition 2.1.7 (symplectomorphism)
Let $(V, \Omega), (V', \Omega')$ be symplectic vector spaces. Then a symplectomorphism $\varphi : V \to V'$ is a linear isomorphism, such that $\varphi^*\Omega' = \Omega$, where $\varphi^*\Omega'(u, v) = \Omega'(\varphi(u), \varphi(v))$.

2.1.2 Symplectic manifolds

Let M be a manifold, $\omega \in \Omega^2(M)$ be a 2-form.

Definition 2.1.8 (symplectic form)
ω is symplectic if ω is closed, and $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is symplectic for all $p \in M$.

Definition 2.1.9 (symplectic manifold)
A symplectic manifold is a pair (M, ω), where M is a manifold and ω is a symplectic form on M.

Definition 2.1.10 (symplectomorphism)
Let $(M_1, \omega_1), (M_2, \omega_2)$ be symplectic manifolds. Then a diffeomorphism $f : M_1 \to M_2$ is a symplectomorphism if $f^*\omega_2 = \omega_1$.

2.1.3 Canonical and tautological forms

Suppose X is a manifold, $\pi : \mathrm{T}^*X \to X$ is the cotangent bundle of X.

Definition 2.1.11 (cotangent coordinates)
Suppose $x_1, \ldots, x_n$ are local coordinates on X, then $(\mathrm{d}x_1)p, \ldots, (\mathrm{d}x_n)_p$ define a basis for T^*_pX. That is, if $\xi \in \mathrm{T}^*_pX$, then $\xi = \sum_i \xi_i(\mathrm{d}x_i)_p$. We call $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ the cotangent coordinates on T^*X associated to $x_1, \ldots, x_n$.

Definition 2.1.12 (tautological form)
The tautological 1-form α is defined pointwise by

$$\alpha_p = (\mathrm{d}\pi_p)^\vee \xi \in \mathrm{T}^*_p(\mathrm{T}^*X)$$

That is, if $p = (x, \xi) \in \mathrm{T}^*X$, $v \in \mathrm{T}_p(\mathrm{T}^*X)$, then

$$\alpha_p(v) = \xi((\mathrm{d}\pi_p)v)$$

Definition 2.1.13 (canonical form)
The canonical symplectic 2-form ω on T^*X is defined by

$$\omega = -\mathrm{d}\alpha$$

Lemma 2.1.14. In cotangent coordinates,

$$\alpha = \sum_i \xi_i \mathrm{d}x_i \quad \text{and} \quad \omega = \sum_i \mathrm{d}x_i \wedge \mathrm{d}\xi_i$$

2.2 Almost complex structures

2.2.1 Complex structures

Definition 2.2.1 (complex structure)
Let V be a vector space, a complex structure on V is a linear map $J : V \to V$ with $J^2 = -\mathrm{id}_V$. The pair (V, J) is called a complex vector space.

Definition 2.2.2 (compatible)
Let (V, Ω) be a symplectic vector space, a complex structure J on V is compatible with Ω if

$$G_J(u, v) = \Omega(u, Jv)$$

defines an inner product on V.

Proposition 2.2.3. Let (V, Ω) be a symplectic vector space, then there exists a compatible complex structure J on V.

Definition 2.2.4 (almost complex structure)
An almost complex structure on a manifold M is a smooth field of complex structures J on TM, i.e. $J_x : T_xM \to T_xM$ is linear, with $J_x^2 = \mathrm{id}$.

The pair (M, J) is called an almost complex manifold.

Definition 2.2.5 (compatible)
Let (M, ω) be a symplectic manifold. An almost complex structure J on M is compatible with ω if

$$g_x(u, v) = \omega_x(u, J_x v)$$

is a Riemannian metric on M. The triple (ω, g, J), where ω is a symplectic form, g a Riemannian metric, J an almost complex structure is called a compatible triple if $g(u, v) = \omega(u, Jv)$.

Proposition 2.2.6. Let (M, ω) be a symplectic manifold, g a Riemannian metric on M, then there exists a canonical almost complex structure on M which is compatible.

Corollary 2.2.7. Any symplectic manifold admits a compatible almost complex structure.

Proposition 2.2.8. Let (M, ω) be a symplectic manifold, J_0, J_1 almost complex structures compatible with ω. Then we have a smooth family J_t of almost complex structures compatible with ω.

Proposition 2.2.9. If (ω, g, J) is a compatible triple, then we can write any one of them in terms of the other two. That is,

1. $g(u, v) = \omega(u, Jv)$,

2. $\omega(u, v) = g(Ju, v)$,

3. $J(u) = \tilde{g}^{-1}(\tilde{\omega}(u))$,

where $\tilde{\omega}, \tilde{g} : TM \to T^*M$, are linear isomorphisms defined by

$$\tilde{\omega}(u)(v) = \omega(u, v) \quad \text{and} \quad \tilde{g}(u)(v) = g(u, v)$$

Definition 2.2.10 (almost complex submanifold)
A submanifold X of an almost complex manifold (M, J) is an almost complex submanifold if $J(TX) \subseteq TX$.

2.2.2 Complexification

Definition 2.2.11 (complexified tangent bundle)
Let (M, J) be an almost complex manifold, the complexified tangent bundle of M is the bundle $TM \otimes \mathbb{C}$, with fibre $(TM \otimes \mathbb{C})_p = T_pM \otimes \mathbb{C}$. Each fibre is a complex vector space.

Proposition 2.2.12. We can extend J to $TM \otimes \mathbb{C}$ by

$$J(v \otimes c) = Jv \otimes c$$

Definition 2.2.13 (J-(anti-)holomorphic tangent vectors)
Define the J-holomorphic tangent vectors to be the eigenvectors of J with eigenvalue i, and the J-anti-holomorphic tangent vectors to be the eigenvectors of J with eigenvalue $-i$. That is,

$$T_{1,0} = \{v \in TM \otimes \mathbb{C} \mid Jv = iv\}$$
$$T_{0,1} = \{v \in TM \otimes \mathbb{C} \mid Jv = -iv\}$$

Lemma 2.2.14. Define $\pi_{1,0} : TM \to T_{1,0}$ by

$$\pi_{1,0}(v) = \frac{1}{2}(v \otimes 1 - Jv \otimes i)$$

Then $\pi_{1,0}$ defines an isomorphism of vector bundles, with $\pi_{1,0} \circ J = i\pi_{1,0}$. An analogous statement holds for $\pi_{0,1}$.

Corollary 2.2.15. Extending $\pi_{1,0}$ and $\pi_{0,1}$ to $TM \otimes \mathbb{C}$, we have an isomorphism

$$(\pi_{1,0}, \pi_{0,1}) : TM \otimes \mathbb{C} \to T_{1,0} \oplus T_{0,1}$$

A very similar result holds for the cotangent bundle, that is, we have an isomorphism

$$(\pi^{1,0}, \pi^{0,1}) : T^*M \otimes \mathbb{C} \to T^{1,0} \oplus T^{0,1}$$

where $T^{1,0}$ and $T^{0,1}$ are the complex (anti-)linear cotangent vectors.

2.2.3 Differential forms

Fix an almost complex manifold (M, J).

Definition 2.2.16 (forms of type (ℓ, m))
For $\ell, m \geq 0$, define

$$\Lambda^{\ell,m} = (\Lambda^{\ell}T^{1,0}) \wedge (\Lambda^{m}T^{0,1})$$

and the forms of type (ℓ, m) is the space of smooth sections of $\Lambda^{\ell,m}$, denoted by $\Omega^{\ell,m}$.

Definition 2.2.17 (complex valued forms)
Let

$$\Lambda^{k}(T^{*}M \otimes \mathbb{C}) = \Lambda^{k}(T^{1,0} \oplus T^{0,1}) = \bigoplus_{\ell+m=k} \Lambda^{\ell,m}$$

Then a section of $\Lambda^{k}(T^{*}M \otimes \mathbb{C})$ is called a complex valued k-form. The space of all complex values k forms is denoted by $\Omega^{k}(M; \mathbb{C})$.

Proposition 2.2.18.

$$\Omega^{k}(M; \mathbb{C}) = \bigoplus_{\ell+m=k} \Omega^{\ell,m}$$

Definition 2.2.19 (projection maps)
Define the projection maps

$$\pi^{\ell,m} : \Lambda^{\ell+m}(T^{*}M \otimes \mathbb{C}) \to \Lambda^{\ell,m}$$

Definition 2.2.20 (differential operators)
Define the differential operators

$$\partial = \pi^{\ell+1,m} \circ d : \Omega^{\ell,m} \to \Omega^{\ell+1,m}$$
$$\bar{\partial} = \pi^{\ell,m+1} \circ d : \Omega^{\ell,m} \to \Omega^{\ell,m+1}$$

2.2.4 J-holomorphic functions

Let $f : M \to \mathbb{C}$ be smooth complex values, and define $\mathrm{d}f = \mathrm{d}(\operatorname{Re} f) + i\,\mathrm{d}(\operatorname{Im} f)$.

Definition 2.2.21 (J-holomorphic functions)
f is J holomorphic at $p \in M$ if $\mathrm{d}f_p \circ J = i\mathrm{d}f_p$, i.e. $\mathrm{d}f_p$ is complex linear. f is J holomorphic if it is J holomorphic at every point.

Remark 2.2.22. We can define J-anti-holomorphic functions similarly.

2.2.5 Dolbeault cohomology

Lemma 2.2.23. Suppose $d = \partial + \overline{\partial}$. Then

$$\overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0 \quad \text{and} \quad \partial^2 = 0$$

Definition 2.2.24 (Dolbeault cohomology)
The cohomology groups given by $\overline{\partial}$ is called the Dolbeault cohomology groups, denoted by

$$\mathrm{H}^{\ell,m}_{\mathrm{Dolbeault}}(M)$$

2.3 Kähler manifolds

2.3.1 Complex manifolds

2.3.2 Kähler forms

2.3.3 Compact Kähler manifolds

2.4 Moment maps

2.4.1 Hamiltonian and symplectic vector fields

2.4.2 (Hamiltonian) actions

2.4.3 Symplectic reduction

2.4.4 Gauge theory

2.4.5 Existence and uniqueness of moment maps