Coadjoint Orbits of SU(n)

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In this note, we will consider the coadjoint orbits of SU(n), and show that they are Kähler manifolds.

Contents

Notation

Throughout this document, we will use the following notation:

- SU(n) is the group of $n \times n$ unitary matrices with determinant 1.
- $\mathfrak{su}(n)$ is its Lie algebra.
- $\mathrm{Ad} : \mathrm{SU}(n) \to \mathrm{GL}(\mathfrak{su}(n))$ is the adjoint representation.
- $\mathrm{Ad}^* : \mathrm{SU}(n) \to \mathrm{GL}(\mathfrak{su}(n)^*)$ the coadjoint representaton.
- $\mathrm{ad} : \mathfrak{su}(n) \to \mathfrak{gl}(\mathfrak{su}(n))$ the adjoint representation, given by $\mathrm{ad}_X(Y) = [X, Y]$.
- M will be a (co)adjoint orbit, $A \in M$ a diagonal element.
- $\langle \cdot, \cdot \rangle$ will denote both the pairing $\mathfrak{su}(n)^* \times \mathfrak{su}(n) \to \mathbb{R}$ and the inner product on $\mathfrak{su}(n)$,
- $\Phi : \mathfrak{su}(n) \to \mathfrak{su}(n)^*$ is the isomorphism induced by the inner product,
- $\ell_g(h) = gh$ is the left multiplication by g map,
- $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ the Riemannian metric on the (co)adjoint orbit.

1 Adjoint and Coadjoint Orbits

Define the Lie algebra

$$\mathfrak{su}(n) = \{X \in \mathrm{Mat}(n, \mathbb{C}) \mid X^* + X = 0, \mathrm{tr}(X) = 0\} \tag{1}$$

where X^* is the conjugate transpose of X, and with the Lie bracket being the matrix commutator. We can define the adjoint representation of $\mathrm{SU}(n)$ as

$$\begin{aligned} \mathrm{Ad} : \mathrm{SU}(n) &\to \mathrm{GL}(\mathfrak{su}(n)) \\ \mathrm{Ad}_g(X) &= gXg^* \end{aligned}$$

Taking the dual representation, we get the coadjoint representation, which is

$$\begin{aligned} \mathrm{Ad}^* : \mathrm{SU}(n) &\to \mathrm{GL}(\mathfrak{su}(n)^*) \\ \mathrm{Ad}^*_g(\beta)(X) &= \left\langle \beta, \mathrm{Ad}_{g^{-1}}(X) \right\rangle \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is used here to denote the pairing $\mathfrak{su}(n)^* \times \mathfrak{su}(n) \to \mathbb{R}$. We will use the same notation for the inner product on $\mathfrak{su}(n)$, which should not be an issue as the inner product defines a natural isomorphism. Now note that $-\kappa$, where κ is the Killing form, defines an inner product

$$\langle X, Y \rangle = -\mathrm{tr}(XY) = \mathrm{tr}(XY^*)$$

defines an inner product on $\mathfrak{su}(n)$[1], which means that we have a natural isomorphism

$$\begin{aligned} \Phi : \mathfrak{su}(n) &\to \mathfrak{su}(n)^* \\ X &\mapsto \langle X, \cdot \rangle \end{aligned}$$

With this, suppose $\beta = \Phi(B)$, then

$$\mathrm{Ad}^*_g(\beta)(X) = \left\langle B, \mathrm{Ad}_{g^{-1}}(X) \right\rangle = -\mathrm{tr}\left(Bg^{-1}Xg\right) = -\mathrm{tr}\left(BAg^{-1}X\right) = \Phi(\mathrm{Ad}_g(B))(X)$$

Therefore, the following diagram commutes

$$\begin{tikzcd} \mathfrak{su}(n) \arrow[r, "\mathrm{Ad}_g"] \arrow[d, "\Phi"'] & \mathfrak{su}(n) \arrow[d, "\Phi"] \\ \mathfrak{su}(n)^* \arrow[r, "\mathrm{Ad}^*_g"] & \mathfrak{su}(n)^* \end{tikzcd}$$

or equivalently, Φ defines an isomorphism of representations between Ad and Ad^*.

2 Root decomposition

Consider the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ of trace free $n \times n$ complex matrices. Then we have the Cartan subalgebra $\mathfrak{t}$ of diagonal matrices. Let E_{ij} be the standard basis matrices for $\mathrm{Mat}(n, \mathbb{C})$, $B \in \mathfrak{t}$. Say

$$B = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix}$$

Then $[B, E_{ij}] = (b_i - b_j)E_{ij}$. This means that we have the eigendecomposition

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{t} \oplus \bigoplus_{1 \le i,j \le n, i \ne j} \mathbb{C}E_{ij} \tag{2}$$

[1] In fact, $\langle A, B \rangle = \mathrm{tr}(AB^*)$ defines a Hermitian inner product on the space of complex matrices.

In particular, if we restrict this to the subalgebra $\mathfrak{su}(n)$, we get the decomposition

$$\mathfrak{su}(n) = \widetilde{\mathfrak{t}} \oplus \bigoplus_{1\leq i<j\leq n} \big(\mathbb{R}(E_{ij} - E_{ji}) \oplus i\mathbb{R}(E_{ij} + E_{ji})\big) \tag{3}$$

where $\widetilde{\mathfrak{t}} = \mathfrak{t} \cap \mathfrak{su}(n)$ is the subalgebra of $\mathfrak{su}(n)$ of diagonal matrices.

3 Tangent space and Diagonalisation

3.1 Diagonalisation and Stabilisers of the coadjoint action

First of all, we note that elements of $\mathfrak{su}(n)$ are skew-hermitian, hence diagonalisable by an element of $\mathrm{SU}(n)$[2]. With this, we can classify the coadjoint orbits based off a diagonal element in the orbit. Consider

$$A = \begin{pmatrix} i\lambda_1 I_{m_1} & & \\ & \ddots & \\ & & i\lambda_k I_{m_k} \end{pmatrix}$$

where I_m is the $m \times m$ identity matrix, $\lambda_j \in \mathbb{R}$, with $\lambda_1 > \lambda_2 > \cdots > \lambda_k$, $m_1 + \cdots + m_k = n$ and $m_1\lambda_1 + \cdots + m_k\lambda_k = 0$. In this case, we have that the orbit is

$$\mathrm{Orb}(A) \cong \mathrm{SU}(n)/\,\mathrm{Stab}(A)$$

where $\mathrm{Stab}(A)$ is the stabiliser of A under the adjoint action. In this case, we have that the stabiliser is the block diagonal subgroup

$$\mathrm{Stab}(A) = \mathrm{S}\left(\mathrm{U(m_1)} \times \cdots \times \mathrm{U(m_k)}\right)$$

where we consider $\mathrm{U(m_1)} \times \cdots \times \mathrm{U(m_k)} \leq \mathrm{SU(n)}$ as the block diagonal subgroup, and

$$\mathrm{S}\left(\mathrm{U(m_1)} \times \cdots \times \mathrm{U(m_k)}\right) = \left(\mathrm{U(m_1)} \times \cdots \times \mathrm{U(m_k)}\right) \cap \mathrm{SU(n)}$$

the subgroup with determinant 1.

3.2 Tangent space

Let M be an adjoint orbit. We will now focus on the generic case, that is, the eigenvalues of A are distinct. In this case, we have that

$$\mathrm{Stab}(A) = T^{n-1}$$

is the torus of diagonal matrices in $\mathrm{SU}(n)$, and we have a diffeomorphism $M \simeq \mathrm{SU}(n)/T$. Therefore, the tangent space at A is

$$\mathrm{T}_A M = \frac{\mathfrak{su}(n)}{\widetilde{\mathfrak{t}}} = \bigoplus_{1\leq i<j\leq n} \big(\mathbb{R}(E_{ij} - E_{ji}) \oplus i\mathbb{R}(E_{ij} + E_{ji})\big) = \{\mathrm{ad}_A(X) \mid X \in \mathfrak{su}(n)\}$$

where the last equality follows from the fact that

$$\mathrm{ad}_A(E_{ij}) = i(\lambda_i - \lambda_j)E_{ij}$$

and that diagonal matrices commute. Next, for $g \in \mathrm{SU}(n)$, $\mathrm{Ad}_g : \mathfrak{su}(n) \to \mathfrak{su}(n)$ is an invertible linear map, hence a diffeomorphism. In particular, this means that if $B = \mathrm{Ad}_g(A)$, then

$$\mathrm{T}_B M = \mathrm{Ad}_g(\mathrm{T}_A M) = \left\{\mathrm{Ad}_g(\mathrm{ad}_A(X)) \mid X \in \mathfrak{su}(n)\right\} = \{\mathrm{ad}_B(X) \mid X \in \mathfrak{su}(n)\} \tag{4}$$

using the fact that

$$\mathrm{ad}_{\mathrm{Ad}_g(A)} = \mathrm{Ad}_g \circ \mathrm{ad}_A \circ \mathrm{Ad}_{g^*}$$

[2]From standard linear algebra arguments, we know that they are $\mathrm{U}(n)$-diagonalisable. But if PBP^{-1} is diagonal, then so is $(\lambda P)B(\lambda P)^{-1}$, and by choosing λ appropriately, $\lambda \in \mathrm{SU}(n)$.

which we will show in the proof of lemma 4.2, and that Ad_{g^*} is a bijection. Moreover, if $\beta = \Phi(B)$, then

$$\begin{aligned}
\langle (\mathrm{ad}_X)^*(\beta), Y \rangle &= \langle \beta, \mathrm{ad}_X(Y) \rangle \\
&= \langle B, [X, Y] \rangle \\
&= -\operatorname{tr}(BXY - BYX) \\
&= -\operatorname{tr}(BXY - XBY) \\
&= \langle [B, X], Y \rangle \\
&= \langle -\mathrm{ad}_X(B), Y \rangle
\end{aligned}$$

Therefore, we have that the following diagram commutes.

$$\begin{tikzcd}
\mathfrak{su}(n) \arrow[r, "-\mathrm{ad}_X"] \arrow[d, "\Phi"'] & \mathfrak{su}(n) \arrow[d, "\Phi"] \\
\mathfrak{su}(n)^* \arrow[r, "\mathrm{ad}_X^*"'] & \mathfrak{su}(n)^*
\end{tikzcd}$$

Thus, we have the tangent space to the corresponding coadjoint orbit is

$$T_B\widetilde{M} = \{\mathrm{ad}_X^*(B) \mid X \in \mathfrak{su}(n)\}$$

4 Kirillov-Kostant-Souriau symplectic form

This section is from [2] Chapter 14. The proof is slightly modified, since here we have an explicit isomorphism between the adjoint and coadjoint representation, which simplifies some of the arguments.

Theorem 4.1. [Kirillov-Kostant-Souriau, [2, Theorem 14.4.1]] Let $M \subseteq \mathfrak{su}(n)^*$ be a coadjoint orbit. Define the 2-form ω on M by

$$\omega_\mu(\mathrm{ad}_X^*(\mu), \mathrm{ad}_Y^*(\mu)) = -\langle \mu, [X, Y] \rangle$$

Then ω is a symplectic form on M.

4.1 ω is well defined

First of all, we show that ω is well defined. That is, it is independent of the choice of $X, Y \in \mathfrak{su}(n)$.

Suppose $Z \in \mathfrak{su}(n)$ is such that $\mathrm{ad}_Z^*(\mu) = \mathrm{ad}_X^*(\mu)$. Then we must have that

$$\langle \mu, [X, Y] \rangle = \langle \mu, [Z, Y] \rangle$$

for all $Y \in \mathfrak{su}(n)$.

4.2 ω is non-degenerate

Suppose we have $X \in \mathfrak{su}(n)$ such that

$$\omega_\mu(\mathrm{ad}_X^*(\mu), \mathrm{ad}_Y^*(\mu)) = \langle \mu, [X, Y] \rangle = 0$$

for all Y. But this is the same as $\mathrm{ad}_X^*(\mu) = 0$. Therefore, ω is non-degenerate.

4.3 ω is closed

First of all, we will need some preliminary results.

Lemma 4.2.

$$\mathrm{ad}^*_{\mathrm{Ad}_g X} = \mathrm{Ad}^*_g \circ \mathrm{ad}^*_X \circ \mathrm{Ad}^*_{g^*}$$

Proof. We will prove the corresponding statement for Ad and ad, and the result will follow by conjuation with the isomorphism Φ.

$$\mathrm{Ad}_g \circ \mathrm{ad}_X \circ \mathrm{Ad}_{g^*}(Y) = gXg^*Ygg^* - gg^*YgXg^* = [gXg^*, Y] = \mathrm{ad}_{\mathrm{Ad}_g X}(Y)$$

$\square$

Lemma 4.3.

$$\mathrm{Ad}_g([X, Y]) = [\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)]$$

Proof. Expand using the definition of Ad. $\square$

Lemma 4.4. $\mathrm{Ad}^*_g : M \to M$ preserves ω, that is,

$$(\mathrm{Ad}^*_g)^*\omega = \omega$$

Proof. Evaluating $\mathrm{ad}^*_{\mathrm{Ad}_g X} = \mathrm{Ad}^*_g \circ \mathrm{ad}^*_X \circ \mathrm{Ad}^*_{g^{-1}}$ at $\nu = \mathrm{Ad}^*_g(\mu)$, we get

$$\mathrm{ad}^*_{\mathrm{Ad}_g X}(\nu) = \mathrm{Ad}^*_g \circ \mathrm{ad}^*_X(\mu) = \mathrm{d}_\mu \mathrm{Ad}^*_g \circ \mathrm{ad}^*_X(\mu)$$

Therefore,

$$\begin{aligned}
((\mathrm{Ad}^*_g)^*\omega)_\mu(\mathrm{ad}^*_X(\mu), \mathrm{ad}^*_Y(\mu)) &= \omega_\nu(\mathrm{d}_\mu \mathrm{Ad}^*_g \cdot \mathrm{ad}^*_X(\mu), \mathrm{d}_\mu \mathrm{Ad}^*_g \cdot \mathrm{ad}^*_Y(\mu)) \\
&= \omega_\nu(\mathrm{ad}^*_{\mathrm{Ad}_g X}(\nu), \mathrm{ad}^*_{\mathrm{Ad}_g Y}(\nu)) \\
&= -\left\langle \nu, [\mathrm{Ad}_g X, \mathrm{Ad}_g Y] \right\rangle \\
&= -\left\langle \nu, \mathrm{Ad}_g([X, Y]) \right\rangle \\
&= -\left\langle \mathrm{Ad}^*_{g^{-1}}(\nu), [X, Y] \right\rangle \\
&= -\langle \mu, [X, Y] \rangle \\
&= \omega_\mu(\mathrm{ad}^*_X(\mu), \mathrm{ad}^*_Y(\mu))
\end{aligned}$$

$\square$

For $\nu \in \mathfrak{su}(n)^*$, define the left-invariant one-form

$$\nu_\ell(g) = (\mathrm{d}_g \ell_{g^{-1}})^*(\nu)$$

for $g \in \mathrm{SU}(n)$. Similarly, for $X \in \mathfrak{su}(n)$, let X_ℓ be the corresponding left invariant vector field on G. Then $\nu_\ell(X_\ell) = \langle \nu, X \rangle$ at all $g \in \mathrm{SU}(n)$.

Fix $\nu \in M$, and consider the map $\pi : \mathrm{SU}(n) \to M$, defined by

$$\pi(g) = \mathrm{Ad}^*_g(\nu)$$

We can use this to pullback $\sigma = \pi^*\omega$ to a two form on $\mathrm{SU}(n)$.

Lemma 4.5. σ is left invariant. That is, $\ell^*_g\sigma = \sigma$ for all $g \in \mathrm{SU}(n)$.

Proof. First, notice that $\pi \circ \ell_g = \mathrm{Ad}_g^* \circ \pi$, since

$$\pi(\ell_g(h)) = \mathrm{Ad}_{gh}^*(\nu) = \mathrm{Ad}_g^* \circ \mathrm{Ad}_h^*(\nu) = \mathrm{Ad}_g^*(\pi(h))$$

With this,

$$\ell_g^*\sigma = \ell_g^*\pi^*\omega = (\pi \circ \ell_g)^*\omega = (\mathrm{Ad}_g^* \circ \pi)^*\omega = \pi^*(\mathrm{Ad}_g^*)^*\omega = \pi^*\omega = \sigma$$

□

Lemma 4.6. $\sigma(X_\ell, Y_\ell) = -\langle \nu_\ell, [X_\ell, Y_\ell] \rangle$.

Proof. By left invariance of both sides, suffices to show that the result holds at e. First notice that

$$\mathrm{d}_I\pi(Y) = -\mathrm{ad}_Y^*(\nu)$$

Therefore, π is a submersion at e. By definition of the pullback,

$$\begin{aligned}
\sigma_I(X, Y) &= (\pi^*\omega)_I(X, Y) \\
&= \omega_{\pi(I)}(\mathrm{d}_e\pi \cdot X, \mathrm{d}_e\pi \cdot Y) \\
&= \omega_\nu(\mathrm{ad}_X^*(\nu), \mathrm{ad}_Y^*(\nu)) \\
&= -\langle \nu, [X, Y] \rangle
\end{aligned}$$

Hence

$$\sigma(X_\ell, Y_\ell)_I = \sigma_I(X, Y) = -\langle \nu, [X, Y] \rangle = -\langle \nu_\ell, [X_\ell, Y_\ell] \rangle_I$$

□

Now for a one form α, we have that

$$\mathrm{d}\alpha(X, Y) = X[\alpha(Y)] - Y[\alpha(X)] - \alpha([X, Y])$$

where for a smooth function $f : M \to \mathbb{R}$, and a vector field X on M, $X[f] := \mathrm{d}f(X)$ is a smooth function $M \to \mathbb{R}$.

Since $\nu_\ell(X_\ell)$ is constant, $Y_\ell[\nu_\ell(X_\ell)] = 0$. Similarly, $X_\ell[\nu_\ell(Y_\ell)] = 0$. Therefore, we have that

$$\mathrm{d}\nu_\ell(X_\ell, Y_\ell) = -\nu_\ell([X_\ell, Y_\ell]) = \sigma(X_\ell, Y_\ell)$$

Now suppose U, V are vector fields on $\mathrm{SU}(n)$. We want to show that $\sigma(U, V) = \mathrm{d}\nu_\ell(U, V)$. As σ is left invariant,

$$\begin{aligned}
\sigma(U, V)_g &= (\ell_{g^{-1}}^*\sigma)_g(U_g, V_g) \\
&= \sigma_I(\underbrace{\mathrm{d}\ell_{g^{-1}} \cdot U_g}_{=:X}, \underbrace{\mathrm{d}\ell_{g^{-1}} \cdot V_g}_{=:Y}) \\
&= \sigma_I(X, Y) \\
&= \mathrm{d}\nu_\ell(X_\ell, Y_\ell)_I \\
&= (\ell_g^*\mathrm{d}\nu_\ell)(X_\ell, Y_\ell)_I \\
&= (\mathrm{d}\nu_\ell)_g(\mathrm{d}\ell_g(X_\ell)_I, \mathrm{d}\ell_g(Y_\ell)_I) \\
&= (\mathrm{d}\nu_\ell)_g(\mathrm{d}\ell_g X, \mathrm{d}\ell_g Y) \\
&= (\mathrm{d}\nu_\ell)_g(U_g, V_g) \\
&= \mathrm{d}\nu_\ell(U, V)_g
\end{aligned}$$

With this, $\mathrm{d}\sigma = \mathrm{d}^2\nu_\ell = 0$. Hence $\pi^*\mathrm{d}\omega = \mathrm{d}(\pi^*\omega) = \mathrm{d}\sigma = 0$. Since $\pi \circ \ell_g = \mathrm{Ad}_g^* \circ \ell_g$, and π is a submersion at I, it is in fact a submersion everywhere. Moreover, π is surjective, by definition.

For $\mu \in M$, and $X, Y \in T_\mu M$, we have that

$$\mathrm{d}\omega_\mu(X, Y, Z) = \mathrm{d}\omega_{\pi(g)}(\mathrm{d}\pi(U), \mathrm{d}\pi(V), \mathrm{d}\pi(W)) = (\pi^*\mathrm{d}\omega)_g(U, V, W) = 0$$

where $g \in \mathrm{SU}(n)$ is such that $\pi(g) = \mu$, which exists by surjectivity, and $U, V, W \in T_g \mathrm{SU}(n)$ such that $\mathrm{d}\pi(U) = X$, $\mathrm{d}\pi(V) = Y$ and $\mathrm{d}\pi(W) = Z$, which exists as π is a submersion. Thus, as $\mu \in M$ is arbitrary, ω is closed.

4.4 ω on adjoint orbits

Using the isomorphism Φ, theorem 4.1 and the computation for ad^*_X, we get the following result.

Theorem 4.7. Let $M \subseteq \mathfrak{su}(n)$ be an adjoint orbit. Define the 2-form ω on M by

$$\omega_A([A, B], [A, C]) = -\langle A, [B, C]\rangle = \mathrm{tr}(A[B, C])$$

Then ω is a symplectic form on M.

This will be convenient for us since we can compute the right hand side directly from the matrices.

5 Kähler structure

In this section, we construct the Kähler structure on adjoint orbits of $\mathrm{SU}(n)$. In principle, we only need two of (ω, g, J) as we can recover the third. See Cannas da Silva [1, §13.2] for a table which summarises the relations and the required conditions if we only have two of the three.

We have already constructed the Kirillov-Kostant-Souriau symplectic form ω. In the following, we will construct the Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$[3] and the almost complex structure J. Once we have shown that these form a compatible triple, since ω is closed and J comes from a diffeomorphism between the adjoint orbit and a complex manifold, we will have that ω is in fact a Kähler form.

5.1 At a diagonal element

Let M be an adjoint orbit. Recall from section 3 that the stabiliser of A is the torus $T \cong T^{n-1}$ of diagonal matrices in $\mathrm{SU}(n)$.

$$\mathrm{T}_A M = \bigoplus_{1 \le i < j \le n} \left(\mathbb{R}(E_{ij} - E_{ji}) \oplus i\mathbb{R}(E_{ij} + E_{ji})\right) \cong \frac{\mathfrak{su}(n)}{\mathfrak{t}} \cong \mathrm{T}_{[1]}\left(\frac{\mathrm{SU}(n)}{T^{n-1}}\right)$$

where the isomorphism is induced by the quotient map

$$\begin{aligned} \pi : \mathrm{SU}(n) &\to M \\ g &\mapsto \mathrm{Ad}_g(A) = gAg^* \end{aligned}$$

Let $e_{ij} = E_{ij} - E_{ji}$ and $f_{ij} = i(E_{ij} + E_{ji})$.

Lemma 5.1.
1. For any $X, Y, Z \in \mathfrak{su}(n)$, $\langle X, [Y, Z]\rangle = \langle [X, Y], Z\rangle$,
2. $[A, E_{ij}] = i(\lambda_i - \lambda_j)E_{ij}$
3. $E_{ij}E_{kl} = \delta_{jk}E_{il}$
4. $\langle \cdot, \cdot \rangle$ is $\mathbb{C}$-bilinear.

Proof. Expand. □

[3] Quite often g will be used for an element of $\mathrm{SU}(n)$, and so we will use $\langle\langle \cdot, \cdot \rangle\rangle$ to denote the inner product.

Using these, we have that

$$\begin{aligned}
\langle A, [E_{ij}, E_{kl}] \rangle &= \langle [A, E_{ij}], E_{kl} \rangle \\
&= i(\lambda_i - \lambda_j) \langle E_{ij}, E_{kl} \rangle \\
&= -i(\lambda_i - \lambda_j)\delta_{jk} \operatorname{tr}(E_{il}) \\
&= -i(\lambda_i - \lambda_j)\delta_{jk}\delta_{il} \\
&= i(\lambda_j - \lambda_i)\delta_{il}\delta_{jk}
\end{aligned}$$

and so,

$$\begin{aligned}
\langle A, [e_{ij}, e_{kl}] \rangle &= \langle A, [E_{ij}, E_{kl}] \rangle - \langle A, [E_{ji}, E_{kl}] \rangle - \langle A, [E_{ij}, E_{lk}] \rangle + \langle A, [E_{ji}, E_{lk}] \rangle \\
&= i(\lambda_j - \lambda_i)\delta_{il}\delta_{jk} - i(\lambda_i - \lambda_j)\delta_{jl}\delta_{ik} - i(\lambda_j - \lambda_i)\delta_{ik}\delta_{jl} + i(\lambda_i - \lambda_j)\delta_{jk}\delta_{il} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
-\langle A, [f_{ij}, f_{kl}] \rangle &= \langle A, [E_{ij}, E_{kl}] \rangle + \langle A, [E_{ji}, E_{kl}] \rangle + \langle A, [E_{ij}, E_{lk}] \rangle + \langle A, [E_{ji}, E_{lk}] \rangle \\
&= i(\lambda_j - \lambda_i)\delta_{il}\delta_{jk} + i(\lambda_i - \lambda_j)\delta_{jl}\delta_{ik} + i(\lambda_j - \lambda_i)\delta_{ik}\delta_{jl} + i(\lambda_i - \lambda_j)\delta_{jk}\delta_{il} \\
&= 0
\end{aligned}$$

Finally, we have that

$$\begin{aligned}
-i\langle A, [e_{ij}, f_{kl}] \rangle &= \langle A, [E_{ij}, E_{kl}] \rangle - \langle A, [E_{ji}, E_{kl}] \rangle + \langle A, [E_{ij}, E_{lk}] \rangle - \langle A, [E_{ji}, E_{lk}] \rangle \\
&= i(\lambda_j - \lambda_i)\delta_{il}\delta_{jk} - i(\lambda_i - \lambda_j)\delta_{jl}\delta_{ik} + i(\lambda_j - \lambda_i)\delta_{ik}\delta_{jl} - i(\lambda_i - \lambda_j)\delta_{jk}\delta_{il} \\
&= 2i(\lambda_j - \lambda_i)(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) \\
&= 2i(\lambda_j - \lambda_i)\delta_{ik}\delta_{jl}
\end{aligned}$$

where the last line is because we require $i < j$ and $k < l$. Hence we have that

$$\begin{aligned}
\omega_A(e_{ij}, e_{kl}) &= 0 \\
\omega_A(f_{ij}, f_{kl}) &= 0 \\
\omega_A(e_{ij}, f_{ij}) &= 2(\lambda_i - \lambda_j) \\
\omega_A(e_{ij}, f_{kl}) &= 0 \text{ for } (i, j) \neq (k, l)
\end{aligned}$$

We can then define an an almost complex structure on $\mathrm{T}_A M$ by

$$J_A(e_{ij}) = f_{ij} \quad J_A(f_{ij}) = -e_{ij} \tag{5}$$

Moreover, we can define an inner product on $\mathrm{T}_A M$ by

$$\langle\langle e_{ij}, e_{ij} \rangle\rangle_A = \langle\langle f_{ij}, f_{ij} \rangle\rangle_A = 2(\lambda_i - \lambda_j)$$

and requiring e_{ij}, f_{ij} to form an orthogonal basis. This is positive definite since we required $\lambda_i > \lambda_j$ for $i < j$. Using this, we find that

$$\omega_A(e_{ij}, f_{ij}) = 2(\lambda_i - \lambda_j) = \langle\langle f_{ij}, f_{ij} \rangle\rangle_A = \langle\langle J_A(e_{ij}), f_{ij} \rangle\rangle_A$$

and J defines an isometry. Hence $(\omega_A, \langle\langle \cdot, \cdot \rangle\rangle_A, J_A)$ define a compatible triple on the vector space $\mathrm{T}_A M$.

5.2 Complex quotient

Let P be the subgroup of lower triangular matrices in $\mathrm{SL}(n, \mathbb{C})$. Consider the composition $\varphi : \mathrm{SU}(n) \to \mathrm{SL}(n, \mathbb{C})/P$ given by the composition

$$\mathrm{SU}(n) \hookrightarrow \mathrm{SL}(n) \twoheadrightarrow \mathrm{SL}(n, \mathbb{C})/P$$

Suppose $\varphi(g) = \varphi(h)$. That is, $gP = hP$. This is true if and only if there exists $p \in P$, such that $h = gp$. In this case, $p = g^{-1}h \in \mathrm{SU}(n)$, therefore, $p \in \mathrm{SU}(n) \cap P = T$, since $p^* = p^{-1}$ is also lower triangular. This means that φ induces a homeomorphism $\mathrm{SU}(n)/T \cong \mathrm{SL}(n, \mathbb{C})/P$. The right hand side is a complex manifold ($\mathrm{SL}(n, \mathbb{C})$) quotiented by a complex Lie group P, so it is a complex manifold. Using the above, we can get a complex structure on $\mathrm{SU}(n)/T \cong M$.

Using the root decompositions eq. (2) for $\mathfrak{sl}(n, \mathbb{C})$ and eq. (3) for $\mathfrak{su}(n)$, we can see that the almost complex structure we defined in eq. (5) is the same as the action of multiplication by i.

5.3 At a general $B \in M$

Fix $g \in \mathrm{SU}(n)$, and let $B = \mathrm{Ad}_g(A)$. Let $\varphi : \mathrm{SU}(n)/T \to \mathrm{SL}(n, \mathbb{C})/P$ be the diffeomorphism from above, which is given by $\varphi([g]) = [\![g]\!]$, where $[g] = gT \in \mathrm{SU}(n)/T$ and $[\![g]\!] = gP \in \mathrm{SL}(n, \mathbb{C})/P$.

First of all, $\mathrm{SL}(n, \mathbb{C})$ is a complex Lie group, so left multiplication is holomorphic. That is, $\mathrm{d}\ell_g(iv) = i\mathrm{d}\ell_g(v)$. If $\widetilde{J}$ denotes the complex structure on $\mathrm{SL}(n, \mathbb{C})$, then we have that

$$\widetilde{J}_g = \mathrm{d}\ell_g \circ \widetilde{J}_I \circ \mathrm{d}\ell_{g^{-1}}$$

ℓ_g descends to a biholomorphism on $\mathrm{SL}(n, \mathbb{C})/P \cong \mathrm{SU}(n)/T$, and so the corresponding almost complex structure $\overline{J}$ on $\mathrm{SU}(n)/T$ is given by

$$\overline{J}_{[g]} = \mathrm{d}\ell_g \circ \overline{J}_{[I]} \circ \mathrm{d}\ell_{g^*} \tag{6}$$

Since the diffeomorphism $\mathrm{SU}(n)/T$ is induced by the map $\pi(g) = \mathrm{Ad}_g(A)$, we have that

$$\begin{aligned}
J_B &= \mathrm{d}\pi \circ \overline{J}_{[g]} \circ \mathrm{d}\pi^{-1} \\
&= \mathrm{d}\pi \circ \mathrm{d}\ell_g \circ \overline{J}_{[I]} \circ \mathrm{d}\ell_{g^*} \circ \mathrm{d}\pi^{-1} \\
&= \mathrm{d}\left(\pi \circ \ell_g \circ \pi^{-1}\right) \circ J_A \circ \mathrm{d}\left(\pi \circ \ell_{g^*} \circ \pi^{-1}\right)
\end{aligned}$$

From the proof of lemma 4.5, we have that $\pi \circ \ell_g = \mathrm{Ad}_g \circ \pi$. Moreover, since Ad_g is a linear map, $\mathrm{d}\,\mathrm{Ad}_g = \mathrm{Ad}_g$. Therefore, we have that the almost complex structure is given by

$$J_B = \mathrm{Ad}_g \circ J_A \circ \mathrm{Ad}_{g^*}$$

We want to show that this is compatible with the Kirillov-Kostant-Souriau symplectic form.

Recall from eq. (4) that

$$\mathrm{T}_B M = \mathrm{Ad}_g(\mathrm{T}_A M)$$

Then we have that

$$\begin{aligned}
\omega_B([B, X], [B, Y]) &= -\langle B, [X, Y]\rangle \\
&= -\left\langle \mathrm{Ad}_g(A), \mathrm{Ad}_g([\mathrm{Ad}_{g^*}(X), \mathrm{Ad}_{g^*}(Y)])\right\rangle \\
&= -\left\langle A, [\mathrm{Ad}_{g^*}(X), \mathrm{Ad}_{g^*}(Y)]\right\rangle \\
&= \omega_A([A, \mathrm{Ad}_{g^*}(X)], [A, \mathrm{Ad}_{g^*}(X)]) \\
&= \omega_A(\mathrm{Ad}_{g^*}([B, X]), \mathrm{Ad}_{g^*}([B, Y]))
\end{aligned}$$

Note this also follows from lemma 4.5 where we showed $\pi^*\omega$ is left invariant. Therefore, the Riemannian metric is given by

$$\langle\!\langle X, Y\rangle\!\rangle_B = \left\langle\!\left\langle \mathrm{Ad}_{g^*}(X), \mathrm{Ad}_{g^*}(Y)\right\rangle\!\right\rangle_A$$

References

[1] Ana Cannas da Silva. *Lectures on Symplectic Geometry*. Publisher: Springer Berlin, Heidelberg. July 2001. DOI: 10.1007/978-3-540-45330-7.

[2] Jerrold E. Marsden and Tudor S. Ratiu. *Introduction to Mechanics and Symmetry*. Publisher: Springer New York. Apr. 1999. DOI: 10.1007/978-0-387-21792-5.