

Computation of moment map

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1 HyperKähler moment maps

On \mathbb{H}^N , we can construct a hyperKähler structure, using the standard metric given by

$$\langle u, v \rangle = \bar{u}^\top v$$

where \bar{u} denotes the (elementwise) quaternionic conjugate of u . The complex structures are given by right multiplication by $-i, -j, -k$ respectively. Let $\omega_I, \omega_J, \omega_K$ be the corresponding Kähler forms, and $\eta = \omega_I i + \omega_J j + \omega_K k$.

Let H be a subgroup of $\mathrm{Sp}(N)$. Then H acts on \mathbb{H}^N preserving the hyperKähler structure. In this case, a hyperKähler moment map is a map $\mu : \mathbb{H}^N \rightarrow \mathfrak{h}^* \otimes \mathrm{Im}(\mathbb{H})$, which is equivariant with respect to the H , and with $d(\mu^X) = X \lrcorner \eta$

In particular, in [1], we make the choice

$$\mu^X(q) = -\bar{q}^\top X q = -\langle q, X q \rangle$$

2 $\mathrm{U}(n)$ action

Choose a sequence (V_0, \dots, V_k) of Hermitian vector spaces, with $\dim_{\mathbb{C}}(V_i) = n_i$, $n_0 = 0, n_k = n$. Let

$$M = \bigoplus_{i=0}^{k-1} (\mathrm{Hom}(V_i, V_{i+1}) \oplus \mathrm{Hom}(V_{i+1}, V_i))$$

and we write a point of M as (α_i, β_i) , where

$$V_0 \xrightleftharpoons[\beta_0]{\alpha_0} V_1 \xrightleftharpoons[\beta_1]{\alpha_1} \dots \xrightleftharpoons[\beta_{k-1}]{\alpha_{k-1}} V_k$$

Note that $\langle \alpha, \beta \rangle = \mathrm{tr}(\alpha \beta^*)$ defines a Hermitian metric on $\mathrm{Hom}(V, W)$, and hence on M , we have the metric

$$\left\langle \left\langle (\alpha_i, \beta_i), (\tilde{\alpha}_i, \tilde{\beta}_i) \right\rangle \right\rangle = \sum_{i=0}^{k-1} \left(\langle \alpha_i, \tilde{\alpha}_i \rangle + \langle \beta_i, \tilde{\beta}_i \rangle \right)$$

The complex structures are

$$I(\alpha_i, \beta_i) = (i\alpha_i, i\beta_i) \quad J(\alpha_i, \beta_i) = (-\beta_i^*, \alpha_i^*)$$

The Lie group $G = \mathrm{U}(n_1) \times \mathrm{U}(n_{k-1})$ acts on M via

$$\begin{aligned} \alpha_i &\mapsto g_{i+1} \alpha_i g_i^{-1} = g_{i+1} \alpha_i g_i^* \\ \beta_i &\mapsto g_i \beta_i g_{i+1}^{-1} = g_i \beta_i g_{i+1}^* \end{aligned}$$

Now notice that $\langle \alpha, \beta \rangle$ as above induces an isomorphism $\mathfrak{u}(m) \cong \mathfrak{u}(m)^*$, via $X \mapsto \langle X, \cdot \rangle$.

3 Moment map

Now let $X_i \in \mathfrak{u}(n_i)$, and let $X = (0, \dots, X_i, \dots, 0) \in \mathfrak{u}(n_1) \oplus \dots \oplus \mathfrak{u}(n_{k-1})$. Let $q = (\alpha_i, \beta_i) \in M$. Then the action of X is

$$Xq = (0, \dots, X_i \alpha_{i-1}, -\alpha_i X_i, \dots, 0, 0, \dots, -\beta_{i-1} X_i, X_i \beta_i, \dots, 0)$$

In particular, we have that

$$\begin{aligned} \langle\langle q, Xq \rangle\rangle &= \langle \alpha_{i-1}, X_i \alpha_{i-1} \rangle - \langle \alpha_i, \alpha_i X_i \rangle - \langle \beta_{i-1}, \beta_{i-1} X_i \rangle + \langle \beta_i, X_i \beta_i \rangle \\ &= \text{tr}(\alpha_{i-1} \alpha_{i-1}^* X_i^* - \alpha_i X_i^* \alpha_i^* - \beta_{i-1} X_i^* \beta_{i-1}^* + \beta_i \beta_i^* X_i^*) \\ &= \text{tr}((\alpha_i^* \alpha_i - \beta_i \beta_i^* + \beta_{i-1}^* \beta_{i-1} - \alpha_{i-1} \alpha_{i-1}^*) X_i) \end{aligned}$$

which gives us

$$\mu_r = (\alpha_{i-1} \alpha_{i-1}^* - \beta_{i-1}^* \beta_{i-1} + \beta_i \beta_i^* - \alpha_i^* \alpha_i)$$

Next, we can take

$$\begin{aligned} \langle\langle q, X \cdot (-J)(q) \rangle\rangle &= -\langle \alpha_{i-1}, -X_i \beta_{i-1}^* \rangle + \langle \alpha_i, -\beta_i X_i^* \rangle + \langle \beta_{i-1}, \alpha_{i-1}^* X_i \rangle - \langle \beta_i, \alpha_i^* X_i \rangle \\ &= \text{tr}(\alpha_{i-1} \beta_{i-1} X_i^* - \alpha_i X_i^* \beta_i + \beta_{i-1} X_i^* \alpha_{i-1} - \beta_i X_i^* \alpha_i) \\ &= -2 \text{tr}((\alpha_{i-1} \beta_{i-1} - \beta_i \alpha_i) X_i) \end{aligned}$$

which gives us

$$\mu_c = (\alpha_{i-1} \beta_{i-1} - \beta_i \alpha_i)$$

Note also that $IX = XI$ and $JX = XJ$, and that $I, J, K = IJ$ define isometries with respect to $\langle\langle \cdot, \cdot \rangle\rangle$, so we also have that

$$\langle\langle q, X \cdot (-J)q \rangle\rangle = \langle\langle Jq, Xq \rangle\rangle$$

References

- [1] Piotr Z. Kobak and Andrew Swann. "Classical nilpotent orbits as hyperkähler quotients". In: *Int. J. Math.* 07.02 (Apr. 1996). Publisher: World Scientific Publishing Co., pp. 193–210. ISSN: 0129-167X. doi: 10.1142/S0129167X96000116. URL: <https://www.worldscientific.com/doi/10.1142/S0129167X96000116> (visited on 07/27/2023).