Kähler reduction

Shing Tak Lam

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Throughout,

1. (M, ω, g, I) is a Kähler manifold,
2. G is a compact Lie group acting on M,
3. (M, ω, G, μ) is a Hamiltonian G-space,
4. G acts by biholomorphisms on M,
5. G acts freely on $\mu^{-1}(0)$.

In particular, as $\omega(u, v) = g(I(u), v)$, I is an isometry on M, G acts by isometries on M. Let $Z = \mu^{-1}(0)/G$. Let

$$\begin{tikzcd} \mu^{-1}(0) \arrow[r, hook, "i"] \arrow[d, two heads, "\pi"'] & M \\ Z = \mu^{-1}(0)/G & \end{tikzcd}$$

be the natural inclusion and quotient maps. In particular, note that π is a surjective submersion. Therefore, any tensor α of type $(0, r)$ on Z is determined by its pullback $\pi^*\alpha$.

The Marsden-Weinstein reduction theorem from symplectic geometry states that there exists a symplectic form $\widetilde{\omega}$ on Z, such that

$$\pi^*\widetilde{\omega} = i^*\omega$$

We will now construct the almost complex structure and Riemannian metric on Z.

Since π is a submersion, for $p \in \mu^{-1}(0)$, $z = \pi(p)$, we have

$$\mathrm{d}\pi_p : \mathrm{T}_p\mu^{-1}(0) \twoheadrightarrow \mathrm{T}_z Z$$

Let $V_p = \ker(\mathrm{d}\pi_p)$ be the vertical bundle, and $H_p = V_p^\perp \leq \mathrm{T}_p\mu^{-1}(0)$ be the horizontal bundle. Therefore, we have an isomorphism

$$\mathrm{d}\pi_p|_{H_p} : H_p \xrightarrow{\cong} \mathrm{T}_z Z$$

For brevity, we write this isomorphism as

$$\begin{aligned} \mathrm{d}\pi_p|_{H_p} : H_p &\to \mathrm{T}_z Z \\ v &\mapsto v_* \\ w^* &\leftarrow\!\shortmid w \end{aligned}$$

With this, we can see that

$$\widetilde{\omega}(u, v) = \omega(u^*, v^*)$$

and that

$$\widetilde{g}(u,v) = g(u^*, v^*)$$

defines a Riemannian metric on Z. Therefore, the almost complex structure we want must be given by

$$\widetilde{I}(u) = I(u^*)_*$$

Assuming this is well defined, then we have that

$$\begin{aligned}
\widetilde{\omega}(u,v) &= \omega(u^*, v^*) \\
&= g(I(u^*), v^*) \\
&= g(\widetilde{I}(u)^*, v^*) \\
&= \widetilde{g}(\widetilde{I}(u), v)
\end{aligned}$$

so (ω, g, I) is a compatible triple.

Lemma. I restricts to a map $H_p \to H_p$.

Proof. Let $N_p = \left(\mathrm{T}_p\mu^{-1}(0)\right)^\perp \leq \mathrm{T}_pM$ be the normal bundle of $\mu^{-1}(0) \subseteq M$. This gives us an orthogonal direct sum

$$\mathrm{T}_pM = N_p \oplus V_p \oplus H_p$$

Fix $X \in \mathfrak{g}$. Then for $v \in \mathrm{T}_pM$,

$$g(\mathrm{grad}(\mu^X), v) = \mathrm{d}\mu^X(v) = \omega(X^\#, v) = g(I(X^\#), v)$$

where $\mathrm{grad}(f)$ is the g-dual of $\mathrm{d}f$. In particular, this means that $\mathrm{grad}(\mu^X) = I(X^\#)$. Let $X_1, \ldots, X_k$ be a basis of $\mathfrak{g}$, with corresponding dual basis $\xi^1, \ldots, \xi^k$. Then the moment map can be written as

$$\mu(p) = \mu^{X_1}(p)\xi^1 + \cdots + \mu^{X_k}(p)\xi^k$$

But this means that

$$\left\{\mathrm{grad}(\mu^{X_1}), \ldots, \mathrm{grad}(\mu^{X_k})\right\} = \left\{I(X_1^\#), \ldots, I(X_k^\#)\right\}$$

is a basis of N_p. As $X_1^\#, \ldots, X_k^\#$ is a basis for V_p, we have that I restricts to a map $N_p \oplus V_p \to N_p \oplus V_p$. By orthogonality, this means that I restricts to a map $H_p \to H_p$. □

Therefore, the map $\widetilde{I}$ as above is well defined. Finally, we need to show that we have a Kähler structure. That is, I is integrable.

Lemma. Let M be a manifold, (ω, g, I) a compatible triple on M. Then (M, ω, g, I) is a Kähler manifold if and only if $\nabla I = 0$, where ∇ is the Levi-Civita connection induced by g.

Moreover, we have the expression

$$\nabla I(u) = \nabla(I(u)) - I(\nabla u)$$

and so $\nabla I = 0$ if and only if $\nabla(I(u)) = I(\nabla u)$ for all vector fields u.

Proof. See Huybrechts, §4.A. for the first part. For the second part, see Nicolaescu page 96. □

Lemma. The Levi-Civita connection induced by $\widetilde{g}$ is

$$\widetilde{\nabla}_X Y = \mathrm{pr}_H\left(\nabla_{X^*} Y^*\right)_*$$

for vector fields X, Y on Z, and we extend X^*, Y^* arbitrarily to a neighbourhood of $\mu^{-1}(0) \subseteq M$. In addition, $\mathrm{pr}_H : \mathrm{T}_p M \to H_p$ is the orthogonal projection.

Proof. Omitted. □

Finally, we note that since I respects the orthogonal decomposition

$$\mathrm{T}_p M = \left(N_p \oplus V_p\right) \oplus H_p$$

pr_H and I commute. With this, we can now compute $\widetilde{\nabla}\widetilde{I}$.

$$\begin{aligned}\left(\widetilde{\nabla}_X \widetilde{I}(Y)\right)^* &= \mathrm{pr}_H \left(\nabla_{X^*} \widetilde{I}(Y)^*)\right) \\ &= \mathrm{pr}_H \left(\nabla_{X^*} I(Y^*)\right) \\ &= \mathrm{pr}_H \left(I(\nabla_{X^*} Y^*)\right) \\ &= I \left(\mathrm{pr}_H(\nabla_{X^*} Y^*)\right) \\ &= \widetilde{I}(\widetilde{\nabla}_X Y)^*\end{aligned}$$

Hence we have that

$$\widetilde{\nabla}_X \widetilde{I}(Y) = \widetilde{I}(\widetilde{\nabla}_X Y)$$

for any vector fields X, Y on Z, and so $\widetilde{\nabla}\widetilde{I} = 0$, and $(Z, \widetilde{\omega}, \widetilde{g}, \widetilde{I})$ is a Kähler manifold.