

Homological algebra

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1 Motivation

In many contexts, instead of considering individual modules, we have a series of modules. One example of this is in simplicial homology, where we have a sequence of free modules, and the homology groups are defined in terms of this sequence. We can generalise the constructions from simplicial homology to more general sequences of modules.

2 Chain complexes

Definition 2.1 (Chain complex)

A chain complex (M_\bullet, d_\bullet) of R -modules is a sequence of R -modules M_i , and R -module homomorphisms d_i

$$\cdots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$

such that $d_i \circ d_{i+1} = 0$ for all i .

Here, we allow a complex to be infinite in both directions, although ‘tails’ of zeros are usually omitted.

Definition 2.2 (Boundary, differentials)

The homomorphisms d_i are called boundary, or differentials.

Remark 2.3. The condition that $d_i \circ d_{i+1} = 0$ is equivalent to $\text{im } d_{i+1} \subseteq \ker d_i$.

In fact, the case when equality holds is important enough that we give it a special name.

2.1 Exact sequences

Definition 2.4 (Exact)

A complex (M_\bullet, d_\bullet) is exact at M_i if $\text{im } d_{i+1} = \ker d_i$. If a complex is exact at all M_i , then we call it an exact sequence.

Example 2.5

A complex

$$0 \longrightarrow L \xrightarrow{\alpha} M$$

is exact at L if and only if α is injective, since exactness at L is exactly $\ker \alpha = 0$, which is equivalent to α being injective.

Example 2.6

A complex

$$M \xrightarrow{\beta} N \longrightarrow 0$$

is exact at N if and only if β is surjective, since exactness at N is exactly $\text{im } \beta = N$.

Definition 2.7 (Short exact sequence)

A short exact sequence is an exact complex of the form

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

Remark 2.8. From the previous examples, we know that α must be injective, β must be surjective, and we also have that $\text{im } \alpha = \ker \beta$.

Example 2.9

Let $\varphi : M \rightarrow N$ be a homomorphism. Then we have a short exact sequence

$$0 \longrightarrow \ker \varphi \longrightarrow M \longrightarrow \text{im } \varphi \longrightarrow 0$$

3 Homology

Definition 3.1 (Homology)

The i -th homology of a complex (M_\bullet, d_\bullet) of R -modules is the R -module

$$H_i(M_\bullet, d_\bullet) = \frac{\ker d_i}{\text{im } d_{i+1}}$$

Remark 3.2.

$$H_i(M_\bullet, d_\bullet) = 0 \iff \text{im } d_{i+1} = \ker d_i \iff \text{the complex } (M_\bullet, d_\bullet) \text{ is exact at } M_i$$

Which gives us one way of thinking about the homology of a complex: it is the measure of the failure of the complex from being exact at M_i . Recall that the kernel can be considered to be the measure of how far a homomorphism is from being injective, and the cokernel a measure of how far a homomorphism is from being surjective, the homology can be considered as a generalisation of these.

Example 3.3

Consider the sequence

$$0 \longrightarrow M_1 \xrightarrow{\varphi} M_0 \longrightarrow 0$$

Then $H_1(M_\bullet, d_\bullet) \cong \ker \varphi$, and $H_0(M_\bullet, d_\bullet) \cong \text{coker } \varphi$.

4 Diagram chase and the snake lemma

One proof method in homological algebra is known as a diagram chase, where given a commutative diagram, we use it to construct (or prove the existence of) an object. One example of this is known as the snake lemma.

Lemma 4.1 (Snake lemma)

Consider two short exact sequences linked by homomorphisms.

$$\begin{array}{ccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & N_1 \longrightarrow 0 \\ & & \downarrow \lambda & & \downarrow \mu & & \downarrow \nu \\ 0 & \longrightarrow & L_0 & \xrightarrow{\alpha_0} & M_0 & \xrightarrow{\beta_0} & N_0 \longrightarrow 0 \end{array}$$

Then we have an exact sequence

$$0 \longrightarrow \ker \lambda \longrightarrow \ker \mu \longrightarrow \ker \nu \xrightarrow{\delta} \operatorname{coker} \lambda \longrightarrow \operatorname{coker} \mu \longrightarrow \operatorname{coker} \nu \longrightarrow 0$$

Remark 4.2. Most of the homomorphisms in the above exact sequence are induced by the corresponding maps λ , μ , ν . The one which we need to construct is δ . Refer to the appendices for the construction of the unnamed maps.

Proof. Consider the diagram below.

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \ker \lambda & \longrightarrow & \ker \mu & \longrightarrow & \ker \nu \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & N_1 \longrightarrow 0 \\ & & \downarrow \lambda & & \downarrow \mu & & \downarrow \nu \\ 0 & \longrightarrow & L_0 & \xrightarrow{\alpha_0} & M_0 & \xrightarrow{\beta_0} & N_0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \operatorname{coker} \lambda & \longrightarrow & \operatorname{coker} \mu & \longrightarrow & \operatorname{coker} \nu \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

(A blue curved arrow labeled δ points from $\ker \nu$ to $\operatorname{coker} \lambda$.)

We will assume that the diagram above is exact at all points except $\ker \nu$ and $\operatorname{coker} \lambda$. The proof can be found in the appendices. We will construct the homomorphism δ like so.

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & \cdots & & \cdots & & \ker \nu \\ & & \vdots & & \vdots & & \downarrow \iota_\nu \\ 0 & \cdots & \cdots & & M_1 & \xrightarrow{\beta_1} & N_1 \cdots 0 \\ & & \vdots & & \downarrow \mu & & \downarrow \nu \\ 0 & \cdots & \cdots & & L_0 & \xrightarrow{\alpha_0} & M_0 \xrightarrow{\beta_0} \bullet \cdots 0 \\ & & \vdots & & \downarrow q_\lambda & & \vdots \\ & & \operatorname{coker} \lambda & \cdots & \cdots & & 0 \\ & & \vdots & & \vdots & & \vdots \\ & & 0 & & 0 & & 0 \end{array}$$

(Purple arrows indicate the construction of δ via the maps ι_ν , μ , ν , α_0 , β_0 , and q_λ .)

Let $a \in \ker \nu$. Then let $b = \iota_\nu(a)$. β_1 is surjective, so let $c \in M_1$ such that $\beta_1(c) = b$. Let $d = \mu(c)$. Now $\beta_0(d) = \beta_0(\mu(c)) = \nu(\beta_1(c)) = \nu(b) = \nu(a) = 0$. So $d \in \ker \beta_0 = \operatorname{im} \alpha_0$. Thus, we have $e \in L_0$ such that $\alpha_0(e) = d$. Finally, let $f = q_\lambda(e)$.

We want to set $\delta(a) = f$. To do this, we need to show that this is independent of the choice of elements in the preimage. Since α_0 is injective, e is uniquely determined by d . On the other hand, we need to show that δ is independent of the choice of c .

Suppose we had c' such that $\beta_1(c') = b$ as well. With the same argument as above, we obtain $d' = \mu(c')$, e' such that $\alpha_0(e') = d'$ and $f' = q_\lambda(e')$.

$$\begin{array}{ccccccc}
 0 & \cdots & L_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & \bullet \cdots \cdots \rightarrow 0 \\
 & & \downarrow \lambda & & \downarrow \mu & & \\
 0 & \cdots & L_0 & \xrightarrow{\alpha_0} & \bullet & & \\
 & & \downarrow q_\lambda & & & & \\
 & & \text{coker } \lambda & & & &
 \end{array}$$

Here, note that $\beta_1(c - c') = \beta_1(c') - \beta_1(c) = b - b = 0$, so $c' - c \in \ker \beta_1 = \text{im } \alpha_1$. Let $g \in L_1$ be such that $\alpha_1(g) = c - c'$. Then $\alpha_0(\lambda(g)) = \mu(\alpha_1(g)) = \mu(c - c') = \mu(c) - \mu(c') = d - d'$. As $\alpha(e - \lambda(g)) = d - (d - d') = d' = \alpha(e')$, we get that $e' = e - \lambda(g)$, so $e' - e = \lambda(g)$. With this, $f' - f = q_\lambda(e' - e) = q_\lambda(\lambda(g)) = 0$. So δ is well defined.

Now we will show that the sequence is exact at $\ker \nu$ and $\text{coker } \lambda$.

To start off, let $a \in \ker \nu$ be such that $a \in \text{im } \beta_2 = \beta_1(\ker \mu)$. Then we have some $c \in M_1$ such that $\beta_1(c) = b = \iota_\nu(a)$, and $c \in \ker \mu$. This then means that (using the notation as in the definition of δ), $d = \mu(c) = 0$. α_0 is injective, so $e = 0$. Thus, $f = \delta(a) = 0$.

Conversely, suppose $a \in \ker \delta$. Let b, c, d, e, f be as in the definition of δ . Since $f = 0$, $e \in \ker q_\lambda = \text{im } \lambda$, so we have some $g \in L_1$ such that $\lambda(g) = e$. Then $\beta_1(c - \alpha_1(g)) = \beta_1(c) - \beta_1(\alpha_1(g)) = \beta_1(c) = b$ by exactness at M_1 . Furthermore, $\mu(c - \alpha_1(g)) = d - \mu(\alpha_1(g)) = d - \alpha_0(\lambda(g)) = d - \alpha_0(e) = 0$. So $b \in \beta_1(\ker \mu)$. Thus, the sequence is exact at $\ker \nu$.

Now let $a \in \ker \nu$. We want to show that $\alpha_{-1}(\delta(a)) = 0$. Let b, c, d, e, f be as in the definition of δ . Then $\alpha_{-1}(\delta(a)) = \alpha_{-1}(f) = \alpha_{-1}(q_\lambda(e)) = q_\mu(\alpha_0(e)) = q_\mu(d) = q_\mu(\mu(c)) = 0$. So $\text{im } \delta \subseteq \ker \alpha_{-1}$.

Conversely, suppose $f \in \ker(\alpha_{-1})$.

$$\begin{array}{ccccc}
 & c & \xrightarrow{\beta_1} & b & \\
 & \downarrow \mu & & \downarrow \nu & \\
 e & \xrightarrow{\alpha_0} & d & \xrightarrow{\beta_0} & \nu(b) \\
 \downarrow q_\lambda & & \downarrow q_\mu & & \\
 f & \xrightarrow{\alpha_{-1}} & 0 & &
 \end{array}$$

Then as q_λ is surjective, we have $e \in L_0$ such that $q_\lambda(e) = f$. Let $d = \alpha_0(e)$. Then $q_\mu(d) = q_\mu(\alpha_0(e)) = \alpha_{-1}(q_\lambda(e)) = \alpha_{-1}(f) = 0$. So $d \in \ker q_\mu = \text{im } \mu$. So we have some $c \in M_1$ such that $\mu(c) = d$. Let $b = \beta_1(c)$. Then $\nu(b) = \nu(\beta_1(c)) = \beta_0(\mu(c)) = \beta_0(d) = \beta_0(\alpha_0(e)) = 0$. So $b \in \ker \nu$, and $b = \iota_\nu(a)$ for some $a \in \ker \nu$. By definition, we then have that $f = \delta(a)$. So the sequence is exact at $\text{coker } \lambda$. \square

There is another lemma which is also sometimes referred to as the Snake lemma, which is the following

Lemma 4.3 (Snake lemma, alternative version)

Suppose we have the exact complex

$$\begin{array}{ccccccc}
 & L_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & N_1 & \longrightarrow 0 \\
 & \downarrow \lambda & & \downarrow \mu & & \downarrow \nu & \\
 0 & \longrightarrow & L_0 & \xrightarrow{\alpha_0} & M_0 & \xrightarrow{\beta_0} & N_0
 \end{array}$$

Then we have an exact sequence of the form

$$\ker \lambda \longrightarrow \ker \mu \longrightarrow \ker \nu \xrightarrow{\delta} \text{coker } \lambda \longrightarrow \text{coker } \mu \longrightarrow \text{coker } \nu$$

The proof of this version is similar enough that it is not worth the effort to reproduce here.

The Snake lemma is very powerful, for example it is used to prove the Mayer-Vietoris theorem in simplicial homology. One, more immediate, consequence of the Snake lemma is the following:

Proposition 4.4

With the same setup as in the Snake lemma,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & N_1 & \longrightarrow & 0 \\ & & \downarrow \lambda & & \downarrow \mu & & \downarrow \nu & & \\ 0 & \longrightarrow & L_0 & \xrightarrow{\alpha_0} & M_0 & \xrightarrow{\beta_0} & N_0 & \longrightarrow & 0 \end{array}$$

if μ is surjective and ν is injective, then λ is surjective and ν is an isomorphism.

Proof. If μ is surjective, then $\text{coker } \mu = 0$. Similarly, ν injective means that $\ker \nu = 0$. Putting this into the Snake lemma, we get

$$0 \longrightarrow \ker \lambda \longrightarrow \ker \mu \longrightarrow 0 \xrightarrow{\delta} \text{coker } \lambda \longrightarrow 0 \longrightarrow \text{coker } \nu \longrightarrow 0$$

Exactness implies that $\text{coker } \lambda = 0$ and $\text{coker } \nu = 0$, so λ surjective and ν is an isomorphism. \square

A Maps in the Snake lemma

As the explicit maps between the (co)kernels of the maps are not necessary to prove the Snake lemma, we omitted the details from the statement and proof. However, we will go through the constructions below.

A.1 Vertical maps

First, we will construct the vertical maps to make the exact sequence

$$0 \longrightarrow \ker \lambda \xrightarrow{\iota_\lambda} L_1 \xrightarrow{\lambda} L_0 \xrightarrow{q_\lambda} \text{coker } \lambda \longrightarrow 0$$

For this, we just note that letting $\iota_\lambda : \ker \lambda \rightarrow L_1$ be the inclusion map, and $q_\lambda : L_0 \rightarrow \frac{L_0}{\text{im } \lambda} \cong \text{coker } \lambda$ be the quotient map, we get an exact sequence.

A.2 Maps between kernels

For this, we will need to construct the maps α_2 and β_2 which makes diagram below commute.

$$\begin{array}{ccccc} \ker \lambda & \xrightarrow{\alpha_2} & \ker \mu & \xrightarrow{\beta_2} & \ker \nu \\ \downarrow \iota_\lambda & & \downarrow \iota_\mu & & \downarrow \iota_\nu \\ L_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & N_1 \\ \downarrow \lambda & & \downarrow \mu & & \downarrow \nu \\ L_0 & \xrightarrow{\alpha_0} & M_0 & \xrightarrow{\beta_0} & N_0 \end{array}$$

Since ι_λ , ι_μ and ι_ν are inclusion maps, the natural choice would be $\alpha_2 = \alpha_1|_{\ker \lambda}$ and $\beta_2 = \beta_1|_{\ker \mu}$. We now need to check that the functions have the correct codomains.

Let $v \in \ker \lambda$. Then, $\mu(\alpha_1(v)) = \alpha_0(\lambda v) = \alpha_0(0) = 0$, so $\alpha_1(v) \in \ker \mu$. Similarly, we can check that β_2 is well defined. The final thing we need to check is the exactness of the sequence

$$0 \longrightarrow \ker \lambda \xrightarrow{\alpha_2} \ker \mu \xrightarrow{\beta_2} \ker \nu$$

For this, we note that $\ker \alpha_2 = \{v \in \ker \lambda : \alpha_1(v) = 0\} = \ker \alpha_1 \cap \ker \lambda$. By the exactness of

$$0 \longrightarrow L_1 \xrightarrow{\alpha_1} M_1 \xrightarrow{\beta_1} N_1 \longrightarrow 0$$

at L_1 , we have that $\ker \alpha_1 = 0$. So $\ker \alpha_2 = 0$, and the sequence is exact at $\ker \lambda$.

Similarly, we have $\ker \beta_2 = \ker \beta_1 \cap \ker \mu$, by exactness of the sequence at M_1 , this is $\text{im } \alpha_1 \cap \ker \mu$. From this, we have that $\text{im } \alpha_2 \subseteq \ker \beta_2$. Now let $w \in \ker \beta_2$. Then we have some $u \in L_1$ such that $\alpha_1(u) = w$, and $\mu(\alpha_1(u)) = 0$. We want to show that $u \in \ker \lambda$. For this, we need the exactness of

$$\begin{array}{ccccc} L_1 & \xrightarrow{\alpha_1} & M_1 & & \\ \downarrow \lambda & & \downarrow \mu & & \\ 0 & \longrightarrow & L_0 & \xrightarrow{\alpha_0} & M_0 \end{array}$$

at L_0 . $\alpha_0(\lambda(u)) = \mu(\alpha_1(u)) = 0$. By exactness at L_0 , $\ker \alpha_0 = 0$, so $\lambda(u) = 0$, and $u \in \ker \lambda$. Thus, $\ker \beta_2 \subseteq \text{im } \alpha_2$, and the sequence is exact.

A.3 Maps between cokernels

The idea here is pretty much the same. As the diagram commutes, this forces the maps that we can use, so all we have to do is to check that these are well defined. We need to construct maps α_{-1} and β_{-1} such that the diagram

$$\begin{array}{ccccccc} L_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & N_1 & & \\ \downarrow \lambda & & \downarrow \mu & & \downarrow \nu & & \\ L_0 & \xrightarrow{\alpha_0} & M_0 & \xrightarrow{\beta_0} & N_0 & & \\ \downarrow q_\lambda & & \downarrow q_\mu & & \downarrow q_\nu & & \\ \text{coker } \lambda & \xrightarrow{\alpha_{-1}} & \text{coker } \mu & \xrightarrow{\beta_{-1}} & \text{coker } \nu & & \end{array}$$

commutes. Since $\text{coker } \lambda \cong \frac{L_0}{\text{im } \lambda}$, we will first construct a map $\tilde{\alpha}_{-1} : L_0 \rightarrow \text{coker } \mu$, and then lift to a map $\alpha_{-1} : \text{coker } \lambda \rightarrow \text{coker } \mu$. The natural maps are $\tilde{\alpha}_{-1} = q_\mu \circ \alpha_0$ and $\tilde{\beta}_{-1} = q_\nu \circ \beta_0$. Now we need to show that these maps respect the quotient relation. Suppose $v_1 - v_2 \in \text{im } \lambda$, then we need to show that $\tilde{\alpha}_{-1}(v_1) = \tilde{\alpha}_{-1}(v_2)$. Since $v_1 - v_2 \in \text{im } \lambda$, we have a $w \in L_0$ such that $\lambda(w) = v_1 - v_2$. Then

$$\tilde{\alpha}_{-1}(v_1) - \tilde{\alpha}_{-1}(v_2) = \tilde{\alpha}_{-1}(v_1 - v_2) = \tilde{\alpha}_{-1}(\lambda(w)) = q_\mu(\alpha_0(\lambda(w))) = q_\mu(\mu(\alpha_1(w))) = 0$$

Thus, this gives us a map $\alpha_{-1} : \text{coker } \lambda \rightarrow \text{coker } \mu$. Similarly, we have a map $\beta_{-1} : \text{coker } \mu \rightarrow \text{coker } \nu$. Now we need to check the exactness of the sequence

$$\text{coker } \lambda \xrightarrow{\alpha_{-1}} \text{coker } \mu \xrightarrow{\beta_{-1}} \text{coker } \nu \longrightarrow 0$$

Since q_λ is surjective, and $q_\mu \circ \alpha_0 = \alpha_{-1} \circ q_\lambda$, we have that $\text{im } \alpha_{-1} = q_\mu(\text{im } \alpha_0) = q_\mu(\ker \beta_0)$. For any $w \in q_\mu(\ker \beta_0)$, we have some $u \in \ker(\beta_0)$ such that $q_\mu(u) = w$. Then $\beta_{-1}(w) = \beta_{-1}(q_\mu(u)) = q_\nu(\beta_0(u)) = q_\nu(0) = 0$. So $w \in \ker(\beta_{-1})$.

Conversely, suppose $w \in \ker(\beta_{-1})$. Then there exists $u \in M_0$ such that $q_\mu(u) = w$. We then have that $q_\nu(\beta_0(u)) = \beta_{-1}(q_\mu(u)) = 0$, so $\beta_0(u) \in \text{im } \nu$. This means that we have some $t \in N_1$ such that $\beta_0(u) = \nu(t)$. Since β_1 is surjective, we have $s \in M_1$ such that $t = \beta_1(s)$. Then $\beta_0(u - \mu(s)) = \nu(t) - \beta_0(\mu(s)) = \nu(t) - \nu(\beta_1(s)) = 0$. So $u - \mu(s) \in \ker \beta_0$. Now $\mu(s) \in \text{im } \mu$, so $q_\mu(\mu(s)) = 0$, and $q_\mu(u) = q_\mu(u - \mu(s)) = w$, and $w \in q_\mu(\ker \beta_0)$.

As $\text{im } \beta_{-1} = q_\nu(\text{im } \beta_0) = q_\nu(N_0) = \text{coker } \nu$, the sequence is exact at $\text{coker } \nu$.