Chapter 4. Vector Spaces (1/2)

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4.1. Vector Spaces and Subspaces

Vector spaces

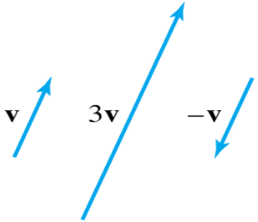
- **Definition:** A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars (real numbers)*, subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.
  1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
  2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
  3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
  4. There is a zero vector 0 in V such that $\mathbf{u}+0=\mathbf{u}$
  5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=0$
  6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
  7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
  8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
  9. $c(d\mathbf{u})=(cd)\mathbf{u}$
  10. $1\mathbf{u}=\mathbf{u}$

- Using these axioms, we can show that
  - the zero vector in Axiom 4 is unique, and
  - the vector $-\mathbf{u}$, called the **negative** of $\mathbf{u}$, in Axiom 5 is unique for each $\mathbf{u}$ in V.
  - *The identity and inverse with respect to vector addition are unique.*

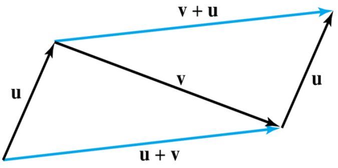
- For each $\mathbf{u}$ in V and scalar c,

$$\begin{aligned} 0\mathbf{u} &= 0 \\ c0 &= 0 \\ -\mathbf{u} &= (-1)\mathbf{u} \end{aligned}$$

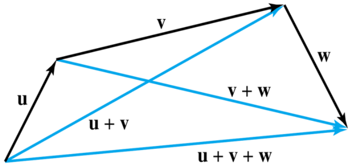
- **Example 2:**
    - Let V be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction.
    - Define addition by the parallelogram rule, and for each $\mathbf{v}$ in V.
    - Define $c\mathbf{v}$ to be the arrow whose length is $|c|$ times the length of $\mathbf{v}$, pointing in the same direction as $\mathbf{v}$ if $c \geq 0$ and otherwise pointing in the opposite direction.
    - See the figure below. Show that V is a vector space.



- **Solution:**
  - The definition of V is geometric, using concepts of length and direction. No x y z-coordinate system is involved. An arrow of zero length is a single point and represents the zero vector.
  - The negative of $\mathbf{v}$ is $(-1)\mathbf{v}$.
  - So Axioms 1, 4, 5, 6, and 10 are evident. See the following figures.



$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.



$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.

Subspaces

- **Definition**: A **subspace** of a vector space V is a subset H of V that has three properties:
  a. The zero vector of V is in H.
  b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
  c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

- **Remark**
  - Properties (a), (b), and (c) guarantee that a subspace H of V is itself a vector space, under the vector space operations already defined in V.
  - Every subspace is a vector space.
  - Conversely, every vector space is a subspace (of itself and possibly of other larger spaces).

A subspace spanned by a set

- The set consisting of only the zero vector in a vector space V is a subspace of V, called the **zero subspace** and written as $\{0\}$.
- As the term **linear combination** refers to any sum of scalar multiples of vectors, and $Span\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ denotes the set of all vectors that can be written as linear combinations of $\mathbf{v}_1, ..., \mathbf{v}_p$.

- **Example 10**: Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V, let $H = Span\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that H is a subspace of V.

- **Solution**

  1. The *zero vector* is in H, since $0 = 0\mathbf{v}_1 + 0\mathbf{v}_2$.
  2. To show that H is *closed under vector addition*, take two arbitrary vectors in H, say. $\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2$ and $\mathbf{w} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$. By Axioms 2, 3, and 8 for the vector space V,

  $$\mathbf{u} + \mathbf{w} = (s_1\mathbf{v}_1 + s_2\mathbf{v}_2) + (t_1\mathbf{v}_1 + t_2\mathbf{v}_2) = (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2$$

  So, $\mathbf{u} + \mathbf{w}$ is in H.
  3. Furthermore, if c is any scalar, then by Axioms 7 and 9,

  $$c\mathbf{u} = c(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) = (cs_1)\mathbf{v}_1 + (cs_2)\mathbf{v}_2$$

  which shows that $c\mathbf{u}$ is in H and H is *closed under scalar multiplication*.

  - Thus, H is a subspace of V.

- **Theorem 1**: If $\mathbf{v}_1, ..., \mathbf{v}_p$ are in a vector space V, then $Span\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ is a subspace of V.

- We call $Span\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ **the subspace spanned** (or **generated**) by $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$.

- Given any subspace H of V, a **spanning** (or **generating**) set for H is a set $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ in H such that $H = \{\mathbf{v}_1, ..., \mathbf{v}_p\}$.

Suggested Exercise

- 4.1.13

4.2 Null spaces, Column spaces, and Linear transformation

Null space of matrix

- **Definition:** The **null space** of an $m \times n$ matrix A, written as $Nul\, A$, is the set of all solutions of the homogeneous equation $A\mathbf{x} = 0$. In set notation,

$$Nul\, A = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = 0\}$$

- **Theorem 2**: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = 0$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$

- **Proof**:
  - $Nul\ A$ is a subset of $\mathbb{R}^n$ because A has n columns.
  - We need to show that $Nul\ A$ satisfies the three properties of a subspace.
    1. 0 is in $Nul\ A$ (trivial solution)
    2. Next, let $\mathbf{u}$ and $\mathbf{v}$ represent any two vectors in $Nul\ A$. Then, $A\mathbf{u} = 0$ and $A\mathbf{u} = 0$. To show that $\mathbf{u} + \mathbf{v}$ is in $Nul\ A$, we must show that $A(\mathbf{u} + \mathbf{v}) = 0$. Using a property of matrix multiplication, we have $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = 0 + 0 = 0$. Thus, $\mathbf{u} + \mathbf{v}$ is in $Nul\ A$, and $Nul\ A$ is closed under vector addition.
    3. Finally, if c is any scalar, then $A(c\mathbf{u}) = c(Au) = c(0) = 0$, which shows that $c\mathbf{u}$ is in $Nul\ A$.

    - Thus, $Nul\ A$ is a subspace of $\mathbb{R}^n$

- **An Explicit Description of $Nul\ A$**
  - There is no obvious relation between vectors in $Nul\ A$ and the entries in A.
  - We say that $Nul\ A$ is defined *implicitly*, because it is defined by a condition that must be checked.
  - No explicit list or description of the elements in $Nul\ A$ is given.
  - *Solving* the equation $A\mathbf{x} = 0$ amounts to producing an *explicit* description of $Nul\ A$

- **Example 3:** Find a spanning set for the null space of the matrix

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

1. The first step is to find the general solution of $A\mathbf{x} = 0$ in terms of free variables. Row reduce the augmented matrix $[A \mid 0]$ to *reduce* echelon form in order to write the basic variables in terms of the free variables:

$$A = \begin{pmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x_1 - 2x_2 - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

2. The general solution is
   - $x_1 = 2x_2 + x_4 - 3x_5$
   - $x_3 = -2x_4 + 2x_5$
   - x_2, x_4, x_5 free.

3. Next, decompose the vector giving the general solution into a linear combination of *vectors where the weights are the free variables*. That is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} \begin{aligned} &= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \\ &= x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w} \end{aligned}$$

4. Every linear combination of **u**, **v**, and **w** is an element of $Nul\ A$. Thus, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for $Nul\ A$. $\square$

- Remark
  1. The spanning set produced by the method in **Example 3** is automatically linearly independent because the free variables are the weights on the spanning vectors.
  2. When $Nul\ A$ contains nonzero vectors, the number of vectors in the spanning set for $Nul\ A$ equals the number of free variables in the equation $A\mathbf{x} = 0$.

Column space of matrix

- **Definition:** The **column space** of an $m \times n$ matrix A, written as $Col\ A$, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1, ..., \mathbf{a}_n]$, then

$$Col\ A = Span\{\mathbf{a}_1, ..., \mathbf{a}_n\}$$

- **Theorem 3**: The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$

- Remark
  - A typical vector in $Col\ A$ can be written as $A\mathbf{x}$ for some $\mathbf{x}$ because the notation $A\mathbf{x}$ stands for a linear combination of the columns of A. That is,

$$Col\ A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$$

  - The notation $A\mathbf{x}$ for vectors in $Col\ A$ also shows that $Col\ A$ is the *range* of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.
  - The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

- **Example 7:** Let

$$A = \begin{bmatrix} 2 & -4 & -2 & -1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

a. Determine if **u** is in $Nul\ A$. Could **u** be in $Col\ A$?
b. Determine if **v** is in $Col\ A$. Could **v** be in $Nul\ A$?

- **Solution to (a)**
  - An explicit description of $Nul\ A$ is not needed here. Simply compute the product $A\mathbf{u}$.

$$Au = \begin{bmatrix} 2 & -4 & -2 & -1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

u is not a solution of $A\mathbf{x} = 0$, so **u** is not in $Nul\ A$. Also, with four entries, **u** could not possibly be in $Col\ A$, since $Col\ A$ is a subspace of $\mathbb{R}^3$.

- **Solution to (b)**
  - Reduce $[A \mid \mathbf{v}]$ to an echelon form.

$$[A \quad v] = \begin{bmatrix} 2 & -4 & -2 & -1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \begin{bmatrix} 2 & -4 & -2 & -1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{v}$ is consistent, so v is in $Col\ A$. With only three entries, $\mathbf{v}$ could not possibly be in $Nul\ A$, since $Nul\ A$ is a subspace of $\mathbb{R}^4$.

Kernel and range of linear transformation

- Subspaces of vector spaces other than $\mathbb{R}^n$ are often described in terms of a linear transformation instead of a matrix.

- **Definition**: A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that
  1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V, and
  2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ and V and all scalars c.

- **Definition**:
  - The **kernel** (or **null space**) of such a T is the set of all $\mathbf{u}$ in V such that $T(u) = 0$ (the zero vector in W).
  - The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.

- **Remark**:
  - The kernel of T is a subspace of V.
  - The range of T is a subspace of W.

Contrast between $Nul\ A$ and $Col\ A$ for an $m \times n$ matrix A

	$Nul\ A$	$Col\ A$
1	$Nul\ A$ is a subspace of $\mathbb{R}^n$	$Col\ A$ is a subspace of $\mathbb{R}^m$
2	$Nul\ A$ is implicitly defined, i.e., you are given only a condition ($A\mathbf{x} = 0$) that vectors in $Nul\ A$ must satisfy.	$Col\ A$ is explicitly defined, *i.e.*, you are told how to build vectors in $Col\ A$
3	It takes time to find vectors in $Nul\ A$. Row operation on $[A \mid 0]$ are required.	It is easy to find vectors in $Col\ A$. The columns of A are displayed; others are formed from them.
4	There is no obvious relation between $Nul\ A$ and the entries in A.	There is an obvious relation between $Col\ A$ and the entries in A, since each column of A is in $Col\ A$.

(continued)

	$Nul\ A$	$Col\ A$
5	A typical vector **v** in $Nul\ A$ has the property that $A\mathbf{v} = 0$.	A typical vector **v** in $Col\ A$ has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6	Given a specific vector **v**, it is easy to tell if **v** is in $Nul\ A$. Just compare $A\mathbf{v}$.	Given a specific vector **v**, it may take time to tell if **v** is in $Col\ A$. Row operation on $[A \mid \mathbf{v}]$ are required.
7	$Nul\ A = \{0\}$ if and only if the equation $A\mathbf{x} = 0$ has only the trivial solution.	$Col\ A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every **b** in $\mathbb{R}^m$.
8	$Nul\ A = \{0\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is *one-to-one*.	$Col\ A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Suggested Excercises

- 4.2.5
- 4.2.17

4.3 Linearly independent sets; Bases

Linearly independent sets; Bases

- An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the vector equation
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = 0 \tag{1}$$
has *only* the trivial solution, $c_1 = 0, \dots, c_p = 0$.

- The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if (1) has a nontrivial solution, *i.e.*, if there are some weights, $c_1, \dots, c_p$, *not all zero*, such that (1) holds. In such a case, (1) is called a **linear dependence relation** among $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

- **Theorem 4:** An indexed set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq 0$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

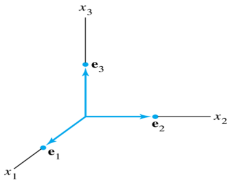
- **Definition:** Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a basis for H if
  i) $\mathcal{B}$ is a linearly independent set, and
  ii) The subspace spanned by $\mathcal{B}$ coincides with H; that is, $H = Span\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$
- **Remark:**
  - The definition of a basis applies to the case when $H = V$, because any vector space is a subspace of itself. Thus, a basis of V is a linearly independent set that spans V.
  - When $H \neq V$, condition ii) includes the requirement that each of the vectors $\mathbf{b}_1, \dots, \mathbf{b}_p$ must belong to H, because $Span\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ contains $\mathbf{b}_1, \dots, \mathbf{b}_p$.

Standard basis

- Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the columns of the $n \times n$ matrix, I_n.
- That is,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

- The set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is called the **standard basis** for $\mathbb{R}^n$. See the following figure.



The standard basis for $\mathbb{R}^3$.

The spanning set theorem

- **Theorem 5:** Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set in V, and let $H = Span\{\mathbf{v}_1, \dots, \mathbf{v}_p$.
  a. If one of the vectors in S – say, $\mathbf{v}_k$ – is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.
  b. If $H \neq \{0\}$, some subset of S is a basis for H.

- **Proof for** a.
  - By rearranging the list of vectors in S, if necessary, we may suppose that $\mathbf{v}_p$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$ – say,

$$v_p = a_1 v_1 + \cdots + a_{p-1} v_{p-1} \tag{3}$$

  - Given any $\mathbf{x}$ in H, we may write

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_{p-1}\mathbf{v}_{p-1} + c_p\mathbf{v}_p \tag{4}$$

    for suitable scalars $c_1, c_2, \dots, c_p$.
  - Substituting the expression for $\mathbf{v}_p$ from (3) into (4), it is easy to see that $\mathbf{x}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_{p-1}$.
  - Thus, $\{\mathbf{v}_1, \dots, \mathbf{v}_{p-1}\}$ spans H, because $\mathbf{x}$ was an arbitrary element of H.

- **Proof for** b.
  - If the original spanning set S is linearly independent, then it is already a basis for H.
  - Otherwise, one of the vectors in S depends on the others and can be deleted, by part (a).
  - So long as there are two or more vectors in the spanning set, we can repeat this process until the spanning set is linearly independent and hence is a basis for H.
  - If the spanning set is eventually reduced to one vector, that vector will be nonzero (and hence linearly independent) because $H \neq \{0\}$.

- **Example 7:** Let

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$$

and $H = Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Note that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, and show that $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = Span\{\mathbf{v}_1, \mathbf{v}_2\}$. Then find a basis for the subspace H.

1. **Proof for** $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = Span\{\mathbf{v}_1, \mathbf{v}_2\}$:
   1.1 $Span\{\mathbf{v}_1, \mathbf{v}_2\} \subset Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$
      - Every vector in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$ belongs to H because $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\mathbf{v}_3$.

   1.2 $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subset Span\{\mathbf{v}_1, \mathbf{v}_2\}$
      - Now let $\mathbf{x}$ be any vector in H – say, $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$.
      - Since $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, we may substitute

$$\begin{aligned} x &= c_1v_1 + c_2v_2 + c_3v_3 \\ &= c_1v_1 + c_2v_2 + c_3(5v_1 + 3v_2) \end{aligned}$$

      - Thus $\mathbf{x}$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$, so every vector in H already belongs to $Span\{\mathbf{v}_1, \mathbf{v}_2\}$

   - We conclude that H and $Span\{\mathbf{v}_1, \mathbf{v}_2\}$ are actually the same set of vectors.

2. **Find a basis** for the subspace H:
   - It follows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of H since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

Basis for $Col\ B$

- **Example 8:** Find a basis for $Col\ B$, where

$$B = \left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_5 \end{array}\right] = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Solution**
  - Each non-pivot column of B is a linear combination of the pivot columns. In fact, $\mathbf{b}_2 = 4\mathbf{b}_1$ and $\mathbf{b}_4 = 2\mathbf{b}_2 - \mathbf{b}_3$. By the Spanning Set Theorem, we may discard $\mathbf{b}_2$ and $\mathbf{b}_4$, and $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ will still span $Col\ B$.
  - Let

$$S = \{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

  - Since $\mathbf{b}_1 \neq 0$ and no vector in S is a linear combination of the vectors that precede it, S is linearly independent. (Theorem 4). Thus, S is a basis for $Col\ B$.

Bases for $Nul\ A$ and $Col\ A$

- **Theorem 6:** The pivot columns of a matrix A form a basis for Col A.
- **Proof:**
  - Let B be the reduced echelon form of A. The set of pivot columns of B is linearly independent, for no vector in the set is a linear combination of the vectors that precede it.
  - Since A is row equivalent to B, the pivot columns of A are linearly independent as well, because any linear dependence relation among the columns of A corresponds to a linear dependence relation among the columns of B.
  - For this reason, every non-pivot column of A is a linear combination of the pivot columns of A.
  - Thus the non-pivot columns of a may be discarded from the spanning set for $Col\ A$, by the Spanning Set Theorem.
  - This leaves the pivot columns of A as a basis for $Col\ A$.

- **Warning:** The pivot columns of a matrix A are evident when A has been reduced only to echelon form. But, be careful to use the pivot columns of A itself for the basis of $Col\ A$. Row operations can change the column space of a matrix. The columns of an echelon form B of A are often not in the column space of A.

Two Views of a Basis

- When the Spanning Set Theorem is used, the deletion of vectors from a spanning set must stop when the set becomes linearly independent.
- If an additional vector is deleted, it will not be a linear combination of the remaining vectors, and hence the smaller set will no longer span V.
- Thus a basis is a spanning set that is as small as possible.
- A basis is also a linearly independent set that is as large as possible.
- If S is a basis for V, and if S is enlarged by one vector — say, $\mathbf{w}$ — from V, then the new set cannot be linearly independent, because S spans V, and $\mathbf{w}$ is therefore a linear combination of the elements in S.
- Sim: **"Basis is small enough to be linearly independent, but basis is large enough to span the space."**

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