Chapter 7. Symmetric Matrices and Quadratic Forms

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1 *7.1. Diagonalization of Symmetric Matrices*

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Symmetric Matrix

- A **symmetric matrix** is a matrix A such that $A^T = A$.

- Such a matrix is necessarily square.

- Its main diagonal entries are arbitrary, but its other entries occur in pairs—on opposite sides of the main diagonal.

- **Theorem 1:** If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

- **Theorem 2:** An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is symmetric matrix.

- **Example 3:**: Orthogonally diagonalize the matrix $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, whose

  characteristic equation is $-\lambda^3 + 12\lambda^2 - 21\lambda - 98 = -(\lambda - 7)^2(\lambda + 2) = 0$

- **Solution:**
  - The usual calculations produce bases for the eigenspaces:
  $$\lambda = 7 : \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}; \ \lambda = -2 : \mathbf{v}_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$
  - Although $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, they are not orthogonal. The projection of $\mathbf{v}_2$ onto $\mathbf{v}_1$ is $\frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1$. The component of $\mathbf{v}_2$ orthogonal to $\mathbf{v}_1$ is
  $$\mathbf{z}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1/2}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$$
  - Then $\{\mathbf{v}_1, \mathbf{z}_2\}$ is an orthogonal set in the eigenspace for $\lambda = 7$
  - (Note that $\mathbf{z}_2$ is linear combination of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, so $\mathbf{z}_2$ is in the eigenspace.)

- (solution continued:)
  - Since the eigenspace is two-dimensional (with basis $\mathbf{v}_1$, $\mathbf{v}_2$), the orthogonal set $\{\mathbf{v}_1, \mathbf{z}_2\}$ is an *orthogonal basis* for the eigenspace, by the Basis Theorem.
  - Normalize $\mathbf{v}_1$ and $\mathbf{z}_2$ to obtain the following orthonormal basis for the eigenspace for $\lambda = 7$:

  $$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$$

  - An orthonormal basis for the eigenspace for $\lambda = -2$ is

  $$\mathbf{u}_3 = \frac{1}{\|2\mathbf{v}_3\|} 2\mathbf{v}_3 = \frac{1}{3} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

  - By Theorem 1, $\mathbf{u}_3$ is orthogonal to the other eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Hence $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.
  - Let $P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}$. Then P orthogonally

  diagonalizes A, and $A = PDP^{-1}$.

Suggested Excercises

- 7.5.1
- 7.5.3
- 7.6 (p.450)

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