Chapter 3. Determinants

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3.1. Introduction to Determinants

Definition

- For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1i}\, det\, A_{1i}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, ..., a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned} detA &= a_{11}detA_{11} - a_{12}detA_{12} + \cdots + (-1)^{1+n}a_{1n}detA_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j}\, a_{1j}\, det(A_{1j}) \end{aligned}$$

- **Example 1.** Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

- **Solution:** Compute

$$\begin{aligned} detA &= a_{11}detA_{11} - a_{12}detA_{12} + a_{13}detA_{13} \\ &= 1 \cdot det \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix} \\ &= 1(0-2) - 5(0-0) + 0(-4-0) = -2 \end{aligned}$$

- Another common notation for the determinant of a matrix uses a pair of vertical lines in place of brackets, i.e. $det\ A = |A|$
- Thus the calculation in Example 1 can be written as

$$det\ A = 1 \cdot \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \cdot \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = \cdots = -2$$

- To state the next theorem, it is convenient to write the definition of $det\, A$ in a slightly different form. Given $A = [a_{ij}]$, the $(i, j)-$**cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} det\, A_{ij} \qquad (4)$$

- Then

$$det\, A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

- This formula is called a **cofactor expansion across the first row** of A.

- **Theorem 1:** The determinant of an $n \times n$matrix A can be computed by a cofactor across any row or down any column.
  - The expansion across the i-th row using the cofactors in (4) is

$$det\, A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

  - The cofactor expansion down the j th column is

$$detA = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

- **Example 2.** Use a cofactor expansion across the third row to compute $det\ A$, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

- **Solution:** Compute

$$\begin{aligned} detA &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= (-1)^{3+1}a_{31}detA_{31} + (-1)^{3+2}a_{32}detA_{32} + (-1)^{3+3}a_{33}detA_{33} \\ &= 0 \cdot \begin{bmatrix} 5 & 0 \\ 4 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + 0\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \\ &= 0 + 2(-1) + 0 = -2 \end{aligned}$$

- **Theorem 2:** If A is a triangular matrix, then $det\ A$ is the product of the entries on the main diagonal of A.

Suggested Exercises

- 3.1.4
- 3.1.10

3.2. Properties of Determinants

- The Theorem 3 below answers to a question "How does an elementary row operation affect determinant?"

- **Theorem 3:** Let A be a square matrix
  - *a) (Replacement)* If a multiple of one row of A is added to another row to produce a matrix B, then det B = det A.
  - *b) (Interchange)* If two rows of A are interchanged to produce B, then $det\ B = -1 \cdot det\ A$.
  - *c) (Scaling)* If one row of A is multiplied by k to produce B, then $det\ B = k \cdot det\ A$.

- **Example 1** Compute $det\ A$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

- **Solution:**
  - The strategy is to reduce A to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries. The first two row replacements in column 1 do not change the determinant:

$$detA = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$$

  - An interchange of rows 2 and 3 reverses the sign of the determinant, so

$$det\ A = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15$$

- **Theorem 4:** A square matrix A is invertible if and only if $det\ A \neq 0$.

- **Example 3.** Compute det A, where $A = \begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$

- **Solution**
  - Add 2 times row 1 to row 3 ($R3 \leftarrow R3 + 2R1$) to obtain

$$detA = det \begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix} = 0$$

  - because the second and third rows of the second matrix are equal.

Column Operations

- **Theorem 5:** If A is a matrix, then $detA^T = det\ A$.
- **Proof**
  - The theorem is obvious for $n = 1$.
  - Suppose the theorem is true for $k \times k$ determinants and let $n = k + 1$. Then the cofactor of a_{1j} in A equals the cofactor of a_{j1} in A^T, because the cofactors involve $k \times k$ determinants. Hence the cofactor expansion of $det\ A$ along the first row equals the cofactor expansion of $det\ A^T$ down the first column. That is, A and A^T have equal determinants.
  - Thus the theorem is true for $n = 1$, and the truth of the theorem for one value of k implies its truth for the next value of $k + 1$. By the principle of *mathematical induction*, the theorem is true for all $n \geq 1$.

Determinants and Matrix Products

- **Theorem 6:** If A and B are matrices, then $det\ AB = (det\ A)(det\ B)$

- **Example 5** Verify Theorem 6 for $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.

- **Solution**
  - $$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

    and

    $$det\ AB = 25 \cdot 13 - 20 \cdot 14 = 325 - 280 = 45$$

  - On the other hand, since $det\ A = 9$ and $det\ B = 5$,

    $$(det\ A)(det\ B) = 9 \cdot 5 = 45 = det\ AB$$

Suggested Exercises

- 3.2.5
- 3.2.9

3.3. Cramer's Rule, Volume, and Linear Transformations

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