Ch 1. Linear Equations (1/2)

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1.1. Systems of Linear Equations

A few terminology

- A *linear equation* (1차방정식, 선형방정식) for the variable $x_1, \ldots, x_n$ is an equation in the form:

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b,$$

  where b and the coefficient (계수) $a_1, a_2, \ldots, a_n$ are numbers, usually known in advance.
- A *system of linear equations* (or a *linear system*) is a collection of one or more linear equations involving the same variables — say, $x_1, \ldots, x_n$.
- A *solution* to the system is a list of numbers for $x_1, \ldots, x_n$ that makes every equation to be true.
- The set of all possible solutions is called the *solution set* of the linear system.
- Two linear systems are called *equivalent* if they have the same solution set.

Number of solutions to a linear system

- A linear system may have
  1. no solution
  2. exactly one solution
  3. infinitely many solutions
- A linear system is *consistent* if there exists a solution (solvable). In this case, it has either one solution or infinitely many solutions.
- A system is *inconsistent* if it has no solution (insolvable).

A linear system written with a matrix

- The essential information of a linear system can be written with a rectangular array called a *matrix*. Given the system,

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

- Collecting the coefficients of each variable creates the following *coefficient matrix*.

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{pmatrix}$$

- From the coefficient matrix, attaching the constants from the rightmost end of the equations create the following *augmented matrix*.

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{pmatrix}$$

- The basic strategy for solving a linear system is to *replace one system with an equivalent system* (*i.e., one with the same solution set*) *that is easier to solve.*

Solving a linear system.

- **Example 1**: Solve the following system.

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 & = & 0 \\ 2x_2 - 8x_3 & = & 8 \\ -4x_1 + 5x_2 + 9x_3 & = & -9 \end{array} \qquad \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{pmatrix}$$

(*work continued*)

Elementary row operations

- Elementary row operations include the following:
  - (Replacement) Replace one row by the sum of itself and a multiple of another row.
  - (Interchange) Interchange two rows.
  - (Scaling) Multiply all entries in a row by a nonzero constant.
- Two matrices are called *row equivalent* if there is a sequence of elementary row operations that transforms one matrix into the other.
- Row operations are *reversible*.
- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Existence and uniqueness

- Mathematicians are always concerned with the question: "Does there exist a solution?' '. Also, if one exists, they want to know whether it is the only one.
- Existence
    - Whether a solution exists.
    - Regarding solvability of (system of) equation.
    - Concerned with consistency of the (system of) equation.
    - If exists, we say that *there exists a solution*.
    - (Note that the above italicized statement does not rule out the possibility of multiple solutions.)
- Uniqueness
    - Whether a solution is the only solution.
    - Whether any other solution exists.
    - If so, we say that *there exists a unique solution*.
- The statement: "*There exists one and only one solution*' ' implies the existence and uniqueness.

- **Example 3**. Determine if the following system is consistent:

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

- **Solution**

1. The augmented matrix, $[A|\mathbf{b}]$, is

$$\left(\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right)$$

2. To obtain an x_1 in in the first equation, interchange R1 and R2:

$$\left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right)$$

3. To eliminate the $5x_1$ term in the third equation, add -5/2 times R1 to R3.

$$\left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{array}\right)$$

4. Next, use the x_2 term in the second equation to eliminate the (-1/2)x_2 term from the third equation. Add 1/2 times R2 to R3.

$$\left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{array}\right)$$

5. The augmented matrix is now in triangular form. To interpret it correctly, let's go back to equation notation.

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= 5/2 \end{aligned}$$

6. $0 = 5/2$?
$0x_1 + 0x_2 + 0x_3 = 5/2$?
There is no such value of x_1, x_2, x_3 that satisfy this equation. This is never true. Impossible. *No solution*. *Hence, inconsistent*.

1.2. Row Reduction and Echelon Forms

Echelon form

- A *nonzero* row or column in a matrix means a row or column that contains at least one nonzero entry
- A *leading entry* of a row refers to the leftmost nonzero entry (in a nonzero row).
- A rectangular matrix is in *echelon form* (or *row echelon form*) if it has the following three properties:
  1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zeros.
  - (Roughly, *Echelon form* refers to a shape that is as triangular as possible)

Row reduced echelon form

- If a matrix in echolon fom satisfies the following additional conditions, then it is said to be in *reduced echelon form* (or *reduce row echelon form*):
  4. The leading entry in each nonzero row is 1.
  5. Each leadning 1 is the only nonzero entry in its column.
  - (Roughly, *Reduced echelon form* refers to a shape that is as diagonal as possible with diagonal entries are 1)

Uniqueness

- Any nonzero matrix may be *row reduced* (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations.
- However, *the reduced echelon form* from a matrix is unique.

Theorem (Uniqueness of the Reduced Echelon Form.)

Each matrix is row equivalent to one and only one reduced echelon matrix.

Pivot position

- If a matrix A is row equivalent to an echelon matrix U, we call U an *echelon form* (*or row echelon form*) *of* A
- If U is in reduced echelon form, we call U *the reduced echelon form of* A.
- *A pivot position* in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.
- A *pivot column* is a column of A that contains a pivot position.

- **Example 2**. Row reduce the matrix A below to echelon form, and locate the pivot columns of A.

$$A = \begin{pmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{pmatrix}$$

- **Solution**

1. The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or pivot, must be placed in this position. Thus, interchange R1 and R4.

$$\begin{pmatrix} \underline{1} & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}$$

2. Create zeros below the pivot, 1, by adding multiples of the R1 to the rows below, and obtain the next matrix.

$$\begin{pmatrix} \underline{1} & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}$$

3. Next, choose the 2 in R2 as the next pivot. Create zeros below it.

$$\begin{pmatrix} \underline{1} & 4 & 5 & -9 & -7 \\ 0 & \underline{2} & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}$$

4. There is no way to create a leading entry in column 3. However, if we interchange R3 and R4, we can produce a leading entry in column 4.

$$\begin{pmatrix} \underline{1} & 4 & 5 & -9 & -7 \\ 0 & \underline{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{pmatrix}$$

5. The matrix below is in echelon form, this C1, C2, C4 of A are *pivot columns*.

$$\begin{pmatrix} \underline{1} & 4 & 5 & -9 & -7 \\ 0 & \underline{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & \underline{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

6. From the original matrix A, pivots positions (0, -2, 3) are identified. And pivot columns are its corresponding columns, namely, C1, C2, C4.

$$A = \begin{pmatrix} \underline{0} & -3 & -6 & 4 & 9 \\ -1 & \underline{-2} & -1 & 3 & 1 \\ -2 & -3 & 0 & \underline{3} & 1 \\ 1 & 4 & 5 & -9 & -7 \end{pmatrix}$$

7. (Exercise) From the steps 1-5, we have identified an echelon form of A. Continue to find the reduced echelon form of A.

- **Example 3**. Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form:

$$A = \begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

- **Solution**
- **STEP 1:** *Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.*

$$A = \begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

- **STEP 2:** *Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.*
- In this example, we do R1 $\leftrightarrow$ R3. (could have done R1 $\leftrightarrow$ R2 instead)

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

- **STEP 3:** *Use row replacement operations to create zeros in all positions below the pivot.*

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

- **STEP 4:** *Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.*

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

We have reached an echelon form of the full matrix. We need to perform one more step to obtain the reduced echelon form.

- **STEP 5:** Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.*

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

- This is the reduced echelon form of the original matrix.
- The combination of steps 1–4 is called the *forward phase* of the row reduction algorithm.
- Step 5, which produces the unique reduced echelon form, is called the *backward phase*.
- The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.

- The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.
- Suppose, for example, that the augmented matrix of a linear system has been changed into the equivalent *reduced echelon form*.

$$\begin{pmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- There are *three* variables because the augmented matrix has *four* columns. The associated system of equations is

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

- Again, there are three variables, and
  - The variables x_1 and x_2 corresponding to pivot columns in the matrix are called *basic variables*.
  - The other variable (corresponding to non-pivot column), x_3, is called a *free variable*.
- Whenever a system is consistent, as in this example, the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables.

$$
\begin{aligned}
x_1 &= 5x_3 + 1 \\
x_2 &= -x_3 + 4 \\
x_3 &\text{ is free.}
\end{aligned}
$$

- The statement "x_3 is free' ' means that you are free to choose any value for x_3. Once the value of free variable is chosen, values of basic variables are determined.
- Each different choice of x_3 determines a (different) solution of the system, and every solution of the system is determined by a choice of x_3.
- "*There exist infinite number of solution.' ' does not cut out for the best description of solution. Yet, identifying free variables and describing the set of solution using the free variables is the better practice.*
- In the rightabove set of equations for basic variables, what happens if the system is consistent and it has a unique solution?

Parametric descriptions of solutions sets

- The descriptions are *parametric descriptions* of solutions sets in which the free variables act as parameters.
- *Solving a system* amounts to finding a parametric description of the solution set or determining that the solution set is empty.
- Whenever a system is consistent and has free variables, the solution set has many parametric descriptions. From the example,

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

the following description is a totally legit parametric description as well.

$$x_1 \text{ is free; } \; x_2 = \frac{21}{4} - \frac{1}{5}x_1; \; x_3 = \frac{1}{5}x_1 - \frac{1}{5}$$

Existence and uniqueness

- Whenever a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has no parametric representation.
- The discussion of free variable and parametric description leads to another perspective on the existence and uniqueness of solution.

Theorem (Existence and Uniqueness Theorem)

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column—i.e., if and only if an echelon form of the augmented matrix has no row of the form $[0 \ \ldots \ 0 \ b]$ *with* b *nonzero.*
If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

Row reduction to solve a linear system

- Using row reduction to solve a linear system
  - Step 1. Write the augmented matrix of the system.
  - Step 2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
  - Step 3. Continue row reduction to obtain the reduced echelon form.
  - Step 4. Write the system of equations corresponding to the matrix obtained in Step 3.
  - Step 5. Rewrite each nonzero equation from Step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

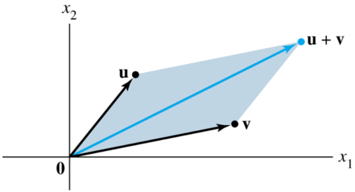
1.3. Vector equations

Example 1

$$\text{Given } \mathbf{u} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}, \text{ find } 4\mathbf{u}, (-3)\mathbf{v}, 4\mathbf{u} - 3\mathbf{v}$$

Geometric description of $\mathbb{R}^2$

- Consider a recangular coordinate system in the plane. Because each point in the plane is determine by an ordered pair of numbers, we can identify a geometric point (a, b) with the column vector $\begin{pmatrix} a \\ b \end{pmatrix}$.
- If $\mathbf{u}$ and $\mathbf{v}$ are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the paralleogram whose other vertices are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{0}$.



Linear combinations

- Given vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, ..., c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_p\mathbf{v_p}$$

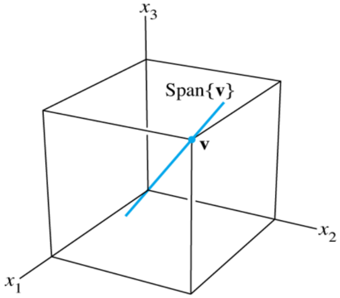
  is called a *linear combination* of $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}$ with *weights* $c_1, c_2, ..., c_p$.
- The weights can be any real numbers, including zero.
- If $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}$ are in $\mathbb{R}^n$, then the set of all linear combination of $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}$ is denoted by $Span\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}\}$, and is called the *subset of* $\mathbb{R}^n$ *spanned* (*or generated*) *by* $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}$. That is, $Span\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_p\mathbf{v_p}$$

  with $c_1, c_2, ..., c_p$ scalars.

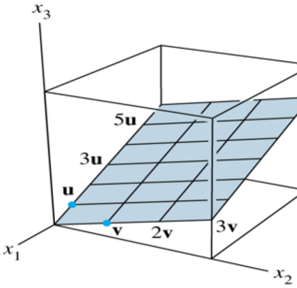
A geometric description of $Span\{\mathbf{v}\}$

- Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$. Then $Span\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$, which is the set of points on the line in $\mathbb{R}^3$ through $\mathbb{v}$ and $\mathbb{0}$.

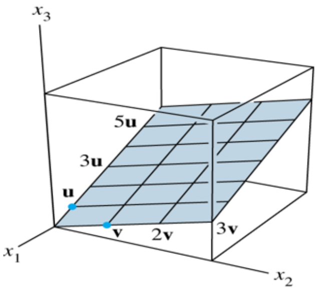


A geometric description of $Span\{\mathbf{u}, \mathbf{v}\}$

- If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$, with $\mathbf{v}$ not a multiple of $\mathbf{u}$, then $Span\{\mathbf{u}, \mathbf{v}\}$ is the plane in $\mathbb{R}^3$ that contains poins of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{0}$.
- In particular, $Span\{\mathbf{u}, \mathbf{v}\}$ contains the line in $\mathbb{R}^3$ through $\mathbf{u}$ and $\mathbf{0}$ and the line through $\mathbf{v}$ and $\mathbf{0}$.



Consistency of $A\mathbf{x} = \mathbf{b}$.



- For a linear system $A\mathbf{x} = \mathbf{b}$, TFAE (The Followings are All Equivalent)
  1. $A\mathbf{x} = \mathbf{b}$ is consistent.
  2.
  3.
  4.

1.4. The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

Computation of $A\mathbf{x}$

Example 4. Compute $A\mathbf{x}$, where

$$A = \begin{pmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

1. Row-perspective - The resulting vector's each element is the inner-product of each row of A and the vector $\mathbf{x}$.

$$\begin{aligned} A\mathbf{x} &= \begin{pmatrix} — & A_{1\bullet} & — \\ — & A_{2\bullet} & — \\ — & A_{3\bullet} & — \end{pmatrix} \begin{pmatrix} | \\ \mathbf{x} \\ | \end{pmatrix} \\ &= \begin{pmatrix} A_{1\bullet} \cdot \mathbf{x} \\ A_{2\bullet} \cdot \mathbf{x} \\ A_{3\bullet} \cdot \mathbf{x} \end{pmatrix} = \begin{pmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{pmatrix} \end{aligned}$$

2. Column-perspective - $A\mathbf{x}$ is the linear combination of column vectors of A, weighted by each element of $\mathbf{x}$.

$$\begin{aligned}
A\mathbf{x} &= \begin{pmatrix} | & | & | \\ A_{\bullet 1} & A_{\bullet 2} & A_{\bullet 3} \\ | & | & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= x_1 A_{\bullet 1} + x_2 A_{\bullet 2} + x_3 A_{\bullet 3} \\
&= x_1 \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ -3 \\ 8 \end{pmatrix} \\
&= \begin{pmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{pmatrix}
\end{aligned}$$

- If $\mathbf{b}$ can be expressed as a linear combination of the column vectors of A, then what?

Theorem (Matrix equation vs vector equation)

If A is an $m \times n$ matrix, with columns $A_{\bullet 1}$, ... , $A_{\bullet n}$, and if $\mathbf{b}$ is in $\mathbb{R}^n$, then the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution set as the vector equation

$$x_1 A_{\bullet 1} + \ldots + x_n A_{\bullet n} = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{pmatrix} | & | & & | & | \\ A_{\bullet 1} & A_{\bullet 2} & \ldots & A_{\bullet n} & \mathbf{b} \\ | & | & & | & | \end{pmatrix}$$

- The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Theorem

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. (Statements are either all true or all false.)

- **a.** For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
- **b.** Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
- **c.** The columns of A span $\mathbb{R}^m$.
- **d.** A has a pivot position in every row.

Suggested Exercises

- 1.1.5, 1.1.16, 1.1.18, 1.1.29
- 1.2.10
- 1.3.14, 1.3.18
- 1.4.11, 1.4.22

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"Man can learn nothing unless he proceeds from the known to the unknown. - Claude Bernard"
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