Chapter 2. Matrix Algebra (1/2)

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2.1. Matrix operation

Matrix operation

- A is an $m \times n$ matrix.
  - That is, a matrix with m rows and n columns
  - Then, the scalar entry in the i-th row and j-th column of A is denoted by a_{ij} and is called the (i, j)-entry of A.
- Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$.

$$
\begin{array}{c c}
 & \begin{array}{ccccc} \phantom{a_{11}} & & \text{Column } j & & \phantom{a_{1n}} \end{array} \\
\text{Row } i & \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = A \\
 & \begin{array}{ccccc} \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & & \mathbf{a}_j & & \mathbf{a}_n \end{array}
\end{array}
$$

Matrix notation.

Matrix operation

- The columns are denoted by $\mathbf{a}_1, \cdots \mathbf{a}_n$, and the matrix A is written as

$$A = \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \\ | & | & & | \end{bmatrix}$$

- The number a_{ij} is the i-th entry (from the top) of the j-th column vector $\mathbf{a}_j$.
- The **diagonal entries** in an $m \times n$ matrix $A = [a_{ij}]$ are a_{11}, a_{22}, a_{33}, ..., and they form the **main diagonal** of A.
- A **diagonal matrix** is a square $n \times n$ matrix whose nondiagonal entries are zero.
- An example of diagonal matrix is the $n \times n$ identity matrix, I_n.

$$I_3 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

Sums and Scalar Multiples

- An $m \times n$ matrix whose entries are all zero is a **zero matrix** and is written as 0.

- The two matrices are **equal** if
  - they have the same size (i.e., the same number of rows and the same number of columns)
  - their corresponding columns are equal,
  - which amounts to saying that their corresponding entries are equal.

- If A and B are $m \times n$ matrices, then the **sum** $A + B$ is the $m \times n$ matrix whose columns are the sums of the corresponding columns in A and B.

Sums and Scalar Multiples

- Since vector addition of the columns is done entrywise, each entry in $A+B$ is the sum of the corresponding entries in A and B.
- The sum $A+B$ is defined only when A and B are the same size.
- **Example 1:** Let $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}, C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$. Find $A+B$ and $A+C$
- **Solution:** $A+B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$ but $A+C$ is not defined because A and C have different sizes.

Sums and Scalar Multiples

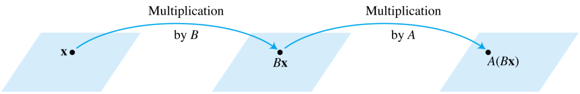
- If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose columns are r times the corresponding columns in A.

- **Theorem 1:** Let A, B and C be matrices of the same size, and let r and s be scalars.
  a) $A + B = B + A$
  b) $(A + B) + C = A + (B + C)$
  c) $A + 0 = A$
  d) $r(A + B) = rA + rB$
  e) $(r + s)A = rA + sA$
  f) $r(sA) = (rs)A$

- Each quantity in Theorem 1 is verified by showing that the matrix on the left side has the same size as the matrix on the right and that corresponding columns are equal.

Matrix Multiplication

- When a matrix B multiplies a vector $\mathbf{x}$, it transforms $\mathbf{x}$ into the vector $B\mathbf{x}$.
- If this vector is then multiplied in turn by a matrix A, the resulting vector is $A(B\mathbf{x})$.
- See the Fig. 2 below.

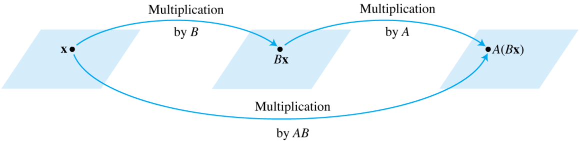


Multiplication by B and then A.

- Thus, $A(B\mathbf{x})$ is produced from x by a *composition of mappings*—the linear transformations.

Matrix Multiplication

- Our goal is to represent this composite mapping as multiplication by a single matrix, denoted by AB, so that $A(B\mathbf{x}) = (AB)\mathbf{x}$.
- See Fig. 3 below



Multiplication by AB.

Matrix Multiplication

- If A is $m \times n$, B is $n \times p$, and $\mathbf{x}$ is in $\mathbb{R}^p$, denote the columns of B by $\mathbf{b}_1, \cdots \mathbf{b}_p$ and the entries in $\mathbf{x}$ by $x_1, \cdots, x_p$
- Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + \cdots + x_p\mathbf{b}_p$$

- By the linearity of multiplication by A,

$$\begin{aligned} A(B\mathbf{x}) &= A(x_1\mathbf{b}_1) + \cdots + A(x_p\mathbf{b}_p) \\ &= x_1 A\mathbf{b}_1 + \cdots + x_p A\mathbf{b}_p \end{aligned}$$

- The vector $A(B\mathbf{x})$ is a linear combination of the vectors $A\mathbf{b}_1, \cdots, A\mathbf{b}_p$, using the entries in $\mathbf{x}$ as weights.
- In matrix notation, this linear combination is written as

$$A(B\mathbf{x}) = [A\mathbf{b}_1 \;\; A\mathbf{b}_2 \cdots A\mathbf{b}_p]\mathbf{x}$$

- Thus, multiplication by $[A\mathbf{b}_1 \; A\mathbf{b}_2 \; \cdots \; A\mathbf{b}_p]$ transforms $\mathbf{x}$ into $A(B\mathbf{x})$.

Matrix Multiplication

- **Definition:** If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$.

- That is,
$$AB = A[\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \cdots \;\; \mathbf{b}_p] = [A\mathbf{b}_1 \;\; A\mathbf{b}_2 \;\; \cdots \;\; A\mathbf{b}_p]$$

- *Multiplication of matrices corresponds to composition of linear transformations.*

Matrix Multiplication

- **Example 3:** Compute AB, where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.
- **Solution:** Write $B = \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix}$, and compute:

$$A\mathrm{b}_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}, \quad A\mathrm{b}_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix}, \quad A\mathrm{b}_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$

  - Then

$$AB = A\begin{bmatrix} \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

$A\mathrm{b}_1$ $A\mathrm{b}_2$ $A\mathrm{b}_3$

Matrix Multiplication

- Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

Row—column rule for computing AB

- If a product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B.
- If $(AB)_{ij}$ denotes the (i, j)-entry in AB and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Properties of matrix multiplication

- **Theorem 2:** Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

  a) $A(BC) = (AB)C$ (associative law of multiplication)
  b) $A(B + C) = AB + AC$ (left distributive law)
  c) $(B + C)A = BA + CA$ (right distributive law)
  d) $r(AB) = (rA)B = A(rB)$ for any scalar r
  e) $I_m A = A = AI_n$ (identity for matrix multiplication)

- **Proof of (a):**
  - Property (a) follows from the fact that matrix multiplication corresponds to composition of linear transformations (which are functions)
  - It is known that the composition of functions is associative.
  - Let $C = [\mathbf{c}_1 \cdots \mathbf{c}_p]$, by the definition of matrix multiplication,

$$BC = [B\mathbf{c}_1 \cdots B\mathbf{c}_p]$$
$$A(BC) = [A(B\mathbf{c}_1) \cdots A(B\mathbf{c}_p)]$$

  - The definition of AB makes $A(B\mathbf{x}) = (AB)\mathbf{x}$ for all $\mathbf{x}$, so

$$A(BC) = [(AB)\mathbf{c}_1 \cdots (AB)\mathbf{c}_p] = (AB)C$$

Properties of matrix multiplication

- The left-to-right order in products is critical because AB and BA are usually not the same. Because the columns of AB are linear combinations of the columns of A, whereas the columns of BA are constructed from the columns of B.
- The position of the factors in the product AB is emphasized by saying that A is *right-multiplied* by B or that B is *left-multiplied* by A.
- If $AB = BA$, we say that A and B **commute** with one another.
- **Warnings:**
  1. In general, $AB \neq BA$.
  2. The cancellation laws do not hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$. *(True only if* $A{-}1$ *exists)*
  3. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$. In other words, $AB = 0 \nRightarrow A = 0$ or $B = 0$.

Powers of a matrix

- If A is an $n \times n$matrix and if k is a positive integer, then A^k denotes the product of k copies of A:

$$A^k = \underbrace{A \ \cdots \ A}_{k}$$

- If A is nonzero and if $\mathbf{x}$ is in $\mathbb{R}^n$, then $A^k\mathbf{x}$ is the result of left-multiplying $\mathbf{x}$ by A repeatedly k times.

- If $k = 0$, then $A^0\mathbf{x}$ should be $\mathbf{x}$ itself.

- Thus, A^0 is interpreted as the identity matrix.

The Transpose of a matrix

- Given an $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Theorem 3: Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a) $(A^T)^T = A$
b) $(A + B)^T = A^T + B^T$
c) For any scalar r, $(rA)^T = rA^T$
d) $(AB)^T = B^T A^T$

- The transpose of a product of matrices equals the product of their transposes in the *reverse* order.
- A^T is often denoted as A^t, ${}^tA$, or ${}^TA$ depending on different academic disciplines.

Suggested Exercises

- 2.1.27

2.2. The inverse of a matrix

Matrix operations

- An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \quad and \quad AC = I,$$

where $I = I_n$, the $n \times n$ identity matrix.

- C, an **inverse** of A, is uniquely determined by A, because if B were another inverse of A, then

$$B = BI = B(AC) = (BA)C = IC = C$$

- This unique inverse is denoted by A^{-1}, so that

$$A^{-1}A = I \ \text{ and } \ AA^{-1} = I$$

Matrix operations

- **Theorem 4:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
  - If $ad - bc \neq 0$, then A is invertible and
  $$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
  - If $ad - bc = 0$, then A is not invertible.

- The quantity $ad - bc = 0$ is called the **determinant** of A, and we write $det A = ad - bc$

- This theorem says that a 2×2 matrix A is invertible if and only if $det A \neq 0$

Matrix operations

- **Theorem 5:** If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = b$ has the unique solution $\mathbf{x} = A^{-1}b$.
- **Proof:**
  - Take any $\mathbf{b}$ in $\mathbb{R}^n$. A solution exists because if $A^{-1}b$ is substituted for $\mathbf{x}$, then $A\mathbf{x} = A(A^{-1}b) = (AA^{-1})b = Ib = b$. So, $A^{-1}b$ is a solution.
  - To prove that the solution is unique, we need to show that if $\mathbf{u}$ is any solution, then $\mathbf{u}$ must be $A^{-1}b$. If $Au = b$, we can multiply both sides by A^{-1} and obtain $A^{-1}Au = A^{-1}b$,$Iu = A^{-1}b$, and $u = A^{-1}b$.

Matrix operations

- **Theorem 6:**

  a) If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

  b) If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

  c) If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

Matrix operations

- **Proof of a)**
  - Find a matrix C such that $A^{-1}C = I$ and $CA^{-1} = I$
  - These equations are satisfied with A in place of C. Hence A^{-1} is invertible, and A is its inverse.

- **Proof of b)**
  - Compute: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$
  - A similar calculation shows that $(B^{-1}A^{-1})(AB) = I$.

- **Proof of c)**
  - Use Theorem 3(d), read from right to left, $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$.
  - Similarly, $A^T(A^{-1})^T = I^T = I$.
  - Hence A^T is invertible, and its inverse is $(A^{-1})T$.

Elementary matrices

- The generalization of Theorem 6(b) $(AB)^{-1} = B^{-1}A^{-1}$ is as follows:
  - The product of $n \times n$ invertible matrices is invertible
  - And the inverse is the product of their inverses in the reverse order.

- An invertible matrix A is row equivalent to an identity matrix, and we can find A^{-1} by *watching the row reduction of A to I.*

Elementary matrices

- An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

- **Example 5:** Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

and $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

  - Compute $E_1 A$, $E_2 A$, and $E_3 A$
  - And describe how these products can be obtained by elementary row operations on A.

Elementary matrices

- **Solution:**
  - Verify that
    $$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}, E_2A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix},$$
    $$E_3A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$
  - Description
    - Addition of -4 times row 1 of A to row 3 produces E_1A (R3 ◎R3 - 4R1)
    - An interchange of rows 1 and 2 of A produces E_2A (R1 ◎R2)
    - Multiplication of row 3 of A by 5 produces E_3A (R3 ◎5 ×R3)
- Remark
  - Left-multiplication (that is, multiplication on the left) by E_1 in Example 1 has the same effect on any $3 \times n$ matrix.
  - Since $E_1 \cdot I = E_1$, we see that E_1 *itself* is produced by this same row operation on the identity.

Elementary matrices

- Example 5 illustrates the following general fact about elementary matrices.
  - If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.
  - Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Elementary matrices

- **Theorem 7:** An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.
- **Proof:**
  - Suppose that A is invertible. Then, since the equation $A\mathbf{x} = b$ has a solution for each $\mathbf{b}$ (Theorem 5), A has a pivot position in every row. Because A is square, the n pivot positions must be on the diagonal, which implies that the reduced echelon form of A is I_n. That is, $A \sim I_n$.
  - Now suppose, conversely, that $A \sim I_n$. Then, since each step of the row reduction of A corresponds to left-multiplication by an elementary matrix, there exist elementary matrices $E_1, ..., E_p$ such that $A \sim E_1 A \sim E_2(E_1 A) \sim \cdots \sim E_p(E_{p-1} ... E_1 A) = I_n$.
  - That is, $E_p..E_1 A = I_n$. Since the product $E_p...E_1$ of invertible matrices is invertible,

$$\begin{aligned}(E_p...E_1)^{-1}(E_p...E_1)A &= (E_p...E_1)^{-1} I_n \\ A &= (E_p...E_1)^{-1}\end{aligned}$$

- **(Proof continued:)**
  - Thus, A is invertible, as it is the inverse of an invertible matrix (Theorem 6). Also, $A^{-1} = [(E_p ... E_1)^{-1}]^{-1} = E_p ... E_1$ Then, $A^{-1} = E_p ... E_1 \cdot I_n$, which says that A^{-1} results from applying $E_1, ..., E_p$ successively to I_n. This is the same sequence that reduced A to I_n.

Algorithm for finding A^{-1}

- Row reduce the augmented matrix$[A \;\; I]$. If A is row equivalent to I, then $[A \;\; I]$ is row equivalent to $[I \;\; A^{-1}]$. Otherwise, A does not have an inverse.

Algorithm for finding A^{-1}

- **Example 2:** Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

- **Solution:**

$$
\begin{aligned}
[A|I] &= \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]
\end{aligned}
$$

Algorithm for finding A^{-1}

- **(Solution continued:)**
  - Theorem 7 shows, since $A \sim I$, that A is invertible, and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

  - Now, check the final answer.

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Another view of matrix inversion

- It is not necessary to check that $A^{-1}A = I$ since A is invertible.

- Denote the columns of I_n by e_1, ..., e_n. Then, row reduction of $[A \;\; I]$ to $[I \;\; A^{-1}]$ can be viewed as the simultaneous solution of the n systems

$$A\mathbf{x} = e_1, \;\; A\mathbf{x} = e_2, \;\; \cdots, A\mathbf{x} = e_n, \tag{1}$$

where the "augmented columns" of these systems have all been placed next to A to form

$$[A \;\; e_1 \;\; e_2 \;\; \cdots \;\; e_n] = [A \;\; I]$$

- The equation $AA^{-1} = I$ and the d

Suggested Exercise

- 2.2.9
- 2.2.17
- 2.2.18
- 2.2.31

2.3. Characterization of invertible matrices

The invertible matrix theorem

- **Theorem 8:** Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

  a) A is an invertible matrix.

  b) A is row equivalent to the $n \times n$ identity matrix.

  c) A has n pivot positions.

  d) The equation $A\mathbf{x} = 0$ has only the trivial solution.

  e) The columns of A form a linearly independent set.

  f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

  g) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

  h) The columns of A span $\mathbb{R}^n$.

  i) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$

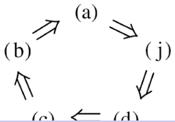
  j) There is an $n \times n$ matrix C such that $CA = I$.

  k) There is an $n \times n$ matrix D such that $AD = I$.

  l) A^T is an invertible matrix.

The proof for the invertible matrix theorem

- If statement (a) is true, then A^{-1} works for C in (j), so (a) $\Rightarrow$ (j).
- Next, (j) $\Rightarrow$ (d).
- Also, (d) $\Rightarrow$ (c).
- If A is square and has n pivot positions, then the pivots must lie on the main diagonal, in which case the reduced echelon form of A is I_n.
- Thus (c) $\Rightarrow$ (b).
- Also, (b) $\Rightarrow$ (a).
- So far, we have completed the following circle.



- Next, (a) $\Rightarrow$ (k) because A^{-1} works for D.
- Also, (k) $\Rightarrow$ (g) and (g) $\Rightarrow$ (a).
- So (k) and (g) are linked to the circle.
- Further, (g), (h), and (i) are equivalent for any matrix.
- Thus, (h) and (i) are linked through (g) to the circle.
- Since (d) is linked to the circle, so are (e) and (f), because (d), (e), and (f) are all equivalent for *any* matrix A.
- Finally, (a) $\Rightarrow$ (l) and (l) $\Rightarrow$ (a).
- This completes the proof.

The invertible matrix theorem

- Theorem 8 could also be written as
  - "The equation $A\mathbf{x} = b$ has a *unique* solution for each $\mathbf{b}$ in $\mathbb{R}^n$."
  - This statement implies (b) and hence implies that A is invertible.
- The following fact follows from Theorem 8.
  - Let A and B be square matrices. If $AB = I$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.
- The Invertible Matrix Theorem divides the set of all $n \times n$ matrices into two disjoint classes.
  - the invertible (nonsingular) matrices
  - the noninvertible (singular) matrices.
- Class property
  - Each statement in the theorem describes a property of every $n \times n$ invertible matrix.
  - The *negation* of a statement in the theorem describes a property of every $n \times n$ singular matrix.
  - For instance, an $n \times n$ singular matrix is *not* row equivalent to I_n, does not have n pivot position, and has linearly *dependent* columns.

The invertible matrix theorem

- **Example 1:** Use the Invertible Matrix Theorem to decide if A is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

- **Solution:**
  - Checking the row equivalance of

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

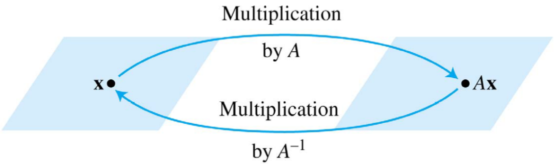
  - So A has three pivot positions and hence is invertible, by the Invertible Matrix Theorem, statement (c).

The invertible matrix theorem

- The Invertible Matrix Theorem *applies only to square matrices*.
- For example, if the columns of a 4×3 matrix are linearly independent, we cannot use the Invertible Matrix Theorem to conclude anything about the existence or nonexistence of solutions of equation of the form $A\mathbf{x} = b$.

Invertible linear transformation

- Matrix multiplication corresponds to composition of linear transformations.
- When a matrix A is invertible, the equation $A^{-1}A\mathbf{x} = \mathbf{x}$ can be viewed as a statement about linear transformations.



- See the above figure. A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.

Invertible linear transformation

- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{2}$$

$$T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{3}$$

Invertible linear transformation

- **Theorem 9:** Let be $T : \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying equation (2) and (3).

- **Proof:**
  - Suppose that T is invertible. Then, (4) shows that T is onto $\mathbb{R}^n$, for if **b** is in $\mathbb{R}^n$ and $\mathbf{x} = S(b)$, then $T(\mathbf{x}) = T(S(b)) = b$, so each **b** is in the range of T. Thus A is invertible, by the Invertible Matrix Theorem, statement (i).
  - Conversely, suppose that A is invertible, and let $S(\mathbf{x}) = A^{-1}\mathbf{x}$. Then, S is a linear transformation, and S satisfies (2) and (3). For instance, $S(T(\mathbf{x})) = S(A\mathbf{x}) = A^{-1}(A\mathbf{x}) = \mathbf{x}$. Thus, T is invertible.

Suggested Exercies

- 2.3.11

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