Mathematics for AI

Homework 5

Read these first:

i. To achieve the full score, you need to write your solutions using LaTeX. If you choose to write your solutions on paper or in a word processing software (e.g., MS Word, LibreOffice), you can receive up to 90% of the score.

ii. If writing on paper, you must use a scanner or a camera scanning app (e.g., CamScanner) to scan the document and submit it as a *single PDF file*. Ensure your answers are written neatly, organized, and legible on paper.

iii. When using LaTeX, follow one of these two conventions:

   (a) Represent scalars with italic letters (a, A), vectors with bold lowercase letters ($\mathbf{a}$, using `\mathbfa`), and matrices with bold uppercase letters ($\mathbf{A}$, using `\mathbfA`), or

   (b) Represent scalars with italic letters (a, A), vectors with bold letters ($\mathbf{a}$), and matrices with typewriter uppercase letters ($\mathtt{A}$, using `\mathttA`).

iv. Your LaTeXdocument must include a *title*, a *date*, and your name as the *author*.

v. If writing on paper, submit a single PDF file; do not send multiple image files.

vi. If using LaTeX, submit the `.tex` source file (along with any other required source files) in addition to the PDF file.

Here is a short tutorial on LaTeX:
`https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes`

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Homework 5

Fall 2024

Convex sets

1. Let $C \subseteq \mathbb{R}^n$ be a convex set, with $x_1, \ldots, x_k \in C$, and let $\theta_1, \ldots, \theta_k \in \mathbb{R}$ satisfy $\theta_i \geq 0$, $\theta_1 + \cdots + \theta_k = 1$. Show that $\theta_1 x_1 + \cdots + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) *Hint. Use induction on k.*

   [Boyd, S.(2004). *Convex Optimization.* Cambridge University Press. (Exercise 2.1)]

2. Let $C \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbb{R}^n \mid x^T A x + b^T x + c \leq 0\},$$

with $A \in S^n$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

   (a) Show that C is convex if $A \succeq 0$.

   (b) Show that the intersection of C and the hyperplane defined by $g^T x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda g g^T \succeq 0$ for some $\lambda \in \mathbb{R}$.

   Are the converses of these statements true?

   [Boyd, S.(2004). *Convex Optimization.* Cambridge University Press. (Exercise 2.10)]

Convex functions

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex, and $a, b \in \operatorname{dom} f$ with $a < b$.

   (a) Show that

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

   for all $x \in [a, b]$.

   (b) Show that

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(x)}{b - x}$$

   for all $x \in (a, b)$. Draw a sketch that illustrates this inequality.

(c) Suppose f is differentiable. Use the result in (b) to show that

$$f'(a) \leq \frac{f(b) - f(a)}{b - a} \leq f'(b).$$

Note that these inequalities also follow from (3.2):

$$f(b) \geq f(a) + f'(a)(b - a), \quad f(a) \geq f(b) + f'(b)(a - b).$$

(d) Suppose f is twice differentiable. Use the result in (c) to show that $f''(x) \geq 0$ and $f''(b) \geq 0$.

[Boyd, S.(2004). *Convex Optimization*. Cambridge University Press. (Exercise 3.1)]

4. For each of the following functions determine whether it is convex, concave.

(a) $f(x) = e^x - 1$ on $\mathbb{R}$.

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$.

(e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$.

(f) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $\mathbb{R}^2_{++}$.

[Boyd, S.(2004). *Convex Optimization*. Cambridge University Press. (Exercise 3.16)]

Duality

5. A simple example Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x^2 + 1 \\ \text{subject to} & (x - 2)(x - 4) \leq 0, \end{array}$$

with variable $x \in \mathbb{R}$.

(a) **Analysis of primal problem.** Give the feasible set, the optimal value, and the optimal solution.

(b) **Lagrangian and dual function.** Plot the objective x^2+1 versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^* \geq \inf_x L(x, \lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.

(c) **Lagrange dual problem.** State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ^*. Does strong duality hold?

[Boyd, S.(2004). *Convex Optimization*. Cambridge University Press. (Exercise 5.1)]

6. Consider the equality constrained least-squares problem

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & Gx = h, \end{array}$$

where $A \in \mathbb{R}^{m \times n}$ with $\operatorname{rank} A = n$, and $G \in \mathbb{R}^{p \times n}$ with $\operatorname{rank} G = p$.

Give the KKT conditions, and derive expressions for the primal solution $x^\star$ and the dual solution $\nu^\star$.

[Boyd, S.(2004). *Convex Optimization.* Cambridge University Press. (Exercise 5.27)]