DGM - Homework 1

Spring 2025

Question 1: Consider a cylindrical water well with a radius of 1 meter. We drop stones from the top-centre of the well. But due to air flow inside the well the stones will not exactly land on the centre at the bottom. The bottom of the well is a circular region represented as

$$C = \{(x, y) \,|\, x^2 + y^2 \leq 1\}. \tag{1}$$

Where the stones fall follows a probabilistic distribution represented by a PDF $f: C \mapsto \mathbb{R}$ with the following form:

$$f(x, y) = \alpha \exp(\sqrt{x^2 + y^2}) \tag{2}$$

1. What is the value of α? (Hint: You may use polar coordinates.)
2. What is the probability of a stone landing in the region

$$S = \{(x, y) \,|\, \sqrt{x^2 + y^2} \leq \frac{1}{4}\}. \tag{3}$$

3. What is the probability of a stone landing in the region

$$S = \{(x, y) \,|\, \frac{1}{2} \leq \sqrt{x^2 + y^2} \leq \frac{3}{4}\}. \tag{4}$$

4. Are the random variables X and Y independent? Give an explaination without obtaining the marginal or conditional distributions?

5. Define the polar coordinates by random variables $R = \sqrt{X^2 + Y^2}$ and $\Theta = \text{atan2}(Y, X)$. What is the PDF $f_{R,\Theta}(r, \theta)$? Remember that the PDF must satisfy

$$\int_{(r,\theta)\in S} f_{R,\Theta}(r, \theta)\, dr d\theta = \Pr((R, \Theta) \in S).$$

6. Are the random variables R and Θ independent? Why?

Question 2: Consider the following joint probability mass function (PMF) $p_{I,N} \colon \{2, 3, 4, \ldots\} \times \{2, 3, 4, \ldots\} \to [0, 1]$:

$$p_{I,N}(i, n) = P(I = i, N = n) = \alpha \left(\frac{1}{i}\right)^n \tag{5}$$

where $i \in \{2, 3, \cdots\}$ and $n \in= \{2, 3, \cdots\}$.

1. Derive the value of α?

2. Obtain the formulae for the marginal distributions $p_I(i)$ and $p_N(n)$.

3. Obtain the formulae for the conditional distributions $p_I(i \mid N = n)$ and $p_N(n \mid I = i)$.

4. Are the random variables I and N independent? Why?

You may write some of the probabilities in terms of the *Riemann zeta function*:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{6}$$

Question 3: Consider the following joint PMF $f_{M,N} \colon \{1, 2, 3\} \times \{1, 2, 3\}$:

m	n	$f_{M,N}(m, n)$
-1	1	0.04
-1	2	0.06
0	1	0.05
0	2	0.08
1	1	0.30
1	2	0.47

1. Derive $f_M(m)$, $f_N(n)$, $f_M(m \mid N = n)$ and $f_N(n \mid M = m)$.

2. Are M and N independent? Why?

Question 4: Assume that the joint distribution

$$P(a,b,c) = \Pr(A=a, B=b, C=c)$$

can be written as

$$P(a,b,c) = \phi(a,b)\,\psi(a,c),$$

where $\phi(a,b) > 0$ and $\psi(a,c) > 0$ for all values of a, b, c.

1. Prove that B and C are conditionally independent given A.

Question 5: Consider the random variables A, B, C takeing on values in sets $D_A = \{0,1,2\}$, $D_B = \{1,2,3,4\}$, $D_C = \{2,3\}$, respectively. Derive the minimum number of *free* parameters to fully identify each of the following distributions using *table representation*. For each case, explain your response.

- $P(A,B,C)$
- $P(A \mid B)$
- $P(B \mid A)$
- $P(B \mid A)$
- $P(A \mid B,C)$
- $P(C \mid A,B)$
- $P(A,B \mid C)$
- $P(A,B,C)$ when we know that A, B, and C are mutually independent.
- $P(A,B,C)$ when we know that B is independent of C given A.