Criminal Pursuit on a Preferential Attachment Tree

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Part I: The Preferential Attachment Tree

We model the growth of a criminal syndicate with a recursive preferential attachment tree, where preference is given to nodes closer to leaves or leaves themselves. We represent the kingpin by a root node.

The Growth Process

Let $\mathbf{T}^0$ be a rooted seed tree and $\mathbf{T}^t$ be the tree at time t. To obtain $\mathbf{T}^t$ from $\mathbf{T}^{t-1}$, we add k new nodes to $\mathbf{T}^{t-1}$. A leaf is a node without any children; these are the street criminals. Let $d(j,t)$ denote the minimum distance from node j to a leaf taken over all leaves in the tree. Each new node attaches to an existing node j on $\mathbf{T}^{t-1}$ with probability proportional to weight $w(j,t-1)$. The weight $w(j,t)$ of node j at time t is defined as:

$$w(j,t) := \frac{1}{d(j,t)+1}$$

Node Density

We found the node density depends on the distance to the root. Figure [1] suggests a shifted gamma gamma (γ-)density approximates this relationship well. The shifted γ-density is given by:

$$\rho_{\alpha,\beta,s}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}(x-s)^{\alpha-1}e^{-\beta(x-s)}.$$

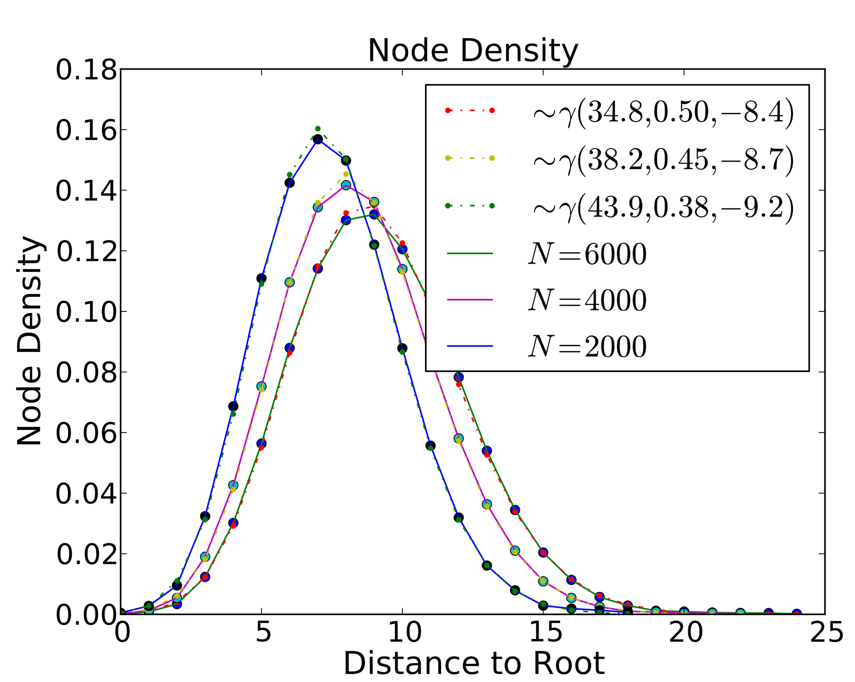

Figure 1: The number of nodes N was $2,000$; $4,000$; $6,000$ and the arrival rate was $k=4$. We averaged over 20 experiments. $\mathbf{T}^0$ was a complete tertiary tree of height 3. We fit the data with a shifted γ-density.

Degree Distribution

A key feature of the Barabási-Albert preferential attachment model [3] is its heavy-tailed degree distribution, which obeys a power-law. From the data seen in Figure [2], we conjecture the degrees in this model are also heavy-tailed, and further, obeys an exponential law.

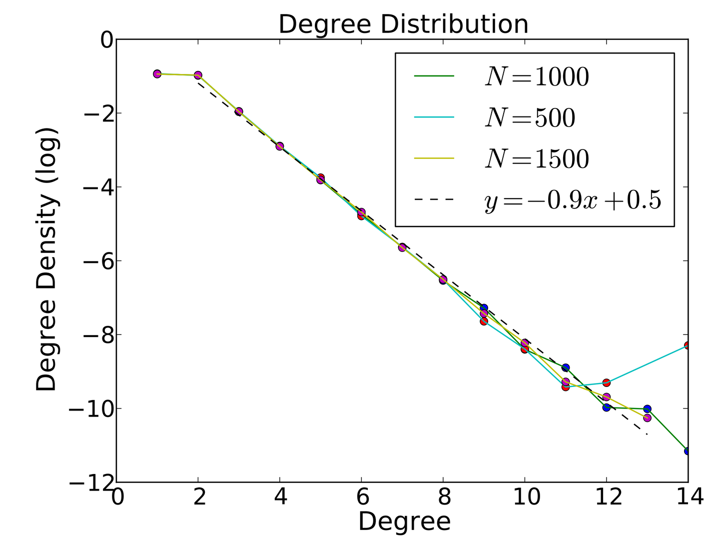

Figure 2: The degree distribution appears to be exponentially distributed.

Rate of Leaf Addition

Leaves have the highest probability of recruitment and their distribution has important theoretical consequences. Figure [3] suggests the rate of leaf addition is constant.

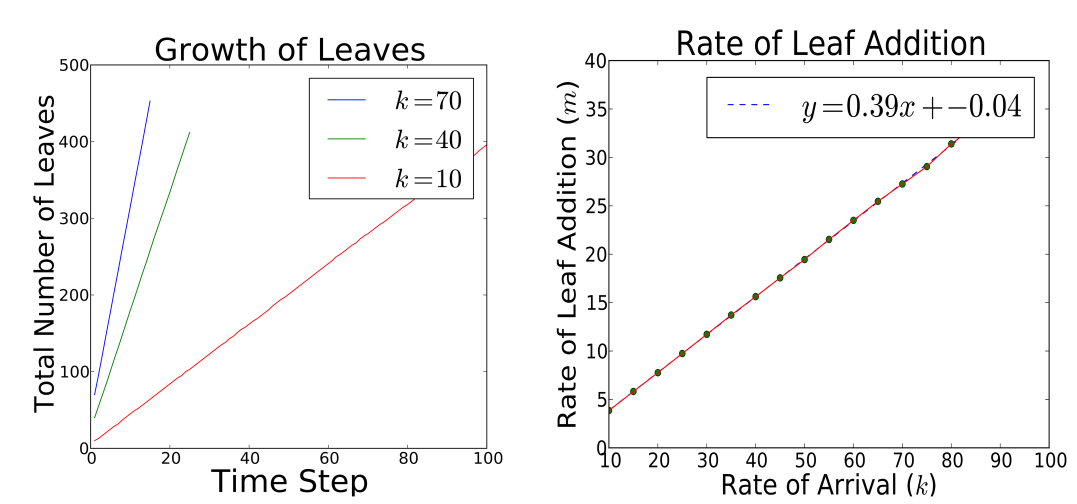

Figure 3: $N=1,000$ and k is varied. The relationship between number of leaves and time (left) appears linear; moreover, the slope of this relation appears to linearly depend on k (right).

Conjectures on Growth

1. The node density is given by a shifted gamma-density $\gamma(\alpha,\beta,s)$ depending on N, k, and $\mathbf{T}^0$.
2. The degree distribution as $N\to\infty$ is exponential and depends on k and $\mathbf{T}^0$.
3. If $\ell(t) = \#$ of leaves at time t, then $\partial\ell/\partial t$ is constant and depends on k alone.

Part II: Introducing Pursuit

We now model the police pursuit of the kingpin on the preferential attachment tree. We search for the police officer's optimal strategy given that her knowledge of the network is local.

The Pursuit Process

Let $\mathbf{T}^0$ be a rooted seed tree. We define a recursive process to obtain $\mathbf{T}^t$ from $\mathbf{T}^{t-1}$.

- $\mathbf{T}^{t-1/2}$: We place an officer uniformly at random on the set of leaves. The officer can either:
  - move in a self-avoiding random walk (investigate) or
  - remove a node and all of its children (arrest).

  The pursuit ends if the officer reaches the root (kingpin caught and game ends), reaches another leaf, or arrests.
- $\mathbf{T}^t$: we conduct preferential attachment with k new nodes.

Police Strategies

1. The officer investigates p times and then arrests ($S_A(p)$).
2. The officer always investigates. She is successful only if she reaches the root (S_I).
3. The officer investigates unless reaching a node of degree at least q, in which case she arrests ($S_D(q)$).

Pursuit Simulations

For strategy $S_A(0)$, the officer always fails when $k>1$. See Figure [4] for a plot of when the police win with strategy $S_A(1)$ as the arrival rate k increases.

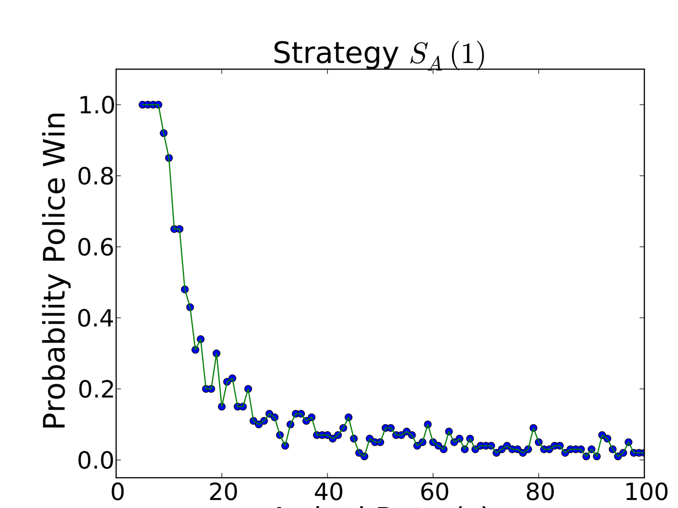

Figure 4: We experimentally determined the probability of winning with strategy $S_A(1)$. The strategy is successful for $k\le 8$. In particular, we estimate $\text{Beat}(S_A(1))\approx 8$.

Beat(S)

Let S be the officers strategy whether to investigate or arrest. Let:

$$\text{Beat}(S) := \max\left(k \,\middle|\, {}^{\text{police eliminate syndicate}}_{\text{in finite time}}\right)$$

The greater $\text{Beat}(S)$, the more effective a strategy S is against a growing criminal syndicate. See Figure [4] and [5] for examples.

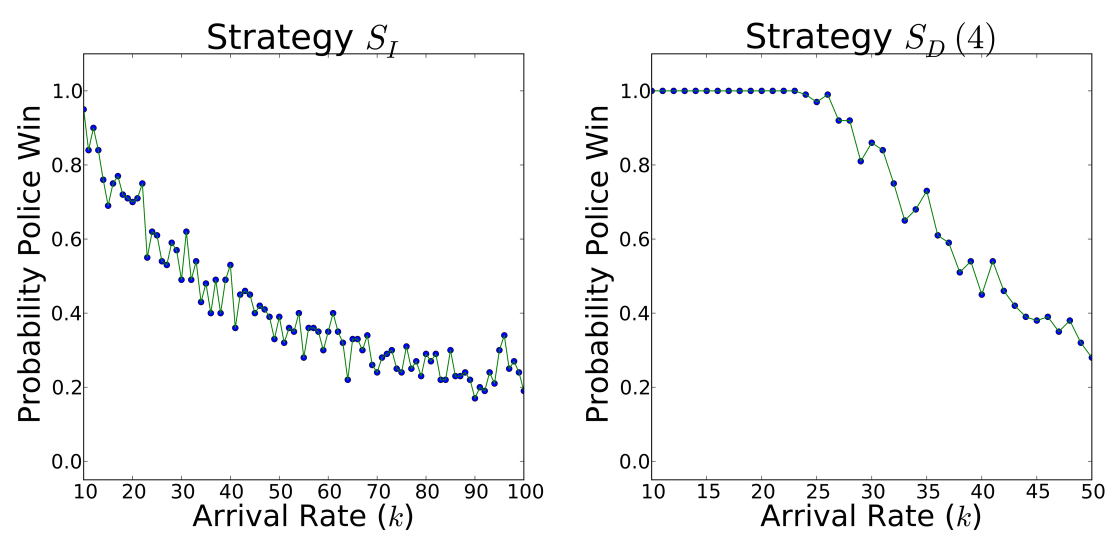

Figure 5: The probability of winning with $S_D(4)$ and S_I. For S_I, there persists a non-zero probability that the root is caught for large k. The right plot suggests $\text{Beat}(S_D(4))\approx 24$.

Conjecture on Pursuit

For every p, there exists a q such that

$$\text{Beat}(S_D(q)) > \max(\text{Beat}(S_A(p)), \text{Beat}(S_I)).$$

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Bibliography

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