Dynamic Preferential Tree Growth And Criminal Pursuit

Part I: Dynamic Preferential Tree Growth

Let $\mathbf{T}^t$ be the tree at time t. We set $\mathbf{T}^0 =$ root, dubbed Pablo.

- The process is recursive. To obtain $\mathbf{T}^t$ from $\mathbf{T}^{t-1}$, k new nodes are added to $\mathbf{T}^{t-1}$. Each new node will be attached to an existing node j in $\mathbf{T}^{t-1}$ with probability proportional to weight $w(j, t-1)$. We set:

  $$\boxed{w(j, t-1) = \text{the weight of node j} = \frac{1}{d(j, \mathcal{L}_{t-1}) + 1}}$$

  where $\mathcal{L}_{t-1} =$ set of leaves at time $t-1$ and d denotes the minimum distance via a directed path in $\mathbf{T}^{t-1}$. In this case, the path's from node j to the set $\mathcal{L}_{t-1}$.

Degree Distribution

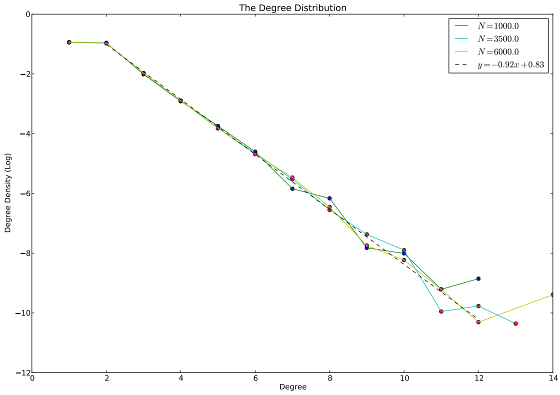

Figure : The Degree Density is given by $\mathbf{P}(\text{node has degree } d)$

Node Density

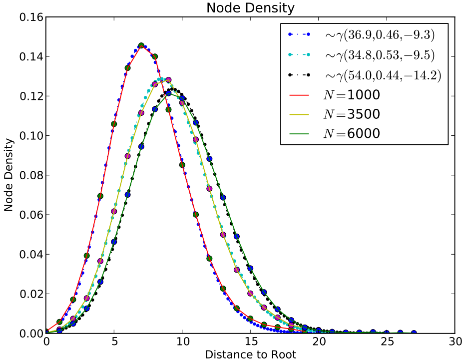

Figure : Node density is the probability that a randomly selected node is at a given height. We performed a single experiment for the above data. $\gamma(\alpha, \beta, s)$ is the fitted gamma distribution with translation parameter s.

The Gamma Distribution

- Let $S_{\alpha,\beta}$ be γ distributed random variable. The density of the gamma distribution is given by:

$$\rho_{\alpha,\beta}(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

- If $\alpha = n \in \mathbb{N}$ and $\beta = \lambda \in \mathbb{R}^{+}$, then the interpretation of $S_{n,\lambda}$ is as follows: if $X_1, \ldots, X_n$ are iid exponential random variables with parameter λ, then:

$$S_{n,\lambda} = X_1 + \ldots + X_n$$

- In the current situation, we add a parameter s, namely the gamma distribution is of the form $S_{\alpha,\beta,s} := S_{\alpha,\beta} + s$, where s is a constant. This simply translates the density function.

Conjectures on Growth

1. The Node Density is a gamma distributed random variable $S_{\alpha,\beta,s}$ where α, β, s depend on the rate of arrival k, the maximum number of nodes before the process stops N, and the weight function w.
2. The degree distribution of the tree obeys an exponential law where $\mathbf{P}(\text{node has degree } d) = e^{-md}$, with $m > 0$.

An Example of Growth

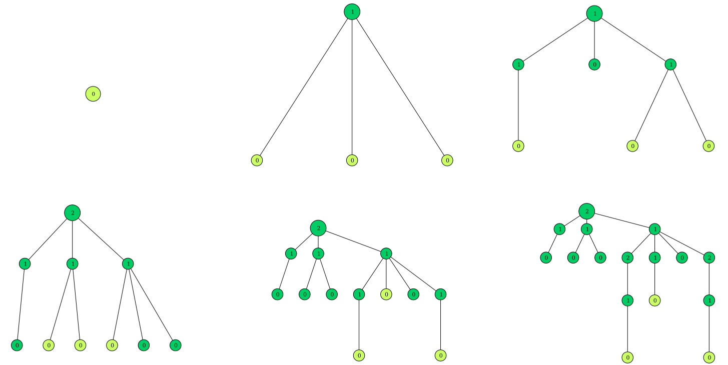

Figure : We see the growth for time steps $0 \leq t \leq 5$ and $k = 3$. We have labeled $d(j, \mathcal{L})$ on each node. The yellow nodes are the new nodes added at the time step.

Part II: The Criminal Pursuit

We begin with uniform complete k-ary tree as the seed. We set $\mathbf{T}^0 = \text{seed}$. Let $\mathbf{T}^t$ be the tree at time t.

1. $\mathbf{T}^{t+1/2}$: We place $k - k^*$ officers on the leaves where k is the arrival rate specified in the model and $k^* \geq 0$ (note when $k^* < 0$, the police always have a winning strategy). The officer can:
   - Move in a self-avoiding walk (investigate)
   - Remove a node and all of its children (arrest)

   If the officer reaches the root, the game is over. Should the officer reach another leaf in his walk, he must give up his investigation and wait until a future round.
2. $\mathbf{T}^{t+1}$: the dynamic growth described earlier is performed with arrival rate k.

Conjectures about Pursuit

1. Let S be a police strategy regarding whether to investigate or arrest. Let:

$$\mathrm{Beat}(S) := \max\left(k^* \middle| \text{police arrests Pablo in finite time}\right)$$

   Let S_1 be the strategy of arresting only and S_2 be the strategy of investigating only. Then, there exists a $S' = C(S_1, S_2)$, a combination of S_1 and S_2 such that

$$\mathrm{Beat}(C(S_1, S_2)) > \max(\mathrm{Beat}(S_1), \mathrm{Beat}(S_2))$$

   Moreover $C(S_1, S_2)$ depends only on *local* information (degree) and the length of the walk.

2. There exists a bad strategy S such that:

$$\mathrm{Beat}(S) < \min(\mathrm{Beat}(S_1), \mathrm{Beat}(S_2))$$