Solving Heterogeneous Agent Model: Continuous vs. Discrete Time

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Today's Goal

- Introduce: What are we solving for the heterogeneous agent model?
- What KLI has been working for? $\rightarrow$ Continuous time model
- Note: You can find the computational materials from the below

  Ben Moll for continuous time:
  `https://benjaminmoll.com/codes/`

  Bayesian estimation of HANK:
  `https://github.com/BASEforHANK`

Solving DSGE Models

- Representative Agent Model
  - Solving one agent of \$100 problem = Solving 100 agent with sum \$100 problem
  - We can solve it by linearization $\rightarrow$ Solving the linear system difference(discrete)/differential(continuous) time model
    $\rightarrow$ Solutions: Deviations from the steady-state
- Heterogeneous Agent Model
  - Solving one agent of \$100 problem $\neq$ Solving 100 agent with sum \$100 problem (This happens when the market is incomplete)
  - We need to solve the problem of each agent defined by state variables

Solving Heterogeneous Agent's Model: Discrete vs. Continuous

1. Discrete
    - Traditional.
    - Easy to code: Model structure $\approx$ Numerical algorithm
    - Slower than the continuous time
2. Continuous
    - Relatively new in a sense of solving HA models
    - For coding: We need more math for solving differential equations
    - By construction, faster than the discrete time

Solving Heterogeneous Agent's Model: Infinite vs. Finite

1. Infinite
   - Assume agents live forever
   - Efficient & Parsimonious for analyzing business-cycle/economic growth
2. Finite
   - Assume agents live finite. Usually in the life-cycle model
   - Appropriate for answering micro question
   - Seems better as it is more realistic, but still hard for solving for business-cycle questions
     $\rightarrow$ Due to the relationship between stationary distribution & aggregate shocks. Explained it in a next slide

Solving Heterogeneous Agent's Model: Without & With Aggregate Shocks - 1

- In other words, (time-invariant) stationary distribution vs. state-dependent distribution
- Distribution?
  - Usually, about the cross-sectional (given time, agent's) distribution
  - Example: Distribution of wealth for Tom, Jerry, Kim and Lee
- Stationary distribution?
  - Over time, each agent's state can be changed, but the overall cross-sectional distribution is not changed
  - Today: Tom - $100, Jerry - $200, Kim - $50 and Lee - $ 150
  - Tomorrow: Tom - $200, Jerry - $100, Kim - $150 and Lee - $ 50

Solving Heterogeneous Agent's Model: Without & With Aggregate Shocks - 2

1. Without aggregate shock
   - Stationary distribution
   - Can work with relatively more well-defined model $\rightarrow$ Easier to find a fixed point for both value functions and distribution
   - For life-cycle model: Comparing to the infinite horizon model, we have one more state-variable(age) $\rightarrow$ Hard to compute the distribution wrt aggregate shock

2. With aggregate shock
   - Intuition: Aggregate(Macro) shock $\rightarrow$ Change distribution over time $\rightarrow$ Hard to find the stationary distribution AND fixed point $\rightarrow$ Hard to solve the value function/bellman equation
   - We need it for the business-cycle issue

Bellman equation: Discrete Time

Given prices $\{w_t, r_t\}$ and the distribution $\mu_t = \mu(a_t, x_t)$, the each agent with wealth a_t and productivity x_t solves the Bellman equation optimally:

$$V(a_t, x_t; \mu_t) = \max_{\{c_t, a_{t+1}\}} \left\{ u(c_t) + \beta \int V(a_{t+1}, x_{t+1}; \mu_{t+1}) dF(x_{t+1}|x_t) \right\} \quad (1)$$

subject to

$$\text{Budget Constraint: } c_t + a_{t+1} = w_t x_t + (1 + r_t) a_t$$
$$\text{AR(1) Process: } \log x_{t+1} = \rho_x \log x_t + \sigma_\varepsilon \varepsilon_{t+1},\ \varepsilon \sim \mathbb{N}(0, 1)$$
$$\text{Distribution: } \Gamma \mu(a_t, x_t) = \mu(a_{t+1}, x_{t+1})$$

Prices are pinned down to

$$r_t = F_K(K_t, L_t) \quad (2)$$
$$w_t = F_L(K_t, L_t) \quad (3)$$

$$\text{where } K_t = \int \int a_t d\mu(a_t, x_t) \& L_t = \int \int x_t d\mu(a_t, x_t) = \int x_t dG(x_t) \quad (4)$$

Bellman equation: Discrete Time, Stationary distribution

Given prices $\{w, r\}$, the each agent solves the Bellman equation optimally:

$$V(a, x) = \max_{\{c,a'\}} \left\{ u(c) + \beta \int V(a', x') dF(x'|x) \right\} \tag{5}$$

subject to

$$c + a' = wx + (1 + r)a$$
$$\log x' = \rho_x \log x + \sigma_\epsilon \varepsilon'$$

How to Solve?

1. Grid search (Brute force method): Always works
2. Golden section: Most popular
3. Endogenous Grid Method: Great way in a case it works. Extremely fast & accurate, theoretically good. But does not work if we want to solve it for the labor supply

Algorithm

1. Guess for r^0 or K^0 $\rightarrow$ Once we guess for $r(K)$, then $K(r)$ and w are automatically obtained

2. Given guessed price r, solve for the value function (5). In the dynamic programming for consumption-saving choice problem, we usually use
   a. Grid search (Brute force method): Always works
   b. Golden section: Most popular
   c. Endogenous Grid Method: Extremely fast & accurate but restrictive

3. Find the stationary distribution $\mu(a, x)$. We usually use
   a. Model simulation(intuitive, less accurate)
   b. Young's method

4. Using the $\mu(a, x)$ or K above, evaluate $r^1 = A\alpha(L/K)^{1-\alpha}$

5. If $||r^0 - r^1|| < \varepsilon$, stop. If not, update and go to 1.

Continuous Time: Hamilton-Jacobi-Bellman Equation

Once we take a limit $\Delta t \to 0$, we have

$$\rho v(a,x) = \max_c \left\{ u(c) + v_a(a,x)da + v_x(a,x)m(x) + \frac{\sigma^2(x)}{2} v_{xx} v(a,x) \right\} \quad (6)$$

where

$$da = wx + ra - c$$

Thus, we have

$$u'(c) = v_a(a,x)$$

Kolmogrov Forward Equation

In a stationary equilibrium, we have

$$0 = -\partial_a \left[s(a,x)\mu(a,x)\right] - \partial_x \left[m(x)\mu(a,x)\right] + \frac{1}{2}\partial_{xx}\left[\sigma^2(x)\mu(a,x)\right] \quad (7)$$

Good property of HJB equation & KFE: Solving HJB equations means solving KFE either

To be fair, finding the stationary distribution $\mu(a,x)$ is also not very hard in the discrete time

Comparison: Cons and Pros of Continuous Time

- Cons
  - We need to take some time to derive & interpret the HJB
- Pros
  - Supper fast
  - Optimal saving: Over-period problem in discrete time vs. Within-period problem in continuous time
  - Over period means: We need to compute both $v(a, x)$ AND $v(a', x)$ wrt a' choice. It becomes more crucial problem if we want to find endogenous labor supply

Labor Supply

1. Discrete time
    - For each a', find the optimal labor supply $n \rightarrow$ Extremely slow due to computational burden
2. Continuous time
    - If utility function is differentiable & no non-covexity issue $\rightarrow$ Almost no additional computational burden
    - We need to use the root finding only at the borrowing limit

Labor supply in the HJB equation

$$\rho v(a,x) = \max_{c,n} \left\{ u(c,n) + v_a(a,x)da + v_x(a,x)m(x) + \frac{\sigma^2(x)}{2} v_{xx} v(a,x) \right\} \tag{8}$$

where

$$da = wxn + ra - c$$

Thus, we have

$$u_c(c,n) = v_a(a,x)$$
$$u_n(c,n) = wx$$

Computational Time: Consumption-Saving Choice Model

	Continuous Time (Finite Difference Method)		Discrete Time (Endogenous Grid Method)	
# of Grids	Speed(sec)	Error (%)	Speed(sec)	Error (%)
100	0.12	0.53	3.24	0.08
1000	0.65	0.05	14.75	0
10000	7.92	0	428.96	-

Table: Computational Time for Consumption-Saving Choice Problem: Continuous Time vs. Discrete Time with EGM method. Reference: Achdou et al.(2022) Online Appendix.

Computational Time: Consumption-Saving-Labor Choice Model

	Continuous Time (Finite Difference Method)	Discrete Time (Golden Section Method)
# of Grids	Speed(sec)	Speed(sec)
100	0.23	6.94
1000	0.35	67.60
10000	1.89	739.14

Table: Computational Time for Consumption-Saving Choice Problem (Two-State Productivity): Continuous Time vs. Discrete Time. My own computation

Note: In the continuous time model with the labor supply, it has a numerical instability at the borrowing limit. But it also arises in the discrete time model either.

Wrap Up

1. I have worked with the discrete time. But I have decided to use the continuous time as it is fast
   a. Main reason: It makes dynamic problem be static
   b. Rendahl (2022): We can obtain the similar computational efficiency using the improved Howard's improvement algorithm

2. Always better? Personally, I do not think so yet
   a. Time frequency interpretation: Discrete time is more directly intutive
   b. Bayesian estimation of HANK: With the discrete time

3. Note that the previous results do not contain the time of calibration/estimation
   a. Even calibration requires the minimization (MATLAB: fminsearch) process. This is why I pursue speed, speed and speed for our digital twin platform project

4. So what? $\rightarrow$ I will use whatever faster & more efficient

KLI Plan

1. For HANK, we should consider the model of aggregate shocks
2. DSGE part
    a. Build one-asset Krusell and Smith model
    b. Begin to work for BASE
3. Data part
    a. Estimate wage stickiness
    b. Low frequency income data: Estimate income process (with KIEP)