

# SOLVING HETEROGENEOUS AGENT MODEL: CONTINUOUS VS. DISCRETE TIME

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# TODAY'S GOAL

- Introduce: What are we solving for the heterogeneous agent model?
- What KLI has been working for? → Continuous time model
- Note: You can find the computational materials from the below

Ben Moll for continuous time:

<https://benjaminmoll.com/codes/>

Bayesian estimation of HANK:

<https://github.com/BASEforHANK>

# SOLVING DSGE MODELS

- Representative Agent Model

- Solving one agent of \$100 problem = Solving 100 agent with sum \$100 problem
- We can solve it by linearization → Solving the linear system difference(discrete)/differential(continuous) time model  
→ Solutions: Deviations from the steady-state

- Heterogeneous Agent Model

- Solving one agent of \$100 problem  $\neq$  Solving 100 agent with sum \$100 problem (This happens when the market is incomplete)
- We need to solve the problem of each agent defined by state variables

# SOLVING HETEROGENEOUS AGENT'S MODEL: DISCRETE VS. CONTINUOUS

## ① Discrete

- Traditional.
- Easy to code: Model structure  $\approx$  Numerical algorithm
- Slower than the continuous time

## ② Continuous

- Relatively new in a sense of solving HA models
- For coding: We need more math for solving differential equations
- By construction, faster than the discrete time

# SOLVING HETEROGENEOUS AGENT'S MODEL: INFINITE VS. FINITE

## ① Infinite

- Assume agents live forever
- Efficient & Parsimonious for analyzing business-cycle/economic growth

## ② Finite

- Assume agents live finite. Usually in the life-cycle model
- Appropriate for answering micro question
- Seems better as it is more realistic, but still hard for solving for business-cycle questions
  - Due to the relationship between stationary distribution & aggregate shocks. Explained it in a next slide

# SOLVING HETEROGENEOUS AGENT'S MODEL: WITHOUT & WITH AGGREGATE SHOCKS - 1

- In other words, (time-invariant) stationary distribution vs. state-dependent distribution
- Distribution?
  - Usually, about the cross-sectional (given time, agent's) distribution
  - Example: Distribution of wealth for Tom, Jerry, Kim and Lee
- Stationary distribution?
  - Over time, each agent's state can be changed, but the overall cross-sectional distribution is not changed
  - Today: Tom - \$100, Jerry - \$200, Kim - \$50 and Lee - \$ 150
  - Tomorrow: Tom - \$200, Jerry - \$100, Kim - \$150 and Lee - \$ 50

# SOLVING HETEROGENEOUS AGENT'S MODEL: WITHOUT & WITH AGGREGATE SHOCKS - 2

## ① Without aggregate shock

- Stationary distribution
- Can work with relatively more well-defined model → Easier to find a fixed point for both value functions and distribution
- For life-cycle model: Comparing to the infinite horizon model, we have one more state-variable(age) → Hard to compute the distribution wrt aggregate shock

## ② With aggregate shock

- Intuition: Aggregate(Macro) shock → Change distribution over time → Hard to find the stationary distribution AND fixed point → Hard to solve the value function/bellman equation
- We need it for the business-cycle issue

## BELLMAN EQUATION: DISCRETE TIME

Given prices  $\{w_t, r_t\}$  and the distribution  $\mu_t = \mu(a_t, x_t)$ , the each agent with wealth  $a_t$  and productivity  $x_t$  solves the Bellman equation optimally:

$$V(a_t, x_t; \mu_t) = \max_{\{c_t, a_{t+1}\}} \left\{ u(c_t) + \beta \int V(a_{t+1}, x_{t+1}; \mu_{t+1}) dF(x_{t+1}|x_t) \right\} \quad (1)$$

subject to

$$\text{BUDGET CONSTRAINT: } c_t + a_{t+1} = w_t x_t + (1 + r_t) a_t$$

$$\text{AR(1) PROCESS: } \log x_{t+1} = \rho_x \log x_t + \sigma_\varepsilon \varepsilon_{t+1}, \varepsilon \sim \mathbb{N}(0, 1)$$

$$\text{DISTRIBUTION: } \Gamma \mu(a_t, x_t) = \mu(a_{t+1}, x_{t+1})$$

Prices are pinned down to

$$r_t = F_K(K_t, L_t) \quad (2)$$

$$w_t = F_L(K_t, L_t) \quad (3)$$

$$\text{where } K_t = \int \int a_t d\mu(a_t, x_t) \& L_t = \int \int x_t d\mu(a_t, x_t) = \int x_t dG(x_t) \quad (4)$$



## BELLMAN EQUATION: DISCRETE TIME, STATIONARY DISTRIBUTION

Given prices  $\{w, r\}$ , the each agent solves the Bellman equation optimally:

$$V(a, x) = \max_{\{c, a'\}} \left\{ u(c) + \beta \int V(a', x') dF(x'|x) \right\} \quad (5)$$

subject to

$$\begin{aligned} c + a' &= wx + (1 + r)a \\ \log x' &= \rho_x \log x + \sigma_\epsilon \epsilon' \end{aligned}$$

# HOW TO SOLVE?

- 1 Grid search (Brute force method): Always works
- 2 Golden section: Most popular
- 3 Endogenous Grid Method: Great way in a case it works. Extremely fast & accurate, theoretically good. But does not work if we want to solve it for the labor supply

# ALGORITHM

- ➊ Guess for  $r^0$  or  $K^0 \rightarrow$  Once we guess for  $r(K)$ , then  $K(r)$  and  $w$  are automatically obtained
- ➋ Given guessed price  $r$ , solve for the value function (5). In the dynamic programming for consumption-saving choice problem, we usually use
  - Ⓐ Grid search (Brute force method): Always works
  - Ⓑ Golden section: Most popular
  - Ⓒ Endogenous Grid Method: Extremely fast & accurate but restrictive
- ➌ Find the stationary distribution  $\mu(a, x)$ . We usually use
  - Ⓐ Model simulation(intuitive, less accurate)
  - Ⓑ Young's method
- ➍ Using the  $\mu(a, x)$  or  $K$  above, evaluate  $r^1 = A\alpha(L/K)^{1-\alpha}$
- ➎ If  $||r^0 - r^1|| < \varepsilon$ , stop. If not, update and go to 1.

# CONTINUOUS TIME: HAMILTON-JACOBI-BELLMAN EQUATION

Once we take a limit  $\Delta t \rightarrow 0$ , we have

$$\rho v(a, x) = \max_c \left\{ u(c) + v_a(a, x)da + v_x(a, x)m(x) + \frac{\sigma^2(x)}{2} v_{xx} v(a, x) \right\} \quad (6)$$

where

$$da = wx + ra - c$$

Thus, we have

$$u'(c) = v_a(a, x)$$

# KOLMOGROV FORWARD EQUATION

In a stationary equilibrium, we have

$$0 = -\partial_a [s(a, x)\mu(a, x)] - \partial_x [m(x)\mu(a, x)] + \frac{1}{2}\partial_{xx} [\sigma^2(x)\mu(a, x)] \quad (7)$$

Good property of HJB equation & KFE: Solving HJB equations means solving KFE either

To be fair, finding the stationary distribution  $\mu(a, x)$  is also not very hard in the discrete time

# COMPARISON: CONS AND PROS OF CONTINUOUS TIME

- Cons

- We need to take some time to derive & interpret the HJB

- Pros

- Supper fast
  - Optimal saving: Over-period problem in discrete time vs. Within-period problem in continuous time
  - Over period means: We need to compute both  $v(a, x)$  AND  $v(a', x)$  wrt  $a'$  choice. It becomes more crucial problem if we want to find endogenous labor supply

## ① Discrete time

- For each  $a'$ , find the optimal labor supply  $n \rightarrow$  Extremely slow due to computational burden

## ② Continuous time

- If utility function is differentiable & no non-convexity issue  $\rightarrow$  Almost no additional computational burden
- We need to use the root finding only at the borrowing limit

## LABOR SUPPLY IN THE HJB EQUATION

$$\rho v(a, x) = \max_{c, n} \left\{ u(c, n) + v_a(a, x)da + v_x(a, x)m(x) + \frac{\sigma^2(x)}{2} v_{xx} v(a, x) \right\} \quad (8)$$

where

$$da = wxn + ra - c$$

Thus, we have

$$u_c(c, n) = v_a(a, x)$$

$$u_n(c, n) = wx$$



# COMPUTATIONAL TIME: CONSUMPTION-SAVING CHOICE MODEL

Continuous Time (Finite Difference Method)			Discrete Time (Endogenous Grid Method)	
# of Grids	Speed(sec)	Error (%)	Speed(sec)	Error (%)
100	0.12	0.53	3.24	0.08
1000	0.65	0.05	14.75	0
10000	7.92	0	428.96	-

**Table:** Computational Time for Consumption-Saving Choice Problem: Continuous Time vs. Discrete Time with EGM method. Reference: Achdou et al.(2022) Online Appendix.

## COMPUTATIONAL TIME: CONSUMPTION-SAVING-LABOR CHOICE MODEL

Continuous Time (Finite Difference Method)		Discrete Time (Golden Section Method)
# of Grids	Speed(sec)	Speed(sec)
100	0.23	6.94
1000	0.35	67.60
10000	1.89	739.14

**Table:** Computational Time for Consumption-Saving Choice Problem (Two-State Productivity): Continuous Time vs. Discrete Time. My own computation

# WRAP UP

- ❶ I have worked with the discrete time. But I have decided to use the continuous time as it is fast
  - ❶ Main reason: It makes dynamic problem be static
  - ❷ Rendahl (2022): We can obtain the similar computational efficiency using the improved Howard's improvement algorithm
- ❷ Always better? Personally, I do not think so yet
  - ❶ Time frequency interpretation: Discrete time is more directly intuitive
  - ❷ Bayesian estimation of HANK: With the discrete time
- ❸ Note that the previous results do not contain the time of calibration/estimation
  - ❶ Even calibration requires the minimization (MATLAB: fminsearch) process. This is why I pursue speed, speed and speed for our digital twin platform project
- ❹ So what? → I will use whatever faster & more efficient

- ① For HANK, we should consider the model of aggregate shocks
- ② DSGE part
  - ⓐ Build one-asset Krusell and Smith model
  - ⓑ Begin to work for BASE
- ③ Data part
  - ⓐ Estimate wage stickiness
  - ⓑ Low frequency income data: Estimate income process (with KIEP)