Supplement Online Appendix to "Granular Origins of Agglomeration"

Shinnosuke Kikuchi
UCSD

Daniel G. O'Connor
Princeton

October 23, 2025

This document contains additional theoretical results, proofs, and computational details for "Granular Origins of Agglomeration". In Section 1, we show the estimates of our key labor supply elasticity by subsamples for robustness. In Section 2, we show that the entry game in the main text is equivalent to an alternate entry game where firms see what sectors and ex-ante shocks other firms have received before entering. In Section 3, we spell out the details of the quantitative model with granular firms in more detail, characterize the equilibrium, and prove the main theoretical results. We present the proofs of the technical lemmas in Section 4. As in the main text, we use $\overline{x}$ to denote the value of a variable x in the absence of any ex-post shocks and $\hat{x}$ to denote log deviations from that value.

1 Robustness of Estimation

Here, we examine whether the estimated short-run labor supply elasticity across firms, κ, varies systematically with the size of the local labor market (LLM). In our main analysis, we showed that firms' employment responses to idiosyncratic shocks depend on their relative size within the market—larger firms expand more when productive because they can attract workers from other firms in the same market. This mechanism implies that what matters is a firm's size relative to its local labor pool, not the absolute size of the labor market itself.

Nonetheless, larger LLMs may exhibit stronger aggregate employment responses to wage changes for reasons unrelated to our mechanism, such as greater worker mobility, more diverse industry composition, or different adjustment frictions. To ensure that our results are not driven by such differences across locations, we re-estimate κ separately for large and small LLMs.

Table 1 reports the results. We classify LLMs based on their annual number of firms between 2002 and 2019, defining large LLMs as those that maintained at least 11 firms (i.e., more than 10) in every year of the sample period, and small LLMs as the remainder. This classification yields 1,631 large and 14,041 small markets out of 15,672 in total, and group assignments are fixed over time according to this criterion. Column (1) of Table 1 replicates the full-sample estimate from the main text, while Columns (2)–(3) show the results for the two subsamples.

The estimated elasticities are 2.53 for large markets and 2.03 for small markets. Although the point estimate is slightly higher for large markets, the difference is not statistically significant. This similarity confirms that the core mechanism—how firms of different relative sizes adjust employment in response to shocks—operates independently of overall market size. Changing the cutoff around this threshold does not materially affect the results.[1]

Table 1: Estimation of Short-run Labor Supply Elasticity across Firms: Subsample by Market Size

Dep. Var.: Log Employment Growth	All LLM (1)	Large LLM (2)	Small LLM (3)
Log Wage Growth	2.48	2.53	2.03
	(0.73)	(0.94)	(0.48)
Observations	1,519,077	974,639	544,438
Unique Num. of LLMs	15,672	1,631	14,041
1st Stage F-Stat.	14.26	8.75	24.36
Covariates	✓	✓	✓
Weighted	✓	✓	✓

Note: This table shows the estimates of short-run labor supply elasticities across firms, κ, by subsamples. The specification is the same as Column (2) of the Table in the main text. Column (1) reports results for the full sample, which replicates the result in the main text. Columns (2)–(3) split local labor markets by annual firm counts over 2002–2019. Column (2) includes markets that maintained *at least 11 firms* (i.e., more than 10) in *every* year from 2002 to 2019 (i.e., $\min_{t\in[2002,2019]} \#\text{firms}_{ist} \geq 11$); Column (3) contains the remainder. Group assignments are fixed over time for each market based on this criterion.

[1] Changing cutoffs around these values does not statistically differentiate the estimates for the subsamples.

2 Alternate Entry Condition

The free entry condition in the main text,

$$\psi_i = \frac{\mathbb{E}\left[\int_0^1 \sum_{n \in \mathcal{N}_{is}} \pi_{isn}(\omega) ds\right]}{m_i}. \tag{1}$$

imagines a staged entry game. All firms choose whether or not to enter the market, and then all of those firms are randomly assigned sectors and ex-ante productivity shocks. In this appendix, we consider an alternate entry game where a potential entrant sees the number of firms in each sector and the ex-ante shocks they have received and decides whether or not to enter. We will characterize this alternate free entry condition and prove that it is the same condition as the main text free entry condition (1).

Under the alternate entry game, the potential entrant can see that there is a mass $p(N, m_i)$ of sectors with N firms, where $p(N, m_i) = m_i^N e^{-m_i}/N!$ is the probability mass function (pmf). Therefore, the probability that the firm enters a sector s with N firms already in it is $p(N, m_i)$. The firm internalizes that if it were to enter such a sector s, that sector would not have N firms, but instead $N + 1$ firms. Nothing else about the equilibrium would change as that sector is small relative to the entire location so the expected profits of the firm would be the average profits of a firm in a sector with $N + 1$ firms, $\pi^e_{i,N+1} \equiv \mathbb{E}[\pi_{isn}(\omega)|N_{is} = N + 1]$, where expectations are taken over ex-ante and ex-post shocks.

Therefore, the free entry condition can be written,

$$\psi_i = \sum_{N=0}^{\infty} \pi^e_{i,N+1} p(N, m_i). \tag{2}$$

Now we will show that this free entry condition (2) is equivalent to (1). Taking the original free entry condition, notice that,

$$\begin{aligned}\frac{\mathbb{E}\left[\int_0^1 \sum_{n \in \mathcal{N}_{is}} \pi_{isn}(\omega) ds\right]}{m_i} &= \frac{\sum_{N=0}^{\infty} \mathbb{E}\left[\sum_{n \in \mathcal{N}_{is}} \pi_{isn}(\omega)|N_{is} = N\right] p\left(N, m_i\right)}{m_i} \\ &= \sum_{N=0}^{\infty} \pi^e_{i,N} \frac{N}{m_i} p(N, m_i).\end{aligned}$$

Taking the alternate free entry condition, we have

$$\sum_{N=0}^{\infty} \pi^e_{i,N+1} p(N, m_i) = \sum_{N=0}^{\infty} \pi^e_{i,N+1} \frac{N+1}{m_i} \frac{m_i^{N+1} e^{-m_i}}{(N+1)!}$$
$$= \sum_{N=1}^{\infty} \pi^e_{i,N} \frac{N}{m_i} p(N, m_i) = \sum_{N=0}^{\infty} \pi^e_{i,N} \frac{N}{m_i} p(N, m_i),$$

proving that the two conditions are equivalent.

3 Details of the Quantitative, Granular Model of Economic Geography

In this section, we lay out the details of the quantitative model and prove that the theoretical results continue to hold in this more general setting.

3.1 Environment

The country is made up of I regions, indexed by $i \in \mathcal{I} \equiv \{1, \ldots, I\}$. There is a mass ℓ of workers and a continuum of sectors $s \in [0, 1]$. The sectors produce perfectly substitutable goods but hire in distinct sectoral labor markets.

Timing. There are four periods $t \in \{-1, 0, 1, 2\}$. In period -1, workers decide where to live and then stay there for the remaining periods, determining population ℓ_i. A mass m_i of firms pay a fixed cost of the traded final good in order to enter in period 0. Each firm is then randomly assigned a sector s and gets an ex-ante productivity draw z from some known distribution.

After observing those initial productivity draws, a representative worker freely allocates her labor L_{isn} across the sectors and firms in period 1. Then, in period 2, the state of the world $\omega \in \Omega$ is revealed. This determines the short-run productivity shocks to each firm. The worker can then reallocate her labor across the firms and sectors subject to moving costs. Firms then produce and sell their goods.

Workers. The fundamental utility of living in location i, u_i is

$$u_i = \overline{u}_i c_i,$$

where $\overline{u}_i$ is the local amenities and c_i is consumption of the freely traded final good. Each worker has an idiosyncratic preference for each location ε_i, so that the utility the worker gets from living in location i is $u_i \varepsilon_i$. We assume that ε_i are distributed Fréchet with shape parameter $\theta > 0$.

Once the worker has decided on a location i, she needs to make her labor supply decisions. She is a risk-neutral representative agent, endowed with one unit of labor that she supplies to the market inelastically. In period 1, the worker freely allocates her units of labor across sectors and firms. In particular, she chooses her vector of labor supply $\boldsymbol{L_i} \equiv \{L_{isn}\}_{s,n}$ in the set of feasible labor allocations $\mathcal{L}$,

$$\boldsymbol{L_i} \in \mathcal{L} \equiv \left\{ \boldsymbol{L_i'} | \int_0^1 \sum_{n \in \mathcal{N}_{is}} L'_{isn} ds \leq 1 \right\}.$$

In period 2, the state of the world ω is revealed. This determines the ex-post productivity shocks for all firms. The worker then reallocates her labor across firms, choosing a vector of labor supply $\boldsymbol{L_i(\omega)} \equiv \{L_{isn}(\omega)\}_{s,n}$ in the set of feasible labor allocations $\mathcal{L}_\Omega(\boldsymbol{L_i})$ which depends on the worker's labor choices in period 1. The set is given by

$$\begin{aligned} \mathcal{L}_\Omega(\boldsymbol{L_i}) \equiv \Bigg\{ \boldsymbol{L_i(\omega)} | 1 = \left(\int_0^1 L_{is}^{-\frac{1}{\nu}} L_{is}(\omega)^{\frac{1+\nu}{\nu}} ds \right)^{\frac{\nu}{1+\nu}}, \\ L_{is}(\omega) = \left(\sum_{n \in \mathcal{N}_{is}} \left(\frac{L_{isn}}{L_{is}} \right)^{-\frac{1}{\kappa}} L_{isn}(\omega)^{\frac{1+\kappa}{\kappa}} \right)^{\frac{\kappa}{1+\kappa}} \Bigg\}, \end{aligned}$$

where $L_{is} = \sum_{n \in \mathcal{N}_{is}} L_{isn}$.

Firms. Firms are the same as in the main text, so we do not restate the assumptions here.

Market Clearing. Total expected production in location i is still

$$Y_i = \mathbb{E}\left[\int_0^1 \sum_{n \in \mathcal{N}_{is}} z_{isn} a_{isn}(\omega) \ell_{isn}(\omega)^{1-\eta} ds \right].$$

The final goods are freely traded, so the market-clearing condition holds at the national level,

$$\sum_{i \in \mathcal{I}} c_i \ell_i + \psi_i m_i = \sum_{i \in \mathcal{I}} Y_i. \tag{3}$$

In the labor market, labor demand needs to equal the individual labor supplied by each worker multiplied by the number of workers. However, this needs to hold for each individual firm as labor is imperfectly substitutable across firms,

$$\ell_{isn}(\omega) = L_{isn}(\omega)\ell_i, \quad \forall s, n, \omega. \tag{4}$$

3.2 Market Structure and Equilibrium

Labor Supply Decision. We will characterize the worker's decision using backward induction, starting with the labor supply decision in periods 1 and 2, and then characterizing the migration decision in period -1 in the next section.

Conditional on living in location i, workers choose their labor allocation across firms and sectors in periods 1 and 2 to maximize their expected utility, taking wages as given. We normalize the price of the final goods to 1, so workers solve the problem.

$$\boldsymbol{L_i}, \boldsymbol{L_i}(\boldsymbol{\omega}) \in \underset{\boldsymbol{L}_{\boldsymbol{i}}' \in \mathcal{L}, \boldsymbol{L}_{\boldsymbol{i}}'(\boldsymbol{\omega}) \in \mathcal{L}_\Omega(\boldsymbol{L}_{\boldsymbol{i}}')}{\operatorname{argmax}} \mathbb{E}\left[\int_0^1 \sum_{n \in \mathcal{N}_{is}} w_{isn}(\omega) L_{isn}'(\omega) ds\right], \tag{5}$$

where $w_{isn}(\omega)$ is the equilibrium wage for firm n in sector s in state of the world ω. We will denote the maximum of (5) by w_i.

Writing out the maximization problem, we get

$$\max_{\boldsymbol{L}_{\boldsymbol{i}}', \boldsymbol{L}_{\boldsymbol{i}}'(\boldsymbol{\omega})} \mathbb{E}\left[\int_0^1 \sum_{n \in \mathcal{N}_{is}} w_{isn}(\omega) L_{isn}'(\omega) ds\right]$$

such that

$$L_{is} = \sum_{n \in \mathcal{N}_{is}} L_{isn} \tag{λ_{is}}$$

$$1 = \int_0^1 L_{is} ds \tag{λ_i}$$

$$L_{is}(\omega)^{\frac{1+\kappa}{\kappa}} = \sum_{n \in \mathcal{N}_{is}} \left(\frac{L_{isn}}{L_{is}}\right)^{-\frac{1}{\kappa}} L_{isn}(\omega)^{\frac{1+\kappa}{\kappa}} \tag{$\lambda_{is}(\omega)$}$$

$$1 = \int_0^1 (L_{is})^{-\frac{1}{\nu}} L_{is}(\omega)^{\frac{1+\nu}{\nu}} ds. \tag{$\lambda_i(\omega)$}$$

Taking the first order conditions, we find that

$$
\begin{aligned}
w_{isn}(\omega) &= \lambda_{is}(\omega)\left(\frac{L_{isn}}{L_{is}}\right)^{-\frac{1}{\kappa}}\frac{1+\kappa}{\kappa}L_{isn}(\omega)^{\frac{1}{\kappa}} \\
\lambda_{is}(\omega) &= \lambda_i(\omega)\left(L_{is}\right)^{-\frac{1}{\nu}}\frac{\kappa}{1+\kappa}\frac{1+\nu}{\nu}L_{is}(\omega)^{\frac{1}{\nu}-\frac{1}{\kappa}} \\
\lambda_{is} &= \mathbb{E}\left[\lambda_{is}(\omega)\frac{1}{\kappa}\left(\frac{L_{isn}}{L_{is}}\right)^{-\frac{1}{\kappa}}L_{isn}^{-1}L_{isn}(\omega)^{\frac{1+\kappa}{\kappa}}\right] \\
\lambda_i &= \lambda_{is} + \mathbb{E}\left[\lambda_i(\omega)\frac{1}{\nu}\left(L_{is}\right)^{-\frac{1}{\nu}}L_{is}^{-1}L_{is}(\omega)^{\frac{1+\nu}{\nu}}\right] - \frac{1}{\kappa}L_{is}^{-1}\mathbb{E}\left[\lambda_{is}(\omega)L_{is}(\omega)^{\frac{1+\kappa}{\kappa}}\right].
\end{aligned} \tag{6}
$$

One can rewrite the short-run labor supply decision in the more familiar form

$$
\frac{L_{isn}(\omega)}{L_{isn}} = \left(\frac{w_{isn}(\omega)}{w_{is}(\omega)}\right)^{\kappa}\frac{L_{is}(\omega)}{L_{is}}, \quad \frac{L_{is}(\omega)}{L_{is}} = \left(\frac{w_{is}(\omega)}{w_i(\omega)}\right)^{\nu},
$$

where

$$
w_{is}(\omega) \equiv \left(\sum_{n\in\mathcal{N}_{is}}\frac{L_{isn}}{L_{is}}w_{isn}(\omega)^{1+\kappa}\right)^{\frac{1}{1+\kappa}}, \quad w_i(\omega) \equiv \left(\int_0^1 L_{is}w_{is}(\omega)^{1+\nu}ds\right)^{\frac{1}{1+\nu}}.
$$

Migration Decision. Each worker chooses the location that maximizes their utility. Therefore, the population in location i satisfies

$$
\ell_i = \int_{\mathbb{R}^{\mathbb{I}}}\mathbb{1}_{i\in\operatorname{argmax}_{i'}\bar{u}_{i'}w_{i'}}dG(\varepsilon)\cdot\ell, \tag{7}
$$

where G is the joint distribution of ε. This is a standard problem in the literature with a Fréchet distribution. It implies that $\ell_i = (u_i/u)^{\theta}\cdot\ell$, where $u = (\sum_i (u_i)^{\theta})^{\frac{1}{\theta}}$.

Labor Demand - Competitive. We will consider two different conduct assumptions on firms after they enter. The first assumption is that they are competitive. Then each active firm maximizes profits, taking wages and prices as given,

$$
\ell_{isn}(\omega) \in \underset{\ell'}{\operatorname{argmax}} \quad z_{isn}a_{isn}(\omega)(\ell')^{1-\eta} - w_{isn}(\omega)\ell'. \tag{8}
$$

This implies that

$$
z_{isn}a_{isn}(\omega)\ell_{isn}(\omega)^{-\eta} = w_{isn}(\omega). \tag{9}
$$

Labor Demand - Cournot. Under Cournot competition, the firm takes as given the labor decisions of the other firms in its own sector. We assume that the firm then takes as given the workers' other options in other sectors. In the math, that will imply that the firm will take $\lambda_i(\omega)$ and λ_i in equation (6) as given. Combining some of the first-order necessary conditions of the worker's problem, we can write the firm problem, using $\boldsymbol{x}_{isn} \equiv \{w_{isn}(\omega), \ell_{isn}(\omega), \ell_{is}(\omega), \ell_{isn}, \ell_{is}\}$, as,

$$
\begin{aligned}
\boldsymbol{x}_{isn} \in & \underset{\boldsymbol{x}'_{isn}}{\operatorname{argmax}} \quad \mathbb{E}\left[z_{isn}a_{isn}(\omega)\ell'_{isn}(\omega)^{1-\eta} - w'_{isn}(\omega)\ell'_{isn}(\omega)\right] \\
& \text{s.t.} \quad (\ell'_{is})^{-\frac{1}{\kappa}}\ell'_{is}(\omega)^{\frac{1+\kappa}{\kappa}} = \left(\ell'_{isn}\right)^{-\frac{1}{\kappa}}\ell'_{isn}(\omega)^{\frac{1+\kappa}{\kappa}} + \sum_{n'\neq n}\left(\ell_{isn'}\right)^{-\frac{1}{\kappa}}\ell_{isn'}(\omega)^{\frac{1+\kappa}{\kappa}} \\
& \qquad \ell'_{is} = \ell'_{isn} + \sum_{n'\neq n}\ell_{isn'} \\
& \qquad w'_{isn}(\omega) = \lambda_i(\omega)\frac{1+\nu}{\nu}\left(\frac{\ell_{is}(\omega)}{\ell_{is}}\right)^{\frac{1}{\nu}-\frac{1}{\kappa}}\left(\frac{\ell_{isn}(\omega)}{\ell_{isn}}\right)^{\frac{1}{\kappa}} \\
& \qquad \lambda_i(1+\kappa)\nu = \mathbb{E}\left[\lambda_i(\omega)(1+\nu)\left(\frac{\ell_{is}(\omega)}{\ell_{is}}\right)^{\frac{1}{\nu}-\frac{1}{\kappa}}\left(\frac{\ell_{isn}(\omega)}{\ell_{isn}}\right)^{\frac{1+\kappa}{\kappa}}\right] \\
& \qquad\qquad + \mathbb{E}\left[\lambda_i(\omega)(\kappa-\nu)\left(\frac{\ell_{is}(\omega)}{\ell_{is}}\right)^{\frac{1+\nu}{\nu}}\right],
\end{aligned} \tag{10}
$$

where we have transformed the per capita variables into the total amount of labor.

Taking the first order conditions and simplifying gives the following necessary conditions,

$$
\begin{aligned}
0 &= \mathbb{E}[w_{isn}(\omega)\ell_{isn}(\omega)] - \ell_{isn}\mathbb{E}\left[\Lambda_{isn}(\omega)\left(\left(\frac{\ell_{isn}(\omega)}{\ell_{isn}}\right)^{\frac{1+\kappa}{\kappa}} - \left(\frac{\ell_{is}(\omega)}{\ell_{is}}\right)^{\frac{1+\kappa}{\kappa}}\right)\right] \\
&\quad - (1-\eta)\mathbb{E}\left[z_{isn}a_{isn}(\omega)\ell_{isn}(\omega)^{1-\eta}\right] \\
0 &= \frac{1+\kappa}{\kappa}w_{isn}(\omega)\ell_{isn}(\omega) - \frac{1+\kappa}{\kappa}\Lambda_{isn}(\omega)\ell_{isn}(\omega)\left(\frac{\ell_{isn}(\omega)}{\ell_{isn}}\right)^{\frac{1}{\kappa}} \\
&\quad - \frac{\Lambda^w_{isn}}{\lambda_i(\omega)}\frac{1+\kappa}{\kappa}\nu w_{isn}(\omega)\frac{\ell_{isn}(\omega)}{\ell_{isn}} - (1-\eta)z_{isn}a_{isn}(\omega)\ell_{isn}(\omega)^{1-\eta} \\
0 &= -\left(\frac{1}{\nu}-\frac{1}{\kappa}\right)\left[w_{isn}(\omega)\ell_{isn}(\omega) - \frac{\Lambda^w_{isn}}{\lambda_i(\omega)}\nu w_{isn}(\omega)\frac{\ell_{isn}(\omega)}{\ell_{isn}}\right. \\
&\quad \left. - \kappa(1+\nu)\Lambda^w_{isn}\left(\frac{\ell_{is}(\omega)}{\ell_{is}}\right)^{\frac{1+\nu}{\nu}}\right] - \Lambda_{isn}(\omega)\ell_{is}\frac{1+\kappa}{\kappa}\left(\frac{\ell_{is}(\omega)}{\ell_{is}}\right)^{\frac{1+\kappa}{\kappa}}
\end{aligned} \tag{11}
$$

Entry. Entry is the same as in the baseline model. Thus, free entry implies that average profits are equal to the cost of entry, (1).

3.3 Theoretical Results for the Competitive Case

In this section, we prove the propositions of the main text in the more general case with $\kappa, \nu \in (0, \infty)$.

Proposition 1. *In any stable equilibrium, the average wage is increasing in the number of workers, i.e.* $\frac{d \log w_i}{d \log \ell_i} > 0$*, if and only if the employment weighted covariance between log firm productivity and log firm employment is increasing in m, i.e.* $\frac{\partial \log \Phi(m)}{\partial \log m}\big|_{m=m_i} > 0$.

Proof. The proof exactly follows the argument in the main text after proving the more general version of Lemma 1, so we do not reproduce it here. □

Proposition 2. *If labor is more substitutable across firms within a sector than across sectors,* $\kappa > \nu$*, then the average wage is increasing in the number of workers, i.e.,* $\frac{d \log w_i}{d \log \ell_i} > 0$ *if and only if idiosyncratic shocks have a positive variance,* $\sigma_N^2 > 0$*. Moreover, the agglomeration benefits converge to zero as the size of the market goes to infinity, i.e.* $\frac{d \log w_i}{d \log \ell_i} \to 0$ *as* $m_i \to \infty$.

Proof. Taking a log first order approximation to the labor supply FOCs (6) after substituting out $\lambda_{is}(\omega)$ and λ_{is} implies,

$$\hat{w}_{isn}(\omega) = \hat{\lambda}(\omega) + \left(\frac{1}{\nu} - \frac{1}{\kappa}\right)\left(\hat{\ell}_{is}(\omega) - \hat{\ell}_{is}\right) + \frac{1}{\kappa}\left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{is}\right)$$
$$\hat{\bar{\lambda}}_i = \frac{1+\nu}{1+\kappa}\mathbb{E}\left[\hat{w}_{isn}(\omega)\right] + \frac{\kappa-\nu}{1+\kappa}\mathbb{E}\left[\hat{\lambda}(\omega)\right] + \frac{\kappa}{1+\kappa}\frac{1+\nu}{\nu}\mathbb{E}\left[\hat{\ell}_{isn}(\omega) - \hat{\ell}_{isn}\right],$$

where we use the fact that $\hat{L}_{isn}(\omega) = \hat{\ell}_{isn}(\omega)$.

Taking a log first-order approximation to the labor constraints embedded in $\mathcal{L}$ and $\mathcal{L}_\Omega(\cdot)$ implies,

$$\hat{\ell}_{is}(\omega) = \sum_{n \in \mathcal{N}_{is}} \frac{\bar{\ell}_{isn}}{\bar{\ell}_{is}}\hat{\ell}_{isn}(\omega), \qquad 0 = \int_0^1 \frac{\bar{\ell}_{is}}{\ell_i}\hat{\ell}_{is}(\omega)ds,$$
$$\hat{\ell}_{is} = \sum_{n \in \mathcal{N}_{is}} \frac{\bar{\ell}_{isn}}{\bar{\ell}_{is}}\hat{\ell}_{isn}, \qquad 0 = \int_0^1 \frac{\bar{\ell}_{is}}{\ell_i}\hat{\ell}_{is}ds.$$

And the labor demand curve implies that $\hat{a}_{isn}(\omega) - \eta\hat{\ell}_{isn}(\omega) = \hat{w}_{isn}(\omega)$. First note that taking expectations $\mathbb{E}[\hat{a}_{isn}(\omega)] - \eta\mathbb{E}[\hat{\ell}_{isn}(\omega)] = \mathbb{E}[\hat{w}_{isn}(\omega)]$ which implies $\mathbb{E}[\hat{w}_{isn}(\omega)] =$

$-\eta\mathbb{E}[\hat{\ell}_{isn}(\omega)]$. Therefore,

$$\begin{aligned}
-\hat{\lambda}_i(\omega) &= \left(\frac{1}{\nu}-\frac{1}{\kappa}\right)\left(\hat{\ell}_{is}(\omega)-\hat{\ell}_{is}\right)+\frac{1}{\kappa}\left(\hat{\ell}_{isn}(\omega)-\hat{\ell}_{is}\right)+\eta\hat{\ell}_{isn}(\omega) \\
-\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\hat{\lambda}(\omega)ds &= \left(\frac{1}{\nu}-\frac{1}{\kappa}\right)\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\left(\hat{\ell}_{is}(\omega)-\hat{\ell}_{is}\right)ds \\
&\quad+\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\left(\frac{1}{\kappa}\left(\hat{\ell}_{isn}(\omega)-\hat{\ell}_{is}\right)+\eta\hat{\ell}_{isn}(\omega)\right)ds \\
\hat{\lambda}(\omega) &= 0.
\end{aligned}$$

Then

$$\begin{aligned}
\hat{\lambda}_i &= \frac{1+\nu}{1+\kappa}\mathbb{E}\left[\hat{w}_{isn}(\omega)\right]+\frac{\kappa-\nu}{1+\kappa}\mathbb{E}\left[\hat{\lambda}(\omega)\right]+\frac{\kappa}{1+\kappa}\frac{1+\nu}{\nu}\mathbb{E}\left[\hat{\ell}_{isn}(\omega)-\hat{\ell}_{isn}\right] \\
\hat{\lambda}_i &= \frac{1+\nu}{1+\kappa}\left(\frac{\kappa}{\nu}-\eta\right)\mathbb{E}\left[\hat{\ell}_{isn}(\omega)\right]-\frac{\kappa}{1+\kappa}\frac{1+\nu}{\nu}\hat{\ell}_{isn}.
\end{aligned}$$

Again, taking a weighted sum across all firms implies that $\hat{\lambda}_i = 0$. Since $\kappa/\nu > 1$ and $\eta \in (0,1)$, it then follows that $\hat{\ell}_{isn} = \mathbb{E}[\hat{\ell}_{isn}(\omega)] = \mathbb{E}[\hat{w}_{isn}(\omega)] = 0$ because otherwise it is not possible for $0 = \left(\frac{\kappa}{\nu}-\eta\right)\mathbb{E}\left[\hat{\ell}_{isn}(\omega)\right]-\frac{\kappa}{\nu}\hat{\ell}_{isn}$ and the labor constraints to hold.

Therefore, we can solve for sectoral labor

$$\begin{aligned}
\hat{a}_{isn}(\omega)-\eta\hat{\ell}_{isn}(\omega) &= \left(\frac{1}{\nu}-\frac{1}{\kappa}\right)\hat{\ell}_{is}(\omega)+\frac{1}{\kappa}\hat{\ell}_{isn}(\omega) \\
\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\hat{a}_{isn}(\omega) &= \left(\frac{1}{\nu}-\frac{1}{\kappa}\right)\hat{\ell}_{is}(\omega)+\left(\frac{1}{\kappa}+\eta\right)\hat{\ell}_{is}(\omega) \\
\frac{1}{\eta+\frac{1}{\nu}}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\hat{a}_{isn}(\omega) &= \hat{\ell}_{is}(\omega).
\end{aligned}$$

And individual firm labor is

$$\hat{\ell}_{isn}(\omega) = \frac{1}{\eta+\frac{1}{\kappa}}\left(\hat{a}_{isn}(\omega)-\frac{\frac{1}{\nu}-\frac{1}{\kappa}}{\eta+\frac{1}{\nu}}\sum_{n'\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn'}}{\overline{\ell}_{is}}\hat{a}_{isn'}(\omega)\right).$$

Therefore,

$$\mathbb{E}[\hat{a}_{isn}(\omega)\hat{\ell}_{isn}(\omega)] = \mathbb{E}\left[\hat{a}_{isn}(\omega)\frac{1}{\eta+\frac{1}{\kappa}}\left(\hat{a}_{isn}(\omega)-\frac{\frac{1}{\nu}-\frac{1}{\kappa}}{\eta+\frac{1}{\nu}}\sum_{n'\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn'}}{\overline{\ell}_{is}}\hat{a}_{isn'}(\omega)\right)\right]$$

$$= \frac{1}{\eta + \frac{1}{\kappa}} \left(\sigma_S^2 + \sigma_N^2 \right) - \frac{\frac{1}{\nu} - \frac{1}{\kappa}}{\left(\eta + \frac{1}{\kappa}\right)\left(\eta + \frac{1}{\nu}\right)} \left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \sigma_N^2 + \sigma_S^2 \right)$$

$$= \frac{1}{\eta + \frac{1}{\nu}} \sigma_S^2 + \frac{\eta + \frac{1}{\nu} - \left(\frac{1}{\nu} - \frac{1}{\kappa}\right) \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}}{\left(\eta + \frac{1}{\kappa}\right)\left(\eta + \frac{1}{\nu}\right)} \sigma_N^2.$$

Then we can calculate $\Phi(m)$,

$$\Phi(m) = \mathbb{E}[a_{isn}(\omega)] + \frac{1-\eta}{2} \int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \mathbb{E}\left[\hat{a}_{isn}(\omega) \hat{\ell}_{isn}(\omega)\right]$$

$$= \mathbb{E}[a_{isn}(\omega)] + \frac{1-\eta}{2} \left(\frac{1}{\eta + \frac{1}{\nu}} \sigma_S^2 + \frac{\eta + \frac{1}{\nu} - \left(\frac{1}{\nu} - \frac{1}{\kappa}\right) \int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n \in \mathcal{N}_{is}} \left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 ds}{\left(\eta + \frac{1}{\kappa}\right)\left(\eta + \frac{1}{\nu}\right)} \sigma_N^2 \right).$$

Therefore,

$$\frac{\partial \log \Phi(m)}{\partial \log m} = -\frac{1-\eta}{2} \frac{\frac{1}{\nu} - \frac{1}{\kappa}}{\left(\eta + \frac{1}{\kappa}\right)\left(\eta + \frac{1}{\nu}\right)} \frac{\partial}{\partial \log m} \left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n \in \mathcal{N}_{is}} \left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 ds \right] \frac{\sigma_N^2}{\Phi(m)}$$

By Lemma 5, $\frac{\partial}{\partial \log m} \left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n \in \mathcal{N}_{is}} \left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 ds \right] < 0$. Then by Proposition 1, the first result will follow. Similarly, by Lemma 5, $\frac{\partial}{\partial \log m} \left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n \in \mathcal{N}_{is}} \left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 ds \right] \to 0$ as $m \to \infty$ so

$$\frac{d \log w_i}{d \log \ell_i} = \frac{\frac{1}{1-\eta} \frac{\partial \log \Phi(m_i)}{\partial \log m_i}}{1 - \frac{1}{1-\eta} \frac{\partial \log \Phi(m_i)}{\partial \log m_i}} \to 0,$$

as $m \to \infty$. □

Proposition 3. *If idiosyncratic shocks have a positive variance, $\sigma_N^2 > 0$, the optimal policy features a subsidy on firm entry proportional to firm profits given by $\tau_i = \frac{1}{\eta} \frac{\partial \log \Phi(m)}{\partial \log m}\big|_{m=m_i} > 0$. Furthermore, the optimal subsidy converges to zero as the size of the market goes to infinity, i.e. $\tau_i \to 0$ as $m_i \to \infty$.*

Proof. The proof exactly follows the argument in the main text after proving the more general version of Lemma 1 so we do not reproduce it here. □

3.4 Characterizing Imperfect Competition

In this subsection, we characterize the equilibrium under Cournot Competition. The expression for production given in Lemma 6 holds no matter how firms behave. Therefore, we go through and characterize what total wage payments are, which then implies profits.

Taking a second-order approximation to total wage payments and the firm FOCs associated with Cournot competition implies the next proposition.

Lemma 7. *If firms compete à la Cournot, total wage compensation in location i can be written,*

$$w_i \ell_i = (1-\eta) z_i (m_i)^{\eta} (\ell_i)^{1-\eta} \left(\tilde{\Phi}(m) + \Psi^c(m) \right),$$

where $\tilde{\Phi}(m)$ *is defined as in Lemma 6 and*

$$\Psi^c(m) \equiv \frac{1+\kappa}{\kappa} \int_{\mathcal{S}} \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_n \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \mathbb{E}\left[\frac{\hat{\Lambda}_{isn}(\omega)}{w_i} \left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{isn} - \hat{\ell}_{is}(\omega) + \hat{\ell}_{is} \right) \right] ds.$$

Proof. First, we note that because workers are freely mobile across all sectors in period 1, the equilibrium with no ex-post shocks and Cournot competition is the same as the equilibrium with no ex-post shocks and competitive firms. Doing a second-order approximation around the point with no ex-post shocks implies

$$\begin{aligned}
&\mathbb{E}\left[\int_0^1 \sum_{n \in \mathcal{N}_{is}} w_{isn}(\omega) \ell_{isn}(\omega) \right] \\
&\quad = \overline{w}_i \overline{\ell}_i \left(\int_0^1 \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\ell_i} \left(1 + \mathbb{E}[\hat{w}_{isn}(\omega)] + \mathbb{E}[\hat{\ell}_{isn}(\omega)] + \frac{1}{2} \mathbb{E}[(\hat{w}_{isn}(\omega) + \hat{\ell}_{isn}(\omega))^2] \right) \right)
\end{aligned}$$

We then take a second-order log approximation to the firm FOCs in equation (11). However, with no shocks $\Lambda_{isn}(\omega) = 0$, so we leave that in levels. The first equation becomes

$$\begin{aligned}
\mathbb{E}[\hat{a}_{isn}(\omega)] + (1-\eta)\mathbb{E}[\hat{\ell}_{isn}(\omega)] + \frac{1}{2}\mathbb{E}\left[(\hat{a}_{isn}(\omega) + (1-\eta)\hat{\ell}_{isn}(\omega))^2 \right] \\
= \mathbb{E}[\hat{w}_{isn}(\omega)] + \mathbb{E}[\hat{\ell}_{isn}(\omega)] + \frac{1}{2}\mathbb{E}\left[(\hat{w}_{isn}(\omega) + \hat{\ell}_{isn})^2 \right] \\
- \frac{1+\kappa}{\kappa} \mathbb{E}\left[\frac{\hat{\Lambda}_{isn}(\omega)}{w_i} \left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{isn} - \hat{\ell}_{is}(\omega) + \hat{\ell}_{is} \right) \right].
\end{aligned}$$

We also need the second-order approximation to the labor constraints embedded in $\mathcal{L}$ and $\mathcal{L}_\Omega(\cdot)$,

$$
\begin{aligned}
\hat{\ell}_{is}(\omega)+&\frac{1}{2}\frac{\kappa}{1+\kappa}\left(-\frac{1}{\kappa}\hat{\ell}_{is}+\frac{1+\kappa}{\kappa}\hat{\ell}_{is}(\omega)\right)^2+\frac{1}{1+\kappa}\hat{\ell}_{is}^2 \\
&=\sum_{n\in\mathcal{N}_{is}}\frac{\ell_{isn}}{\overline{\ell}_{is}}\left(\hat{\ell}_{isn}(\omega)+\frac{1}{2}\frac{\kappa}{1+\kappa}\left(-\frac{1}{\kappa}\hat{\ell}_{isn}+\frac{1+\kappa}{\kappa}\hat{\ell}_{isn}(\omega)\right)^2+\frac{1}{1+\kappa}\hat{\ell}_{isn}^2\right) \\
0&=\int_0^1\overline{\ell}_{is}\left(\hat{\ell}_{is}(\omega)+\frac{1}{2}\frac{\nu}{1+\nu}\left(-\frac{1}{\nu}\hat{\ell}_{is}+\frac{1+\nu}{\nu}\hat{\ell}_{is}(\omega)\right)^2+\frac{1}{1+\nu}\hat{\ell}_{is}^2\right).
\end{aligned}
$$

So then we can write

$$
\begin{aligned}
&\int_0^1\frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\left(\mathbb{E}\left[\hat{w}_{isn}(\omega)\right]+\mathbb{E}\left[\hat{\ell}_{isn}(\omega)\right]+\frac{1}{2}\mathbb{E}[(\hat{w}_{isn}(\omega)+\hat{\ell}_{isn}(\omega))^2]\right)ds \\
&=\frac{1}{2}\int_0^1\frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\mathbb{E}\left[(\hat{a}_{isn}(\omega)+(1-\eta)\hat{\ell}_{isn}(\omega))^2\right]ds \\
&\quad+\frac{1+\kappa}{\kappa}\int_0^1\frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\mathbb{E}\left[\frac{\hat{\Lambda}_{isn}(\omega)}{w_i}\left(\hat{\ell}_{isn}(\omega)-\hat{\ell}_{isn}-\hat{\ell}_{is}(\omega)+\hat{\ell}_{is}\right)\right]ds \\
&\quad+\int_0^1\frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\left(\mathbb{E}[\hat{a}_{isn}(\omega)]+(1-\eta)\mathbb{E}[\hat{\ell}_{isn}(\omega)]\right)ds \\
&=\mathbb{E}[\hat{a}_{isn}(\omega)]+\frac{1}{2}\mathbb{E}[\hat{a}_{isn}(\omega)^2]+(1-\eta)\int_0^1\frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\mathbb{E}[\hat{a}_{isn}(\omega)\hat{\ell}_{isn}(\omega)]ds \\
&\quad-\eta\frac{1-\eta}{2}\int_0^1\frac{\overline{\ell}_{is}}{\ell}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\mathbb{E}[\hat{\ell}_{isn}(\omega)^2]ds \\
&\quad-\frac{1}{\kappa}\frac{1-\eta}{2}\int_0^1\frac{\overline{\ell}_s}{\ell}\sum_{n\in\mathcal{N}_s}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\mathbb{E}\left[\left(\hat{\ell}_{sn}(\omega)-\hat{\ell}_{isn}\right)^2\right]ds \\
&\quad-\left(\frac{1}{\nu}-\frac{1}{\kappa}\right)\frac{1-\eta}{2}\int_0^1\frac{\overline{\ell}_{is}}{\ell}\mathbb{E}\left[\left(\hat{\ell}_{is}(\omega)-\hat{\ell}_{is}\right)^2\right]ds \\
&\quad+\frac{1+\kappa}{\kappa}\int_0^1\frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\mathbb{E}\left[\frac{\hat{\Lambda}_{isn}(\omega)}{w_i}\left(\hat{\ell}_{isn}(\omega)-\hat{\ell}_{isn}-\hat{\ell}_{is}(\omega)+\hat{\ell}_{is}\right)\right]ds.
\end{aligned}
$$

This completes the proof. □

Proposition 7 gives wages completely in second-order terms. Therefore, following Benigno and Woodford (2003) and Benigno and Woodford (2012), we can then approximate the labor constraints and firm first-order conditions with a log first-order approximation.

We then computationally evaluate $\tilde{\Phi}^c(m)$ and $\Psi^c(m)$ and use those functions in calculating the equilibrium.

4 Proofs of Technical Lemmas

In this section, we prove the Technical Lemmas used in the main text and Section 3 for the more general case with $\kappa, \nu \in (0, \infty)$. To get the results for the results in the main text, one simply needs to take the limit as $\kappa \to \infty$ and $\nu \to 0$.

Lemma 1. *If firms are competitive conditional on entry, the regional production function is $Y_i(\ell, m) = z_i m^\eta \ell^{1-\eta} \Phi(m)$, where $z_i \equiv \mathbb{E}[z_{isn}^{1/\eta}]^\eta$ and $\Phi(m)$ is given by,*

$$\Phi(m) \equiv \mathbb{E}[a_{sn}(\omega)] + \frac{1-\eta}{2} \int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n \in \mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \operatorname{Cov}\left(\log a_{sn}(\omega), \log \ell_{sn}(\omega)\right) ds. \tag{12}$$

Proof. By Lemma 6, the regional production can be written $Y_i(\ell, m) = z_i m^\eta \ell^{1-\eta} \tilde{\Phi}(m)$ where

$$\begin{aligned}
\tilde{\Phi}(m) \equiv \mathbb{E}[a_{sn}(\omega)] &+ (1-\eta) \int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n \in \mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathbb{E}[\hat{a}_{sn}(\omega) \hat{\ell}_{sn}(\omega)] ds \\
&- \eta \frac{1-\eta}{2} \int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n \in \mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathbb{E}[\hat{\ell}_{sn}(\omega)^2] ds \\
&- \frac{1}{\kappa} \frac{1-\eta}{2} \int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n \in \mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathbb{E}\left[\left(\hat{\ell}_{sn}(\omega) - \hat{\ell}_{sn}\right)^2\right] ds \\
&- \left(\frac{1}{\nu} - \frac{1}{\kappa}\right) \frac{1-\eta}{2} \int_0^1 \frac{\overline{\ell}_s}{\ell} \mathbb{E}\left[\left(\hat{\ell}_s(\omega) - \hat{\ell}_s\right)^2\right] ds
\end{aligned}$$

Therefore, the planner is looking to maximize this second-order production function subject to the labor constraints. Following Benigno and Woodford (2003) and Benigno and Woodford (2012), we can do a linear approximation to the labor constraints, embedded in the sets $\mathcal{L}$ and $\mathcal{L}_\Omega(\cdot)$,

$$\begin{aligned}
\hat{\ell}_{is}(\omega) &= \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \hat{\ell}_{isn}(\omega), & 0 &= \int_0^1 \frac{\overline{\ell}_{is}}{\ell_i} \hat{\ell}_{is}(\omega) ds, \\
\hat{\ell}_{is} &= \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \hat{\ell}_{isn}, & 0 &= \int_0^1 \frac{\overline{\ell}_{is}}{\ell_i} \hat{\ell}_{is} ds.
\end{aligned}$$

To simplify the explication, we rewrite the maximization problem with vector notation as

$$
\begin{aligned}
\max_{x} \quad & ax - \frac{1}{2}x'bx \\
\text{s.t.} \quad & cx = 0
\end{aligned}
$$

where x is the vector of labor supply decisions, a is the vector of productivity shocks so that

$$
ax = (1-\eta)\int_0^1 \frac{\overline{\ell}_s}{\ell}\sum_{n\in\mathcal{N}_s}\frac{\overline{\ell}_{sn}}{\overline{\ell}_s}\mathbb{E}[\hat{a}_{sn}(\omega)\hat{\ell}_{sn}(\omega)]ds,
$$

b is the self-adjoint operator (i.e., symmetric matrix) representing the loss function so that

$$
\begin{aligned}
x'bx = {} & \eta(1-\eta)\int_0^1 \frac{\overline{\ell}_s}{\ell}\sum_{n\in\mathcal{N}_s}\frac{\overline{\ell}_{sn}}{\overline{\ell}_s}\mathbb{E}[\hat{\ell}_{sn}(\omega)^2]ds \\
& + \frac{1}{\kappa}(1-\eta)\int_0^1 \frac{\overline{\ell}_s}{\ell}\sum_{n\in\mathcal{N}_s}\frac{\overline{\ell}_{sn}}{\overline{\ell}_s}\mathbb{E}\left[\left(\hat{\ell}_{sn}(\omega)-\hat{\ell}_{sn}\right)^2\right]ds \\
& + \left(\frac{1}{\nu}-\frac{1}{\kappa}\right)(1-\eta)\int_0^1 \frac{\overline{\ell}_s}{\ell}\mathbb{E}\left[\left(\hat{\ell}_s(\omega)-\hat{\ell}_s\right)^2\right]ds,
\end{aligned}
$$

and c is the matrix representing the linear constraints. Forming the Lagrangian, we have

$$
ax - \frac{1}{2}x'bx - \lambda cx.
$$

Taking the FOCs, we get

$$
a - (bx)' - \lambda c = 0.
$$

Therefore, $x = b^{-1}(a-\lambda c)' = b^{-1}(a' - c'\lambda')$. Using the constraint, we can solve for λ:

$$
\begin{aligned}
0 &= cx \\
&= cb^{-1}(a'-c'\lambda') \\
\lambda' &= (cb^{-1}c')^{-1}cb^{-1}a'.
\end{aligned}
$$

Then, to complete the proof, we will show that $ax = x'bx$.

$$
\begin{aligned}
x'bx &= \left(b^{-1}(a'-c'(cb^{-1}c')^{-1}cb^{-1}a')\right)' bb^{-1}(a'-c'(cb^{-1}c')^{-1}cb^{-1}a') \\
&= (a - ab^{-1}c'(cb^{-1}c')^{-1}c)b^{-1}(a'-c'(cb^{-1}c')^{-1}cb^{-1}a') \\
&= ab^{-1}a' - ab^{-1}c'(cb^{-1}c')^{-1}cb^{-1}a' - ab^{-1}c'(cb^{-1}c')^{-1}cb^{-1}a' \\
&\quad + ab^{-1}c'(cb^{-1}c')^{-1}cb^{-1}c'(cb^{-1}c')^{-1}cb^{-1}a'
\end{aligned}
$$

$$= ab^{-1}a' - ab^{-1}c'(cb^{-1}c')^{-1}cb^{-1}a',$$

where we use the fact that b and $cb^{-1}c'$ are self adjoint and therefore b^{-1} and $(cb^{-1}c')^{-1}$ are self adjoint. Similarly,

$$\begin{aligned} ax &= ab^{-1}(a' - c'(cb^{-1}c')^{-1}cb^{-1}a') \\ &= x'bx. \end{aligned}$$

Therefore, $ax - \frac{1}{2}x'bx = \frac{1}{2}ax$, completing the proof. $\square$

Lemma 2. *For firms competing $f \in \{p, c\}$, the regional production function is $Y_i^f(\ell, m) = z_i m^\eta \ell^{1-\eta} \Phi^f(m)$, where $z_i \equiv \mathbb{E}[z_{isn}^{1/\eta}]^\eta$. Furthermore, $\Phi^f(m) \leq \Phi^p(m)$ for all f and m.*

Proof. This result follows immediately from Lemma 6 and noting that, as perfect competition is efficient, it must produce the most conditional on the number of firms and workers. $\square$

Lemma 3. *For firms competing in Cournot, total wage compensation in location i can be written,*

$$w_i \ell_i = (1-\eta) z_i (m_i)^\eta (\ell_i)^{1-\eta} \left(\Phi^c(m_i) + \Psi^c(m_i) \right),$$

where $\Psi^c(m_i) \leq 0$.

Proof. This result follows from Lemma 7 and noting that wages must have a markdown. $\square$

Lemma 4. *Expected sectoral HHI, weighted by average productivity shock, converges to 0 as the number of firms goes to infinity. More precisely, $\psi_N \to 0$ as $N \to \infty$ where $\psi_N \equiv \mathbb{E}\left[\frac{\sum_{n \in \mathcal{N}} z_{isn}^{1/\eta}}{N z_i^{1/\eta}} \frac{\sum_{n \in \mathcal{N}} (z_{isn}^{1/\eta})^2}{\left(\sum_{n \in \mathcal{N}} z_{isn}^{1/\eta} \right)^2} \middle| N \right]$.*

Proof. Note that because $1 - F_{iz}$ is regularly varying, there exists α and slowly varying function L[2] such that $1 - F_{iz}(x) = x^{-\alpha} L(x)$.

Then there are two cases: $\mathbb{E}[z_{isn}^{2/\eta}]$ exists and $\mathbb{E}[z_{isn}^{2/\eta}]$ does not exist. We will take each case in turn.

Suppose that $\mathbb{E}[z_{isn}^{2/\eta}]$ exists. Then we can write ψ_N,

$$\psi_N = \frac{1}{z_i^{1/\eta}} \mathbb{E}\left[\frac{1}{N} \frac{N}{\sum_{n \in \mathcal{N}} z_{isn}^{1/\eta}} \frac{\sum_{n \in \mathcal{N}} (z_{isn}^{1/\eta})^2}{N} \middle| N \right].$$

[2] A function is slowly varying if for every $a > 0$, $\frac{L(ax)}{L(x)} \to 1$ as $x \to \infty$.

By the strong law of large numbers, $\frac{1}{N} \to 0$, $\frac{N}{\sum_{n\in\mathcal{N}} z_{isn}^{1/\eta}} \to \frac{1}{\mathbb{E}[z_{isn}^{1/\eta}]}$, and $\frac{\sum_{n\in\mathcal{N}}(z_{isn}^{1/\eta})^2}{N} \to \mathbb{E}[z_{isn}^{2/\eta}]$ almost surely. Therefore, the integrand converges to 0 almost surely, and $\psi_N \to 0$.

Suppose that $\mathbb{E}[z_{isn}^{2/\eta}]$ does not exist. Then we can rewrite ψ_N as

$$\psi_N = \mathbb{E}\left[\frac{a_N}{N^2}\frac{\sum_{n\in\mathcal{N}}(z_{isn}^{1/\eta})^2}{a_N}\frac{N}{\sum_n z_{isn}^{1/\eta}}\right],$$

where a_N is defined so that $\mathbb{P}(z_{isn}^{2/\eta} > a_N) = N^{-1}$. By Lévy's theorem,[3] $\frac{1}{a_N}\left(\sum_n (z_{isn}^{1/\eta})^2\right)$ converges in distribution to a non-degenerate distribution, $\frac{N}{\sum_n z_{isn}^{1/\eta}} \to \frac{1}{\mathbb{E}[z_{isn}^{1/\eta}]}$ almost surely. That simply leaves a_N/N^2. Note that

$$\begin{aligned}\frac{a_N}{N^2} &= a_N \mathbb{P}\left(z_{isn}^{2/\eta} > a_N\right)^2 \\ &= a_N a_N^{-\alpha} L(a_N^{1/2})^2.\end{aligned}$$

This converges to 0 as $a_N \to \infty$ if $\alpha > 1$. But note that since the mean exists, α must be greater than 1. Further, note that $a_N \to \infty$ as $N \to \infty$; otherwise, the variance would exist. Thus, the integrand must converge to 0 in distribution and $\psi_N \to 0$. $\square$

Lemma 5. *Average sectoral HHI has the following properties:*

(i) $\frac{\partial}{\partial \log m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 ds\right] < 0$; *and*

(ii) $\frac{\partial}{\partial \log m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 ds\right] \to 0$ *as* $m \to \infty$.

Proof. Recall that $\overline{\ell}_{isn} = \left(\frac{(1-\eta)z_{isn}}{\overline{w}_i}\right)^{\frac{1}{\eta}}$ where $\overline{w}_i = (1-\eta)\ell_i^{-\eta} m_i^{\eta} z_i$. Therefore,

$$\begin{aligned}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 ds &= \int_0^1 \frac{\ell_i m_i^{-1} z_i^{-\frac{1}{\eta}}\sum_{n\in\mathcal{N}_{is}} z_{isn}^{\frac{1}{\eta}}}{\ell_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{z_{isn}^{\frac{1}{\eta}}}{\sum_{n'\in\mathcal{N}_{is}} z_{isn'}^{\frac{1}{\eta}}}\right)^2 ds \\ &= \int_0^1 \frac{N_{is}}{m_i}\frac{\sum_{n\in\mathcal{N}_{is}} z_{isn}^{1/\eta}}{N_{is} z_i^{1/\eta}}\frac{\sum_{n\in\mathcal{N}_{is}}(z_{isn}^{1/\eta})^2}{\left(\sum_{n\in\mathcal{N}_{is}} z_{isn}^{1/\eta}\right)^2} ds\end{aligned}$$

We can then decompose this into an expectation over the number of firms in the sector and, conditional number of firms, the expected productivity shocks. We will denote $\psi_N =$

[3] See Durrett (2019).

$\mathbb{E}\left[\frac{\sum_{n\in\mathcal{N}} z_{isn}^{1/\eta}}{N z_i^{1/\eta}} \frac{\sum_{n\in\mathcal{N}} (z_{isn}^{1/\eta})^2}{\left(\sum_{n\in\mathcal{N}} z_{isn}^{1/\eta}\right)^2} | N\right]$. Then we have

$$\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 = \sum_{N=0}^{\infty} \frac{N}{m} \psi_N \frac{m^N e^{-m}}{N!}.$$

Then we can take the derivative with respect to m,

$$\begin{aligned}
\frac{\partial}{\partial m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2\right] &= \frac{\partial}{\partial m}\left[\sum_{N=1}^{\infty} \frac{N}{m} \psi_N \frac{m^N e^{-m}}{N!}\right] \\
&= \sum_{N=1} (N-1) N \psi_N \frac{m^{N-2} e^{-m}}{N!} - \sum_{N=1} N \psi_N \frac{m^{N-1} e^{-m}}{N!} \\
&= \sum_{N=2} \psi_N \frac{m^{N-2} e^{-m}}{(N-2)!} - \sum_{N=2} \psi_{N-1} \frac{m^{N-2} e^{-m}}{(N-2)!} \\
&= \sum_{N=2} (\psi_N - \psi_{N-1}) \frac{m^{N-2} e^{-m}}{(N-2)!} < 0,
\end{aligned}$$

where the inequality comes from the fact that ψ_N is decreasing, so $\psi_N - \psi_{N-1} < 0$. Therefore, HHI is decreasing.

We next turn to proving Lemma 5.2, that $m\frac{\partial}{\partial m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2\right]$ converges to zero as $m \to \infty$.

We start by showing that $\frac{\partial}{\partial m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2\right] \to 0$. Take $\epsilon > 0$. Since $\psi_N \to 0$, there is some $\overline{N}$ such that for $N \geq \overline{N}$, $\psi_N < \frac{\epsilon}{2}$. Notice that $\frac{m^{x-2} e^{-m}}{(x-2)!} \in (0,1)$. And also notice that $\frac{m^{x-2} e^{-m}}{(x-2)!} \to 0$ as $m \to \infty$.

Thus, there is a $\overline{m}$ such that for $m > \overline{m}$, $\frac{m^{x-2} e^{-m}}{(x-2)!} < \frac{1}{\psi_1}\frac{\epsilon}{2}$ for all $x \leq \overline{N}$. Therefore, for $m > \overline{m}$,

$$\begin{aligned}
\frac{\partial}{\partial m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2\right] &= \sum_{N=2}^{\infty} (\psi_N - \psi_{N-1}) \frac{m^{x-2} e^{-m}}{(x-2)!} \\
&> \sum_{N=2}^{\overline{N}} (\psi_N - \psi_{N-1}) \frac{m^{x-2} e^{-m}}{(x-2)!} + \sum_{N=\overline{N}+1}^{\infty} (\psi_N - \psi_{N-1}) \\
&= \sum_{N=2}^{\overline{N}} (\psi_N - \psi_{N-1}) \frac{m^{x-2} e^{-m}}{(x-2)!} - \psi_{\overline{N}}
\end{aligned}$$

$$
\begin{aligned}
&> \sum_{N=2}^{\overline{N}} (\psi_N - \psi_{N-1}) \frac{1}{\psi_1} \frac{\epsilon}{2} - \frac{\epsilon}{2} \\
&= -\frac{\epsilon}{2} - \frac{\epsilon}{2}.
\end{aligned}
$$

Furthermore, the second derivative is positive for sufficiently large m,

$$
\frac{\partial^2}{\partial m^2} \left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n \in \mathcal{N}_{is}} \left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \right)^2 \right] = \sum_{N=3}^{\infty} \left((\psi_N - \psi_{N-1}) - (\psi_{N-1} - \psi_{N-2}) \right) \frac{m^{N-3} e^{-m}}{(N-3)!} > 0,
$$

because ψ_N is convex for sufficiently large N. Therefore, we have a function $f(x)$ such that $f(x) \to 0$, $f'(x) \to 0$ and $f''(x) > 0$. It then follows that $xf'(x) \to 0$. That is $m \frac{\partial}{\partial m} \left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n \in \mathcal{N}_{is}} \left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \right)^2 \right] \to 0$.

To see this, suppose that $xf'(x)$ does not converge to zero. Then there is a $\epsilon > 0$ and a sequence of $x_n \to \infty$ such that $x_n f'(x_n) < -\epsilon$. As $f''(x) > 0$, it follows that $f'(x) < f'(x_n) < -\frac{\epsilon}{x_n}$ for all $x < x_n$. Therefore,

$$
\begin{aligned}
f(x) &= f(0) + \int_0^x f'(t) dt \\
&< f(0) + \sum_{n=1}^{N: x_N < x} \int_{x_{n-1}}^{x_n} f'(x_n) dt + \int_{X_N}^{x} f'(x) dt \\
&< f(0) - \sum_{n=1}^{N: x_N < x} (x_n - x_{n-1}) \frac{\epsilon}{x_n} + (x - x_N) f'(x) \\
&\to f(0) - \sum_{n=1}^{\infty} (x_n - x_{n-1}) \frac{\epsilon}{x_n}
\end{aligned}
$$

But then

$$
\sum_{n=1}^{\infty} (x_n - x_{n-1}) \frac{\epsilon}{x_n} \approx \epsilon \int_0^{\infty} \frac{1}{x} dx \to \infty.
$$

Therefore, this contradicts, $f(x) \to 0$ so $xf'(x)$ must converge to 0. □

Lemma 6. *The regional production function is $Y_i(\ell, m) = z_i m^{\eta} \ell^{1-\eta} \tilde{\Phi}(m)$, where $z_i \equiv \mathbb{E}[z_{isn}^{1/\eta}]^{\eta}$ and $\tilde{\Phi}(m)$ is given by,*

$$
\tilde{\Phi}(m) \equiv \mathbb{E}[a_{sn}(\omega)] + (1 - \eta) \int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n \in \mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathbb{E}[\hat{a}_{sn}(\omega) \hat{\ell}_{sn}(\omega)] ds
$$

$$
\begin{aligned}
&- \eta \frac{1-\eta}{2} \int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n \in \mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathbb{E}[\hat{\ell}_{sn}(\omega)^2] ds \\
&- \frac{1}{\kappa} \frac{1-\eta}{2} \int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n \in \mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathbb{E}\left[\left(\hat{\ell}_{sn}(\omega) - \hat{\ell}_{sn} \right)^2 \right] ds \\
&- \left(\frac{1}{\nu} - \frac{1}{\kappa} \right) \frac{1-\eta}{2} \int_0^1 \frac{\overline{\ell}_s}{\ell} \mathbb{E}\left[\left(\hat{\ell}_s(\omega) - \hat{\ell}_s \right)^2 \right] ds. \qquad (13)
\end{aligned}
$$

Proof. Expected production is $\int_0^1 \sum_{n \in \mathcal{N}_{is}} z_{isn} a_{isn}(\omega) \ell_{isn}(\omega)^{1-\eta} ds$. We will do a second order approximation around the point $\log \tilde{a}_{isn}(\omega) = \log \tilde{A}_{is}(\omega) = 0$.

We start by characterizing the solution at that point. We find that there is some $\overline{w}_i$ such that

$$
\overline{w}_i = (1-\eta) z_{isn} (\overline{\ell}_{isn})^{-\eta},
$$

so that $\overline{\ell}_{isn} = \left(\frac{(1-\eta) z_{isn}}{\overline{w}_i} \right)^{\frac{1}{\eta}}$. Then labor clearing requires

$$
\ell_i = \int_0^1 \ell_i \overline{L}_{is} ds = \int_0^1 \sum_{n \in \mathcal{N}_{is}} \overline{\ell}_{isn} ds = \int_0^1 \sum_{n \in \mathcal{N}_{is}} \left(\frac{(1-\eta) z_{isn}}{\overline{w}_i} \right)^{\frac{1}{\eta}} ds.
$$

This implies that $\overline{w}_i = (1-\eta) \ell_i^{-\eta} m_i^{\eta} \mathbb{E}\left[z_{isn}^{1/\eta} \right]^{\eta}$. Then production is

$$
\overline{Y}_i = \int_0^1 \sum_{n \in \mathcal{N}_{is}} z_{isn} \left(\frac{(1-\eta) z_{isn}}{\overline{w}_i} \right)^{\frac{1-\eta}{\eta}} ds = z_i \ell_i^{1-\eta} m_i^{\eta},
$$

where $z_i = \mathbb{E}\left[z_{isn}^{1/\eta} \right]^{\eta}$.

Taking the log second-order approximation to production gives

$$
\begin{aligned}
Y_i \approx z_i (m_i)^{\eta} (\ell_i)^{1-\eta} \int_0^1 \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_i} \Big(1 &+ \left(\hat{a}_{isn}(\omega) + (1-\eta) \hat{\ell}_{isn}(\omega) \right) \\
&+ \frac{1}{2} \left(\hat{a}_{isn}(\omega) + (1-\eta) \hat{\ell}_{isn}(\omega) \right)^2 \Big) ds.
\end{aligned}
$$

To transform this to be completely second order, we do a second-order approximation to the labor constraints,

$$-\frac{1}{\kappa}\hat{\ell}_{is}+\frac{1+\kappa}{\kappa}\hat{\ell}_{is}(\omega)+\frac{1}{2}\left(-\frac{1}{\kappa}\hat{\ell}_{is}+\frac{1+\kappa}{\kappa}\hat{\ell}_{is}(\omega)\right)^2 = \sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\left[-\frac{1}{\kappa}\hat{\ell}_{isn}+\frac{1+\kappa}{\kappa}\hat{\ell}_{isn}(\omega)+\frac{1}{2}\left(-\frac{1}{\kappa}\hat{\ell}_{isn}+\frac{1+\kappa}{\kappa}\hat{\ell}_{isn}(\omega)\right)^2\right],$$

$$0=\int_0^1\frac{\overline{\ell}_{is}}{\ell_i}\left[-\frac{1}{\nu}\hat{\ell}_{is}+\frac{1+\nu}{\nu}\hat{\ell}_{is}(\omega)+\frac{1}{2}\left(-\frac{1}{\nu}\hat{\ell}_{is}+\frac{1+\nu}{\nu}\hat{\ell}_{is}(\omega)\right)^2\right]ds,$$

$$\hat{\ell}_{is}+\frac{1}{2}\hat{\ell}_{is}^2=\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\left(\hat{\ell}_{isn}+\frac{1}{2}\hat{\ell}_{isn}^2\right),$$

and

$$0=\int_0^1\frac{\overline{\ell}_{is}}{\ell_i}\left(\hat{\ell}_{is}+\frac{1}{2}\hat{\ell}_{is}^2\right)ds,$$

where we use the fact $\hat{\ell}_{isn}(\omega)=\hat{L}_{isn}(\omega)$.

Then we can transform $\int_0^1\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_i}\hat{\ell}_{isn}(\omega)ds$ to second order. That is,

$$\begin{aligned}
\int_0^1\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_i}\hat{\ell}_{isn}(\omega)ds &= -\frac{1}{2}\frac{\kappa}{1+\kappa}\int_0^1\frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\left(-\frac{1}{\kappa}\hat{\ell}_{isn}+\frac{1+\kappa}{\kappa}\hat{\ell}_{isn}(\omega)\right)^2 ds\\
&\quad+\frac{1}{2}\frac{\kappa}{1+\kappa}\int_0^1\frac{\overline{\ell}_{is}}{\overline{\ell}_i}\left(-\frac{1}{\kappa}\hat{\ell}_{is}+\frac{1+\kappa}{\kappa}\hat{\ell}_{is}(\omega)\right)^2 ds\\
&\quad+\frac{1}{1+\kappa}\int_0^1\frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\hat{\ell}_{isn}ds-\frac{1}{1+\kappa}\int_0^1\frac{\overline{\ell}_{is}}{\ell_i}\hat{\ell}_{is}ds\\
&\quad+\int_0^1\frac{\overline{\ell}_{is}}{\ell_i}\hat{\ell}_{is}(\omega)ds\\
&= -\frac{1}{2}\frac{\kappa}{1+\kappa}\int_0^1\frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\left(-\frac{1}{\kappa}\hat{\ell}_{isn}+\frac{1+\kappa}{\kappa}\hat{\ell}_{isn}(\omega)\right)^2 ds\\
&\quad+\frac{1}{2}\frac{\kappa}{1+\kappa}\int_0^1\frac{\overline{\ell}_{is}}{\overline{\ell}_i}\left(-\frac{1}{\kappa}\hat{\ell}_{is}+\frac{1+\kappa}{\kappa}\hat{\ell}_{is}(\omega)\right)^2 ds\\
&\quad-\frac{1}{2}\frac{\nu}{1+\nu}\int_0^1\frac{\overline{\ell}_{is}}{\ell_i}\left(-\frac{1}{\nu}\hat{\ell}_{is}+\frac{1+\nu}{\nu}\hat{\ell}_{is}(\omega)\right)^2 ds\\
&\quad+\frac{1}{1+\nu}\int_0^1\frac{\overline{\ell}_{is}}{\ell_i}\hat{\ell}_{is}ds\\
&\quad+\frac{1}{1+\kappa}\int_0^1\frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\hat{\ell}_{isn}ds-\frac{1}{1+\kappa}\int_0^1\frac{\overline{\ell}_{is}}{\ell_i}\hat{\ell}_{is}ds
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\frac{\kappa}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \left(-\frac{1}{\kappa}\hat{\ell}_{isn} + \frac{1+\kappa}{\kappa}\hat{\ell}_{isn}(\omega)\right)^2 ds \\
&\quad + \frac{1}{2}\frac{\kappa}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \left(-\frac{1}{\kappa}\hat{\ell}_{is} + \frac{1+\kappa}{\kappa}\hat{\ell}_{is}(\omega)\right)^2 ds \\
&\quad - \frac{1}{2}\frac{\nu}{1+\nu}\int_0^1 \frac{\overline{\ell}_{is}}{\ell_i} \left(-\frac{1}{\nu}\hat{\ell}_{is} + \frac{1+\nu}{\nu}\hat{\ell}_{is}(\omega)\right)^2 ds \\
&\quad - \frac{1}{2}\frac{1}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\ell_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \hat{\ell}_{isn}^2 ds \\
&\quad + \frac{1}{2}\frac{1}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\ell_i} \hat{\ell}_{is}^2 ds \\
&\quad - \frac{1}{2}\frac{1}{1+\nu}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \hat{\ell}_{is}^2 ds \\
&= -\frac{1}{2}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \hat{\ell}_{isn}(\omega)^2 ds \\
&\quad - \frac{1}{2}\frac{1}{\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{isn}\right)^2 ds \\
&\quad - \frac{1}{2}\left(\frac{1}{\nu} - \frac{1}{\kappa}\right)\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_{is}} \left(\hat{\ell}_{is}(\omega) - \hat{\ell}_{is}\right)^2 ds.
\end{aligned}
$$

Substituting this into the expression for the log second-order approximation to production gives

$$
\begin{aligned}
\frac{Y_i}{z_i(m_i)^\eta(\ell_i)^{1-\eta}} &\approx 1 + \mathbb{E}[\hat{a}_{isn}(\omega)] + \frac{1}{2}\mathbb{E}[\hat{a}_{isn}(\omega)^2] \\
&\quad + (1-\eta)\int_0^1 \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\ell_i} \hat{a}_{isn}(\omega)\hat{\ell}_{isn}(\omega) ds \\
&\quad + \frac{(1-\eta)^2}{2}\int_0^1 \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\ell_i} \hat{\ell}_{isn}(\omega)^2 ds \\
&\quad - \frac{1-\eta}{2}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \hat{\ell}_{isn}(\omega)^2 ds \\
&\quad - \frac{1-\eta}{2}\frac{1}{\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{isn}\right)^2 ds \\
&\quad - \frac{1-\eta}{2}\left(\frac{1}{\nu} - \frac{1}{\kappa}\right)\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_{is}} \left(\hat{\ell}_{is}(\omega) - \hat{\ell}_{is}\right)^2 ds \\
&\approx \mathbb{E}[a_{isn}(\omega)] + (1-\eta)\int_0^1 \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\ell_i} \hat{a}_{isn}(\omega)\hat{\ell}_{isn}(\omega) ds
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1-\eta}{2}\eta \int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \hat{\ell}_{isn}(\omega)^2 ds \\
&- \frac{1-\eta}{2}\frac{1}{\kappa} \int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{isn}\right)^2 ds \\
&- \frac{1-\eta}{2} \left(\frac{1}{\nu} - \frac{1}{\kappa}\right) \int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_{is}} \left(\hat{\ell}_{is}(\omega) - \hat{\ell}_{is}\right)^2 ds,
\end{aligned}
$$

where we use the fact that to log second order, $\mathbb{E}[a_{sn}(\omega)] \approx 1 + \mathbb{E}[\hat{a}_{sn}(\omega)] + \frac{1}{2}\mathbb{E}[\hat{a}_{sn}(\omega)^2]$.
$\square$

References

Benigno, P. and Woodford, M. (2003). Optimal monetary and fiscal policy: A linear-quadratic approach. *NBER Macroeconomics Annual*, 18:271–333.

Benigno, P. and Woodford, M. (2012). Linear-quadratic approximation of optimal policy problems. *Journal of Economic Theory*, 147(1):1–42.

Durrett, R. (2019). *Probability: theory and examples*, volume 49. Cambridge university press.