COMP0011 – Mathematics and Statistics

Louis Parlant

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Part 1 : Algebra

Chapter 1 : Complex Numbers

I. Introduction

- Why complex numbers?

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  Algebraic argument: some equations must have solutions but have no solutions among real numbers.

  Practical argument: complex numbers make our life easier in many areas (trigonometry, physics, engineering)

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real part
imaginary part

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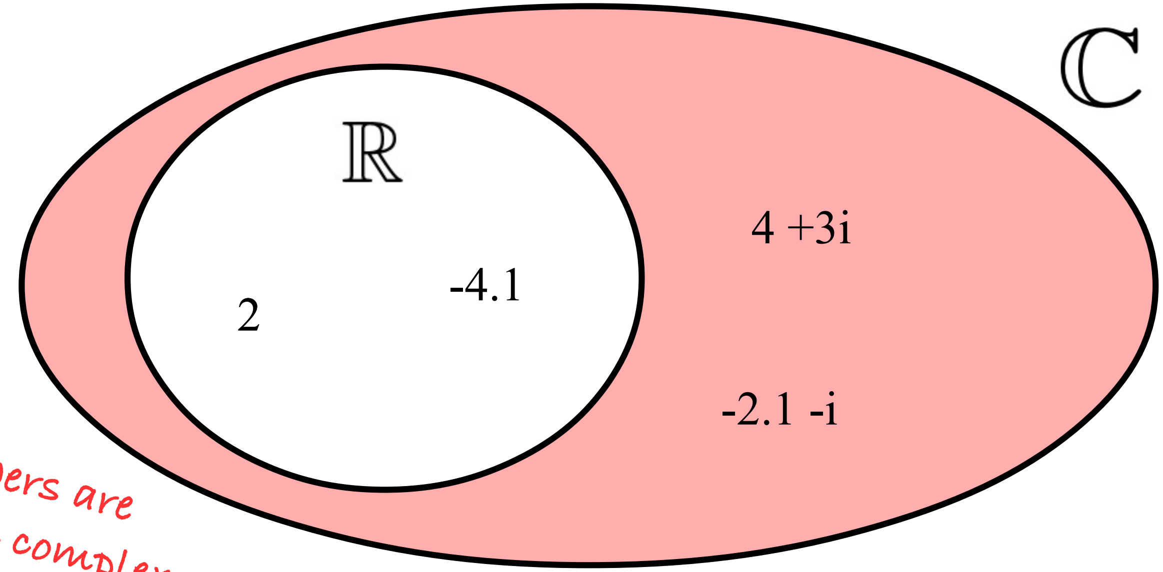
- Foundation of complex numbers: $i = \sqrt{-1}$
- A complex number can be written as $z = \underbrace{x}_{\text{real part}} + i\underbrace{y}_{\text{imaginary part}} \qquad x, y \in \mathbb{R}$
- The set of all complex numbers is denoted by $\mathbb{C}$
- Examples: $2 + 3i, \quad 5i, \quad -5.2, \quad -6 + 12i$

I. Introduction

- Sets of complex and real numbers



Real numbers are included in complex numbers

II. Basic Complex Arithmetic

- Addition:

$$(2 + 3i) + (5 - 8i) =$$

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- Addition:

$$(2+3i)+(5-8i)=(2+5)+(3+(-8))i=7-5i$$

Add up the real parts and imaginary parts separately

II. Basic Complex Arithmetic

- Multiplication:

$$(3 + 4i) \times (-2 + 3i) =$$

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- **Multiplication:**

$$
\begin{aligned}
(3+4i)\times(-2+3i) &= (3\times(-2)) + (3\times 3i) + (4i\times(-2)) + (4i\times 3i) \\
&= -6 + 9i - 8i + 12i^2 \\
&= -6 + 9i - 8i - 12 \\
&= -18 + i
\end{aligned}
$$

II. Basic Complex Arithmetic

- Conjugate of a complex number: $$\overline{x + iy} = x - iy$$

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- Conjugate of a complex number: $\overline{x + iy} = x - iy$

- Remark: $z \times \overline{z} = (x + iy) \times (x - iy) = x^2 + y^2$

Real number

II. Basic Complex Arithmetic

- Division:

$$\frac{2+3i}{5-4i} =$$

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- Division:

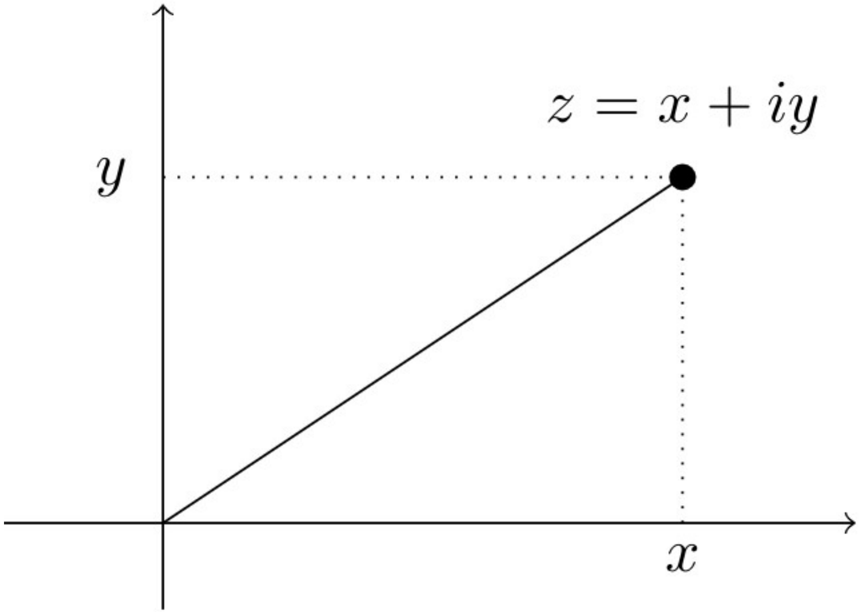
$$\frac{2+3i}{5-4i} = \frac{2+3i}{5-4i} \times \frac{5+4i}{5+4i} = \frac{(2+3i)(5+4i)}{(5-4i)(5+4i)} = \frac{-2+23i}{25+16} = \frac{-2}{41} + \frac{23}{41}i$$

use the conjugate of the denominator

III. Geometric representation

- A complex number is represented by a point in the 2D plane

- Cartesian coordinates



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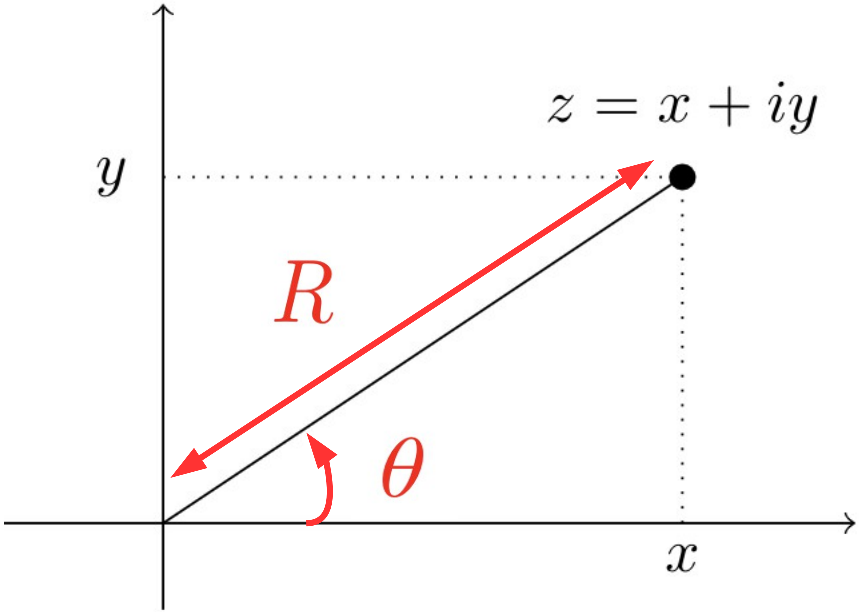
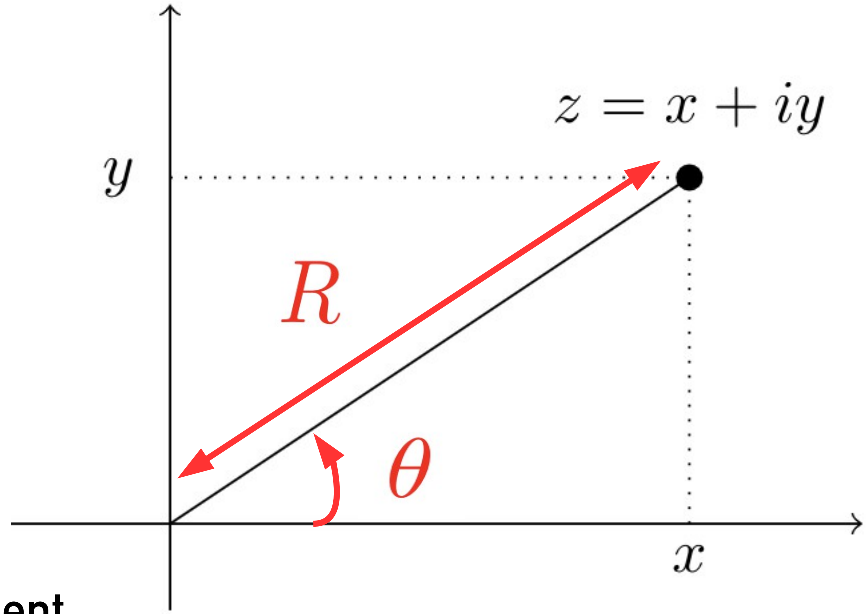


III. Geometric representation

- A complex number is represented by a point in the 2D plane

- Cartesian coordinates or polar coordinates

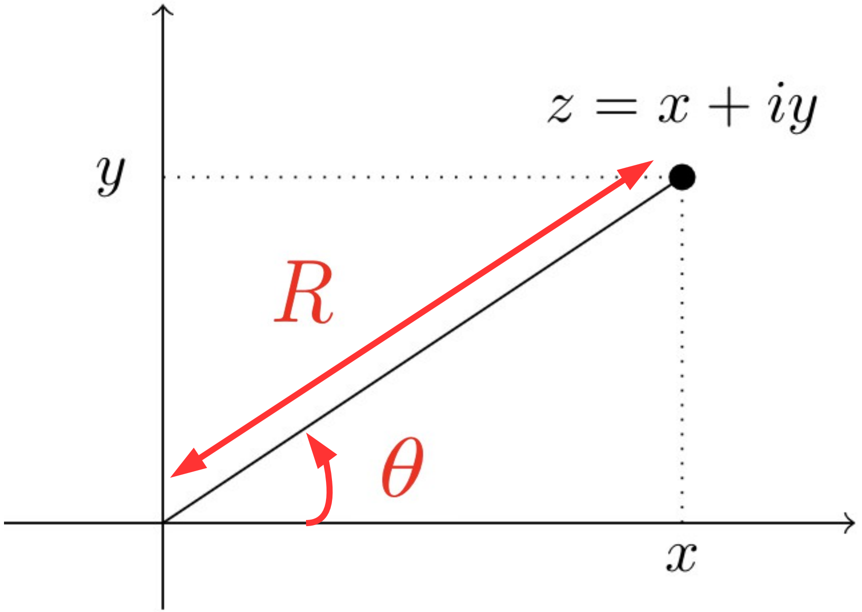
- R is called absolute value, θ argument
  or modulus



III. Geometric representation

$$x = R \times cos(\theta)$$
$$y = R \times sin(\theta)$$

$$|z| = R = \sqrt{x^2 + y^2}$$
$$\theta = arg(z)$$

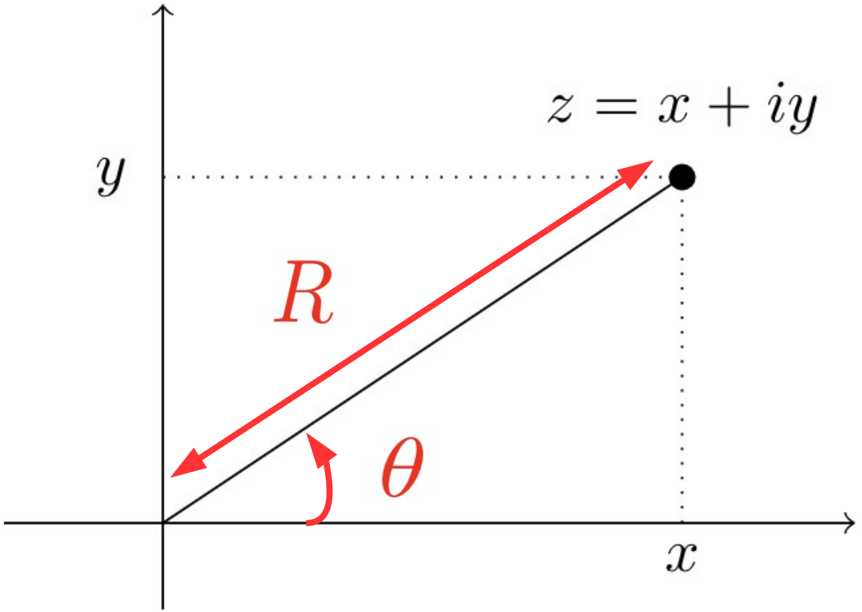


IV. Exponential form

A new notation!

$$z = R \times e^{i\theta}$$

$$R, \theta \in \mathbb{R};\ R \geq 0$$

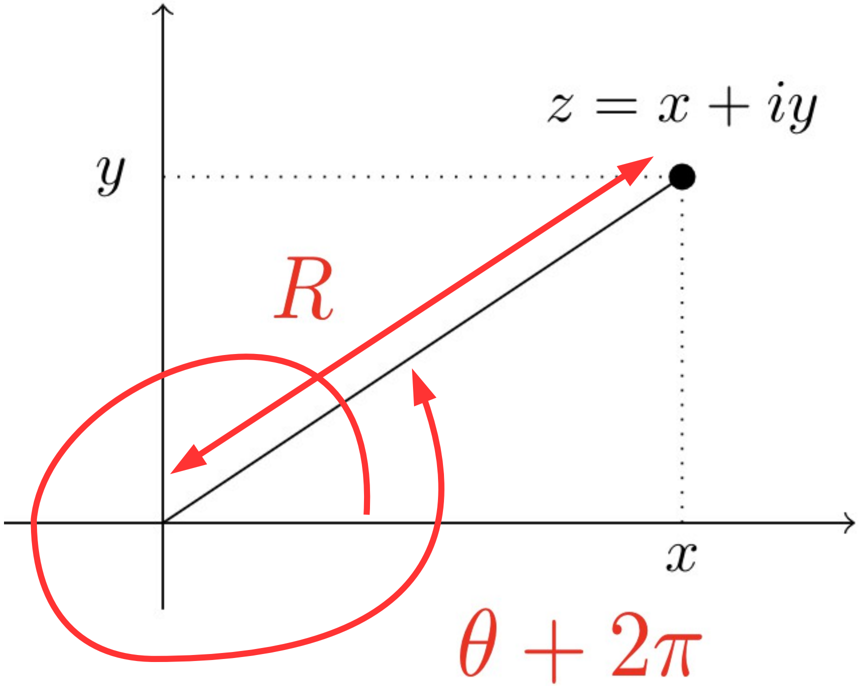


IV. Exponential form

- Two complex numbers whose arguments differ by 2π are considered equal.

- Example:

$$3e^{i\frac{\pi}{4}} = 3e^{i\left(\frac{\pi}{4}+2\pi\right)} = 3e^{i\frac{9\pi}{4}}$$

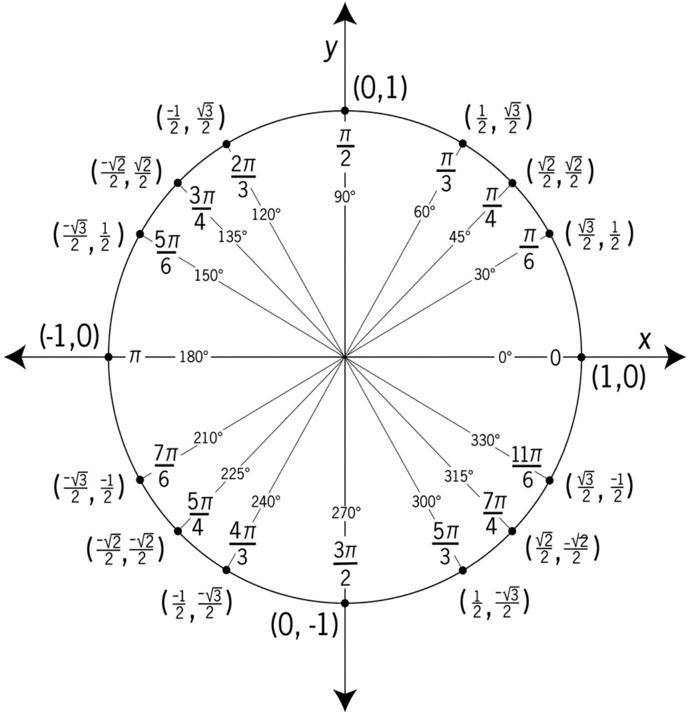


IV. Exponential form

- Examples:

$$z = \frac{\sqrt{3}}{2} + \frac{1}{2}i =$$

$$z' = \sqrt{2} - i\sqrt{2} =$$

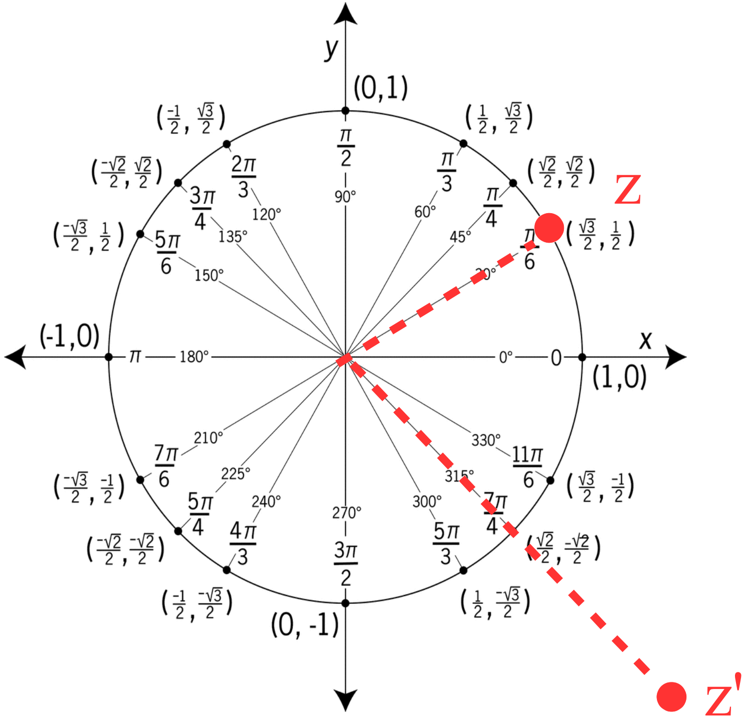


IV. Exponential form

- Examples:

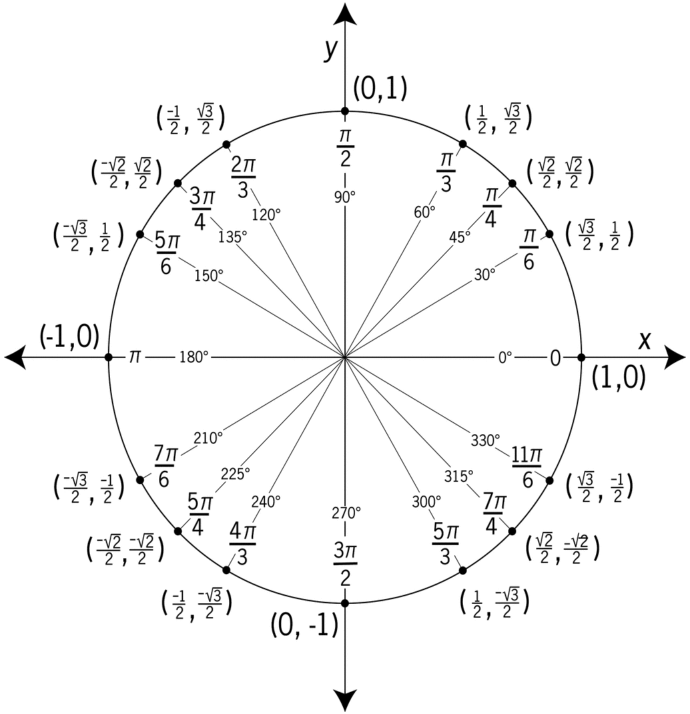
$$z = \frac{\sqrt{3}}{2} + \frac{1}{2}i = 1 \times e^{i\frac{\pi}{6}}$$

$$z' = \sqrt{2} - i\sqrt{2} = 2 \times e^{-i\frac{\pi}{4}}$$



IV. Exponential form

- It is important to know the coordinates of points on the circle corresponding to classic angles



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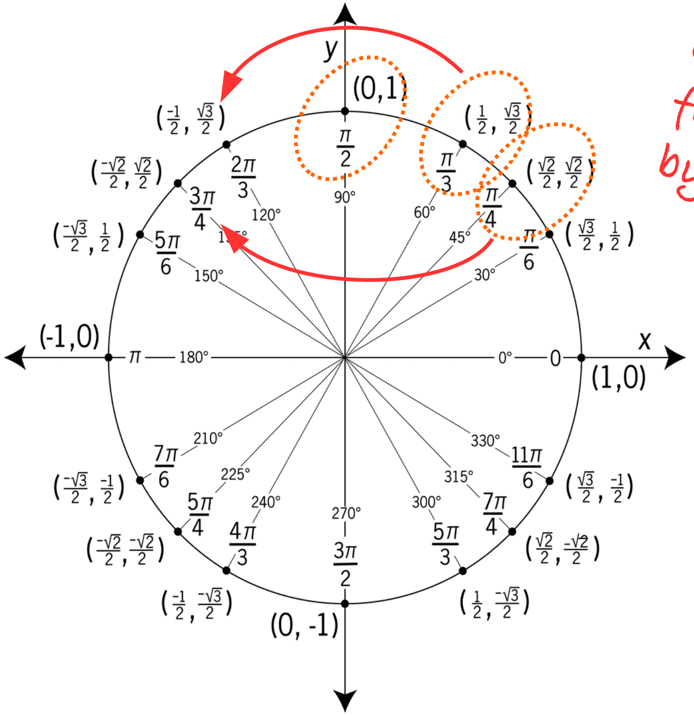

Learn these and find the others by symmetry

IV. Exponential form

- Multiplication of complex numbers is easier in exponential form:

$$\left(1 \times e^{i\frac{\pi}{6}}\right) \times \left(2 \times e^{-i\frac{\pi}{4}}\right) = 2 \times e^{i\frac{\pi}{6}+(-i\frac{\pi}{4})}$$

$$= 2 \times e^{i(\frac{\pi}{6}-\frac{\pi}{4})}$$

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$$= 2 \times e^{i\left(\frac{\pi}{6} - \frac{\pi}{4}\right)}$$

Multiply the moduli and
add up the arguments

IV. Exponential form

- Division of complex numbers is easier in exponential form:

$$\frac{\left(1 \times e^{i\frac{\pi}{6}}\right)}{\left(2 \times e^{-i\frac{\pi}{4}}\right)} = \frac{1}{2} \times \frac{e^{i\frac{\pi}{6}}}{e^{-i\frac{\pi}{4}}}$$

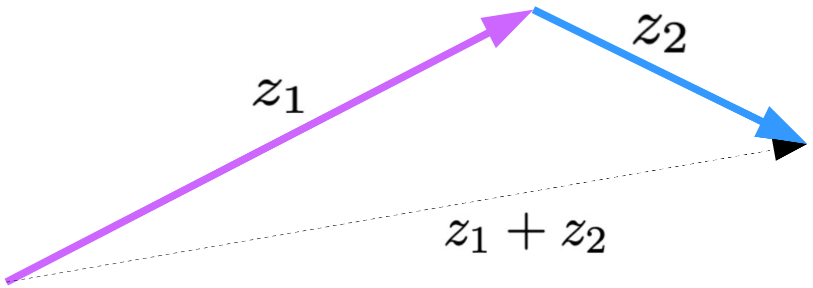
$$= \frac{1}{2} \times e^{i\frac{\pi}{6}+i\frac{\pi}{4}}$$

$$= \frac{1}{2} \times e^{i\frac{5\pi}{12}}$$

IV. Exponential form

- The absolute value (or modulus) verifies the following:

Triangular Equality $\boxed{|z_1 + z_2| \leq |z_1| + |z_2|}$



V. Roots of unity

- Let $n \in \mathbb{N}$. How to find all complex numbers z such that $z^n = 1$?

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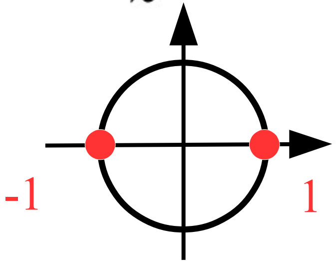
- Simple cases first: $n=2$

  There are 2 complex numbers z such that $z^2 = 1$: 1 and -1

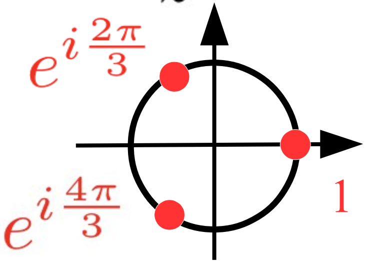


V. Roots of unity

- Let $n \in \mathbb{N}$. How to find all complex numbers z such that $z^n = 1$?

- Simple cases first: n=3

  There are 3 complex numbers z such that $z^3 = 1$: $1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}$



proof

$$(e^{i\frac{2\pi}{3}})^3 = (e^{i\times 2\pi}) = 1$$

V. Roots of unity

- Let $n \in \mathbb{N}$. How to find all complex numbers z such that $z^n = 1$?

- In general: assume that z is a solution. We write it as $z = R \times e^{i\theta}$

  Then we have:

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This complex equality implies equalities for:

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- The argument of z

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This complex equality implies equalities for:

- The absolute value of z:

$$|z^n| = |1| \Rightarrow R^n = 1 \Rightarrow R = 1$$

- The argument of z:

$$arg(z^n) = arg(1) = 0[2\pi] \Rightarrow n\theta = 0 + 2k\pi \Rightarrow \theta = \frac{2k\pi}{n}$$

$$k \in \mathbb{N}$$

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Need to consider the argument up to increments of 2π

- The argument of z:

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- Summary:

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- Note that $e^{i\frac{2k\pi}{n}} = e^{i\frac{2k\pi}{n}+2\pi} = e^{i\frac{2(k+n)\pi}{n}}$

- We can consider only the first n values of k : there are exactly n distinct solutions, which we denote by $w_1, \ldots, w_n$.

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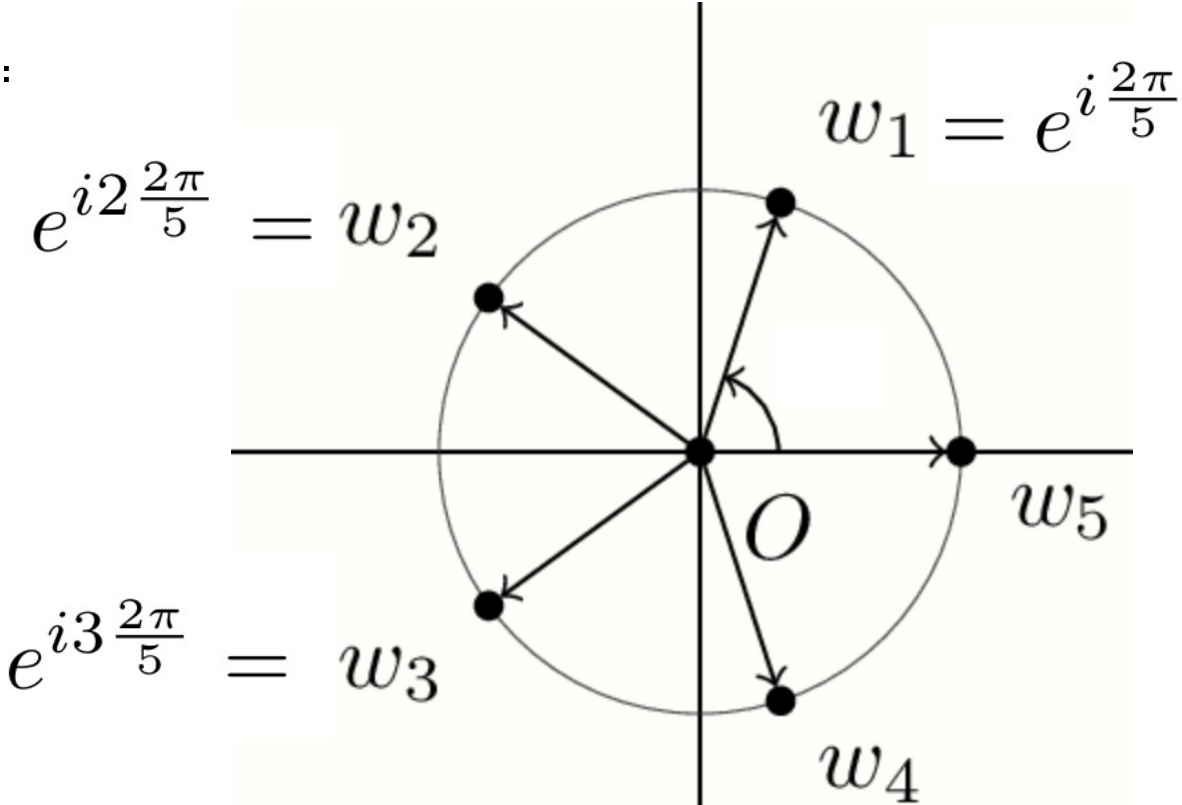
This one is always equal to 1

V. Roots of unity

- Geometric interpretation:
  Example: $n=5$

$$z^5 = 1$$



V. Roots of unity

- Geometric interpretation:
  Example: $n=5$

$$z^5 = 1$$

- Finding the n-th roots of unity amounts to subdividing the unit disc in n equal parts

VI. More complex equations

- Let's use our knowledge of complex numbers to solve equations in $\mathbb{C}$.

- What are the complex numbers z such that $\overline{z} = z - i$?

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- What are the complex numbers z such that $\overline{z} = z - i$?
- We write $z = x + iy$,
  if it verifies the equation then we must have: $x - iy = x + iy - i$

$$-2iy = -i$$

$$y = \frac{1}{2}$$

VI. More complex equations

- Converse: it is easy to show that for any $x \in \mathbb{R}$,

  $z = x + \frac{1}{2}i$ is a solution of $\overline{z} = z - i$.

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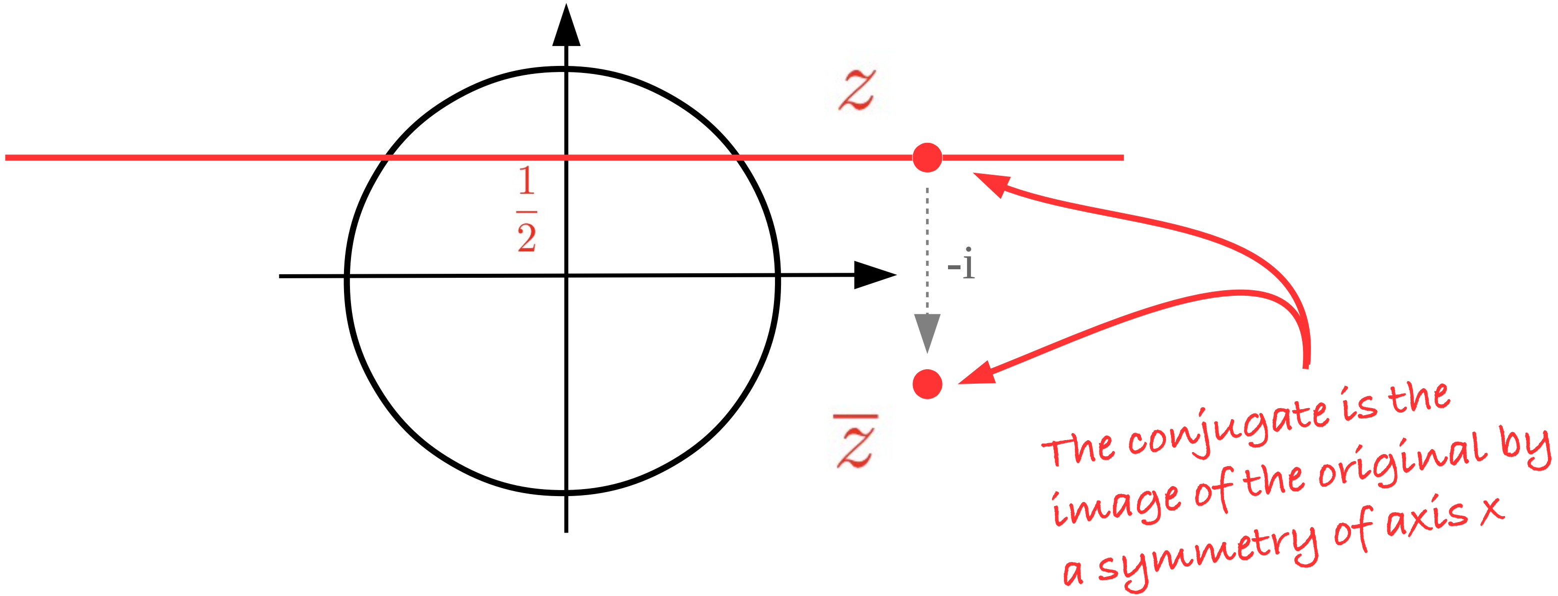
- What is the geometrical location (*locus*) of the set of solutions?

VI. More complex equations

$$\overline{z} = z - i \qquad \textbf{Solutions: } z = x + \frac{1}{2}i \qquad x \in \mathbb{R}$$



VI. More complex equations

- Let's focus on another equation:
- What are the complex numbers z such that $arg(z^2) = \frac{\pi}{2}$?

VI. More complex equations

$$arg(z^2) = \frac{\pi}{2}$$

- Let's write $z = Re^{i\theta}$. Then we have $z^2 = R^2 e^{i\theta \times 2}$.
- Therefore z verifies the equation if $\theta \times 2 = \dfrac{\pi}{2} + 2k\pi$ $\quad k \in \mathbb{N}$

$$\Rightarrow \theta = \frac{\pi}{4} + k\pi$$

VI. More complex equations

Note that squaring doubles the argument

$$arg(z^2) = \frac{\pi}{2}$$

- Let's write $z = Re^{i\theta}$. Then we have $z^2 = R^2 e^{i\theta \times 2}$.
- Therefore z verifies the equation if $\theta \times 2 = \frac{\pi}{2} + 2k\pi$ $\quad k \in \mathbb{N}$

$$\Rightarrow \theta = \frac{\pi}{4} + k\pi$$

VI. More complex equations

$$arg(z^2) = \frac{\pi}{2}$$

- Converse: it is easy to show that for any $k \in \mathbb{N}$, for any $R \geq 0$,

  the complex number $z = Re^{i(\frac{\pi}{4}+k\pi)}$ is a solution of the equation.

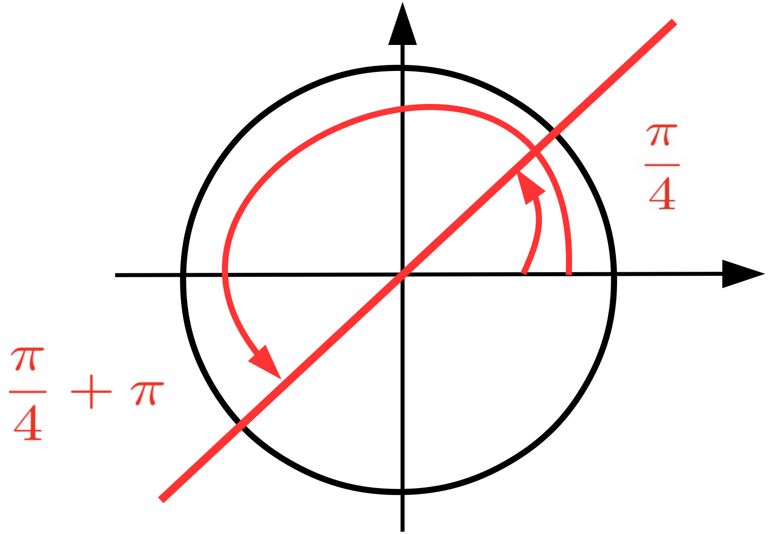
VI. More complex equations

$$arg(z^2) = \frac{\pi}{2}$$

- Solutions: $z = Re^{i(\frac{\pi}{4}+k\pi)}$ for $k \in \mathbb{N}$ $R \geq 0$
- What is the geometric location (*locus*) of the set of solutions?

VI. More complex equations

$arg(z^2) = \frac{\pi}{2}$ Solutions: $z = Re^{i(\frac{\pi}{4}+k\pi)}$ for $k \in \mathbb{N}$ $R \geq 0$



VII. Conclusions

- Different representations for complex numbers
- Relations with trigonometry
- Geometric interpretation
- Solving complex equations