ECON8000: Quantitative Skills for Economics
Lecture 4: Topology and Metric Spaces II

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Compact Sets

If X is a topological space, a set $A \subset X$ is compact if, whenever $\{U_i\}$ is a collection of open sets such that $A \subset \bigcup_i U_i$, there are $U_{i_1}, \ldots, U_{i_k}$ such that $A \subset U_{i_1} \cup \cdots \cup U_{i_k}$. That is, *every open cover of A has a finite subcover.*

This is not an intuitive definition. In spite of its enormous importance, the concept only started to emerge around 1910. But in comparison with "closed and bounded" (the equivalent definition for $\mathbb{R}^n$) and "every sequence has a convergent subsequence" (the equivalent definition for metric spaces) it is often easier to verify when doing a proof, due to its primitive and general character.

Basic Results for Compactness

Theorem: If X is a topological space, $K \subset X$ is compact, and $C \subset K$ is closed, then C is compact.

Proof. If $\{U_i\}$ is an open cover of C, then $\{U_i\}$ together with $X \setminus C$ is an open cover of K, so there are $i_1, \ldots, i_k$ such that $K \subset U_{i_1} \cup \cdots \cup U_{i_k} \cup (X \setminus C)$, and thus $C \subset U_{i_1} \cup \cdots \cup U_{i_k}$.

Basic Results for Compactness

Theorem: If X and Y are topological spaces, $f : X \to Y$ is continuous, and $K \subset X$ is compact, then $f(K)$ is compact.

Proof. Let $\{V_i\}$ be a collection of open subsets of Y that cover $f(K)$. By continuity each $f^{-1}(V_i)$ is open, so $\{f^{-1}(V_i)\}$ is an open cover of K. Therefore because K is compact, there are finitely many $i_1, \ldots, i_k$ such that

$$K \subset f^{-1}(V_{i_1}) \cup \cdots \cup f^{-1}(V_{i_k}),$$

which means that $f(K) \subset V_{i_1} \cup \cdots \cup V_{i_k}$.

Compact Subsets of $\mathbb{R}$

Lemma: For any $a < b$, $[a, b]$ is compact.

Proof. Let $\{U_i\}$ be an open cover of $[a, b]$. Let S be the set of $c \in [a, b]$ such that $[a, c]$ is contained in the union of finitely many of the U_i. There is some i such that $a \in U_i$, so $S \neq \emptyset$. Let ℓ be the least upper bound of S. If $\ell < b$, then $(\ell - \varepsilon, \ell + \varepsilon) \subset U_i$ for some i and $\varepsilon > 0$, and for some $c \in (\ell - \varepsilon, \ell)$ there is a finite cover of $[a, c]$, which combines with U_i to give a finite cover $[a, \ell + \varepsilon)$, contradicting the definition of ℓ. Thus $\ell = b$. Again $(b - \varepsilon, b] \subset U_i$ for some i and $\varepsilon > 0$, and for some $c \in (\ell - \varepsilon, \ell)$ there is a finite cover of $[a, c]$, which combines with U_i to give a finite cover $[a, b]$.

Theorem: A set $K \subset \mathbb{R}$ is compact if and only if it is closed and bounded.

Proof. A closed bounded set is compact because it is a closed subset of a bounded interval. To prove the converse suppose that K is compact. If K was unbounded $\{ (-n, n) : n \in \mathbb{N} \}$ would be an open cover with no finite subcover. If K was not closed there would be a number $t \in \overline{K} \setminus K$, so t would be contained in every closed set that contains K. Therefore there would be no $\varepsilon > 0$ such that $(t - \varepsilon, t + \varepsilon) \cap K = \emptyset$, so

$$\{ (-\infty, t - \varepsilon) \cup (t + \varepsilon, \infty) : \varepsilon > 0 \}$$

would be an open cover of K without a finite subcover.

Bonus Theorem for Economists

Theorem: If X is a compact topological space and $f : X \to \mathbb{R}$ is continuous, then f is bounded and there is an $x \in X$ such that $f(x) = \max_{x' \in X} f(x')$.

Proof. $f(X)$ is compact.

Compact Sets in $\mathbb{R}^n$

Heine-Borel Theorem: A set $C \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. If C is bounded, then it is contained in some rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Each $[a_i, b_i]$ is compact, so the last result implies that R is compact. If C is also closed, it is a closed subset of the compact set R, so it is compact.

If C is unbounded, then it cannot be compact since

$$\{ \mathbf{B}_n(0) : n \in \mathbb{N} \}$$

is an open cover without a finite subcover. If C is not closed there is a point $x \in \overline{C} \setminus C$ such that every open set containing x has a nonempty intersection with C, but then

$$\{ \mathbb{R}^n \setminus \overline{\mathbf{B}_\varepsilon(x)} : \varepsilon > 0 \}$$

is an open cover of C without a finite subcover.

Nice Properties of Metric Spaces

Recall that a metric space is a pair (X, d) where X is a set and $d : X \times X \to [0, \infty)$ is a function such that, for all $x, y, z \in X$, $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$, and $d(x, z) \leq d(x, y) + d(y, z)$.

Definition

A topological space X is Hausdorff, or a Hausdorff space, if, for all distinct $x, y \in X$, there are open $U, V \subset X$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

If X is a metric space, it is Hausdorff because we can set $U = \mathbf{B}_{\frac{1}{2}d(x,y)}(x)$ and $V = \mathbf{B}_{\frac{1}{2}d(x,y)}(y)$.

Theorem: If X is a Hausdorff space and $C \subset X$ is compact, then C is closed.

Proof. It suffices to show that each $x \in X \setminus C$ is contained in an open set that does not intersect C, since then $X \setminus C$ is the union of these open sets and is consequently open. For each $y \in C$ there are open sets U_y and V_y that are disjoint (that is, $U_y \cap V_y = \emptyset$) such that $x \in U_y$ and $y \in V_y$. Since C is compact, it is covered by some finite collection $V_{y_1}, \ldots, V_{y_k}$, and $\bigcap_{i=1}^{k} U_{y_i}$ is open and does not intersect C.

Normal

Definition

A topological space X is normal, or a normal space, if, for any closed disjoint C and D there are disjoint open sets U and V such that $C \subset U$ and $D \subset V$.

A metric space X is normal because we can set

$$U = \bigcup_{x \in C} \mathbf{B}_{\frac{1}{2}d(x,D)}(x) \quad \text{and} \quad V = \bigcup_{y \in D} \mathbf{B}_{\frac{1}{2}d(y,C)}(y).$$

Convergent Sequences

Let (X, d) be a metric space. A sequence in X is, formally, a function from $\mathbb{N}$ to X, but we usually think of it as a countable list $x_1, x_2, \ldots$ of elements of X. Often we write (x_i) to denote such a sequence. Such a sequence converges to a point x if $d(x_i, x) \to 0$; we write $x_i \to x$ to indicate that this is the case. If there is some (necessarily unique) x such that $x_i \to x$, then we say that (x_i) is convergent. A point x is a limit point of a set X if there is a sequence (x_i) with $x_i \in X$ for all i and $x_i \to x$.

Proposition: Boundary

Proposition: For any $S \subset X$, $\overline{S}$ is the set of limit points of S.

Proof. If x is a limit point of S, then for every $\varepsilon > 0$ there is some element of S in $\mathbf{B}_\varepsilon(x)$, so $x \in \overline{S}$. On the other hand if $x \in \overline{S}$, then S intersects every $\mathbf{B}_\varepsilon(x)$, and we can create a sequence in S converging to x by choosing $x_1 \in \mathbf{B}_1(x)$, $x_2 \in \mathbf{B}_{1/2}(x)$, $x_3 \in \mathbf{B}_{1/3}(s)$, etc.

Cauchy Sequences and Completeness

Let (X, d) be a metric space. A Cauchy sequence in X is a sequence $x_1, x_2, \ldots$ that is eventually arbitrarily close to itself, in the sense that for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq N$. Clearly a convergent sequence is Cauchy. The space X is complete if every Cauchy sequence is convergent. We now give a result whose import is that every metric space is a subspace of a unique complete space. In a sense, the containing complete space is always there, even if you try to ignore it.

To state the theorem we need some additional terminology. If X is a topological space, a set $S \subset X$ is dense if $\overline{S} = X$. If (X, d) and (Y, e) are metric spaces, a function $f : X \to Y$ is an isometry if $e(f(x), f(x')) = d(x, x')$ for all $x, x' \in X$.

The Contraction Mapping Theorem

- Economic equilibria are typically fixed points.
- For this reason the result we describe now, which is also known as the Banach fixed point theorem, is especially significant for us.
- Let (X, d) be a metric space, and let $f : X \to X$ be a function.
- A point x is a fixed point of f if $f(x) = x$.
- We say that f is a contraction if there is a constant $\alpha \in (0, 1)$, called the modulus of contraction, such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$.

The Contraction Mapping Theorem

Theorem: If X is nonempty and complete and $f : X \to X$ is a contraction, then f has a unique fixed point.

Proof. Let α be the modulus of contraction. Uniqueness is easy: if x and x' are fixed points, then

$$d(x, x') = d(f(x), f(x')) \leq \alpha d(x, x'),$$

so $d(x, x') = 0$ and thus $x = x'$.

To prove existence, let x be any point in X, and define the sequence $x_0, x_1, x_2, \ldots$ be setting $x_0 = x$ and $x_i = f(x_{i-1})$ for $i \geq 1$. By induction $d(x_i, x_{i+1}) \leq \alpha^i d(x_0, x_1)$, so when $m < n$ we have

$$d(x_m, x_n) \leq \Big(\sum_{i=m}^{n-1} \alpha_i \Big) d(x_0, x_1) \leq \frac{\alpha^m}{1-\alpha} d(x_1, x_2).$$

Thus (x_i) is Cauchy, and has a limit x^* because X is complete. We have

$$d(f(x^*), x_{i+1}) \leq \alpha d(x^*, x_i) \to 0,$$

so $x_i \to f(x^*)$ and thus $f(x^*) = x^*$.