ECON8000: Quantitative Skills for Economics
Lecture 3: Topology and Metric Spaces I

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Vector Spaces

A vector space over a field F is a triple $(V, +, \cdot)$, where $+ : V \times V \to V$ and $\cdot : F \times V \to V$ are operations called vector addition and scalar multiplication, such that:

- $(u+v)+w = u+(v+w)$ for all $u, v, w \in V$.
- $u+v = v+u$ for all $u, v \in V$.
- There is a $0 \in V$ such that $v+0 = v$ for all $v \in V$.
- For each $v \in V$ there is $-v \in V$ such that $v+(-v) = 0$.
- $\alpha(\beta v) = (\alpha\beta)v$ for all $\alpha, \beta \in F$ and $v \in V$.
- $1v = v$ for all $v \in V$.
- $(\alpha+\beta)v = \alpha v + \beta v$ for all $\alpha, \beta \in F$ and $v \in V$.
- $\alpha(u+v) = \alpha u + \alpha v$ for all $\alpha \in F$ and $u, v \in V$.

Examples: a) F itself; b) $\mathbb{Q}(\sqrt{2})$ is a vector space over $\mathbb{Q}$; c) $\mathbb{R}$ is a vector space over $\mathbb{Q}$; d) $C([0,1])$ is the set of continuous functions $f : [0,1] \to \mathbb{R}$.

Inner Products

If V is vector space over an ordered field F, an inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \to F$ such that:

- $\langle v, w \rangle = \langle w, v \rangle$ for all $v, w \in V$.
- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
- $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$ for all $\alpha \in F$ and all $v, w \in V$.
- $\langle v, v \rangle \geq 0$ for all $v \in V$, and $\langle v, v \rangle = 0$ if and only if $v = 0$.

Example: On F^n the standard inner product is $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$.

An inner product space is a vector space endowed with an inner product. We define the norm of $v \in V$ to be

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

The Cauchy-Schwartz Inequality

Let V be an inner product space.

Theorem: $|\langle v, w\rangle| \leq \|v\| \cdot \|w\|$ for all $v, w \in V$, with equality if and only if $w = \alpha v$ or $v = \alpha w$ for some $\alpha \in F$.

Proof. If $w = 0$, then $\langle v, w\rangle = 0$ (proof?) and $\|v\| \cdot \|w\| = 0$, and $w = 0v$. Therefore suppose that $w \neq 0$. If $\lambda = \langle v, w\rangle / ||w||^2$, then

$$0 \leq \|v - \lambda w\|^2 = \langle v - \lambda w, v - \lambda w\rangle = \langle v, v\rangle - \lambda\langle v, w\rangle - \lambda\langle w, v\rangle + \lambda^2\langle w, w\rangle$$

$$= \|v\|^2 - 2\langle v, w\rangle^2/\|w\|^2 + \langle v, w\rangle^2/\|w\|^2 = \|v\|^2 - \langle v, w\rangle^2/\|w\|^2.$$

If this holds with equality, then $\langle v - \lambda w, v - \lambda w\rangle = 0$ and thus $v = \lambda w$. On the other hand if $v = \alpha w$, then $\|v\| = \sqrt{\alpha^2\|w\|^2} = |\alpha| \cdot \|w\|$, so $|\langle v, w\rangle| = \left|\alpha\|w\|^2\right| = \|v\| \cdot \|w\|$.

Normed Spaces

Let V be a vector space over an order field F. A norm on V is a function $\|\cdot\| : V \to F$ such that:

a) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

b) $\|\alpha v\| = |\alpha| \cdot \|v\|$ for all $\alpha \in F$ and $v \in V$.

c) $\|v\| \geq 0$ with equality if and only if $v = 0$.

Examples: On F^n the city block norm is $\|x\|_1 = |x_1| + \cdots + |x_n|$, and the sup norm is $\|x\|_\infty = \max_i |x_i|$.

A normed space is a vector space over an ordered field that is endowed with a norm.

If V is an inner product space, the norm derived from the inner product is, in fact, a norm: c) is one of the conditions in the definition of an inner product, for b) we have

$$\|\alpha v\| = \sqrt{\langle \alpha v, \alpha v \rangle} = \sqrt{\alpha^2 \cdot \|v\|^2} = |\alpha| \cdot \|v\|,$$

and the Cauchy-Schwartz inequality gives

$$(\|v + w\|)^2 = \langle v + w, v + w \rangle = \|v\|^2 + 2\langle v, w \rangle + \|w\|^2$$

$$\leq \|v\|^2 + 2\|v\| \cdot \|w\| + \|w\|^2 = (\|v\| + \|w\|)^2.$$

Metric Spaces

A metric space is a pair (X, d) where X is a set and $d : X \times X \to [0, \infty)$ is a function, called a metric, such that:

- $d(x, y) = 0$ if and only if $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$. (Symmetry)
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. (Triangle Inequality)

If V is a normed space and we set $d(v, w) = \|v - w\|$, then d is a metric: $d(v, w) = 0$ if and only if $v = w$ because $\|v\| = 0$ if and only if $v = 0$, $d(v, w) = d(w, v)$ because

$$\|-v\| = |-1| \cdot \|v\| = \|v\|,$$

and

$$d(u, w) = \|u - w\| \leq \|u - v\| + \|v - w\| = d(u, v) + d(v, w).$$

If $x \in X$ and $\varepsilon > 0$, the ε-ball centered at x is

$$\mathbf{B}_\varepsilon(x) = \{\, y \in X : d(x,y) < \varepsilon \,\}.$$

We extend this notation to balls around sets: if $S \subset X$ and $\varepsilon > 0$, then

$$\mathbf{B}_\varepsilon(S) = \bigcup_{x \in S} \mathbf{B}_\varepsilon(x) = \{\, y \in X : d(x,y) < \varepsilon \text{ for some } x \in S \,\}.$$

A set $U \subset X$ is open if, for every $x \in U$, there is some $\varepsilon > 0$ such that $\mathbf{B}_\varepsilon(x) \subset U$. We observe that $\emptyset$ and X itself are open. If U and V are open, then $U \cap V$ is open: if $x \in U \cap V$, then there are $\varepsilon_U, \varepsilon_V > 0$ such that $\mathbf{B}_{\varepsilon U}(x) \subset U$ and $\mathbf{B}_{\varepsilon_V}(x) \subset V$, so $\mathbf{B}_{\min\{\varepsilon_U, \varepsilon_V\}}(x) \subset U \cap V$. If I is an arbitrary index set and $\{U_i\}_{i \in I}$ is a collection of open set, then $\bigcup_i U_i$ is open.

Topological Spaces

A topological space is a pair (X, τ) in which X is a set and τ is a set of subsets of X, called open sets, such that:

- $\emptyset \in \tau$ and $X \in \tau$.
- If $U, V \in \tau$, then $U \cap V \in \tau$.
- For any $\mathcal{C} \subset \tau$, $\bigcup_{U \in \mathcal{C}} U \in \tau$.

We say that τ is a topology for X.

Roughly speaking, we are going to work backwards through these different types of spaces, from most general to increasingly structured. In this way we will see each of the concepts that we bring to bear on $\mathbb{R}^n$, such as convexity or limit, at the maximum level of generality, which is its most natural setting, and we will get some initial taste of the organization of modern mathematics. It should be understood that we are skimming the surface, since each of these topics could easily be studied in a semester length course.

Topological Spaces

Recall that a topological space is a pair (X, τ) in which X is a set and τ is a set of subsets of X, called open sets, such that:

- $\emptyset \in \tau$ and $X \in \tau$.
- If $U, V \in \tau$, then $U \cap V \in \tau$.
- For any $\mathcal{C} \subset \tau$, $\bigcup_{U \in \mathcal{C}} U \in \tau$.

We say that τ is a topology for X.

Example: If $X = \{a, b\}$ and $\tau = \{\emptyset, \{a\}, X\}$, then τ is a topology for X that cannot be derived from any metric on X. (Why?)

We will not be interested in topologies that are not derived from metrics, even though they are very important in some branches of mathematics. The reason we study topological spaces is that defining concepts at this level of generality (when possible) is clarifying, and proofs that use only topological concepts are usually simpler than those that mix in metric space concepts.

Interior, Closure, and Boundary

Let (X, τ) be a topological space. A set $C \subset X$ is closed if its complement $C^c = X \setminus C$ is open, i.e., $C^c \in \tau$. Note that $\emptyset$ and X are closed, the union of two closed sets is closed, and the intersection of an arbitrary collection of closed sets is closed.

Let $S \subset X$ be a set.

- The interior $\operatorname{int} S$ of S is the largest open set contained in S. (This makes sense because the union of all open sets contained in S is open.)
- The closure $\overline{S}$ of S is the smallest closed set containing S. (This makes sense because the intersection of all closed sets containing S is closed.)
- The boundary or frontier of S is $\partial S = \overline{S} \setminus \operatorname{int} S$.

We say that S is a neighborhood of a point x if there is an open set U such that $x \in U \subset S$.

Continuous Functions

Let (X, τ) and (Y, σ) be topological spaces. A function $f : X \to Y$ is continuous if, for all $V \in \sigma$, $f^{-1}(V) \in \tau$, i.e., *preimages of open sets are open.*

Consider a function $f : \mathbb{R} \to \mathbb{R}$.

- If f is continuous in the ε-δ sense, then, for any open $V \subset \mathbb{R}$, $f^{-1}(V)$ is open because if $x \in f^{-1}(V)$, then $\mathbf{B}_\varepsilon(f(x)) \subset V$ for some $\varepsilon > 0$, so $\mathbf{B}_\delta(x) \subset f^{-1}(\mathbf{B}_\varepsilon(f(x))) \subset f^{-1}(V)$ for some $\delta > 0$.
- Suppose that, for every open $V \subset \mathbb{R}$, $f^{-1}(V)$ is open. If $x \in \mathbb{R}$, and $\varepsilon > 0$, then $\mathbf{B}_\varepsilon(f(x))$ is open, so $f^{-1}(\mathbf{B}_\varepsilon(f(x)))$ is open and consequently contains $\mathbf{B}_\delta(x)$ for some $\delta > 0$.

If $f : X \to Y$ is a continuous bijection and f^{-1} is continuous, then f is a homeomorphism. Topologically, X and Y are “the same” space.

Compositions Preserve Continuity

Recall that our main ways of constructing new functions from given functions are composition, restriction to a subdomain, and cartesian products. We now check that the first two of these preserve continuity, and we will eventually see that the third does as well.

Proposition: If X, Y, and Z are topological spaces and $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then $g \circ f$ is continuous.

Proof. If $U \subset Z$ is open, then $g^{-1}(U)$ is open because g is continuous, and $f^{-1}(g^{-1}(U))$ is continuous because f is continuous, but $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$, so we have shown that $g \circ f$ is continuous.

Connected Sets

If X is a topological space, a set $A \subset X$ is connected if there do not exist open sets $U, V \subset X$ such that $U \cap V = \emptyset$, $A \cap U \neq \emptyset \neq A \cap V$, and $A \subset U \cup V$. Intuitively, A "doesn't fall apart into multiple pieces."

Theorem: If X and Y are topological spaces, $f : X \to Y$ is continuous, and $A \subset X$ is connected, then $f(A)$ is connected.

Proof. Aiming at a contradiction, suppose that $f(A)$ is not connected, so there are open sets $U, V \subset Y$ such that $U \cap V = \emptyset$, $f(A) \cap U \neq \emptyset \neq f(A) \cap V$, and $f(A) \subset U \cup V$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open, by continuity, $A \cap f^{-1}(U) \neq \emptyset \neq A \cap f^{-1}(V)$, and $A \subset f^{-1}(U) \cup f^{-1}(V)$. This contradicts the assumption that A is connected.

Connected Components

Let X be a topological space. For each $x \in X$, the union of all connected sets that contain x is the connected component of x. The next result shows that the connected component is connected, so it is the largest connected set containing x.

Proposition: If $\{A_i\}_{i \in I}$ is a collection of connected sets, any two of which have a nonempty intersection, then $A = \bigcup_i A_i$ is connected.

Proof. Aiming at a contradiction, suppose that there are open sets U and V with $U \cap V = \emptyset$, $A \cap U \neq \emptyset \neq A \cap V$, and $A \subset U \cup V$. For each i, $A_i \subset U \cup V$, so either $A_i \subset U$ or $A_i \subset V$. There must be an i with $A_i \subset U$ and a j with $A_j \subset V$, but then $A_i \cap A_j \subset U \cap V = \emptyset$, contrary to hypothesis.

Connected Components are Closed

Proposition: If X is a topological space and $A \subset X$ is connected, then $\overline{A}$ is connected.

Proof. For the sake of producing a contradiction suppose not, so there are open sets U and V such that $U \cap V = \emptyset$, $\overline{A} \cap U \neq \emptyset \neq \overline{A} \cap V$, and $\overline{A} \subset U \cup V$. It cannot be the case that $A \cap U \neq \emptyset \neq A \cap V$, so without loss of generality we may assume that $A \subset U$. Then $X \setminus V$ is a closed set that contains A, so $\overline{A} \subset X \setminus V$, which contradicts $\overline{A} \cap V \neq \emptyset$.

For any $x \in X$, the connected component of x is a connected set that contains all connected sets that contain x, so it contains its closure, which is to say it is closed.

Thus the set of connected components of points in X is a partition of X into closed sets.

Closed Intervals are Connected

Lemma: If $a, b \in \mathbb{R}$ and $a \leq b$, then $[a, b]$ is connected.

Proof. Aiming at a contradiction, suppose that $[a, b]$ is not connected, so there are (relatively) open $U, V \subset [a, b]$ with $U \cap V = \emptyset$, $U \neq \emptyset \neq V$, and $U \cup V = [a, b]$. Without loss of generality suppose that $a \in U$. Let

$$S = \{\, t \in [a, b] : [a, t] \subset U \,\}.$$

Then S is nonempty because it contains a, and it is bounded above by b, so it has a least upper bound ℓ. If $\ell \in U$, then $[a, \ell] \subset U$ and $(\ell - \varepsilon, \ell + \varepsilon) \subset U$ for some $\varepsilon > 0$, but then $[a, \ell + \varepsilon) \subset S$, which contradicts the definition of ℓ. On the other hand, if $\ell \in V$, then $(\ell - \varepsilon, \ell + \varepsilon) \subset V$ for some $\varepsilon > 0$, and $\ell - \varepsilon$ is an upper bound of S, again contradicting the definition of ℓ.

Compact Sets

If X is a topological space, a set $A \subset X$ is compact if, whenever $\{U_i\}$ is a collection of open sets such that $A \subset \bigcup_i U_i$, there are $U_{i_1}, \ldots, U_{i_k}$ such that $A \subset U_{i_1} \cup \cdots \cup U_{i_k}$. That is, *every open cover of A has a finite subcover.*

This is not an intuitive definition. In spite of its enormous importance, the concept only started to emerge around 1910. But in comparison with "closed and bounded" (the equivalent definition for $\mathbb{R}^n$) and "every sequence has a convergent subsequence" (the equivalent definition for metric spaces) it is often easier to verify when doing a proof, due to its primitive and general character.

Basic Results for Compactness

Theorem: If X is a topological space, $K \subset X$ is compact, and $C \subset K$ is closed, then C is compact.

Proof. If $\{U_i\}$ is an open cover of C, then $\{U_i\}$ together with $X \setminus C$ is an open cover of K, so there are $i_1, \ldots, i_k$ such that $K \subset U_{i_1} \cup \cdots \cup U_{i_k} \cup (X \setminus C)$, and thus $C \subset U_{i_1} \cup \cdots \cup U_{i_k}$.

Theorem: If X and Y are topological spaces, $f : X \to Y$ is continuous, and $K \subset X$ is compact, then $f(K)$ is compact.

Proof. Let $\{V_i\}$ be a collection of open subsets of Y that cover $f(K)$. By continuity each $f^{-1}(V_i)$ is open, so $\{f^{-1}(V_i\}$ is an open cover of K. Therefore there are $i_1, \ldots, i_k$ such that

$$K \subset f^{-1}(V_{i_1}) \cup \cdots \cup f^{-1}(V_{i_k}),$$

which means that $f(K) \subset V_{i_1} \cup \cdots \cup V_{i_k}$.

Compact Subsets of $\mathbb{R}$

Lemma: For any $a < b$, $[a, b]$ is compact.

Proof. Let $\{U_i\}$ be an open cover of $[a, b]$. Let S be the set of $c \in [a, b]$ such that $[a, c]$ is contained in the union of finitely many of the U_i. There is some i such that $a \in U_i$, so $S \neq \emptyset$. Let ℓ be the least upper bound of S. If $\ell < b$, then $(\ell - \varepsilon, \ell + \varepsilon) \subset U_i$ for some i and $\varepsilon > 0$, and for some $c \in (\ell - \varepsilon, \ell)$ there is a finite cover of $[a, c]$, which combines with U_i to give a finite cover $[a, \ell + \varepsilon)$, contradicting the definition of ℓ. Thus $\ell = b$. Again $(b - \varepsilon, b] \subset U_i$ for some i and $\varepsilon > 0$, and for some $c \in (\ell - \varepsilon, \ell)$ there is a finite cover of $[a, c]$, which combines with U_i to give a finite cover $[a, b]$.

Theorem: A set $K \subset \mathbb{R}$ is compact if and only if it is closed and bounded.

Proof. A closed bounded set is compact because it is a closed subset of a bounded interval. To prove the converse suppose that K is compact. If K was unbounded $\{(-n, n) : n \in \mathbb{N}\}$ would be an open cover with no finite subcover. If K was not closed there would be a number $t \in \overline{K} \setminus K$, so t would be contained in every closed set that contains K. Therefore there would be no $\varepsilon > 0$ such that $(t - \varepsilon, t + \varepsilon) \cap K = \emptyset$, so

$$\{(-\infty, t - \varepsilon) \cup (t + \varepsilon, \infty) : \varepsilon > 0\}$$

would be an open cover of K without a finite subcover.

Bonus Theorem for Economists: If X is a compact topological space and $f : X \to \mathbb{R}$ is continuous, then f is bounded and there is an $x \in X$ such that $f(x) = \max_{x' \in X} f(x')$.

Proof. $f(X)$ is compact.