ECON8000: Quantitative Skills for Economics
Lecture 6: Functions

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Functions: The Environment

We will study functions $f : S \to \mathbb{R}$ where $S \subset \mathbb{R}$ will usually be either open or compact. Of course such a function is *continuous* if $f^{-1}(U)$ is an open subset of S whenever $U \subset \mathbb{R}$ is open.

The Intermediate Value Theorem

Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) \leq f(b)$ $(f(b) \leq f(a))$, then $[f(a), f(b)] \subset f([a, b])$ $([f(b), f(a)] \subset f([a, b]))$.

Differentiation

Let $I \subset \mathbb{R}$ be an open interval, and let $f : I \to \mathbb{R}$ be a function. It is *differentiable* at $x \in I$ if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists, in which case the limit is called the *derivative* of f at x, and is denoted by $f'(x)$ or $f^{(1)}(x)$. We say that f is *continuously differentiable* if it is differentiable and $f' : I \to \mathbb{R}$ is continuous. Inductively, for $k \geq 2$ we say that f is k *times continuously differentiable* if it is $k-1$ times continuously differentiable and $f^{(k-1)}$ is continuously differentiable, in which case its derivative is denoted by $f^{(k)}$. If f is k times continuously differentiable for every k it is C^{∞}.

The Chain Rule

Theorem: If $f : (a, b) \to (c, d)$ is differentiable at t and $g : (c, d) \to \mathbb{R}$ is differentiable at $f(t)$, then

$$(g \circ f)'(t) = g'(f(t)) \cdot f'(t).$$

Proof. Clearly $(g \circ f)'(t)$ and $f'(t)$ are zero if f is constant on a neighborhood of t, and otherwise

$$\lim_{h \to 0} \frac{g(f(t+h)) - g(f(t))}{h}$$
$$= \lim_{h \to 0} \frac{g(f(t+h)) - g(f(t))}{f(t+h) - f(t)} \cdot \frac{f(t+h) - f(t)}{h} = g'(f(t)) \cdot f'(t).$$

The Product Rule

Theorem: If $f, g : (a, b) \to \mathbb{R}$ are continuous and differentiable at t, then

$$(fg)'(t) = f'(t)g(t) + f(t)g'(t).$$

Proof. We have

$$\lim_{h\to 0} \frac{f(t+h)g(t+h) - f(t)g(t)}{h} = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \cdot g(t+h)$$

$$+ \lim_{h\to 0} f(t) \cdot \frac{g(t+h) - g(t)}{h}.$$

The Mean Value Theorem

Theorem: If $f : (a, b) \to \mathbb{R}$ is differentiable at t and $f(t) = \max_s f(s)$ or $f(t) = \min_s f(s)$, then $f'(t) = 0$.

Proof. If $f(t) = \max_s f(s)$, then

$$\lim_{h \to 0^-} \frac{f(t+h) - f(t)}{h} \geq 0 \geq \lim_{h \to 0^+} \frac{f(t+h) - f(t)}{h}.$$

The proof when $f(t) = \min_s f(s)$ is similar.

Rolle's Theorem

Rolle's Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous and differentiable at every point in (a, b), and $f(a) = f(b)$, then there is a $t \in (a, b)$ such that $f'(t) = 0$.

Proof. Since $[a, b]$ is compact there are t_-, t_+ such that $f(t_-) = \min_s f(s)$ and $f(t_+) = \max_s f(s)$. The last result gives the desired conclusion when $t_- \in (a, b)$ or $t_+ \in (a, b)$. If $t_-, t_+ \in \{a, b\}$, then $f(t) = f(a)$ for all t and $f'(t) = 0$ for all $t \in (a, b)$.

In applications the most convenient form of Rolle's theorem is:

Mean Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous and differentiable at every point in (a, b), then there is a $t \in (a, b)$ such that

$$f'(t) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let $g : [a, b] \to \mathbb{R}$ be the function

$$g(t) = f(t) - \frac{f(b) - f(a)}{b - a}(t - a).$$

Applying Rolle's theorem to g gives a t such that

$$0 = g'(t) = f'(t) - \frac{f(b) - f(a)}{b - a}.$$

Maximization

If $f : (a, b) \to \mathbb{R}$ is a function, a point t is a *strict local maximum* if there is an $\delta > 0$ such that $f(t) > f(s)$ for all $s \in (t - \delta, t) \cup (t, t + \delta)$.

Theorem: Let $f : (a, b) \to \mathbb{R}$ by C^2, and let t be a point in (a, b). If t is a local maximum, then $f'(0) = 0$ and $f''(0) \leq 0$. If $f'(0) = 0$ and $f''(t) < 0$, then t is a local maximum.

The Lagrangean

The following result may seem a bit silly, but the one dimensional case is our model for higher dimensional problems, so we should be sure to understand it.

Theorem: Suppose that $a < 0 < b$ and $u : (a, b) \to \mathbb{R}$ is C^1. Let

$$\mathcal{L}(t; \lambda) = u(t) + \lambda t.$$

If t^* solves that problem of maximizing $u(t)$ subject to $t \geq 0$, then there is a λ^* such that $\frac{\partial \mathcal{L}}{\partial t}(t^*; \lambda^*) = 0$. In addition, $\lambda^* \geq 0$ and $\lambda^* t^* = 0$. (This final condition is called *dual slackness*.)

Proof. If $t^* > 0$, then $u'(t^*) = 0$ and we can set $\lambda^* = 0$. If $t^* = 0$, then $u'(0) \leq 0$, and we can set $\lambda^* = -u'(0)$.

Convex and Concave Functions

Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is *convex* (*concave*) if, for all $s, t \in I$ and $\alpha \in [0, 1]$,

$$f((1-\alpha)s + \alpha t) \leq (\geq)(1-\alpha)f(s) + \alpha f(t).$$

We will only discuss convex functions, but everything we say applies equally to concave functions, with obvious modifications. We say that f is *strictly convex* if the inequality above holds strictly whenever $s \neq t$ and $\alpha \in (0, 1)$.

Theorem: Continuous Convex Function

Theorem: If $I \subset \mathbb{R}$ is an open interval and $f : I \to \mathbb{R}$ is convex, then f is continuous.

Proof. Fix $a, b \in I$ with $a < b$ and $\delta > 0$ such that $a - \delta, b + \delta \in I$. Let M and m be upper and lower bounds for f on $[a - \delta, b + \delta]$. If $a \leq s < t \leq b$, then

$$f(t) \leq \tfrac{\delta}{t-s+\delta} f(s) + \tfrac{t-s}{t-s+\delta} f(t+\delta) = f(s) + \tfrac{t-s}{t-s+\delta}(f(t+\delta) - f(s))$$

$$\leq f(s) + \tfrac{M-m}{\delta}(t-s)$$

and

$$f(s) \leq \tfrac{t-s}{t-s+\delta} f(s-\delta) + \tfrac{\delta}{t-s+\delta} f(t) = f(t) + \tfrac{t-s}{t-s+\delta}(f(s-\delta) - f(t))$$

$$\leq f(t) + \tfrac{M-m}{\delta}(t-s)$$

Therefore

$$|f(t) - f(s)| \leq \tfrac{M-m}{\delta}(t-s).$$

Lemma: Convex Function

Lemma: If $f : I \to \mathbb{R}$ is convex and $a, b \in I$, then there are m and M such that $m \leq f(t) \leq M$ for all $t \in [a, b]$.

Proof. The claim is trivial when $a = b$, so without loss of generality assume that $a < b$.

- Fix $s \in (a, b)$.
- For $t \in (a, s)$ we have $f(s) \leq \frac{b-s}{b-t} f(t) + \frac{s-t}{b-t} f(b)$ and thus $f(t) \geq \frac{b-t}{b-s} f(s) - \frac{s-t}{b-s} f(b)$.
- Similarly, for $t \in (s, b)$ we have $f(t) \geq \frac{t-a}{s-a} f(s) - \frac{t-s}{s-a} f(a)$.

Jensen's Inequality

Theorem: If $I \subset \mathbb{R}$ is an interval, $f : I \to \mathbb{R}$ is a convex function, $k \geq 2$ is an integer, $t_1, \ldots, t_k \in I$, $\alpha_1, \ldots, \alpha_k > 0$, and $\sum_i \alpha_i = 1$, then

$$f(\alpha_1 t_1 + \cdots + \alpha_k t_k) \leq \alpha_1 f(t_1) + \cdots + \alpha_k f(t_k).$$

Jensen's Inequality: Proof

Proof. We argue by induction on k. When $k = 2$ the asserted inequality is just the definition of convexity. Assuming the inequality has been established with $k-1$ in place of k, we apply the definition of convexity, then the induction hypothesis, to compute that

$$f(\alpha_1 t_1 + \cdots + \alpha_k t_k) = f((1-\alpha_k)(\tfrac{\alpha_1}{1-\alpha_k} t_1 + \cdots + \tfrac{\alpha_{k-1}}{1-\alpha_k} t_{k-1}) + \alpha_k t_k)$$

$$\leq (1-\alpha_k) f(\tfrac{\alpha_1}{1-\alpha_k} t_1 + \cdots + \tfrac{\alpha_{k-1}}{1-\alpha_k} t_{k-1}) + \alpha_k f(t_k)$$

$$\leq (1-\alpha_k)(\tfrac{\alpha_1}{1-\alpha_k} f(t_1) + \cdots + \tfrac{\alpha_{k-1}}{1-\alpha_k} f(t_{k-1})) + \alpha_k f(t_k) = \sum_i \alpha_i f(t_i).$$