ECON8000: Quantitative Skills for Economics
Lecture 2: Logic and Set Theory 2

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New Functions from Old

Definition

Informally, if $f : X \to Y$ and $g : Y \to Z$ are functions, their composition if the function $g \circ f : X \to Z$ taking each x to $g(f(x))$. Formally, if $f = (X, Y, G)$ and $g = (Y, Z, H)$ are functions, their composition is $g \circ f = (X, Z, J)$ where J is the set of $(x, z) \in X \times Z$ such that there is a $y \in Y$ such that $(x, y) \in G$ and $(y, z) \in H$.

Definition

If $f = (X, Y, G)$ is a function and $W \subset X$, the restriction of f to W is

$$f|_W = (W, Y, G \cap (W \times Y)).$$

That is, for each $w \in W$ we have $f|_W(w) = f(w)$.

Definition

If $f = (X, Y, G)$ and $f' = (X', Y', G')$ are functions, we can define a cartesian product function

$$f \times f' = (X \times X', Y \times Y', \{\, ((x, x'), (y, y')) : (x, y) \in G, (x', y') \in G' \,\}).$$

That is, $f \times f'$ takes (x, x') to $(f(x), f'(x'))$.

Definition

If $f = (X, Y, G)$ and $g = (X, Z, H)$ is a function, we can define a different cartesian product function $f \times g : X \to Y \times Z$ by setting $(f \times g)(x) = (f(x), g(x))$.

Exercise

If $f \times g = (X, Y \times Z, J)$, *what is* J*?*

Injections, Surjections, and Bijections

Definition

A function $f : X \to Y$ is said to be one-to-one, or injective, or an injection, if $f(x) \neq f(x')$ for all distinct $x, x' \in X$.

Example

Let $A := [0,1] \cup [3,4]$. If $f : A \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 2x & x \in [0,1], \\ 7 - x & x \in [3,4], \end{cases}$$

then f is one-to one.

Example

Every strictly increasing (or strictly decreasing) function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one —constant functions are not.

Definition

A function $f : X \to Y$ is said to be onto, or surjective, or a surjection, if $f(X) = Y$.

Definition

A function $f : X \to Y$ is said to be bijective, or a bijection (or a one-to-one correspondence) if $f^{-1}(y)$ is a singleton for every $y \in Y$ (in other words, if f is both onto and one-to-one).

Example

$f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3$

Definition

If $f = (X, Y, G)$ is a bijection, its inverse is

$$f^{-1} = (Y, X, \{ (y, x) : (x, y) \in G \}).$$

(The proper interpretation of the symbol f^{-1} must be inferred from context.)

Set Valued Functions and Correspondences

Definition

Formally a set-valued function is an ordered triple $F = (X, Y, G)$ in which X and Y are sets, $G \subset X \times Y$. Informally a set-valued function $F : X \to Y$ from X to Y is a rule that assigns a subset $F(x) = \{\, y : (x, y) \in G \,\}$ of Y to each $x \in X$. If $F(x) \neq \emptyset$ for all $x \in X$, then F is a correspondence.

Example

Let $F : [0, 1] \to [-1, 1]$ be $F(x) = \{\left(1 - x^2\right)^{1/2}, -\left(1 - x^2\right)^{1/2}\}$.

Example

If $f : X \to Y$ is a function, then we may regard $f^{-1} : Y \to X$ as a set valued function that is a correspondence if f is surjective.

Example

If $f : X \times Y \to \mathbb{R}$ is a function, then $F : X \to Y$ and $G : X \to Y$ given by

$$F(x) = \arg\min_{y \in Y} f(x, y) = \{\, y \in Y : f(x, y) \leq f(x, y') \text{ for all } y' \in Y \,\}$$

and

$$G(x) = \arg\max_{y \in Y} f(x, y) = \{\, y \in Y : f(x, y) \geq f(x, y') \text{ for all } y' \in Y \,\}$$

are set valued functions that are correspondences under certain conditions.

Finite, Countable, and Uncountable Sets

Definition

Two sets X and Y have the same cardinality if there is a bijection $b : X \to Y$. This is an equivalence relation. (That is, "have the same cardinality" is a reflexive, symmetric, transitive relation.) The cardinality of X is its equivalence class, which is denoted by $|X|$. If X is finite, we identify $|X|$ with its number of elements. We say that X is countably infinite if there exists a bijective function $b : X \to \mathbb{N}$. A set is countable if it is either finite or countably infinite, and otherwise it is uncountable.

Definition

A set X has at least as many elements as a set Y, written as $|X| \geq |Y|$, if there is an injection $f : Y \to X$.

A composition of two injections is an injection, so this is a transitive relation. A binary relation R is *complete* if, for any two objects X and Y in its domain, either XRY or YRX.

Countably Infinite Sets

Theorem: If $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$ are countably infinite, then $X \times Y$ is countably infinite.

Proof. We count $X \times Y$ by "snaking around:"
$(x_1, y_1) \to 1, (x_2, y_1) \to 2, (x_1, y_2) \to 3, (x_1, y_3) \to 4, (x_2, y_2) \to 5, (x_3, y_1) \to 6, \ldots.$

Some Countable and Uncountable Sets

Theorem: $\mathbb{R}$ is uncountable.

Proof. (Cantor's diagonal argument) Suppose that the interval $(0,1)$ is countable, so we can list its elements as $s_1, s_2, \ldots$. We construct a number $s = 0.d_1d_2\ldots$ by letting d_1 be an element of $\{1, \ldots, 8\}$ that is different from the first digit of s_1, letting d_2 be an element of $\{1, \ldots, 8\}$ that is different from the second digit of s_2, and so forth. Then s is an element of $(0,1)$ that is different from every s_i, which is a contradiction.

Intervals in $\mathbb{R}$

Some examples below will use intervals in $\mathbb{R}$, so we define them. Given $a, b \in \mathbb{R}$ such that $a < b$, we call a set described as follows

$$\{x \in \mathbb{R} : a < x < b\}$$

an open interval and we denote it by (a, b), i.e.,

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

Analogously define closed intervals, semi-open intervals:

$$[a, b]\,;\ (a, b]\,;\ \ [a, b)\,.$$

We may also consider sets such as
- $(a, \infty) := \{x \in \mathbb{R} : x > a\}$
- $[a, \infty) := \{x \in \mathbb{R} : x \geq a\}$
- $(-\infty, a) := \{x \in \mathbb{R} : x < a\}$
- $(-\infty, a] := \{x \in \mathbb{R} : x \leq a\}$

Countable and Uncountable Set Operations

Unions, intersections, and cartesian products can be taken over countable or uncountable (see below) index sets. They may be denoted by $\bigcup_{i=1}^{\infty} X_i$, $\bigcap_{i=1}^{\infty} X_i$, or $\prod_{i=1}^{\infty} X_i$ in the countable case, and by $\bigcup_{i \in I} X_i$, $\bigcap_{i \in I} X_i$, or $\prod_{i \in I} X_i$ when I might be uncountable.

Example

For $i = 1, \ldots, k$ let $S_i = [1/i, 1]$. We have the following finite union of sets:

$$A = \overset{k}{\underset{i=1}{\cup}} S_i, = S_1 \cup S_2 \cup \ldots \cup S_k = \{1\} \cup [1/2, 1] \ldots \cup [1/k, 1] = [1/k, 1].$$

Example

We may also consider a countably infinite union of sets:

$$B = \overset{\infty}{\underset{i=1}{\cup}} S_i, = \{1\} \cup [1/2, 1] \cup [1/3, 1] \cup \ldots = (0, 1].$$

Let $I = [2, 3]$, and for each $x \in I$ let $S_x := [1/x, 1]$. We may consider the uncountable unions of sets:

$$C = \underset{x \in I}{\cup} S_x, = [1/3, 1].$$

Intersections may also be taken over a finite, infinite, or uncountable collection of sets. Let $\widetilde{A} = \overset{k}{\underset{i=1}{\cap}} S_i$, $\widetilde{B} = \overset{\infty}{\underset{i=1}{\cap}} S_i$, and $\widetilde{C} = \underset{x \in I}{\cap} S_x$.

Practice problem: obtain $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$.

Practice problem: show that $\overset{\infty}{\underset{i=1}{\cap}} (0, 1/i) = \emptyset$ and $\overset{\infty}{\underset{i=1}{\cap}} (-1/i, 1/i) = \{0\}$.

Collection of Sets and Closure Under Operations

Sometimes we are interested in a certain class (or collection) of sets. For example, the collection of sets $\mathcal{C}$ is the collection of all finite semi-open intervals of real numbers which are closed on the right side, i.e.,

$$\mathcal{C} := \{(a, b] \subset \mathbb{R} : -\infty < a \leq b < \infty\}.$$

Note: the empty set is an element of $\mathcal{C}$; we obtain it when $a = b$.

Very often we are interested in whether an operation (and or repeated operations) of sets in a collection (union, intersection, etc.) always yields another element of that collection. If this is the case we say that such a collection is closed under that operation.

Example

$\mathcal{C}$ may be closed under finite intersection but not infinite intersection.
$\mathcal{C}$ may be closed under countable intersection, but not uncountable intersection.
$\mathcal{C}$ may be closed under arbitrary intersection.

Fields

In preparation for introducing the set of real numbers, we define a more general notion.

Definition

A field is an ordered triple $(k, +, \cdot)$ in which k is a set and $+ : k \times k \to k$ and $\cdot : k \times k \to k$ are functions (that we will write in the usual way) such that:

(a) $(x + y) + z = x + (y + z)$ for all $x, y, z \in k$.

(b) $x + y = y + x$ for all $x, y \in k$.

(c) there is a $0 \in k$ such that $0 + x = x$ for all $x \in k$.

(d) for each $x \in k$ there is a $-x \in k$ such that $x + (-x) = 0$.

(e) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in X$.

(f) $x \cdot y = y \cdot x$ for all $x, y \in k$.

(g) there is a $1 \in k \setminus \{0\}$ such that $1 \cdot x = x$ for all $x \in X$.

(h) for each $x \in X \setminus \{0\}$ there is an $x^{-1} \in X \setminus \{0\}$ such that $x \cdot x^{-1} = 1$.

Of course $\mathbb{Q}$ and $\mathbb{R}$ are fields, and the set $\mathbb{C}$ of complex numbers is another example you hopefully know already. A slightly more complicated example is $\mathbb{Q}(\sqrt{2}) = \{ r + s\sqrt{2} : r, s \in \mathbb{Q} \}$. (How does division work?)

Exercise

Let p be a prime number. Two integers a and b are congruent mod p if $a - b$ is divisible by p. Show that "congruent mod p" is an equivalence relation. Denoting the equivalence class of a by $[a]$, define addition and multiplication of equivalence classes by $[a] + [b] = [a + b]$ and $[a] \cdot [b] = [a \cdot b]$. Show that these operations are well defined in the sense that the definitions are independent of the choices of representatives. Show that with these operations $F_p = \{ [a] : a \in \mathbb{Z} \}$ is a field, called the integers mod p.

Ordered Fields

An ordered field is a quadruple $(F, +, \cdot, <)$ in which $(F, +, \cdot)$ is a field and $<$ is a binary relation on F such that:

- For each $x \in F$, exactly one of $x < 0$, $x = 0$, $0 < x$ is true.
- If $x, y > 0$, then $x + y > 0$ and $x \cdot y > 0$.
- If $x > y$, then $x + z > y + z$ for all z.

Evidently $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{R}$ are ordered fields, and $\mathbb{C}$ and F_p are not.

If $x < 0$, then $0 = x + (-x) < 0 + (-x) = -x$, so either $x \geq 0$ or $-x \geq 0$. We define $|x|$ to be x if $x \geq 0$ and $-x$ otherwise.

Of course these properties of the real numbers are familiar, and we are stretching things out a bit so as to not have one big long list of axioms for $\mathbb{R}$. What is possibly new are F_p and the terms "field" and "ordered field."

The Real Numbers

The real numbers is the unique (this needs to be proved) ordered field $(\mathbb{R}, +, \cdot, <)$ such that:

- (LUB Axiom) If a nonempty set $S \subset \mathbb{R}$ is bounded above, then it has a least upper bound: there is a b such that $b \geq s$ for all $s \in S$ and $b \leq b'$ for all b' such that $b' \geq s$ for all $s \in S$.
- (Archimedian Axiom) For any $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that $1/n < \varepsilon$.

Assume that we already know what $\mathbb{Q}$ is. A **Dedekind cut** is a partition $\{B, T\}$ of $\mathbb{Q}$ such that $b < t$ for all $b \in B$ and $t \in T$. One way to construct $\mathbb{R}$ is, roughly, to let it be the set of Dedekind cuts. (If $r \in \mathbb{Q}$, we need to let $((-\infty, r), [r, \infty))$ and $((-\infty, r], (r, \infty))$ represent the same number.) Think about how addition and multiplication of Dedekind cuts should be defined, and how to verify all the axioms.

Using the LUB Axiom

How can we use the LUB Axiom to prove that there is a unique $\sqrt{2} \in \mathbb{R}$ such that $\sqrt{2} > 0$ and $(\sqrt{2})^2 = 2$?

- Let $S = \{ t \in \mathbb{R} : t^2 < 2 \}$.
  - This is nonempty (e.g., $0 \in S$) and bounded above by (for example) 2.
- Let b be the least upper bound of S.
  - If $b^2 < 2$, then $(b + 1/n)^2 < 2$ for sufficiently large $n \in \mathbb{N}$, contradicting the assumption that b is an upper bound of S.
  - If $b^2 > 2$, then, for sufficiently large $n \in \mathbb{N}$, $(b - 1/n)^2 > 2 > s^2$ and thus $b - 1/n > s$, for all $s \in S$ contradicting the assumption that b is the LUB of S.
  - Thus $b^2 = 2$.
- If $0 < b' < b$, then ${b'}^2 < b^2$, and if $b < b'$, then $b^2 < {b'}^2$, so there is no other positive number whose square is 2.

The Archimedian axiom is much more subtle. What happens without it is studied in nonstandard analysis.

Infimum, Supremum, Minimum, Maximum

Let S be a nonempty subset of $\mathbb{R}$.

- The infimum of S, denoted by $\inf S$, is the greatest lower bound of S if S is bounded below, and otherwise it is $-\infty$.
- The supremum of S, denoted by $\sup S$, is the least upper bound of S if S is bounded above, and otherwise it is ∞.
- The minimum of S, denoted by $\min S$, is the greatest lower bound of S if S is bounded below and its greatest lower bound is an element of S, and otherwise it is not defined.
- The maximum of S, denoted by $\max S$, is the greatest lower bound of S if S is bounded below and its greatest lower bound is an element of S, and otherwise it is not defined.

When you write, for example, "Let $a = \min S$," you are both *asserting* that $\min S$ is defined and you are *defining* a to be that number.

$\mathbb{R}^n$

$\mathbb{R}^n$ is the set of ordered n-tuples of real numbers $x = (x_1, \ldots, x_n)$. There are the following operations:

- For $x, y \in \mathbb{R}^n$ we let $x + y = (x_1 + y_1, \ldots, x_n + y_n)$.
- For $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ we let $\alpha x = (\alpha x_1, \ldots, \alpha x_n)$.
- For $x, y \in \mathbb{R}^n$ we let $\langle x, y \rangle = x \cdot y = \sum_i x_i \cdot y_i$.

This is the domain of primary interest, but we study it from the perspective of more abstract structures such as vector spaces, inner product spaces, normed spaces, metric spaces, and topological spaces. This approach clarifies the underlying logic and points to many interesting additional structures and topics of inquiry.