ECON8000: Quantitative Skills for Economics
Lecture 10: Stationary Equilibrium

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Stationary Equilibrium

Definition. A Markovian strategy σ for the SDP is defined to be a strategy where for each T, σ_t depends on h_t only through t and the period-t state under h_t, $s_t[h_t]$.

Definition. A stationary strategy is a Markovian strategy $\{\pi_t\}$ which satisfies the further condition that $\pi_t = \pi_\tau = \pi$ for all t and τ.

Thus, in a stationary strategy, the action taken in any period t depends only on the state at the beginning of that period, and not even on the value of t. It is usual to denote such a strategy by $\pi^{(\infty)}$, but for notational simplicity we denote it by function π.

Definition. A stationary optimal strategy is a stationary strategy that is also an optimal strategy.

Assumptions

We have already assumed that:

1. r is bounded on $S \times A$
2. r is continuous on $S \times A$.
3. f is continuous on $S \times A$.
4. Φ is a continuous, compact-value correspondence on S.

Theorem. Suppose the SDP $\{S, A, \Phi, f, r\delta\}$ satisfies Assumptions 1 to 4. Then there exists a stationary optimal policy π^*. Furthermore, the value function $V = W(\pi^*)$ is continuous on S, and is the unique bounded function that satisfies the Bellman equation at each $s \in S$:

$$\begin{aligned} W(\pi^*)(s) &= \max_{a \in \Phi(s)} \{r(s,a) + \delta W(\pi^*)(f(s,a))\} \\ &= r(s, \pi^*(s)) + \delta W(\pi^*)[f(s, \pi^*(s))] \end{aligned}$$

Optimal Growth Model

1. There is a single good which may be consumed or invested.
2. The conversion of investment to output takes one period and is achieved through a production function $f : \mathbb{R}_+ \to \mathbb{R}_+$.
3. x_t denotes period-t investment
4. y_{t+1} denote the output available on period-$(t+1)$, given $f(x_t)$.
5. Agents beings with an initial endowment of $y = y_0 \in \mathbb{R}_{++}$.
6. In each period, agent observes available stock y_t and decides on the division of this stock between period-t consumption c_t and period-t investment x_t.
7. Consumption of c_t in period t gives utility $u(c_t)$: $u : \mathbb{R}_+ \to \mathbb{R}$.
8. Agent discounts future utility by discount factor $\delta \in [0, 1)$ and wishes to maximize total discounted utility from lifetime consumption.

Problem to Solve

Thus, the problem is to solve:

$$\max \sum_{t=0}^{\infty} \delta^t u(c_t)$$

$$s.t \quad y_0 = y$$

$$y_{t+1} = f(x_t) \quad t = 0, 1, 2 \ldots$$

$$c_t + x_t = y_t \quad t = 0, 1, 2 \ldots$$

$$c_t, x_t \geq 0 \quad t = 0, 1, 2 \ldots$$

Environment

- $S^* = \mathbb{R}_+$ is the state space.
- $A^* = \mathbb{R}_+$ action space.
- $\Phi(y) = [0, y]$ is the feasible action correspondence taking states $y \in S^*$ into the set of feasible action $[0, y] \subset A^*$ at y.
- $r(y, c) = u(c)$, the reward from taking action $c \in \Phi(y)$ at the state $y \in S^*$.
- $F(y, c) = f(y - c)$ is the transition function taking current state-action pairs (y, c) into future states $F(y, c)$.
- The tuple $\{S^*, A^*, \Phi, r, F, \delta\}$ now defines a stationary discounted programming problem, which represents the optimal growth model.

Existence of Optimal Strategies

- Notice that u may be unbounded on $\mathbb{R}_+$.
- Rather than imposing unboundedness, we consider more natural and plausible restriction which will ensure that we may restrict S^* and A^* to compact intervals in $\mathbb{R}_+$, thereby obtaining boundedness of u from its continuity.

Suppose that production function f satisfies the following conditions:

1. $f(0) = 0$ (no free production)
2. f is continuous and nondecreasing on $\mathbb{R}_+$ (continuity and monotonicity)
3. There is $\bar{x} > 0$ such that $f(x) \leq x \quad \forall x \geq \bar{x}$ (unproductivity at high investment levels)

1. $u : \mathbb{R}_+ \to \mathbb{R}$ is continuous on $\mathbb{R}_+$
2. The tuple $\{S, A, \Phi, r, F, \delta\}$ now meets the requisite compactness and continuity to guarantee the existence of an optimal strategy.

Theorem. There is a stationary optimal strategy $g : S \to A$ in the optimal growth problem under Assumptions 1 and 2. The value function V is continuous on S and satisfies the Bellman equation at each $y \in S$:

$$\begin{aligned} V(y) &= \max_{c \in [0,y]} \{u(c) + \delta V[f(y-c)]\} \\ &= u(g(y)) + \delta V[f(y - g(y))] \end{aligned}$$

Example

An economy is composed by a continuum of identical agents with preferences:

$$\mathbb{E}\left(\sum_{t=0}^{\infty}\beta^t u(c_t)\right)$$

where c_t is the level of consumption in t, with $u_c > 0$ and $u_{cc} < 0$. The consumption good (also used as capital asset) is produced according to the function $F(K_t, N_t)$, where K_t is capital, which depreciates at a rate $\delta < 1$, N_t is total employment and F exhibits constant returns to scale (Cobb-Douglas, $K^{1-\alpha} \cdot N^{\alpha}$). Each agent has an initial amount of capital K_0 and a unit of time to devote to work. Consumers are the owners of capital which lease it to the representative firm.

Example: Recursive Formulation

(a) Define the recursive problem of the central planner and derive the first order conditions.

Answer. $V(k) = \max_{0 \leq y \leq f(k)} u(f(k) - y) + \beta V(y)$.

The first order condition with respect to y is

$$u'(K(1-\delta) + F(K, N) - y) = \beta V'(y)$$

Example

Assume now that in the same economy a labour tax τ is imposed to provide liquidity to the investment I of a public good, g. This public good depreciates each period at a rate $\lambda < 1$ such that $g_t = g_t(1-\lambda) + I$. Agents value this public good, so the utility function per period is $u(c_t; g_t)$.

(b) Define the recursive problem of the central planer. The wage is given by w. (Hint:
$I = \tau w = \tau F_2(K, N) = \tau \alpha F(K, N)$ if Cobb-Douglas is assumed and $N = 1$).
Note: $F_2(K, 1) = \alpha(K^{1-\alpha} \cdot 1^{\alpha-1}) = \alpha \cdot K^{1-\alpha} = \alpha F(K, 1)$

(c) What happen if $\lambda = 1$. Is g still a state variable?

Answer (b)

Normalizing with $N = 1$, we have:

$$I = \tau w \rightarrow \tau F_2(K,1) = \tau \cdot \alpha F(K,1) \text{(assuming Cobb-Douglas)}$$

Therefore, the recursive problem of the central planner is defined as:

$$V(K,g) = \max_{y} \{ u\left(K(1-\delta) + F\left(K,1\right) - y - \tau \cdot \alpha F(K,1); g\right)$$

$$+\beta V(y,x)\}$$

$$s.t$$

$$x = g(1-\lambda) + \tau \cdot \alpha F(K,1)$$

Answer (c)

If $\lambda = 1$, then $x = \tau\alpha F(K, 1)$ and we don't need g to project x. However, g is still a state variable given that enters in the utility function and its value is determined by the level of capital in the last period.

Set 10-1 Question

Consider the following problem of an individual that maximize its utility subject to a quantity of resources, without discount factor.

$$\max \sum_{t=0}^{T} u(c_t) : 0 < T < \infty$$

such that:

$$\sum_{t=0}^{T} c_t = R_0 > 0$$

where R_0 is the amount of resources to consume in the economy. Assume that $u(c_t) = \sqrt{c_t}$.

(a) Express the dynamic problem at t. (Hint: $R_t = R_{t-1} - C_{t-1}$).

(b) Solve using the Dynamic Programming Algorithm.

Set 10-1 Answer (a)

R_t is our state variable in t. Therefore, dynamic problem at t can be expressed as:

$$J_t(R_t) = \max_{0 \leq c_t \leq R_t} \sqrt{c_t} + J_{t+1}(R_{t+1}) \quad s.t: R_t = R_{t-1} - C_{t-1}$$

Set 10-1 Answer (b)

Let's solve in T:

$$J_T(R_T) = \max_{0 \leq C_T \leq R_T} \sqrt{C_T}$$

Then: $C_T^* = R_T$ and $J_T(R_T) = \sqrt{C_T}$. Therefore, solving in $T-1$:

$$J_{T-1}(R_{T-1}) = \max_{0 \leq C_{T-1} \leq R_{T-1}} \sqrt{C_{T-1}} + J_T(R_T)$$

s.t

$$R_T = R_{T-1} - C_{T-1}$$

Plugging the restriction:

$$J_{T-1}(R_{T-1}) = \max_{0 \leq C_{T-1} \leq R_{T-1}} \sqrt{C_{T-1}} + J_T(R_{T-1} - C_{T-1})$$

Given that $J_T(R_T) = \sqrt{C_T}$,

$$J_{T-1}(R_{T-1}) = \max_{0 \leq C_{T-1} \leq R_{T-1}} \sqrt{C_{T-1}} + \sqrt{R_{T-1} - C_{T-1}}$$

FOC

$$\frac{1}{2\sqrt{C_{T-1}}} - \frac{1}{2\sqrt{R_{T-1} - C_{T-1}}} = 0 \rightarrow C^*_{T-1} = \frac{R_{T-1}}{2}$$

and therefore $J_{T-1}(R_{T-1}) = 2\sqrt{\frac{R_{T-1}}{2}}$.

Solving at $T - 2$

$$J_{T-2}(R_{T-2}) = \max_{0 \leq C_{T-2} \leq R_{T-2}} \sqrt{C_{T-2}} + J_{T-1}(R_{T-1})$$

s.t

$$R_{T-1} = R_{T-2} - C_{T-2}$$

Plugging the restriction:

$$J_{T-2}(R_{T-2}) = \max_{0 \leq C_{T-2} \leq R_{T-2}} \sqrt{C_{T-2}} + J_{T-1}(R_{T-2} - C_{T-2})$$

Given that $J_{T-1}(R_{T-1}) = 2\sqrt{\frac{R_{T-1}}{2}}$:

$$J_{T-2}(R_{T-2}) = \max_{0 \leq C_{T-2} \leq R_{T-2}} \sqrt{C_{T-2}} + 2\sqrt{\frac{R_{T-2} - C_{T-2}}{2}}.$$

FOC

Then

$$\frac{1}{2\sqrt{C_{T-2}}} - \frac{2}{2}\frac{1}{2}\sqrt{\frac{2}{R_{T-2} - C_{T-2}}} = 0 \to C^*_{T-2} = \frac{R_{T-2}}{3}$$

In conclusion,

$$\begin{aligned}
C^*_{T-2} &= \frac{R_{T-2}}{3} \\
C^*_{T-1} &= \frac{R_{T-1}}{2} = \frac{R_{T-2} - C^*_{T-2}}{2} \\
&= \frac{R_{T-2} - \frac{R_{T-2}}{3}}{2} = \frac{R_{T-2}}{3} \\
C^*_T &= R_T = R_{T-1} - C_{T-1} = R_{T-2} - C_{T-2} - C_{T-1} \\
&= R_{T-2} - \frac{R_{T-2}}{3} - \frac{R_{T-2}}{3} = \frac{R_{T-2}}{3}.
\end{aligned}$$

So in general, $C_0 = \ldots = C_T = \frac{R_0}{T+1}$.

Set 10-2 Question

Consider the following problem:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

such that $c_t + k_{t+1} \leq f(k_t)$, $c_t \geq 0$, $k_t \geq 0$ and k_0 fixed and known. Assume that $u' > 0$, $u'' < 0$, $f' > 0$, $f'' < 0$, $f(0) = 0$ and $\lim_{c\to 0} u'(c) = \lim_{k\to 0} f'(k) = \infty$. Additionally, U and f are bounded.

(a) Write Bellman equation.

(b) Show the existence and uniqueness of value function.

(c) Provide a strategy to find the solution.

Set 10-2 Answer (a)

$$V(k) = \max_{0 \leq y \leq f(k)} u(f(k) - y) + \beta V(y)$$

Set 10-2 Answer (b)

Let T an operator which maps from $B(X)$ to $B(X)$, where $B(X)$ is the set of all bounded functions, such that:

$$Th(k) = \max_{0 \leq y \leq f(k)} u(f(k) - y) + \beta h(y)$$

with Blackwell Theorem we can prove that T is a contraction:
Monotonicity Let $h_1(k) \geq h_2(k)$, $\forall k \geq 0$. Then:

$$Th_1(k) = \max_{0 \leq k' \leq f(k)} u(f(k) - k') + \beta h_1(k')$$

$$Th_2(k) = \max_{0 \leq k' \leq f(k)} u(f(k) - k') + \beta h_2(k')$$

Given that $h_1(k) \geq h_2(k)$ $\forall k \geq 0$, then $Th_1(k) \geq Th_2(k)$, $\forall k \geq 0$.

Blackwell's Condition: Monotonicity

Discount Let $a \geq 0$. Then:

$$T(h+a)(k) = \max_{0 \leq k' \leq f(k)} u(f(k) - k') + \beta[h(k') + a]$$

Therefore:

$$T(h+a)(k) = Th(k) + \beta a$$

and T is a contraction. By Contraction Mapping Theorem:

- T has a unique fixed point V
- Let $V_0 \in B(X)$, $d(V, T^n V_0) \leq \beta^n d(V_0, V)$.

Recursive Formulation

Making $TV_0 = V_1$, $TV_1 = V_2$,..., $TV_i = V_{i+1}$ and recall that:

$$Th(k) = \max_{0 \leq k' \leq f(k)} u(f(k) - k') + \beta h(k').$$

Then, let V the fixed point of T:

$$TV(k) = \max_{0 \leq k' \leq f(k)} u(f(k) - k') + \beta h(k') = V(k)$$

and the value function exists and is unique.

Set 10-2 Answer (c)

One possibility is to solve the problem with finite time and apply backward induction to find solutions. Then, analyse each solution when $T \to \infty$ and find a guess for the original problem. Then, prove if the guess is the solution in the infinite time, that is, test if the guess solves Bellman equation for finite horizon.

Set 10-3 Question

An agent that lives infinitely periods has to decide consumption (c_t) and savings s_t each period that maximizes.

$$\sum_{t=0}^{\infty} \beta^t c_t^{\alpha}$$

At $t = 0$, $s_0 > 0$ and for all the other periods, $s_t \geq 0$. The optimal value function is characterized by $V(s) = As^{\alpha}$ which satisfies Bellman equation. The rate of return of savings is $R = 1 + r$, where r is the interest rate.

(a) Show that the optimal solution is of the form $c_t = \phi s_t$. Find the value of ϕ.

(b) Find the value of A.

HINT: Remember that $s_{t+1} = (s_t - c_t)R$.

Set 10-3 Answer (a)

Given the form of the optimal value function, using the Bellman equation:

$$As^{\alpha} = \max_{c} c^{\alpha} + \beta A((s-c)R)^{\alpha}$$

FOC:

$$\alpha c^{\alpha-1} - \beta A \alpha((s-c)R)^{\alpha-1} R = 0$$

Therefore:

$$c = \left[\frac{R(\beta AR)^{1/(\alpha-1)}}{1 + R(\beta AR)^{1/(\alpha-1)}} \right] s$$

which implies that:

$$\phi = \left[\frac{R(\beta AR)^{1/(\alpha-1)}}{1 + R(\beta AR)^{1/(\alpha-1)}} \right]$$

Set 10-3 Answer (b)

Recall that:

$$As^{\alpha} = \max_{c} c^{\alpha} + \beta A((s-c)R)^{\alpha}$$

Then, using the fact that $c = \phi s$

$$As^{\alpha} = (\phi s)^{\alpha} + \beta A((s - \phi s)R)^{\alpha}$$

and:

$$A = \frac{\phi^{\alpha}}{1 - \beta((1-\phi)R)^{\alpha}}$$