ECON8000: Quantitative Skills for Economics
Lecture 8: Optimization 1

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The Lagrangean

Theorem: Suppose that $a < 0 < b$ and $u : (a, b) \to \mathbb{R}$ is C^1. Let

$$\mathcal{L}(t; \lambda) = u(t) + \lambda t.$$

If t^* solves that problem of maximizing $u(t)$ subject to $t \geq 0$, then there is a λ^* such that $\frac{\partial \mathcal{L}}{\partial t}(t^*; \lambda^*) = 0$. In addition, $\lambda^* \geq 0$ and $\lambda^* t^* = 0$. (This final condition is called *dual slackness*.)

Example 1

Suppose you have 80 million AUD to spend on equipment (K) and workforce (L) for a company that has the following production function $Q(K, L) = 50K^2\sqrt{L}$. Suppose that the price of product to be sell is equal to 1 as well as all the price of factors.

(a) Find how much you should spend in K and L.

(b) Calculate the Lagrange multiplier of the optimization problem.

(c) Using (b) estimate the change in the production if you budget to spend is 79 million AUD.

Answer (a)

The problem can be written as:

$$\max_{K,L} 50K^2\sqrt{L} \quad s.t: \quad K + L \leq 80$$

Then:

$$\mathcal{L} = 50K^2\sqrt{L} - \lambda_1(K + L - 80).$$

The Kuhn Tucker conditions are:

$$\begin{align}
\mathcal{L}_K = 0 &: 100K\sqrt{L} - \lambda_1 = 0 \tag{1}\\
\mathcal{L}_L = 0 &: 25\frac{K^2}{\sqrt{L}} - \lambda_1 = 0 \tag{2}\\
\mathcal{L}_{\lambda_1} \geq 0 &: \lambda_1(K + L - 80) = 0 \tag{3}\\
K + L &\leq 80. \tag{4}
\end{align}$$

Answer (a) Continued

Using (1) and (2):

$$4L = K \tag{5}$$

Hence, if $\lambda_1 = 0$ then using (1): $K = L = 0$.
If $\lambda_1 > 0$ then $K + L = 80$. Using (5):

$$L = 16 \quad K = 64$$

which clearly constitute the solution of the problem.

Answer (b) and (c)

By (1):

$$\lambda_1 = 100K\sqrt{L}$$

Using $K = 64$ and $L = 16$:

$$\lambda_1 = 25600.$$

By the setting of the Lagrangean given by $\mathcal{L} = Q(K, L) - \lambda_1(K + L - 80)$, denoted by the changes in the budget by Δ,

$$\mathrm{d}Q(K^*, L^*) = \lambda_1 \cdot \triangle = 25600 \cdot (79 - 80) = -25,600.$$

Example 2

Consider the following maximization problem:

$$\begin{aligned} \max_{x \in \mathbb{R}} \quad & 2x^3 - 3x^2 \\ s.t \quad & (3 - x)^3 \geq 0 \end{aligned}$$

(a) Find all solutions that satisfies the Kuhn Tucker conditions.

(b) Find the solution for the maximization problem.

(c) Are the Kuhn Tucker conditions necessaries for the optimal solution?

Answer (a)

The Lagrangean is given by

$$\mathcal{L} = 2x^3 - 3x^2 + \lambda(3 - x)^3.$$

The Kuhn Tucker conditions are:

$$6x^2 - 6x - 3\lambda(3 - x)^2 = 0 \tag{6}$$
$$(3 - x)^3 \geq 0 \tag{7}$$
$$\lambda(3 - x)^3 = 0 \tag{8}$$

To consider all $\lambda \in \mathbb{R}_+$:

- If $\lambda = 0$, then using (6): $x^2 - x = 0$ which solutions are $x_1 = 0$ and $x_2 = 1$.
- If $\lambda > 0$, using (8), $x = 3$. Replacing in (6) gives value different than zero, violating condition.
- Therefore, the solutions are $x_1 = 0$, $x_2 = 1$, $\lambda = 0$.

Answer (b) and (c)

- We have that candidates of optimal solution were 0, 1 and 3.
- Replacing in the objective function we find that its maximal value is 27 at $x = 3$, which constitutes the solution for this maximization problem.
- As we can notice, the Kuhn Tucker are not necessary conditions given the fact that the solution violates the Kuhn Tucker condition.

Motivation

- In many practical situations we have to make repeated decisions over time.
- For instance, we may have to decide every year how much money to spend and how much to save.
- While spending more money today may make us happier *today*, it will decrease how much we can spend *tomorrow*.
- Dynamic programming provides an approach to study how to balance these conflicting interests.

Finite-Horizon Dynamic Programming

Definition

A Finite Horizon (Markovian) Dynamic Programming Problem (FHDP) is defined by a tuple $\{S, A, T, (r_t, f_t, \Phi_t)_{t=1}^{T}\}$ where:

1. S is the state space of the problem, with generic element s.
2. A is the action space of the problem, with generic element a.
3. T, a positive integer, is the horizon of the problem.
4. For each $t \in \{1, \ldots, T\}$:
   (a) $r_t : S \times A \to \mathbb{R}$ is the period-t reward function
   (b) $f_t : S \times A \to S$ is the period-t transition function
   (c) $\Phi_t : S \to P(A)$ is the period-t feasible action correspondence.

The FHDP has a simple interpretation:

1. The decision-maker begins from some fixed initial state $s_1 = s \in S$.
2. The set of actions available to the decision-maker at this state is given by correspondence $\Phi_1(s_1) \subset A$.
3. When the decision-maker chooses an action $a_1 \in \Phi_1(s)$ two things happen:
   3.1 Decision-maker receives an immediate reward of $r_1(s_1, a_1)$.
   3.2 The state s_2 at the beginning of period 2 is realized as $s_2 = f_1(s_1, a_1)$
4. At this new state, the set of feasible actions is given by $\Phi_2(s_2) \subset A$
5. Then, when decision-maker choose an action $a_2 \in \Phi_2(s_2)$, receives a reward $r_2(s_2, a_2)$ and the period-3 state s_3 is realized as $s_3 = f_2(s_2, a_2)$.
6. Problem proceeds in this way till the terminal date T is reached.

The objective is to choose a plan for taking actions at each point in time in order to maximize the sum of the per-period rewards over the horizon of the model.

Example

$$\max \sum_{t=1}^{T} r_t(s_t, a_t)$$

$$s.t:$$

$$s_1 = s \in S$$

$$s_t = f_{t-1}(s_{t-1}, a_{t-1}), \quad t = 2, \ldots, T$$

$$a_t \in \Phi_t(s_t), \quad t = 1, \ldots, T$$

Histories, Strategies and the Value Function

- A t-history η_t is a vector $\{s_1, a_1, \ldots, s_{t-1}, a_{t-1}, s_t\}$ of the state s_τ in each period τ up to t, the action a_τ taken that period, and the period-t state s_t.
- Let $H_1 = S$, and for $t > 1$, let H_t denote the set of all possible t-histories η_t.
- Given a t-history η_t, we will denote by $s_t[\eta_t]$ the period-t state under the history η_t.
- A strategy σ, for the problem, is a sequence $\{\sigma_t\}_{t=1}^{T}$ where for each t, $\sigma_t : H_t \to A$ specifies the action $\sigma_t(\eta_t) \in \Phi_t(s_t[\eta_t])$ to be taken in period t as a function of the history $\eta_t \in H_t$ up to t
- The requirement that $\sigma_t(\eta_t)$ be an element of $\Phi_t(s_t[\eta_t])$ ensures that the feasibility of actions at all points is built into the definition of a strategy.
- Let Σ denote the set of all strategies σ for the problem.

Each strategy $\sigma \in \Sigma$ gives rise from each initial state $s \in S$ to a unique sequence of states and actions $\{s_t(\sigma, s), a_t(\sigma, s)\}$, and therefore, to a unique sequence of histories $\eta_t(\sigma, s)$, in the obvious recursive manner: we have $s_1(\sigma, s) = s$, and for $t = 1, \ldots, T$,

$$\eta_t(\sigma, s) = \{s_1(\sigma, s), a_1(\sigma, s), \ldots, s_t(\sigma, s)\}$$
$$a_t(\sigma, s) = \sigma_t[\eta_t(\sigma, s)]$$
$$s_{t+1}(\sigma, s) = f_t[s_t(\sigma, s), a_t(\sigma, s)]$$

Thus, given an initial state $s_1 = s \in S$, each strategy σ gives rise to a unique period-t reward $r_t(\sigma)(s)$ from s defined as:

$$r_t(\sigma)(s) = r_t[s_t(\sigma, s), a_t(\sigma, s)]$$

The total reward under σ from the initial state s, denoted $W(\sigma)(s)$, is, therefore, given by:

$$W(\sigma)(s) = \sum_{t=1}^{T} r_t(\sigma)(s)$$

Now, define the function $V : S \to \mathbb{R}$ by:

$$V(s) = \sup_{\sigma \in \Sigma} W(\sigma)(s)$$

which is called the value function of the problem. A strategy σ^* is said to be an optimal strategy if the payoff it generates from any initial state is the supremum over possible payoffs from that state, that is, if:

$$W(\sigma^*)(s) = V(s) \quad \forall s \in S$$

Markovian Strategies

Any τ-history $\eta_\tau = (s_1, a_1, \ldots, s_\tau)$ in any given FHDP $\{S, A, T, (r_t, \Phi_t, f_t)_{t=1}^T\}$ results in another FHDP, namely the $(T-\tau)$-period problem given by:

$$\{S, A, T-\tau, (r_t^*, \Phi_t^*, f_t^*)_{t=1}^{T-\tau}\}$$

whose initial state is $s = s_\tau$ and where, for $t = 1, \ldots, T-\tau$, we have:

$$\begin{aligned} r_t^*(s,a) &= r_{t+\tau}(s,a), \quad (s,a) \in S \times A \\ \Phi_t^*(s) &= \Phi_{t+\tau}(s), \quad s \in S \\ f_t^*(s,a) &= f_{t+\tau}(s,a), \quad (s,a) \in S \times A \end{aligned}$$

We shall call the problem $\{S, A, T-\tau, (r_t^*, \Phi_t^*, f_t^*)_{t=1}^{T-\tau}\}$ the $(T-\tau)$-period continuation problem, and, for notional simplicity, denote it by $\{S, A, T-\tau, (r_t, \Phi_t, f_t)_{t=\tau+1}^{T}\}$:

- All τ-histories η_τ that end in s_τ result in the same continuation $(T-\tau)$-period problem.
- Hence, at any point t in an FHDP, the current state s_t encapsulates all relevant information regarding continuation possibilities from period T onwards, such as the strategies that are feasible in the continuation and the consequent rewards that may be obtained.

Since continuation possibility from a state is not affected by how one arrives at that state, it appears intuitively plausible that there is no gain to be made by conditioning actions on anything more than just the value of the current state and the time period in which this state was reached. This lead us to the notion of a Markovian strategy.

A Markovian Strategy

Definition

A Markovian strategy is a strategy σ in which at each $t = 1, \ldots, T-1$, σ_t depends on the t-history η_t only through t and the value of the period-t state $s_t[\eta_t]$ under η_t.

Such a strategy can be represented simply by a sequence $\{g_1, \ldots, g_T\}$, where for each t, $g_t : S \to A$ specifies the action $g_t(s_t) \in \Phi_t(s_t)$ to be taken in period t, as a function of only the period-t state s_t.

Definition

If a Markovian strategy $\{g_1, \ldots, g_T\}$ is also an optimal strategy, it is called a Markovian optimal strategy.

Existence of an Optimal Strategy

Let a strategy $\sigma = \{\sigma_1, \ldots, \sigma_{t-1}, g_t, \ldots, g_T\}$ to represent the strategy in which the decision-maker acts according to the recommendations of σ for the first $t-1$ periods, and then switches to following the dictates of strategy $\{g_t, \ldots, g_T\}$. The key to proving the existence of an optimal strategy - indeed of a Markovian optimal strategy is the following lemma.

Lemma 1. Let $\sigma = (\sigma_1, \ldots, \sigma_T)$ be an optimal strategy for the FHDP $\{S, A, T, (r_t, \Phi_t, f_t)_{t=1}^T\}$. Suppose that for some $\tau \in \{1, \ldots, T\}$ the $(T - \tau + 1)$-period continuation problem $\{S, A, T - \tau + 1, (r_t, \Phi_t, f_t)_{t=1}^T\}$ admits a Markovian optimal strategy $\{g_\tau, \ldots, g_T\}$. Then the strategy

$$\{\sigma_1, \ldots, \sigma_{\tau-1}, g_\tau, \ldots, g_T\}$$

is an optimal strategy for the original problem.

The importance of Lemma 1 arises from the fact that it enables us to solve for an optimal strategy by the method of backwards induction. That is, we consider the one-period problem in which, from any $s \in S$ we solve:

$$\max r_T(s, a) \quad s.t. \quad a \in \Phi_T(s)$$

Let $g_T^*(s)$ be a solution to this problem at the state s. By Lemma 1, the solution g_T^* can be used in the last period of an optimal strategy, without changing the total rewards. Therefore:

1. We can find the actions in the two-period problem beginning at period-$(T-1)$ that will be optimal for the two-period problem.
2. An induction argument completes the construction of the optimal strategy.

Observation and Assumptions

Observation: Even the one-period problem will not have a solution unless minimal continuity and compactness conditions are met, then we have to impose the following assumptions:

- **Assumption 1** For each t, r_t is continuous and bounded on $S \times A$.
- **Assumption 2** For each t, f_t is continuous on $S \times A$
- **Assumption 3** For each t, Φ_t is continuous, compact-value correspondence on S.

Theorem: Bellman Equation

Theorem. Under assumptions 1 to 3, the dynamic programming problem admits a Markovian optimal strategy. The value function $V_t(\cdot)$ of the $(T - t + 1)$-period continuation problem satisfies for each $t \in \{1, \ldots, T\}$ and $s \in S$, the following condition, known as the Bellman Equation:

$$V_t(s) = \max_{a \in \Phi_t(s)} \{r_t(s, a) + V_{t+1}(f_t(s, a))\}$$

The Consumption-Savings Problem

1. Consumer faces a T-period planning horizon, where T is a finite positive integer.
2. Has an initial wealth of $w \in \mathbb{R}_+$.
3. If he begins the period with w_t and consumes c_t that period, his wealth at the beginning of the next period is $(w_t - c_t)(1 + r)$, where $r \geq 0$ is the interest rate.
4. Consumer utility: $u(c) = \sqrt{c}$

To apply the Theorem

Hence:

- State space S is the set of possible wealth level, which is $\mathbb{R}_+$.
- Action space A is the set of possible consumption levels, which is also $\mathbb{R}_+$.
- Reward function is defined by $r_t(w, c) = u(c)$.
- Transition function is defined by $f_t(w, c) = (w - c)(1 + r)$.
- Correspondence Φ_t is given by $\Phi_t(w) = [0, w]$, for all w.

For ease notation, $k \equiv (1+r)$. Then, in the last period, consumer solves:

$$\max_{c \in [0,w]} \sqrt{c}$$

- $\sqrt{c}$ is strictly increasing, then the unique solution at any w is consume everything.
- Thus, the unique optimal strategy for the one-period problem is $g_T(w) = w$ for all $w \in S$.
- and the one-period value function V_T is given by $V_T(w) = \sqrt{w} \quad w \in S$.

Now let's go to $T-1$. The action $g_{T-1}(w)$ is part of an optimal strategy for the two-period problem iff solves:

$$\max_{c\in[0,w]} \left\{\sqrt{c} + V_T[k(w-c)]\right\}.$$

Using previous result, $g_{T-1}(w)$ must solve:

$$\max_{c\in[0,w]} \left\{\sqrt{c} + \sqrt{k(w-c)}\right\}.$$

This is a strictly convex optimization problem and therefore FOC are sufficient and necessary. Deriving with respect to c gives the solution

$$g_{T-1}(w) = \frac{w}{1+k} \quad \forall w \in S.$$

Substituting this into the maximization problem:

$$V_{T-1}(w) = (1+k)^{1/2} w^{1/2} \quad w \in S.$$

Continuing with this procedure, a patten begins to emerge and we obtain:

$$V_t(w) = (1 + k + \ldots + k^{T-\tau})^{1/2} w^{1/2}$$

and

$$g_t(w) = \frac{w}{1 + k + \ldots + k^{T-t}}.$$

To check that these are correct expressions we use induction. Suppose that these are the forms of V_τ and g_τ for $\tau \in \{t+1, \ldots, T\}$.

V_t and g_t

We will show that V_t and g_t also have these forms.
In period t, at the state w, consumer solves:

$$\max_{c \in [0,w]} \{\sqrt{c} + V_{t+1}[k(w-c)]\}$$

where $V_{t+1}[k(w-c)] = (1 + k + \ldots + k^{T-t-1})^{1/2}[k(w-c)]^{1/2}$ by the induction hypothesis.
This is a strictly convex optimization problem so $g_{T-t}(w)$ is uniquely determined by the solution.
A simple calculation shows it is, in fact, given by:

$$g_t(w) \frac{w}{1 + k + \ldots + k^{T-t}} \quad w \in S$$

and substituting this back into the objective function of the problem, we obtain:

$$V_t(w) =)(l + k + \ldots + k^{T-t})^{1/2} w^{1/2}, \quad w \in S$$

and the induction step is complete.