ECON8000: Quantitative Skills for Economics
Lecture 7: Convexity

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Convex and Concave Functions

Let $U \subset V$ be convex and open, and let $f : U \to \mathbb{R}$ be a function. We say that f is *convex* if

$$f((1-t)x_0 + tx_1) \leq (1-t)f(x_0) + tf(x_1)$$

for all $x_0, x_1 \in U$ and $t \in [0, 1]$. We say that f is *concave* if

$$f((1-t)x_0 + tx_1) \geq (1-t)f(x_0) + tf(x_1)$$

for all $x_0, x_1 \in U$ and $t \in [0, 1]$.

As economists, we maximize utility or profits (engineers minimize loss) so we will focus on the concave case. Needless to say, everything we say pertains to convex functions if you turn it upside down.

Definition: Convex Combination

Let V be a finite dimensional inner product space. A *convex combination* of $x^1, \ldots, x^k \in V$ is a point of the form

$$\alpha_1 x^1 + \cdots + \alpha_k x^k$$

where $\alpha_1, \ldots, \alpha_k \in [0, 1]$ $(\alpha_1, \ldots, \alpha_k \in \mathbb{R})$ and $\sum_i \alpha_i = 1$.

Jensen's Inequality

Jensen's inequality is valid in the multidimensional case.

Theorem: If $U \subset V$ is open and $f : U \to \mathbb{R}$ is concave, then for any $x_1, \ldots, x_k \in U$ and $\alpha_1, \ldots, \alpha_k \in [0, 1]$ with $\sum_i \alpha_i = 1$, then

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) \geq \alpha_1 f(x_1) + \cdots \alpha_k f(x_k).$$

Proof. Since f is concave, its hypograph is convex, and consequently it contains

$$\alpha_1(x_1, f(x_1)) + \cdots + \alpha_k(x_k, f(x_k)) =$$

$$(\alpha_1 x_1 + \cdots + \alpha_k x_k, \alpha_1 f(x_1) + \cdots \alpha_k f(x_k)).$$

The definition of the hypograph now gives the desired inequality.

Definition: Convex Hull

- A set $S \subset V$ is *convex* if it contains all convex combinations of its elements.
- The *convex hull* of a set $S \subset V$ is the set of convex combinations of elements of S.

We think of forming the convex hull of S by wrapping it in cellophane.

Lemma: A set $C \subset V$ is convex if and only if $(1-t)x_0 + tx_1 \in C$ for all $x_0, x_1 \in C$ and all $t \in [0,1]$ ($t \in \mathbb{R}$).

Proof.

- The condition is clearly necessary. If it holds, then a convex combination

  $$\alpha_1 x^1 + \cdots + \alpha_k x^k = \alpha_1 x_1 + (1-\alpha_1)\left(\tfrac{\alpha_2}{1-\alpha_1}x^2 + \cdots + \tfrac{\alpha_k}{1-\alpha_1}x^k\right)$$

  of k points in C is contained in C because (by induction on k) it is a convex combination of a point in C and a convex combination of $k-1$ elements of C.
- (For this to work we need $\alpha_1 \neq 1$, but there is some i such that $\alpha_i \neq 1$, and we can attain $\alpha_1 \neq 1$ by reindexing.)

Corollary: If $\{C_i\}_{i \in I}$ is any collection of convex subsets of V, then $\bigcap_i C_i$ is convex.

An Elementary Result

If C and D are subsets of V, then their *Minkowski sum* is $A + B = \{ x + y : x \in C \text{ and } y \in D \}$. We often write $x + C$ in place of $\{x\} + C$.

Lemma: If C and D are convex subsets of V, then $C + D$ is convex.

Proof.

- If $x_0 + y_0, x_1 + y_1 \in C + D$ and $t \in [0, 1]$, then $(1 - t)x_0 + tx_1 \in C$ and $(1 - t)y_0 + ty_1 \in D$, by convexity.
- So, $(1-t)(x_0+y_0)+t(x_1+y_1) = (1-t)x_0+tx_1+(1-t)y_0+ty_1$ is an element of $C + D$.

Carathéodory's Theorem

Carathéodory's Theorem If V is n-dimensional and x is an element of the convex hull of $S \subset V$, then x is an convex combination of $n + 1$ elements of S.

Linear Independence, Spans, and Bases

There are three key concepts related to a vector space V:

- A set $S \subset V$ is linearly (in)dependent if there (do not) exist distinct $v_1, \ldots, v_k \in S$ and scalars $\alpha_1, \ldots, \alpha_k \in F$, not all of which are 0, such that $\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$.
- The span of a set $S \subset V$ is the set of all linear combinations $\alpha_1 v_1 + \cdots + \alpha_k v_k$, where $v_1, \ldots, v_k \in S$ and $\alpha_1, \ldots, \alpha_k \in F$.
- A set $B \subset V$ is a basis of V if it is linearly independent and its span is all of V.

Theorem on Linearly Independence

Theorem: If V is n-dimensional and $v_1, \ldots, v_n \in V$, then $v_1, \ldots, v_n$ are linearly independent if and only if they span V, so in either case $v_1, \ldots, v_n$ is a basis.

Proof

Proof.

- Let $x = \sum_{i=0}^{k} \alpha_i x_i$ be a minimal (with respect to k) representation of x as a convex combination of elements of S. Then all the α_i are positive.
- If $k > n$, then $x_1 - x_0, \ldots, x_k - x_0$ are linearly dependent, so $\beta_1(x_1 - x_0) + \cdots + \beta_k(x_k - x_0) = 0$ for some $\beta_1, \ldots, \beta_k$, not all of which are zero.
- Let $\beta_0 := -(\beta_1 + \cdots + \beta_k)$, so that $\sum_{i=0}^{k} \beta_i = 0$ and $\sum_{i=0}^{k} \beta_i x_i = 0$.
- There is some t such that $\alpha_i + t\beta_i = 0$ for some i, and of the various such t we take one of smallest absolute value.
- This insures that for other j, $\alpha_j + s\beta_j > 0$ for all s between 0 and t, so that $\alpha_j + t\beta_j \geq 0$.
- Now $x = \sum_{i=0}^{k} \alpha_i x_i$ and $\sum_{i=0}^{k} \beta_i x_i = 0$ imply $x = \sum_{i=0}^{k} (\alpha_i + t\beta_i) x_i$, which contradicts minimality because $\alpha_i + t\beta_i = 0$ for some i and we can reduce k.

The Separating Hyperplane Theorem

The separating hyperplane theorem is due to Minkowski.

Theorem: If $C \subset V$ is a closed convex set that does not contain the origin, then there is an $\nu \in V \setminus \{0\}$ and a number $c > 0$ such that $\langle \nu, x \rangle > c$ for all $x \in C$.

The Separating Hyperplane Theorem: Proof

Proof.

- Let ν be a point in C that is closer to the origin than any other.
- (Such an ν exists because the intersection of C with a sufficiently large closed ball centered at the origin is compact and nonempty, so it has a point that minimizes the distance to the origin.)
- Let $c := \|\nu\|^2/2$.
- Fix $x \in C$, and for $t \in [0, 1]$ let

  $$f(t) = \|(1-t)\nu + tx\|^2 = \langle \nu + t(x-\nu), \nu + t(x-\nu) \rangle.$$

  Then $f'(t) = 2\langle \nu, x-\nu \rangle + 2t\|x-\nu\|^2$.
- Since the line segment between ν and x is in C we have $f(0) \le f(t)$ for all t, so $f'(0) \ge 0$ and thus $\langle x, \nu \rangle \ge \|\nu\|^2 > c > 0$.

Two Fundamental Theorems of Welfare Economics

1. A competitive equilibrium allocation is Pareto efficient.
2. Out of all possible Pareto optimal outcomes, one can achieve any particular one by enacting a lump-sum wealth redistribution and then letting the market take over.

The Second Fundamental Theorem of Welfare Economics

- One of the best known applications of the separating hyperplane theorem in economics is the second fundamental theorem of welfare economics.
- We will study this in the context of an exchange economy. (There will be no production.)

Setting

- There is a finite positive number ℓ of goods or commodities, indexed by h.
- There is an aggregate endowment

$$\omega \in \mathbb{R}^{\ell}_{++} = \{ x \in \mathbb{R}^{\ell} : x_1, \ldots, x_\ell > 0 \}.$$

- There are m consumers indexed by i.
- We assume that for each i, any $x_i \in \mathbb{R}^{\ell}_{+} = \{ x \in \mathbb{R}^{\ell} : x_1, \ldots, x_\ell \geq 0 \}$ is a possible consumption bundles, and it is not possible to consume a negative quantity of any good.
- An m-tuple $x = (x_1, \ldots, x_m) \in (\mathbb{R}^{\ell}_{+})^m$ is an *allocation*.
- It is *feasible* if $\sum_i x_i \leq \omega$.

Preferences

- Each consumer i has a preference relation $\succeq_i$, which is a binary relation on $\mathbb{R}^\ell_+$ meaning "at least as good as for i."
- We write $x' \succ_i x$ to indicate that $x' \succeq_i x$ and not $x \succeq_i x'$, i.e., i strictly prefers x' to x, and $x' \sim_i x$ to indicate that $x' \succeq_i x$ and $x \succeq_i x'$, i.e., i is indifferent between x and x'.

Preferences Continued

We assume that $\succeq_i$ satisfies:

- (Transitivity) For all $x, x', x'' \in \mathbb{R}^\ell_+$, if $x' \succeq_i x$ and $x'' \succeq_i x'$, then $x'' \succeq_i x$.
- (Continuity) For each $x \in \mathbb{R}^\ell$, i's *strict upper contour set* $U_i(x) = \{\, x' \in \mathbb{R}^\ell_+ : x' \succ_i x \,\}$ is (relatively) open.
- (Continuity) Preferences are called to be *continuous* if for any $x \in X_i$, the sets $\{y \in X_i : x \succsim_i y\}$ and $\{y \in X_i : y \succsim_i x\}$ are closed.
- (Strong Monotonicity) For all $x, x' \in \mathbb{R}^\ell_+$, if $x' \geq x$ and $x'_h > x_h$ for some h, then $x' \succ_i x$.
- (Convexity) For all $x \in \mathbb{R}^\ell$, $U_i(x)$ is convex.

The preference relation is *complete* if, for all $x, x' \in \mathbb{R}^\ell_+$, either $x \succeq_i x'$ or $x' \succeq_i x$.

Monotonicity and Locally Non satiated Preference

- An agent's preferences are said to be weakly monotonic if, given a consumption bundle x, the agent prefers all consumption bundles y that have more of all goods. That is, $y \gg x$ implies $y \succ x$.
- An agent's preferences are said to be strongly monotonic if, given a consumption bundle x, the agent prefers all consumption bundles y that have more of at least one good, and not less in any other good. That is, $y \geq x$ and $y \neq x$ imply $y \succ x$.
- An agent's preferences are said to be locally non satiated if when X is the consumption set, then for any $x \in X$ and every $\varepsilon > 0$, there exists a $y \in X$ such that $\|y - x\| \leq \varepsilon$ and y is preferred to x.

Pareto Optimality

- If x and x' are (not necessarily feasible) allocations, x' is a *weak Pareto improvement* of x if $x_i' \succeq_i x_i$ for all i and $x_i' \succ_i x_i$ for some i.
- A feasible allocation x is (strongly) *Pareto optimal* if there is no feasible weak Pareto improvement of x.

Price Equilibrium

A feasible allocation x^* and a price vector $p^* \in \mathbb{R}^\ell_+ \setminus \{0\}$ constitute a *price equilibrium with transfers* if there exist nonnegative wealth levels $w_1, \ldots, w_m$ such that $\sum_i w_i = p^* \cdot \omega$ and each i is maximizing her preference subject to her budget constraint in the sense that $p^* \cdot x_i > w_i$ for all $x_i \in \mathbb{R}^\ell_+$ such that $x_i \succ_i x_i^*$.

Quasi-equilibrium with transfers

A feasible allocation x^* and a price vector $p^* \in \mathbb{R}^\ell_+ \setminus \{0\}$ constitute a *quasi-equilibrium with transfers* if there exist nonnegative wealth levels $w_1, \ldots, w_m$ such that $\sum_i w_i = p^* \cdot \omega$ and for each i, $p^* \cdot x_i \geq w_i$ for all $x_i \in \mathbb{R}^\ell_+$ such that $x_i \succ_i x_i^*$.
In a quasi-equilibrium it is possible that a consumer could buy a strictly better bundle if it would exhaust her income.

Second Fundamental Theorem of Welfare Economics

Second Fundamental Theorem of Welfare Economics: If $x^* \in \mathbb{R}^n$ is a Pareto optimal allocation, then there is a $p^* \in \mathbb{R}^\ell_{++}$ such that p^* and x^* constitute a quasi-equilibrium.

Proof

- Feasibility gives $\sum_i x_i^* \leq \omega$, and in fact this must hold with equality because otherwise (by strong monotonicity) it could not be Pareto optimal.
- Let $V = \sum_i U_i(x_i^*)$. This is a sum of open convex sets, so it is open and convex.
- Since x^* is Pareto optimal, V cannot contain ω, so the separating hyperplane theorem gives a $p^* \in \mathbb{R}^\ell$ such that $p^* \cdot x > p^* \cdot \omega$ for all $x \in V$.
- All the components of p^* must be nonnegative because for any good with a negative price, adding enough of that good to each x_i^*, then adding these bundles, would give a point in V that cost less than ω.
- This implies that $p^* \cdot \omega \geq p^* \cdot \sum_i x_i^*$.

Proof Continued

- If x is an allocation such that $x_i \succeq_i x_i^*$ for all i, then (by monotonicity) there are allocations x' arbitrarily close to x such that $\sum_i x_i' \in V$, and $p^* \cdot x' > p^* \cdot \omega$.
- Thus, we have $p^* \cdot \sum_i x_i \geq p^* \cdot \omega$.
- If, for some i, $x_i \succ_i x_i^*$, then $p^* \cdot (x_i + \sum_{j \neq i} x_j^*) \geq p^* \cdot \omega \geq p^* \cdot (x_i^* + \sum_{j \neq i} x_j^*)$.
- So $p^* \cdot x_i \geq p^* \cdot x_i^*$.
- Thus p^* and x^* constitute a price quasi-equilibrium.

To complete

- Suppose $x_i \succ_i x_i^*$. We now turn to conditions under which $p \cdot x_i \geq w_i$ implies $p \cdot x_i > w_i$.
- For this to be true we need now to assume that the consumption set X is convex and the preference relation $\succsim_i$ is continuous.
- Then, if there exists a consumption vector x_i' such that $x_i' \in X_i$ and $p \cdot x_i' < w_i$, a quasi-equilibrium is a price equilibrium.

General Setting

At each $\theta \in \Theta$,

$$\text{Maximize } f(x) \text{ subject to } x \in \mathcal{D}(\theta).$$

- Θ: Parameter Space
- f: Objective Function
- $\mathcal{D}$: Constraint Set

Let Θ and S be subsets of $\mathbb{R}^\ell$ and $\mathbb{R}^n$, respectively. Let $P(S)$ denotes the power set of S, that is, the set of all nonempty subsets of S.

USC and LSC

- A correspondence $\phi : \Theta \to P(S)$ is said to be upper semicontinuous (hereafter usc) at a point $\theta \in \Theta$ if for all open sets V such that $\phi(\theta) \subset V$, there exists an open set U containing θ such that $\theta' \in U \cap \Theta$ implies $\phi(\theta') \subset V$. We say that ϕ is usc on Θ if ϕ is usc at each $\theta \in \Theta$.
- A correspondence $\phi : \Theta \to P(S)$ is said to be lower semicontinuous (hereafter lsc) at a point $\theta \in \Theta$ if for all open sets V such that $\phi(\theta) \cap V \neq \emptyset$, there exists an open set U containing θ such that $\theta' \in U \cap \Theta$ implies $V \cap \phi(\theta') \neq \emptyset$. We say that ϕ is lsc on Θ if ϕ is lsc at each $\theta \in \Theta$.
- A correspondence $\phi : \Theta \to P(S)$ is said to be continuous at $\theta \in \Theta$ if Φ is both usc and lsc at θ. The correspondence ϕ is continuous on Θ if it is continuous at each $\theta \in \Theta$.

Correspondence and Graph

Let X be a subset of $\mathbb{R}^\ell$ and P be a subset of $\mathbb{R}$. A correspondence $\Phi : X \to P$ is defined to be XXX-valued if for all $x \in X$, $\Phi(x)$ is XXX.

For example, XXX can be compact, single, convex, and non empty.

Given a mapping $f : X \to Y$, the graph of the mapping is the set

$$G(f) = \{(x, f(x)) : x \in X\},$$

which is a subset of $X \times Y$.

Maximum Theorem

Theorem. Let $f : S \times \Theta \to \mathbb{R}$ be a continuous function and $\mathcal{D} : \Theta \to P(S)$ be a compact-valued, continuous correspondence. Let $f^* : \Theta \to \mathbb{R}$ and $\mathcal{D}^* : \Theta \to P(S)$ be defined by

$$\begin{aligned} f^*(\theta) &= \max\{f(x,\theta) : x \in \mathcal{D}(\theta)\} \\ \mathcal{D}^*(\theta) &= \arg\max\{f(x,\theta) : x \in \mathcal{D}(\theta)\} \\ &= \{x \in \mathcal{D}(\theta) : f(x,\theta) = f^*(\theta)\}. \end{aligned}$$

Then f^* is a continuous function on Θ and $\mathcal{D}^*$ is a compact valued, usc correspondence on Θ.

Maximum Theorem under Convexity

Further,

- If f is concave on $S \times \Theta$ and $\mathcal{D}$ has a convex graph, then f^* is a concave function and $\mathcal{D}^*$ is a convex valued usc correspondence.
- If f is strictly concave, then f^* is also strictly concave, and $\mathcal{D}^*$ is a continuous function.

Application in Economics

- There are ℓ commodities.
- We assume that the price of each commodity is always strictly positive, and so $p \in \mathbb{R}^{\ell}_{++}$.
- We also assume that income is strictly positive, and so $I \in \mathbb{R}_{++}$.
- Let $\Theta = \mathbb{R}^{\ell}_{++} \times \mathbb{R}_{++}$.

Budget Correspondence

The budget correspondence $\mathcal{B} : \Theta \to P(S)$ is given by:

$$\mathcal{B}(p, I) = \{x \in \mathbb{R}^{\ell}_{+} : p \cdot x \leq I\}.$$

Define the indirect utility function v and the demand correspondence x by

$$v(p, I) = \max\{u(x) : x \in \mathcal{B}(p, I)\}$$
$$x(p, I) = \{x \in \mathcal{B}(p, I) : u(x) = v(p, I)\}.$$

Theorem on Budget Correspondence

Theorem. The correspondence $\mathcal{B} : \Theta \to P(S)$ is a continuous, compact-valued and convex-valued correspondence.
Proof. Tutorial Submission 9.

Theorem on Demand Correspondence

Theorem. The indirect utility function v and the demand correspondence x satisfy the following properties on Θ:

1. v is a continuous function on Θ and x is a compact-valued usc correspondence on Θ;
2. if u is strictly concave, then x is a continuous function.

Proof: Demand Correspondence

- By assumption, u is continuous on $\Theta \times S$. We are done with the first statement by the Maximum Theorem.
- A single-valued continuous correspondence is a continuous function.
- Suppose u is strictly concave. Since $\mathcal{B}(p, I)$ is a convex set, $x(p, I)$ is single-valued for each (p, I) and is thus a continuous function.

Kakutani's Fixed Point Theorem

Theorem. Let $X \subset \mathbb{R}^n$ be compact and convex. If $\Phi : X \to P(X)$ is a usc correspondence that has nonempty, compact and convex values, then Φ has a fixed point.

The Notation for Normal Form Games

- The set of *agents* is $I = \{1, \ldots, n\}$.
- An *n-person normal form game* is a $2n$-tuple

$$N = \langle S_1, \ldots, S_n; u_1, \ldots, u_n \rangle.$$

  - Each S_i is a nonempty finite set of *pure strategies* for agent i.
  - Let $S = \prod_{i \in I} S_i$ be the set of *pure strategy profiles.*
  - $u_i : S \to \mathbb{R}$ is agent i's *utility* or *payoff function.*
- We adopt the following notation for "everyone but i."
  - $S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$.
  - If $s_i \in S_i$ and $s_{-i} \in S_{-i}$, then (s_i, s_{-i}) is the obvious strategy profile.

Mixed Strategies and Expected Payoffs

- A *mixed strategy* for an agent is a probability distribution over his pure strategies.
- Typically we denote a mixed strategy for i by $\sigma_i \in \Delta(S_i)$.
- $\Sigma = \Delta(S_1) \times \cdots \times \Delta(S_n)$ is the space of *mixed strategy profiles.*
- Let $\Sigma_{-i} = \Delta(S_1) \times \cdots \times \Delta(S_{i-1}) \times \Delta(S_{i+1}) \times \cdots \times \Delta(S_n)$.
- If $\sigma_i \in \Delta(S_i)$ and $\sigma_{-i} \in \Sigma_{-i}$, then (σ_i, σ_{-i}) is the obvious mixed strategy profile.
- The *expected payoff* for agent i at a mixed strategy profile σ is

$$r_i(\sigma) = \sum_{s \in S} \left(\prod_{j \in I} \sigma_j(s_j) \right) \cdot u_i(s) = \sum_{s_i \in S_i} \sigma_i(s_i) \cdot u_i(s_i, \sigma_{-1}).$$

Best Response

- We say that σ_i is a *best response* for i to $\sigma \in \Sigma$ (actually to $\sigma_{-i} \in S_{-i}$) if $r_i(\sigma_i, \sigma_{-i}) \geq r_i(\sigma_i', \sigma_{-i})$ for all $\sigma_i' \in \Delta(S_i)$.
- For $\sigma \in \Sigma$ we let $BR(\sigma) = \prod_{i \in I} BR_i(\sigma)$.
- Formally define:

$$BR_i(\sigma) = \arg\max\{r_i(\hat{\sigma}_i, \sigma_{-i}) | \hat{\sigma}_i \in \Sigma_i\}.$$

- The correspondence

$$BR \colon \Sigma \to \Sigma$$

is called the *best response correspondence.*

Definition of Nash Equilibrium

Definition: A Nash equilibrium of the game N is a mixed strategy vector $\sigma^* \in \Sigma$ such that $\sigma^* \in BR(\sigma^*)$.

Existence of Nash Equilibrium

- Σ is nonempty, compact, and convex.
- Because r_i is continuous, by the Maximum Theorem, BR_i is a nonempty-valued, compact-valued and usc correspondence from Σ to Σ.
- Note that r_i is linear (thus concave) on Σ_i.
- By the Maximum Theorem under Convexity, for each $\sigma \in \Sigma$, $BR_i(\sigma)$ is convex, so $BR(\sigma)$ is convex since it is the cartesian product of convex sets.
- The existence of Nash equilibrium follows from Kakutani's fixed point theorem.