ECON8000: Quantitative Skills for Economics
Lecture 1: Logic and Set Theory 1

Shino Takayama

University of Queensland

February 2020

Lecture 1: Learning Objectives

- Understanding math logics
- Understanding important mathematics concept such as functions

Logical Propositions

A proposition is a claim saying that something may be true or false. Let's consider two propositions P and Q :

P : B is a subset of A and A is finite
Q : B is finite

Given elementary propositions $P, Q, R, \ldots$, compound propositions can be created by repeated applications of:

- *negation*: $\neg P$ means that P is false.
- *conjunction*: $P \wedge Q$ means that both P and Q are true.

In the usual way parentheses are used to remove ambiguities concerning the order in which the operations are applied. We convene that $\neg$ has precedence, so $\neg P \wedge Q$ means $(\neg P) \wedge Q$.

Two Abbreviations

There are two logical symbols that can be defined in terms of negation and conjunction:

- *disjunction*: $P \vee Q$ means $\neg(\neg P \wedge \neg Q)$, i.e., either P or Q is true. (This is "logical or," allowing both P and Q to be true.)
- *implication*: $P \Rightarrow Q$ means $\neg P \vee Q$, i.e., if P is true, then Q is also true, in short "P implies Q".

In the example
P : B is a subset of A and A is finite
Q : B is finite
we have $P \Rightarrow Q$ because if P holds, then Q holds as well.

There are many possible relationships between propositions and now we will describe the associated terminology of some of them.

Sufficient Conditions

Instead of saying that "P implies Q" we could say that "P is a sufficient condition for Q." Furthermore in this kind of propositions P is called the hypothesis of the proposition and Q is called the conclusion of the proposition.

In general $P \Rightarrow Q$, does not imply that $\neg P \Rightarrow \neg Q$.

Example

Let A, B, and C be three sets such that $C = A \cup B$. Consider the following propositions:
P' : A is infinite
Q' : C is infinite

It is clear that $P' \Rightarrow Q'$.

However if P' does not hold it can still happen that Q' does hold: A may be finite and B infinite, and therefore C would be infinite (thus Q' would hold although P' does not).

The Contrapositive of a Proposition

In the last example, if Q' does not hold, then P' does not hold either (if C is not infinite, then A cannot be infinite because $C = A \cup B$).

In general, if the proposition $(P \Rightarrow Q)$ holds, then the proposition

$$\text{not } Q \Rightarrow \text{ not } P$$

always holds and vice versa. This is why, if $(P \Rightarrow Q)$, then we say that "Q is a necessary condition for P". Or, using notation, (not Q) $\Rightarrow$ (not P). This last proposition is called the contrapositive of the proposition $(P \Rightarrow Q)$.

Note: This is useful when we want to prove propositions of the type $(P \Rightarrow Q)$ but it is easier to prove that (not Q) $\Rightarrow$ (not P). Using such argument is called proving by contradiction or proving using a contrapositive argument. This is illustrated in some examples below. We shall come back to this when we write down proofs.

The Converse of a Proposition

Definition

Let P and Q be two propositions, then the proposition $(Q \Rightarrow P)$ is called the converse of the proposition $(P \Rightarrow Q)$.

If a proposition $(P \Rightarrow Q)$ is true, the converse $(Q \Rightarrow P)$ may or may not be true.

Example

Consider the propositions
$P : x \in \mathbb{N}$
$Q : x \in \mathbb{R}$
Here $(P \Rightarrow Q)$ is true, but the converse $(Q \Rightarrow P)$ is not.

Equivalent Propositions

Definition

Let P and Q be two propositions such that both $P \Rightarrow Q$ and $Q \Rightarrow P$, then P is said to be a necessary and sufficient condition for Q. This is denoted by $P \Leftrightarrow Q$ or P *iff* Q.

Notice that if P is a necessary and sufficient condition for Q, then it also holds that Q is a necessary and sufficient condition for P. Then, P and Q are said to be logically equivalent.

Example

P:$A \cap B = \emptyset$ and Q:$B \subset A^c$. Propositions P and Q are equivalent.

Set Theory

We work with an informal and intuitive understanding of a set as a collection of objects, some of which may themselves be sets.

Example

$\{$students sitting in this class today$\} = \{$your names....$\}$

Important examples:
- the set of natural numbers, $\mathbb{N}$;
- the set of rational numbers, $\mathbb{Q}$;
- the set of integers, $\mathbb{Z}$;
- the set of real numbers, $\mathbb{R}$.

Example

An element of set may be a set, as in $\mathbb{R} \cup \{\mathbb{R}\}$.

Typically we are interested in sets such that all of their elements belong to a set that we are familiar with, and also satisfy some specific condition(s).

Example

$A = \{x \in \mathbb{R} : 0 \leq x \leq 2\}$

More generally, $A = \{x : P(x)\}$ is the set of x's that satisfy the property $P(x)$

In such a description of a set, $A = \{\cdots : \cdots\}$,

– on the left hand side of ":" we specify the general set, for example $\mathbb{R}$.

– on the right hand side of ":" we specify the specific condition required to be an element of A, for example, $0 \leq x \leq 2$.

Russell's Paradox

Russell's Paradox: Let P be the set whose elements are the sets S such that S is not an element of itself. Is P an element of itself?

Since the late 19^{th} century mathematicians have understood all objects occurring in mathematics as being constructed using the operations of set theory. For example, we might define the ordered pair (a, b) to be $\{a, \{b\}\}$. Russell's paradox shows that some restrictions need to be imposed (e.g., setting $(a, b) = \{a, \{b\}\}$ wouldn't work if $a = \{b\}$). This is a complicated, tricky subject, so we will generally work with this point of view, but informally, counting on common sense to avoid weird exceptions.

Union, Intersection, and Set Difference

Definition

The union between two sets A and B is the set of elements of A and elements of B, and it is denoted by $A \cup B$.

In other words, $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
(here "or" includes the possibility that x is in both A and B).

Definition

The intersection between two sets A and B is the set of elements in both A and B, and it is denoted by $A \cap B$.

In other words, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Definition

Let S and X be two sets, then S minus X, denoted by $S \setminus X$ or $S - X$, is given by $S \setminus X = \{x : x \in S \text{ and } x \notin X\}$.

Let S and X be two sets such that $S \subset X$, then the complement of S (with respect to X) is the set of elements in X that are not in S and it is denoted by S^c.[1]

The concept of complement is more often used when it is well understood that we are working only with elements of X. For example, if we are working with real numbers; recall $A = \{x \in \mathbb{R} : 0 \leq x \leq 2\}$, then $A \subset \mathbb{R}$ and

$$A^c = \mathbb{R} \setminus A = \{x \in \mathbb{R} : x < 0 \text{ or } x > 2\}.$$

[1] Throughout these lectures $A \subset B$ includes the possibility that $A = B$.

De Morgan's Law

Theorem: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$
(Distributive Laws) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ and
$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

Cartesian Product

Definition

The Cartesian product of two sets X and Y is the set $X \times Y$ of all the ordered pairs (x, y) such that $x \in X$ and $y \in Y$.

(As we saw above, giving a formal definition of an ordered pair can be tricky, and we won't try. Use common sense.)

For example $\mathbb{R}^2$ is the Cartesian product $\mathbb{R} \times \mathbb{R}$.

$\mathbb{R}^n$, for $n = 3, 4, \ldots$ is defined in a similar manner:

$$\begin{aligned} \mathbb{R}^3 &= \mathbb{R}^2 \times \mathbb{R}, \\ \mathbb{R}^4 &= \mathbb{R}^3 \times \mathbb{R}, \ldots \end{aligned}$$

Therefore,

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R},\ i = 1, 2, \ldots, n\}.$$

If $\{X_1, X_2, ..., X_n\}$ is a collection of sets indexed with positive integers from 1 to n, the Cartesian product of this indexed collection of sets is denoted by

$$X_1 \times X_2 \times ... \times X_n$$

or $\underset{i=1}{\overset{n}{\times}} X_i$. Thus $\underset{i=1}{\overset{n}{\times}} X_i$ is the set of tuples $(x_1, x_2, ..., x_n)$ where $x_i \in X_i$ for $i = 1, 2, ..., n$. These tuples often are denoted by x, i.e.,

$$x = (x_1, x_2, ..., x_n)$$

where $x_i \in X_i$. Furthermore, x_i is called the i^{th} coordinate of the point x and X_i is called the i^{th} coordinate set. The set $\{1, 2, ..., n\}$ is said to be an index set.

Relations

Definition

Formally a relation (on a single set X) is an ordered pair (X, R) in which X is a set and $R \subset X \times X$.

We usually write R in place of (X, R) and xRy in place of $(x, y) \in R$. The relation R is:

- reflexive if xRx for all $x \in X$.
- symmetric if, for all $x, y \in X$, xRy if and only if yRx.
- transitive if xRz whenever xRy and yRz.

An equivalence relation is a relation that is reflexive, symmetric, and transitive. If R is an equivalence relation and $x \in X$, the equivalence class of x is $[x] = \{ x' \in X : xRx' \}$. A partition of X is a collection $\{S_i\}_{i \in I}$ of nonempty subsets of X such that $S_i \cap S_j = \emptyset$ for all distinct $i, j \in I$ and $\bigcup_i S_i = X$. If R is an equivalence relation, then its set of equivalence classes is a partition of X.

Functions

Definition

Formally a function is an ordered triple $f = (X, Y, G)$ in which X and Y are sets, $G \subset X \times Y$, and for each $x \in X$ there is exactly one $y \in Y$ (denoted by $f(x)$ and called the image of x under f) such that $(x, y) \in G$. Informally f is a rule assigning an element of Y to each $x \in X$.

- X is the domain of f.
- Y is the range of f.
- G is the graph of f.
- for any $A \subset X$, the image of A under f

$$f(A) = \{y \in Y : y = f(x) \text{ for some } x \in A\}.$$

- $f(X)$ is the image of f.
- for $B \subset Y$, $f^{-1}(B) = \{\, x \in X : f(x) \in B \,\}$
- for $y \in Y$, $f^{-1}(y) = \{\, x \in X : f(x) = y \,\}$ (except when f is bijective, as defined below).