Question 2.17

1 Preliminaries

Consider a Weiner process $B_t \in \mathbb{R}$.

Choose some partition of $[0, t]$ such that for some $n \in \mathbb{N}$,

$$0 = t_0 \le t_1 \le ... \le t_n = t$$

And define

$$\begin{aligned} \triangle t_k &= t_k - t_{k-1} \\ W_k &= B_{t_k} - B_{t_{k-1}} (= \triangle B_k) \end{aligned}$$

By the independent increment property of the Wiener process and the Guassian distribution of each increment, we have

$$W_k \sim N(0, \triangle t_k)$$

Such that

$$\mathbb{E}(W_k) = 0 \tag{1.1}$$

$$\mathbb{E}\left(W_k^2\right) = \triangle t_k \tag{1.2}$$

And since the Moment Generating Function for the distribution of W_k is

$$M(s) = \exp\left(\frac{\triangle t_k}{2} t^2\right)$$

We have

$$\mathbb{E}\left(W_k^4\right) = M^{(4)}(0) = 3(\triangle t_k)^2 \tag{1.3}$$

Furthermore,

$$\sum_{k=1}^{n} \mathbb{E}\left(W_k^2\right) = \sum_{k=1}^{n} \triangle t_k = t$$

2 Part (a)

2.1 We first want to show that $\mathbb{E}\left[\left(\left[\sum_{k=1}^{n} W_k^2\right]-t\right)^2\right]=2\sum_{k=1}^{n}\left(\triangle t_k\right)^2$.

Since $t=\sum_{k=1}^{n}\triangle t_k$, and $\mathbb{E}\left(W_k^2\right)=\triangle t_k$, we have

$$
\begin{aligned}
\mathbb{E}\left[\left(\left[\sum_{k=1}^{n} W_k^2\right]-t\right)^2\right] &= \mathbb{E}\left[\left(\sum_{k=1}^{n}\left[W_k^2-\triangle t_k\right]\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{k=1}^{n}\left[W_k^2-\mathbb{E}\left(W_k^2\right)\right]\right)^2\right] \\
&= \mathbb{E}\left[\sum_{k=1}^{n}\left(W_k^2-\mathbb{E}\left(W_k^2\right)\right)^2\right]+\mathbb{E}\left(\sum_{k=1}^{n}\sum_{j\neq k}^{n}\left[W_k^2-\mathbb{E}\left(W_k^2\right)\right]\left[W_j^2-\mathbb{E}\left(W_j^2\right)\right]\right)
\end{aligned}
$$

Claim: $\mathbb{E}\left(\sum_{k=1}^{n}\sum_{j\neq k}^{n}\left[W_k^2-\mathbb{E}\left(W_k^2\right)\right]\left[W_j^2-\mathbb{E}\left(W_j^2\right)\right]\right)=0$

Since for all $j\neq k$, W_k and W_j are independently determined random variables, it must also be the case that W_k^2 and W_j^2 are independently determined as well, in which case $\mathbb{E}\left(W_k^2W_j^2\right)=\mathbb{E}\left(W_j^2\right)\mathbb{E}\left(W_k^2\right)$, therefore

$$
\begin{aligned}
\mathbb{E}\left(\left[W_k^2-\mathbb{E}\left(W_k^2\right)\right]\left[W_j^2-\mathbb{E}\left(W_j^2\right)\right]\right) &= \mathbb{E}\left(W_k^2W_j^2-W_k^2\mathbb{E}\left(W_j^2\right)-W_j^2\mathbb{E}\left(W_k^2\right)+\mathbb{E}\left(W_k^2\right)\mathbb{E}\left(W_j^2\right)\right) \\
&= \mathbb{E}\left(W_k^2W_j^2\right)-2\mathbb{E}\left(W_j^2\right)\mathbb{E}\left(W_k^2\right)+\mathbb{E}\left(W_k^2\right)\mathbb{E}\left(W_j^2\right) \\
&= \mathbb{E}\left(W_j^2\right)\mathbb{E}\left(W_k^2\right)-\mathbb{E}\left(W_j^2\right)\mathbb{E}\left(W_k^2\right) \\
&= 0
\end{aligned}
$$

We would also then get

$$
\begin{aligned}
\mathbb{E}\left(\sum_{k=1}^{n}\sum_{j\neq k}^{n}\left[W_k^2-\mathbb{E}\left(W_k^2\right)\right]\left[W_j^2-\mathbb{E}\left(W_j^2\right)\right]\right) &= \sum_{k=1}^{n}\sum_{j\neq k}^{n}\mathbb{E}\left(\left[W_k^2-\mathbb{E}\left(W_k^2\right)\right]\left[W_j^2-\mathbb{E}\left(W_j^2\right)\right]\right) \\
&= \sum_{k=1}^{n}\sum_{j\neq k}^{n}0
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbb{E}\left[\left(\left[\sum_{k=1}^{n} W_k^2\right]-t\right)^2\right] &= \mathbb{E}\left[\sum_{k=1}^{n}\left(W_k^2-\mathbb{E}\left(W_k^2\right)\right)^2\right] \\
&= \sum_{k=1}^{n}\mathbb{E}\left[\left(W_k^2-\mathbb{E}\left(W_k^2\right)\right)^2\right] \\
&= \sum_{k=1}^{n}\left(\mathbb{E}\left[W_k^4\right]-\mathbb{E}\left(W_k^2\right)^2\right) \\
&= \sum_{k=1}^{n}\left(3\triangle t_k^2-\triangle t_k^2\right) \\
&= 2\sum_{k=1}^{n}\left(\triangle t_k\right)^2
\end{aligned}
$$

Let $Y^{(n)}(t,\omega) := \sum_{k=1}^{n} W_k^2(\omega)$. Then this result implies that

$$Var\left(Y^{(n)}(t,\cdot)\right) = 2\sum_{k=1}^{n}(\triangle t_k)^2 \approx O(\triangle t_k)$$

And

$$\lim_{\triangle t_k \to 0} Var\left(Y^{(n)}(t,\cdot)\right) = 0$$

2.2 Want to show that $\langle B,B\rangle_t^{(2)}(\omega) = t,\ \ a.s.$

Let $Y^{(n)}(t,\omega) := \sum_{k=1}^{n} W_k^2(\omega)$, taking expectations on both sides,

$$\mathbb{E}\left(Y^{(n)}(t,\cdot)\right) = \sum_{k=1}^{n}\mathbb{E}(W_k) = t$$

And from above result, we have

$$\lim_{\triangle t_k \to 0} Var\left(Y^{(n)}(t,\cdot)\right) = 0$$

Therefore as the mesh of the partition approaches 0, i.e. as $\max_k \triangle t_k \to 0$, we see that $Y(t,\omega)$ converges in distribution to a deterministic variable with value t. Therefore,

$$\langle B,B\rangle_t^{(2)}(\omega) = \lim_{\triangle t_k \to 0} Y^{(n)}(t,\omega) = t,\ \text{almost surely}$$

2.3 Part (b)

All continuous finite variation processes have zero quadratic variation.

If the quadratic variation of a continuous process is finite, then the total variation is not finite, hence infinite.

B_t is continuous, and has a finite quadratic variation of t.

Illustrative demonstration

$\langle B,B\rangle_t(\omega) = \lim_{\triangle t_k \to 0}\sum_{t_k \le t}|W_k|$.

$$\begin{aligned}\mathbb{E}(|W_k|) &= \int_{-\infty}^{\infty}\frac{|x|}{\sqrt{2\pi\triangle t_k}}\exp\left(-\frac{x^2}{2t}\right)dx \\ &= \sqrt{\frac{2\triangle t_k}{\pi}}\end{aligned}$$

$$Var(|W_k|) = \triangle t_k$$

Let $\triangle t_k = \frac{t}{n}$ for some $n \in \mathbb{N}$. As $\triangle t_k \to 0$, we have $n \to \infty$, and

$$\begin{aligned}\lim_{\triangle t_k \to 0}\sum_{t_k \le t}\mathbb{E}|W_k| &= \lim_{n\to\infty} n\cdot\mathbb{E}(|W_k|) \\ &= \lim_{n\to\infty}\sqrt{\frac{2}{\pi}}\sqrt{tn} \\ &= \infty\end{aligned}$$

$$\lim_{\triangle t_k \to 0} Var\left(\sum_{t_k \leq t} |W_k|\right) = t$$

We might (loosely) say that $\langle B, B\rangle_t(\omega)$ is a normally distributed (by CLT) stochastic variable with a very very large (infinite) expected value and variance that is very very small relative to the expected value (finite variance). The probability of $\langle B, B\rangle_t(\omega)$ taking on a small value (finite value) is vanishingly small.