The Ito Integral

Shino Takayama
School of Economics
University of Queensland

Construction of the Ito Integral

* Consider:

$$\frac{dN}{dt} = (r(t) + \text{"}noise\text{"})N(t)$$

* More generally:

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot \text{"}noise\text{"}$$

where b and σ are some given functions.

Case of 1-dimensional noise

- Let W_t be a stochastic process to represent the noise term. Then, we obtain:

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot W_t$$

- What property do we want for W_t? It would be nice if:

1. If $t_1 \neq t_2$, then W_{t_1} and W_{t_2} are independent.
2. $\{W_t\}$ is stationary, i.e. $\{W_{t_1+t}, \cdots, W_{t_k+t}\}$ does not depend on t.
3. $E[W_t] = 0$ for all t.

The difficulty

* There does not exist any "reasonable" stochastic process satisfying the first and second properties.
* Such a process cannot have continuous paths.
* If we require $E[W_t^2]=1$, then the function $(t,\omega)\rightarrow W_t(\omega)$ cannot even be measurable w.r.t. the σ-algebra $\mathcal{B}\times\mathcal{F}$, where $\mathcal{B}$ is the Borel σ-algebra on $[0,\infty]$.
* It is possible to represent W_t as a generalized stochastic process called the <u>white noise process</u>.

White Noise Process

* $\{\epsilon_t\}_t$ is a white noise process if

1. $E(\epsilon_t) = 0 \ \forall t$;
2. $E(\epsilon_t^2) = s^2 < \infty \ \forall t$; and
3. $E(\epsilon_t \epsilon_{t'}) = 0 \ \ \forall t \neq t'$;

where all those expectations are taken prior to t, t'.

It is possible to represent W_t as a generalized stochastic process, "the White noise process."

Approach from Discrete Version

* Rewrite

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot W_t$$

by replacing W_t by a proper stochastic process.

* To do so, we consider the discrete version

$$X_{k+1} - X_k = b(t_k, X_k)\Delta t_k + \sigma(t_k, X_k)W_k\Delta t_k$$

where $X_j = X(t_j)$, $W_k = W_{t_k}$, $\Delta t_k = t_{k+1} - t_k$.

* Regard $W_k\Delta t_k$ as $\Delta V_k = V_{t_{k+1}} - V_{t_k}$.
* V_t should have stationary independent increments with mean 0.
* The only such process with continuous paths is the Brownian motion B_t. So replace $W_k\Delta t_k$ by $\Delta B_k = B_{t_{k+1}} - B_{t_k}$.

From discrete to continuous time

- Sum up and then we obtain:

$$X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j)\Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta B_j$$

- If we can take $\Delta t_j \to 0$, then we obtain:

$$X_k = X_0 + \int_0^t b(s, X_s)ds + \boxed{\int_0^t \sigma(s, X_s)dB_s}$$

First Step to Ito Integral

* Suppose $0 \leq S < T$ and $f(t,\omega)$ is given. We want to define:

$$\int_S^T f(t,\omega)dB_t(\omega).$$

* First assume that f has the form

$$\phi(t,\omega) = \sum_{j\geq 0} e_j(\omega)\cdot\chi_{[\frac{j}{2^n},\frac{j+1}{2^n})}(t)$$

where χ denotes the characteristic (indicator) function and n is a natural number.

Ito and Stratonovich

- The Ito integral – $t_j^* = t_j$
  - We write it as

$$\int_S^T f(t,\omega)\,dB_t(\omega)$$

- The Stratonovich integral – $t_j^* = (t_j + t_{j+1})/2$

$$\int_S^T f(t,\omega)\circ dB_t(\omega)$$

Def: “adapted”

- Let $B_t(\omega)$ be n-dimensional Brownian motion. Then we define $\mathcal{F}_t = \mathcal{F}_t^{(n)}$ to be the σ-algebra generated by the random variables $\{B_i(s)\}_{1 \le i \le n, 0 \le s \le t}$.
- Let $\{\mathcal{N}_t\}_{t \ge 0}$ be an increasing family of σ-algebras of subsets of Ω. A process $g(t,\omega):[0,\infty)\times\Omega \to \Re^n$ is called $\mathcal{N}_t$-adapted if for each $t \ge 0$ the function

$$\omega \to g(t,\omega)$$

is $\mathcal{N}_t$-measurable.

Intuitive Understanding

* One often thinks of $\mathcal{F}_t$ as "the history of B_s up to time t."
* Intuitively, that h is $\mathcal{F}_t$-measurable means that the value of $h(\omega)$ can be decided from the value of $B_s(\omega)$ for $s < t$.
* For example,
  * $h_1(\omega) = B_{t/2}(\omega)$ is $\mathcal{F}_t$-measurable;
  * $h_2(\omega) = B_{2t}(\omega)$ is not $\mathcal{F}_t$ - measurable.
* Note that $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$ and that $\mathcal{F}_t \subset \mathcal{F}$ for all t.

The Class of Functions for Ito Integral

* Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions

$$f(t,\omega):[0,\infty)\times\Omega \to \Re$$

such that

1. $(t,\omega) \to f(t,\omega)$ is $\mathcal{B}\times\mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$.
2. $f(t,\omega)$ is $\mathcal{F}_t$-adapted.
3. $E[\int_S^T f(t,\omega)^2 dt] < \infty$.

Steps to the Ito integral

* We will define the Ito integral $\mathcal{I}(f)(\omega) = \int_S^T f(t,\omega)dB_t(\omega)$ where B_t is a Brownian motion.

The procedure consists of 3 steps:

1. Define $\mathcal{I}(\phi)$ for a simple class of functions ϕ ;
2. Show that each $f \in \mathcal{V}$ can be approximated by such ϕ;
3. Define $\int fdB$ as the limit of $\int \phi dB$ as $\phi \to f$.

We start with "elementary" functions.

Definition [Elementary Functions]. A function $\phi \in \mathcal{V}$ is called elementary if it has the form

$$\phi(t,\omega) = \sum_{j \geq 0} e_j(\omega) \cdot \chi_{[t_j, t_{j+1})}(t).$$

Lemma: The Ito isometry

* If $\phi(t,\omega)$ is bounded and elementary, then

$$E\left[\left(\int_s^t \phi(t,\omega)dB_t(\omega)\right)^2\right] = E\left[\int_s^t \phi(t,\omega)^2 dt\right].$$

Proof: The Ito isometry

* Proof:
* Let $\Delta B_j = B_{t_{j+1}} - B_{t_j}$. Then,

$$E\left[e_i e_j \Delta B_i \Delta B_j\right] = \begin{cases} 0 & \text{if } i \neq j; \\ E[e_j^2]\cdot(t_{j+1} - t_j) & \text{if } i = j \end{cases}.$$

Thus,

$$E\left[\left(\int_S^T \phi \, dB\right)^2\right] = \sum_i \sum_j E\left[e_i e_j \Delta B_i \Delta B_j\right]$$

$$= \sum_j E[e_j^2]\cdot(t_{j+1} - t_j) = E\left[\int_S^T \phi^2 \, dt\right].$$

Bounded Convergence Theorem

- Let $(X_1, X_2, \cdots)$ be a sequence in $\Re$ on a probability space $(\Omega, \mathcal{F}, P)$. Assume that $X = \lim_{n\to\infty} X_n$ exists almost surely.

- Suppose that there exists a finite constant M such that for all $n \geq 1$, $|X_n| \leq M$ almost surely.

- Then,
  - $E(|X|) \leq M$;
  - $\lim_{n\to\infty} E(X_n) = E(X)$; and
  - $\lim_{n\to\infty} E(|X - X_n|) = 0 \cdot$

Step 1: Elementary Functions

* Let $g \in \mathcal{V}$ be bounded and $g(\cdot,\omega)$ continuous for each ω. Then there exist elementary functions $\phi_n \in \mathcal{V}$ such that

$$E\left[\int_S^T (g-\phi_n)^2\,dt\right] \to 0 \quad \text{as } n \to \infty.$$

* Proof:

Define $\phi_n(t,\omega) = \sum_j g(t_j,\omega)\cdot\chi_{[t_j,t_{j+1})}(t)$. Then, ϕ_n is elementary.

Then,

$$\int_S^T (g-\phi_n)^2\,dt \to 0 \quad \text{as } n\to\infty, \text{for each } \omega,$$

because $g(\cdot,\omega)$ is continuous for each ω.

By bounded convergence theorem, the proof is complete.

Step 2: Bounded Functions

* Let $h \in \mathcal{V}$ be bounded. Then, there exist bounded functions $g_n \in \mathcal{V}$ such that $g_n(\cdot,\omega)$ is continuous for all ω and n, and

$$E\left[\int_S^T (h-\phi_n)^2\,dt\right] \to 0 \quad \text{as } n \to \infty.$$

* Proof:

Suppose $|h(t,\omega)| \le M$ for all (t,ω). For each n, let ψ_n be a non-negative, continuous function on $\Re$ such that

(i) $\psi_n(x) = 0$ for all $x \le -(1/n)$ and $x \ge 0$; and

(ii) $\int_{-\infty}^{\infty} \psi_n(x)\,dx = 1$.

Step 2 Continued

- Define
$$g_n(t,\omega) = \int_0^t \psi_n(s-t)h(s,\omega)ds.$$
- Then, $g_n(\cdot,\omega)$ is continuous for each ω and $|g_n(t,\omega)| \le M$.
- Since $h \in \mathcal{V}$, we can show that $g_n(t,\cdot)$ is $\mathcal{F}_t$-measurable.
- Moreover,
$$\int_S^T (g_n(s,\omega) - h(s,\omega))^2 ds \to 0 \text{ as } n \to \infty, \text{ for each } \omega$$

as $\{\psi_n\}_n$ constitutes an approximate identity.

- By Bounded convergence theorem, we obtain
$$E\left[\int_S^T (h(t,\omega) - g_n(t,\omega))^2 dt\right] \to 0 \text{ as } n \to \infty, \text{ for each } \omega.$$

Dominated Convergence Theorem

* Let $f_1, f_2, \cdots$ be measurable functions in $\Re$ on a measure space $(\Omega, \mathcal{F}, \mu)$.
* Suppose that g is a nonnegative measurable function defined on $(\Omega, \mathcal{F}, \mu)$ such that, for each n, $|f_n| \leq g$ a.e..
* If $\int g d\mu < \infty$, then

$$-\infty < \int (\liminf f_n) d\mu \leq \liminf \int f_n d\mu$$

$$\leq \limsup \int f_n d\mu \leq \int (\limsup f_n) d\mu$$

* If $f = \lim f_n$ exists almost everywhere, then
  * $\int |f| d\mu < \infty$;
  * $\lim \int f_n d\mu = \int f d\mu$; and
  * $\lim \int (|f - f_n|) d\mu = 0$.

Here all the limits are taken over n.

Step 3: A Sequence of Bounded Functions

- Let $f \in \mathcal{V}$. Then, there exists a sequence $\{h_n\} \subset \mathcal{V}$ such that h_n is bounded for each n and

$$E\left[\int_S^T (f - h_n)^2 dt\right] \to 0 \quad \text{as } n \to \infty.$$

- Proof:

Let

$$h_n(t,\omega) = \begin{cases} -n & \text{if } f(t,\omega) < -n; \\ f(t,\omega) & \text{if } -n \le f(t,\omega) \le n; \\ n & \text{if } f(t,\omega) > n. \end{cases}$$

The conclusion follows by dominated convergence theorem.

Formally complete the definition

- We are considering:

$$\mathcal{I}(f)(\omega) = \int_S^T f(t,\omega)dB_t(\omega) \text{ for } f \in \mathcal{V}.$$

- Let $f \in \mathcal{V}$. By Step 1-3, we can choose elementary functions ϕ_n such that

$$E\left[\int_S^T | f - \phi_n |^2 \, dt\right] \to 0 \quad \text{as } n \to \infty.$$

Then, define

$$\mathcal{I}(f)(\omega) := \int_S^T f(t,\omega)dB_t(\omega) = \lim_{n\to\infty}\int_S^T \phi_n(t,\omega)dB_t(\omega).$$

The limit exists as $\{\int_S^T \phi_n(t,\omega)dB_t\}$ forms a Cauchy sequence in $L^2(P)$ by the Ito isometry.

Definition: The Ito integral

* Let $f \in \mathcal{V}(S,T)$. Then the Ito integral of f from S to T is defined by

$$\int_S^T f(t,\omega)dB_t(\omega) = \lim_{n\to\infty}\int_S^T \phi_n(t,\omega)dB_t(\omega) .$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$E\left[\int_S^T \left(f(t,\omega) - \phi_n(t,\omega)\right)^2 dt\right] \to 0$$

as $n \to \infty$.

Two corollaries to the Ito isometry

Corollary 3.1.7: For all $f \in \mathcal{V}(S,T)$,

$$E\left[\left(\int_S^T f(t,\omega)dB_t\right)^2\right] = E\left[\int_S^T f^2(t,\omega)dB_t\right].$$

Corollary 3.1.8: If $f(t,\omega) \in \mathcal{V}(S,T)$ and $f_n(t,\omega) \in \mathcal{V}(S,T)$ for $n = 1, 2, \cdots$ and $E\left[\int_S^T \left(f_n(t,\omega) - f(t,\omega)\right)^2 dt\right] \to 0$ as $n \to \infty$, then

$$\int_S^T f_n(t,\omega)dB_t(\omega) \to \int_S^T f(t,\omega)dB_t(\omega) \quad \text{in } L^2(P) \text{ as } n \to \infty.$$

Example 3.1.9.

- Assume $B_0 = 0$. Then, $\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$.
- Proof:

Let $\phi_n(t,\omega) = \sum B_j(\omega) \cdot \chi_{[t_j, t_{j+1})}(s)$ where $B_j = B_{t_j}$. Then,

$$E\left[\int_0^t (\phi_n - B_s)^2 ds\right] = E\left[\sum_j \int_{t_j}^{t_{j+1}} (B_j - B_s)^2 ds\right]$$

$$= \sum_j \int_{t_j}^{t_{j+1}} (s - t_j) ds = \sum_j \frac{1}{2}(t_{j+1} - t_j)^2 \to 0$$

as $\Delta t_j \to 0$.

Example 3.1.9. continued

- By Corollary 3.1.8,

$$\int_0^t B_s dB_s = \lim_{\Delta t_j \to 0} \int_0^t \phi_n dB_s = \lim_{\Delta t_j \to 0} \sum_j B_j \Delta B_j .$$

Now, $\Delta(B_j^2) = B_{j+1}^2 - B_j^2 = (B_{j+1} - B_j)^2 + 2B_j(B_{j+1} - B_j)$

$$= (\Delta B_j)^2 + 2B_j \Delta B_j$$

Since $B_0 = 0$, we obtain

$$B_t^2 = \sum_j \Delta(B_j^2) = \sum_j (\Delta B_j)^2 + 2\sum_j B_j \Delta B_j$$; or

$$\sum_j B_j \Delta B_j = \frac{1}{2} B_t^2 - \frac{1}{2} \sum_j (\Delta B_j)^2$$

Since $\sum_j (\Delta B_j)^2 \to t$ in $L^2(P)$ as $\Delta t_j \to 0$, the result follows.

Theorem 3.2.1

* Let $f, g \in \mathcal{V}(0,T)$ and $0 \le S < U < T$. Then,

1. $\int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t$ for a.a. ω ;

2. $\int_S^T (cf + g) dB_t = c \cdot \int_S^T f dB_t + \int_S^T g dB_t$ for some c and a.a. ω ;

3. $E\left[\int_S^T f \, dB_t \right] = 0$;

4. $\int_S^T f dB_t$ is $\mathcal{F}_T$-measurable.

Def: Filtration and Martingale

- A filtration on $(\Omega, \mathcal{F})$ is a family $\mathcal{M} = \{\mathcal{M}_t\}_{t \geq 0}$ of σ-algebras with $\mathcal{M}_t \subset \mathcal{F}$ such that

$$0 \leq s < t \Rightarrow \mathcal{M}_s \subset \mathcal{M}_t .$$

- An n-dimensional stochastic process $\{M_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, P)$ is called a martingale with respect to a filtration $\{\mathcal{M}_t\}_{t \geq 0}$ and P if

(1) M_t is $\mathcal{M}_t$-measurable for all t;

(2) $E\left[|M_t|\right] < \infty$ for all t; and

(3) $E[M_s | \mathcal{M}_t] = M_t$ for all $s \geq t$.

Example 3.2.3.

* Brownian motion B_t in $\Re^n$ is a martingale w.r.t. the σ -algebras $\mathcal{F}_t$ generated by $\{B_s : s \le t\}$, because

$$E[|B_t|]^2 \le E[|B_t|^2] = |B_0|^2 + nt \ .$$

For $s \ge t$,

$$\begin{aligned} E[B_s \mid \mathcal{F}_t] &= E[B_s - B_t + B_t \mid \mathcal{F}_t] \\ &= E[B_s - B_t \mid \mathcal{F}_t] + E[B_t \mid \mathcal{F}_t] \\ &= 0 + B_t = B_t \qquad . \end{aligned}$$

Theorem 3.2.4. Doob's Martingale Inequality

* If M_t is a martingale such that $t \to M_t$ is continuous a.s., then for all $p \geq 1$, $T \geq 0$ and all $\lambda > 0$,

$$P[\sup_{0 \leq t \leq T} |M_t| \geq \lambda] \leq \frac{1}{\lambda^p} \cdot E[|M_T|^p].$$

We can use this inequality to prove that the Ito integral can be chosen to depend continuously on time.

Borel-Cantelli Lemma

* Let $(A_1, A_2, \cdots)$ be a sequence of events in a probability space $(\Omega, \mathcal{F}, P)$. Assume that for each $i \neq j$, the events A_i

and A_j are either negatively correlated or uncorrelated.

Then, if $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m) = 1$.

Theorem 3.2.5

- Let $f \in \mathcal{V}(S,T)$. There exists a t-continuous stochastic process J_t on $(\Omega, \mathcal{F}, P)$ such that

$$P\left[J_t = \int_0^t f dB \right] = 1 \text{ for all } t, 0 \le t \le T.$$

- Proof: Let $\phi_n = \phi_n(t,\omega) = \sum_j e_j(n) \chi_{[t_j^{(n)}, t_{j+1}^{(n)})}(t)$

be elementary functions such that

$$\lim_{n \to 0} E\left[\int_0^T (f - \phi_n)^2 dt \right] = 0.$$

Sketch of Proof: Step 2

* Let $\begin{cases} I_n(t,\omega) = \int_0^t \phi_n(s,\omega)dB_s(\omega); \\ I_t = I_t(t,\omega) = \int_0^t f(s,\omega)dB_s(\omega); 0 \le t \le T. \end{cases}$

Now, $I_n(\cdot,\omega)$ is continuous for all n.

Moreover, $I_n(t,\omega)$ is a martingale w.r.t. $\mathcal{F}_t$ for all n, because for $s > t$,

$$E[I_n(s,\omega) \mid \mathcal{F}_t] = E[(\int_0^t \phi_n dB + \int_0^s \phi_n dB) \mid \mathcal{F}_t]$$

$$= \int_0^t \phi_n dB + E\left[\sum_{t \le t_j^{(n)} \le t_{j+1}^{(n)} \le s} e_j^{(n)} \Delta B_j \middle| \mathcal{F}_t \right]$$; and

Because Continued

* $$E\left[\sum_{t \le t_j^{(n)} \le t_{j+1}^{(n)} \le s} e_j^{(n)} \Delta B_j \middle| \mathcal{F}_t \right] = \sum_j E[E[e_j^{(n)} \Delta B_j \mid \mathcal{F}_{t_j^{(n)}}] \mid \mathcal{F}_t]$$
  $$= \sum_j E[e_j^{(n)} E[\Delta B_j \mid \mathcal{F}_{t_j^{(n)}}] \mid \mathcal{F}_t] = 0.$$
* $I_n - I_m$ is also an $\mathcal{F}_t$-martingale.

Sketch of Proof: Step 3

* By Theorem 3.2.4,

$$P\left[\sup_{0\le t\le T}\left|I_n(t,\omega)-I_m(t,\omega)\right|>\epsilon\right]\le\frac{1}{\epsilon^2}\cdot E\left[\left|I_n(t,\omega)-I_m(t,\omega)\right|^2\right]$$

$$=\frac{1}{\epsilon^2}\cdot E\left[\int_0^T(\phi_n-\phi_m)^2\,ds\right]\to 0 \quad \text{as} \quad m,n\to\infty.$$

* Hence we may choose a subsequence $n_k\uparrow\infty$ s.t.

$$P\left[\sup_{0\le t\le T}\left|I_{n_{k+1}}(t,\omega)-I_{n_k}(t,\omega)\right|>\epsilon\right]<2^{-k}.$$

Sketch of Proof: Final Step

- By the Borel-Cantelli lemma,

$$P\left[\sup_{0\le t\le T}\left|I_{n_{k+1}}(t,\omega)-I_{n_k}(t,\omega)\right|>2^{-k}\ \text{for infinitely many } k\right]=0.$$

- For a.a. ω, there exists $k_1(\omega)$ such that

$$\sup_{0\le t\le T}\left|I_{n_{k+1}}(t,\omega)-I_{n_k}(t,\omega)\right|\le 2^{-k}\ \text{ for } k\ge k_1(\omega).$$

Corollary 3.2.6

* Let $f \in \mathcal{V}(S,T)$ for all T. Then,

$$M_t(\omega) = \int_0^t f(s,\omega)\,dB_s$$

is a martingale w.r.t. $\mathcal{F}_t$ and

$$P\left[\sup_{0\le t\le T} |M_t| \ge \lambda\right] \le \frac{1}{\lambda^2} \cdot E\left[\int_0^t f(s,\omega)^2\,ds\right]$$

for all $\lambda, T > 0$