The Ito Integral

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Two corollaries to the Ito isometry

Corollary 3.1.7: For all $f \in \mathcal{V}(S,T)$,

$$E\left[\left(\int_S^T f(t,\omega)\,dB_t\right)^2\right] = E\left[\int_S^T f^2(t,\omega)\,dB_t\right].$$

Corollary 3.1.8: If $f(t,\omega) \in \mathcal{V}(S,T)$ and $f_n(t,\omega) \in \mathcal{V}(S,T)$ for $n = 1, 2, \cdots$ and $E\left[\int_S^T \left(f_n(t,\omega) - f(t,\omega)\right)^2 dt\right] \to 0$ as $n \to \infty$, then

$$\int_S^T f_n(t,\omega)\,dB_t(\omega) \to \int_S^T f(t,\omega)\,dB_t(\omega) \quad \text{in } L^2(P) \text{ as } n \to \infty.$$

Example 3.1.9.

- Assume $B_0 = 0$. Then, $\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$.
- Proof:

Let $\phi_n(t,\omega) = \sum B_j(\omega)\cdot\chi_{[t_j,t_{j+1})}(s)$ where $B_j = B_{t_j}$. Then,

$$E\left[\int_0^t (\phi_n - B_s)^2 ds\right] = E\left[\sum_j \int_{t_j}^{t_{j+1}} (B_j - B_s)^2 ds\right]$$

$$= \sum_j \int_{t_j}^{t_{j+1}} (s - t_j) ds = \sum_j \frac{1}{2}(t_{j+1} - t_j)^2 \to 0$$

as $\Delta t_j \to 0$.

Example 3.1.9. continued

* By Corollary 3.1.8,

$$\int_0^t B_s dB_s = \lim_{\Delta t_j \to 0} \int_0^t \phi_n dB_s = \lim_{\Delta t_j \to 0} \sum_j B_j \Delta B_j \,.$$

Now,

$$\Delta(B_j^2) = B_{j+1}^2 - B_j^2 = (B_{j+1} - B_j)^2 + 2B_j(B_{j+1} - B_j)$$
$$= (\Delta B_j)^2 + 2B_j \Delta B_j$$

Since $B_0 = 0$, we obtain

$$B_t^2 = \sum_j \Delta(B_j^2) = \sum_j (\Delta B_j)^2 + 2\sum_j B_j \Delta B_j$$; or

$$\sum_j B_j \Delta B_j = \frac{1}{2} B_t^2 - \frac{1}{2} \sum_j (\Delta B_j)^2$$

Since $\sum_j (\Delta B_j)^2 \to t$ in $L^2(P)$ as $\Delta t_j \to 0$, the result follows.

Theorem 3.2.1

* Let $f, g \in \mathcal{V}(0,T)$ and $0 \le S < U < T$. Then,

1. $\int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t$ for a.a. ω;

2. $\int_S^T (cf + g) dB_t = c \cdot \int_S^T f dB_t + \int_S^T g dB_t$ for some c and a.a. ω;

3. $E\left[\int_S^T f \, dB_t\right] = 0$;

4. $\int_S^T f dB_t$ is $\mathcal{F}_T$-measurable.

Def: Filtration and Martingale

* A filtration on $(\Omega, \mathcal{F})$ is a family $\mathcal{M} = \{\mathcal{M}_t\}_{t\geq 0}$ of σ-algebras with $\mathcal{M}_t \subset \mathcal{F}$ such that

$$0 \leq s < t \Rightarrow \mathcal{M}_s \subset \mathcal{M}_t \, .$$

* An n-dimensional stochastic process $\{M_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$ is called a martingale with respect to a filtration $\{\mathcal{M}_t\}_{t\geq 0}$ and P if

(1) M_t is $\mathcal{M}_t$-measurable for all t;

(2) $E\big[|M_t|\big] < \infty$ for all t; and

(3) $E[M_s \mid \mathcal{M}_t] = M_t$ for all $s \geq t$.

Example 3.2.3.

* Brownian motion B_t in $\Re^n$ is a martingale w.r.t. the σ-algebras $\mathcal{F}_t$ generated by $\{B_s : s \le t\}$, because

$$E[|B_t|]^2 \le E[|B_t|^2] = |B_0|^2 + nt .$$

For $s \ge t$,

$$E[B_s \mid \mathcal{F}_t] = E[B_s - B_t + B_t \mid \mathcal{F}_t]$$

$$= E[B_s - B_t \mid \mathcal{F}_t] + E[B_t \mid \mathcal{F}_t]$$

$$= 0 + B_t = B_t .$$

Theorem 3.2.4. Doob's Martingale Inequality

* If M_t is a martingale such that $t \to M_t$ is continuous a.s., then for all $p \geq 1$, $T \geq 0$ and all $\lambda > 0$,

$$P[\sup_{0 \leq t \leq T} |M_t| \geq \lambda] \leq \frac{1}{\lambda^p} \cdot E[|M_T|^p].$$

We can use this inequality to prove that the Ito integral can be chosen to depend continuously on time.

Borel-Cantelli Lemma

* Let $(A_1, A_2, \cdots)$ be a sequence of events in a probability space $(\Omega, \mathcal{F}, P)$. Assume that for each $i \neq j$, the events A_i

and A_j are either negatively correlated or uncorrelated.

Then, if $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m) = 1$.

Theorem 3.2.5

- Let $f \in \mathcal{V}(S,T)$. There exists a t-continuous stochastic process J_t on $(\Omega, \mathcal{F}, P)$ such that

$$P\left[J_t = \int_0^t f dB \right] = 1 \text{ for all } t, 0 \le t \le T.$$

- Proof: Let $\phi_n = \phi_n(t,\omega) = \sum_j e_j(n) \chi_{[t_j^{(n)}, t_{j+1}^{(n)})}(t)$

be elementary functions such that

$$\lim_{n \to 0} E\left[\int_0^T (f - \phi_n)^2 dt \right] = 0.$$

Sketch of Proof: Step 2

* Let $\begin{cases} I_n(t,\omega) = \int_0^t \phi_n(s,\omega)dB_s(\omega); \\ I_t = I_t(t,\omega) = \int_0^t f(s,\omega)dB_s(\omega); 0 \le t \le T. \end{cases}$

Now, $I_n(\cdot,\omega)$ is continuous for all n.

Moreover, $I_n(t,\omega)$ is a martingale w.r.t. $\mathcal{F}_t$ for all n, because for $s > t$,

$$E[I_n(s,\omega) \mid \mathcal{F}_t] = E[(\int_0^t \phi_n dB + \int_0^s \phi_n dB) \mid \mathcal{F}_t]$$

$$= \int_0^t \phi_n dB + E\left[\sum_{t \le t_j^{(n)} \le t_{j+1}^{(n)} \le s} e_j^{(n)} \Delta B_j \middle| \mathcal{F}_t \right]$$; and

Because Continued

* $$E\left[\sum_{t \le t_j^{(n)} \le t_{j+1}^{(n)} \le s} e_j^{(n)} \Delta B_j \,\middle|\, \mathcal{F}_t \right] = \sum_j E[E[e_j^{(n)} \Delta B_j \mid \mathcal{F}_{t_j^{(n)}}] \mid \mathcal{F}_t]$$
$$= \sum_j E[e_j^{(n)} E[\Delta B_j \mid \mathcal{F}_{t_j^{(n)}}] \mid \mathcal{F}_t] = 0.$$
* $I_n - I_m$ is also an $\mathcal{F}_t$-martingale.

Sketch of Proof: Step 3

- By Theorem 3.2.4,

$$P\left[\sup_{0\le t\le T}\left|I_n(t,\omega)-I_m(t,\omega)\right|>\epsilon\right]\le\frac{1}{\epsilon^2}\cdot E\left[\left|I_n(t,\omega)-I_m(t,\omega)\right|^2\right]$$

$$=\frac{1}{\epsilon^2}\cdot E\left[\int_0^T(\phi_n-\phi_m)^2\,ds\right]\to 0 \qquad \text{as} \quad m,n\to\infty.$$

- Hence we may choose a subsequence $n_k\uparrow\infty$ s.t.

$$P\left[\sup_{0\le t\le T}\left|I_{n_{k+1}}(t,\omega)-I_{n_k}(t,\omega)\right|>\epsilon\right]<2^{-k}.$$

Sketch of Proof: Final Step

* By the Borel-Cantelli lemma,

$$P\left[\sup_{0\le t\le T}\left|I_{n_{k+1}}(t,\omega)-I_{n_k}(t,\omega)\right|>2^{-k}\ \text{for infinitely many } k\right]=0.$$

* For a.a. ω, there exists $k_1(\omega)$ such that

$$\sup_{0\le t\le T}\left|I_{n_{k+1}}(t,\omega)-I_{n_k}(t,\omega)\right|\le 2^{-k}\ \text{for } k\ge k_1(\omega).$$

Corollary 3.2.6

- Let $f \in \mathcal{V}(S,T)$ for all T. Then,

$$M_t(\omega) = \int_0^t f(s,\omega)\,dB_s$$

is a martingale w.r.t. $\mathcal{F}_t$ and

$$P\left[\sup_{0 \le t \le T} |M_t| \ge \lambda\right] \le \frac{1}{\lambda^2} \cdot E\left[\int_0^t f(s,\omega)^2\,ds\right]$$

for all $\lambda, T > 0$

The Ito Formula

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Definition*: 1-dimensional Ito process

* Let B_t be a 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$.
  An Ito process is a stochastic process X_t of the form

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s$$

where $v \in \mathcal{W}_{\mathcal{H}}$, so that

$$P\left[\int_0^t v(s,\omega)^2 ds < \infty \text{ for all } t \geq 0\right] = 1$$

and u is $\mathcal{H}_t$ - adapted and

$$P\left[\int_0^t |u(s,\omega)| ds < \infty \text{ for all } t \geq 0\right] = 1$$

Theorem: 1-dimensional Ito formula

- Let X_t be an Ito process given by $dX_t = udt + vdB_t$. Let $g(t,x) \in C^2\left([0,\infty) \times \Re\right)$. Then, $Y_t = g(t, X_t)$ is again an Ito process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)\cdot(dX_t)^2 \quad (4.1.7)$$

where $(dX_t)^2 = (dX_t)\cdot(dX_t)$ is computed according to the rules

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \quad dB_t \cdot dB_t = dt. \ (4.1.8)$$

Example 4.1.3.

* Consider the integral $I = \int_0^t B_s dB_s$.
* Let $X_t = B_t$ and $g(t,x) = \frac{1}{2}x^2$.
* Then, $Y_t = g(t, B_t) = \frac{1}{2}B_t^2$.
* By Ito formula, $dY_t = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dB_t)^2$.
* Hence, $d(\frac{1}{2}B_t^2) = B_t dB_t + \frac{1}{2}dt.$
* In other words,

$$\frac{1}{2}B_t^2 = \int B_t dB_t + \frac{1}{2}t.$$

Example 4.1.4.

* Consider $\int_0^t s dB_s$.
* Let $Y_t = g(t, B_t) = tB_t$ by setting $g(t, x) = tx$.
* By Ito formula, $dY_t = B_t dt + t dB_t + 0.$
* Thus, we obtain:

$$d(tB_t) = B_t dt + t dB_t.$$

* Hence,

$$tB_t = \int_0^t B_s ds + \int_0^t s dB_s.$$

Theorem: Integration by parts

- Suppose $f(s,\omega)$ is continuous and of bounded variation w.r.t. $s \in [0,t]$ for a.a. ω. Then

$$\int_0^t f(s)dB_s = f(t)B_t - \int_0^t B_s df_s.$$

Sketch of the Proof for Ito Formula

* By substituting $dX_t = udt + vdB_t$ in (4.1.7) and using (4.1.8),

$$g(t,X_t) = g(0,X_0) + \int_0^t \left(\frac{\partial g}{\partial s}(s,X_s) + u_s \frac{\partial g}{\partial x}(s,X_s) + \frac{1}{2} v_s^2 \frac{\partial^2 g}{\partial s^2}(s,X_s) \right) ds$$

$$+ \int_0^t v_s \frac{\partial g}{\partial s}(s,X_s) dB_s \quad \text{where } u_s = u(s,\omega), v_s = v(s,\omega).$$

* We may assume that u_t and v_t are elementary functions.
* Let $R_j = o(|\Delta t_j|^2 + |\Delta X_j|^2)$.
* Then, by Taylor's theorem, we obtain the following:

$$\sum_j \Delta g(t_j, X_j) = \sum_j \frac{\partial g}{\partial t} \Delta t_j + \sum_j \frac{\partial g}{\partial x} \Delta X_j$$

$$+ \frac{1}{2} \sum_j \frac{\partial^2 g}{\partial t^2} (\Delta t_j)^2 + \sum_j \frac{\partial^2 g}{\partial t \partial x} (\Delta t_j)(\Delta X_j) + \frac{1}{2} \sum_j \frac{\partial^2 g}{\partial x^2} (\Delta X_j)^2 + \sum_j R_j.$$

Sketch of the Proof 2

- Observe that:

$$\sum_j \frac{\partial g}{\partial t}\Delta t_j \to \int_0^t \frac{\partial g}{\partial s}(s, X_s)ds \quad \text{and} \quad \sum_j \frac{\partial g}{\partial x}\Delta X_j \to \int_0^t \frac{\partial g}{\partial x}(s, X_s)dX_s$$

- Since u and v are elementary, we obtain:

$$\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta X_j)^2 = \sum_j \frac{\partial^2 g}{\partial x^2}u_j^2(\Delta t_j)^2 + 2\sum_j \frac{\partial^2 g}{\partial x^2}u_j v_j(\Delta t_j)(\Delta B_j)$$

$$+\sum_j \frac{\partial^2 g}{\partial x^2}v_j^2(\Delta B_j)^2 \quad \text{where} \quad u_j = u(t_j,\omega), v_j = v(t_j,\omega).$$

- Note: $E[(\sum_j \frac{\partial^2 g}{\partial x^2}u_j v_j(\Delta t_j)(\Delta B_j))^2] = \sum_j E[(\frac{\partial^2 g}{\partial x^2}u_j v_j)^2](\Delta t_j)^3$

$$\to 0 \text{ as } \Delta t_j \to 0.$$

Sketch of the Proof 3

- The final step:

$$\sum_j \frac{\partial^2 g}{\partial x^2} v_j^2 (\Delta B_j)^2 \to \int_0^t \frac{\partial^2 g}{\partial x^2} v^2 ds \text{ in } L^2(P) \text{ as } \Delta t_j \to 0.$$

- Let $a(t) = \frac{\partial^2 g}{\partial x^2}(t, X_t) v^2(t, \omega), a_j = a(t_j).$
- Consider

$$E[(\sum_j a_j (\Delta B_j)^2 - \sum_j a_j \Delta t_j)^2] =$$

$$= \sum_{i,j} E[a_i a_j ((\Delta B_i)^2 - \Delta t_i)((\Delta B_j)^2 - \Delta t_j)].$$

- If $i \neq j$, by independence the term vanishes.

Sketch of the Proof 4

* Finally,

$$\sum_j E[a_j^2((\Delta B_j)^2 - \Delta t_j)^2] =$$

$$= \sum_j E[a_j^2] \cdot E[(\Delta B_j)^4 - 2(\Delta B_j)^2 \Delta t_j + (\Delta t_j)^2]$$

$$= \sum_j E[a_j^2] \cdot E[3(\Delta t_j)^2 - 2(\Delta t_j)^2 + (\Delta t_j)^2] \to 0 \quad \text{as } \Delta t_j \to 0.$$

* We have used the following formula from Ex 2.8-(b):

$$E[B_t^{2k}] = \frac{(2k)!}{2^k \cdot k!} t^k ; k \in \mathbf{N}.$$

The multi-dimensional Ito formula

* Let $B(t,\omega) = (B_1(t,\omega), \cdots, B_m(t,\omega))$ denote an m-dimensional Brownian motion. Suppose that each of the processes $u_i(t,\omega)$

and $v_{i,j}(t,\omega)$ satisfies the conditions given in Definition*.
Consider the following system:

$$dX(t) = udt + vdB(t)$$

where

$$X(t) = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix}, u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, v = \begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nn} \end{pmatrix}, dB(t) = \begin{pmatrix} dB_1(t) \\ \vdots \\ dB_n(t) \end{pmatrix}$$

Theorem: The general Ito formula

* Let $g(t,\omega) = (g_1(t,\omega), \cdots, g_p(t,\omega))$ be a C^2 map from $[0,\infty) \times \Re^n$ into $\Re^p$. Then, the process

$$Y(t,\omega) = g(t, X(t))$$

is an Ito process, whose k-th component, Y_k, is given by

$$dY_k = \frac{\partial g_k}{\partial t}(t, X)dt + \sum_i \frac{\partial g_k}{\partial x_i}(t, X)dX_i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t, X)dX_i dX_j$$

where $dB_i dB_j = \delta_{ij} dt,\ dB_t dt = dt dB_t = 0$.

The Martingale Convergence Theorem

- Let $X \in L^1(P)$. Let $\{\mathcal{N}_k\}_{k=1}^{\infty}$ be an increasing family of σ-algebras, with $\mathcal{N}_k \subset \mathcal{F}$.
- Define $\mathcal{N}_\infty$ to be the σ-algebra generated by $\{\mathcal{N}_k\}_{k=1}^{\infty}$.
- Then,

$$E[X \mid \mathcal{N}_k] \to E[X \mid \mathcal{N}_\infty] \text{ as } k \to \infty,$$

a.e. P and in $L^1(P)$.

Lemma 4.3.1.

- Fix $T > 0$. The set of random variables

$$\{\phi(B_{t_1},\cdots,B_{t_n}) : t_i \in [0,T], \phi \in C_0^{\infty}(\Re^n), n = 1,2,\cdots\}$$

is dense in $L^2(\mathcal{F}_T, P)$.

Proof:

Let $\{t_i\}_{i=1}^{\infty}$ be a dense subset of $[0,T]$ and for each $n = 1,2,\cdots$, let $\mathcal{H}_n$ be the σ-algebra generated by $B_{t_1}(\cdot),\cdots,B_{t_n}(\cdot)$. Then, clearly $\mathcal{H}_n \subset \mathcal{H}_{n+1}$ and $\mathcal{F}_T$ is the smallest σ-algebra containing all the $\mathcal{H}_n$'s.

Choose $g \in L^2(\mathcal{F}_T, P)$. By the Martingale Convergence Theorem, $g = E[g \mid \mathcal{F}_T] = \lim_{n\to\infty} E[g \mid \mathcal{H}_n]$.

The limit is pointwise a.e. (P) and in $L^2(\mathcal{F}_T, P)$.

The Doob-Dynkin Lemma

* If $X, Y : \Omega \to \Re^n$ are two given functions, then Y is $\mathcal{H}_X$-measurable if and only if there exists a Borel measurable function $g : \Re^n \to \Re^n$ such that $Y = g(X)$.

Lemma 4.3.1 Proof Continued

- By the Doob-Dynkin Lemma, for each n,

$$E[g \mid \mathcal{H}_n] = g_n(B_{t_1}, \cdots, B_{t_n})$$

for some Borel measurable function $g_n : \Re^n \to \Re$.

- Each such $g_n(B_{t_1}, \cdots, B_{t_n})$ can be approximated in $L^2(\mathcal{F}_T, P)$ by functions $\phi_n(B_{t_1}, \cdots B_{t_n})$ where $\phi_n \in C_0^\infty(\Re^n)$ and the result follows.
- $C_0^k(U)$ denotes a set of functions in $C^k(U)$ with compact support in U.

Lemma 4.3.2.

* The linear span of random variables of the type

$$F(\omega) = \exp\left\{ \int_0^T h(t)\,dB_t(\omega) - \frac{1}{2}\int_0^T h^2(t)\,dt \right\} \quad (4.3.1)$$

for $h \in L^2[0,T]$ is dense in $L^2(\mathcal{F}_T^{(n)}, P)$.

Proof:

A real function is said to be analytic if it possesses derivatives of all orders and agrees with its Taylor series in a neighborhood of every point.

Suppose that $g \in L^2(\mathcal{F}_T, P)$ is orthogonal to all functions of the form (4.3.1).

Proof Continued (P2)

* Then, for all $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Re^n$ and all $t_1, \cdots, t_n \in [0, T]$,

$$G(\lambda) := \int_\Omega \exp[\lambda_1 B_{t_1}(\omega) + \cdots + \lambda_n B_{t_n}(\omega)] g(\omega) dP(\omega) = 0.$$

* As $G(\lambda)$ is real analytic in $\lambda \in \Re^n$ and hence G has an analytic extension to the complex space $\mathbf{C}^n$ given by:

$$G(z) := \int_\Omega \exp[z_1 B_{t_1}(\omega) + \cdots + z_n B_{t_n}(\omega)] g(\omega) dP(\omega) = 0$$

for all $z = (z_1, \cdots, z_n) \in \mathbf{C}^n$.

Proof Continued (P3)

* Then, by the inverse Fourier transform theorem, for $\phi \in C_0^\infty(\Re^n)$

$$\int_\Omega \phi(B_{t_1},\cdots,B_{t_n})g(\omega)dP(\omega)$$

$$=\int_\Omega (2\pi)^{-n/2}\left(\int_{\Re^n} \hat{\phi}(y)e^{i(y_1B_{t_1}+\cdots+y_nB_{t_n})}dy\right)g(\omega)dP(\omega)$$

$$=(2\pi)^{-n/2}\int_{\Re^n} \hat{\phi}(y)\left(\int_\Omega e^{i(y_1B_{t_1}+\cdots+y_nB_{t_n})}g(\omega)dP(\omega)\right)dy$$

$$=(2\pi)^{-n/2}\int_{\Re^n} \hat{\phi}(y)G(iy)dy=0.$$

The Ito representation theorem

- Let $F \in L^2(\mathcal{F}_T^{(n)}, P)$. Then, there exists a unique stochastic process $f(t,\omega) \in \mathcal{V}^n(0,T)$ such that

$$F(\omega) = E[F] + \int_0^T f(t,\omega)dB(t)$$

where $B(t) = (B_1(t), \cdots, B_n(t))$ is n-dimensional

Proof: We consider only the case $n = 1$.

First assume that F has the form of

$$F(\omega) = \exp\left\{\int_0^T h(t)dB_t(\omega) - \frac{1}{2}\int_0^T h^2(t)dt\right\}$$

for some $h(t) \in L^2[0,T]$.

Proof Continued (P2)

- Define

$$Y_t(\omega) = \exp\left\{\int_0^t h(s)dB_s(\omega) - \frac{1}{2}\int_0^t h^2(s)ds\right\};\ 0 \le t \le T$$

- Then, by Ito's formula,

$$dY_t = Y_t(h(t)dB_t - \tfrac{1}{2}h^2(t)dt) + \tfrac{1}{2}Y_t(h(t)dB_t)^2 = Y_t h(t)dB_t$$

- Thus,

$$Y_t = 1 + \int_0^t Y_s h(s)dB_s;\ t \in [0,T]$$

- Hence,

$$F = Y_T = 1 + \int_0^T Y_s h(s)dB_s$$

Proof Continued (P3)

* Second consider an arbitrary $F \in L^2(\mathcal{F}_T^{(n)}, P)$.
* Approximate F by linear combinations F_n of functions of the form in the first step.
* For each n,

$$F_n(\omega) = E[F_n] + \int_0^T f_n(s,\omega)dB_s(\omega); \quad \text{where } f_n \in \mathcal{V}(0,T).$$

* By the Ito isometry,

$$E[(F_n - F_m)^2] = E[(E[F_n - F_m] + \int_0^T (f_n - f_m)dB_t)^2]$$

$$= E[F_n - F_m]^2 + 2E[E[F_n - F_m]\int_0^T (f_n - f_m)dB_t] + \int_0^T E(f_n - f_m)^2 dt$$

$$\to 0 \quad \text{as } n, m \to \infty$$

Proof Continued (P4)

- Thus, $\{f_n\}$ is a Cauchy sequence in $L^2([0,T]\times\Omega)$.
- As $f_n \in \mathcal{V}(0,T)$, the limit is also $f \in \mathcal{V}(0,T)$.
- Again, by the Ito isometry,

$$F = \lim_{n\to\infty} F_n = \lim_{n\to\infty} (E[F_n] + \int_0^T f_n dB) = E[F] + \int_0^T f dB,$$

the limit being taken in $L^2(\mathcal{F}_T, P)$.

- This completes the proof.
- The uniqueness follows from the Ito isometry.

The martingale representation theorem

- Let $B(t)=(B_1(t),\cdots,B_n(t))$ be n-dimensional. Suppose M_t is an $\mathcal{F}_t^{(n)}$-martingale w.r.t. P and $M_t \in L^2(P)$ for all $t \geq 0$. Then, there exists a unique stochastic process $g(s,\omega)$ such that $g \in \mathcal{V}_t^{(n)}(0,t)$ for all $t \geq 0$ and

$$M_t(\omega)=E[M_0]+\int_0^t g(s,\omega)dB(s)$$

a.s., for all $t \geq 0$
- Proof for one-dimensional case:

Theorem 4.3.3. is applicable for $T=t$, and $F=M_t$.

Proof (P2)

* Assume $0 \le t_1 < t_2$.
* Then,

$$M_{t_1} = E[M_{t_2} \mid \mathcal{F}_{t_1}] = E[M_0] + E[\int_0^{t_2} f^{(t_2)}(s,\omega)dB_s(\omega) \mid \mathcal{F}_{t_1}]$$

$$= E[M_0] + \int_0^{t_1} f^{(t_2)}(s,\omega)dB_s(\omega).$$

* Compare with $M_{t_1} = E[M_0] + \int_0^{t_1} f^{(t_1)}(s,\omega)dB_s(\omega).$
* Hence, $0 = E[(\int_0^{t_1} (f^{(t_2)} - f^{(t_1)})dB)^2] = \int_0^{t_1} E(f^{(t_2)} - f^{(t_1)})^2 dt.$

Proof (P3)

* Thus, we obtain:

$$f^{(t_1)}(s,\omega) = f^{(t_2)}(s,\omega) \quad \text{for a.a. } (s,\omega) \in [0,t_1] \times \Omega.$$

* Define $f(s,\omega)$ for a.a. $s \in [0,\infty) \times \Omega$ by

$$f(s,\omega) = f^{(N)}(s,\omega) \quad \text{if} \quad s \in [0,N].$$

* We obtain:

$$M_t(\omega) = E[M_0] + \int_0^t f(s,\omega)dB(s) \qquad \forall t \geq 0.$$