Mathematical Preliminaries: Brownian Motion

Shino Takayama

School of Economics

University of Queensland

Def: sigma-algebra

* If Ω is a given set, then a σ-algebra $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:

1. $\varnothing \in \mathcal{F}$;
2. $F \in \mathcal{F} \Rightarrow F^C \in \mathcal{F}$;
3. $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measureable space.

Definition: Probability Space

* A probability measure P on a measurable space $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0,1]$ such that

1. $P(\varnothing) = 0,\ P(\Omega) = 1$;
2. if $A_1, A_2, \cdots \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoin, then

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) \ .$$

The triple $(\Omega, \mathcal{F}, P)$ is called a probability space.

The smallest sigma-algebra

* A probability space is called a complete probability space if $\mathcal{F}$ contains all subsets G of Ω with P-outer measure zero, i.e. with

$$P^*(G) := \inf\{P(F) : F \in \mathcal{F}, G \subset F\} = 0 .$$

* The subsets F of Ω which belong to $\mathcal{F}$ are called $\mathcal{F}$-measurable.
* Let

$$\mathcal{H}_{\mathcal{U}} = \bigcap\{\mathcal{H} : \mathcal{H} \text{ is } \sigma-\text{algebra of } \Omega, \mathcal{U} \subset \mathcal{H}\} .$$

We call $\mathcal{H}_{\mathcal{U}}$ the σ-algebra generated by $\mathcal{U}$.

Random Variable and Expectation

* A random variable X is an $\mathcal{F}$-measurable function where

$$X : \Omega \to \Re^n$$

* Every random variable induces a probability measure μ_X on $\Re^n$, defined by $\mu_X(B) = P(X^{-1}(B))$.
* Then μ_X is called the distribution of X .
* If $\int_\Omega |X(\omega)| dP(\omega) < \infty$, then the number

$$E[X] := \int_\Omega |X(\omega)| dP(\omega) = \int_{\Re^n} x d\mu_X(x)$$

is called the expectation of X with respect to P.

The L^p-spaces

* If $X:\Omega \to \Re^n$ is a random variable and $p \in [1,\infty)$ is a constant, we define the L^p-norm of X denoted by $\|X\|_p$ as follows:

$$\|X\|_p = \|X\|_{L^p(P)} = \left(\int_\Omega |X(\omega)|^p \, dP(\omega) \right)^{\frac{1}{p}} .$$

* If $p = \infty$, set

$$\|X\|_\infty = \|X\|_{L^\infty(P)} = \sup\{ |X(\omega)| : \omega \in \Omega \} .$$

* The corresponding L^p-spaces are defined by

$$L^p(P) = L^p(\Omega) = \{ X:\Omega \to \Re^n \mid \|X\|_p < \infty \} .$$

Independence

* Two subsets $A, B \in \mathcal{F}$ are called independent if
$$P(A \cap B) = P(A) \cdot P(B) .$$
* A collection $\mathcal{A} = \{\mathcal{H}_i : i \in I\}$ of families $\mathcal{H}_i$ of measurable sets is independent if
$$P(H_{i_1} \cap \cdots \cap H_{i_k}) = P(H_{i_1}) \cdots P(H_{i_k})$$
for all choices of $H_{i_1} \in \mathcal{H}_{i_1}, \cdots, H_{i_k} \in \mathcal{H}_{i_k}$ with different indices $i_1, \cdots, i_k$.
* A collection of random variables $\{X_i : i \in I\}$ is independent if the collection of generated σ-algebras $\mathcal{H}_{X_i}$ is independent.

Stochastic Process

* A Stochastic process is a parametrized collection of random variables

$$\{X_t\}_{t \in T}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\Re^n$.

Kolmogorov's extension theorem

* For all $t_1,\cdots,t_k \in T$, $k \in N$, let $\nu_{t_1,\cdots,t_k}$ be probability measures on $\mathfrak{R}^{nk}$ s.t.

(K1) $\nu_{t_{\sigma(1)},\cdots,t_{\sigma(k)}}(F_1 \times \cdots \times F_k) = \nu_{t_1,\cdots,t_k}(F_{\sigma^{-1}(1)} \times \cdots \times F_{\sigma^{-1}(k)})$

for all permutations σ on $\{1,2,\cdots,k\}$ and

(K2) $\nu_{t_1,\cdots,t_k}(F_1 \times \cdots \times F_k) = \nu_{t_1,\cdots,t_k,t_{k+1},\cdots,t_{k+m}}(F_1 \times \cdots \times F_k \times \underbrace{\mathfrak{R}^n \times \cdots \times \mathfrak{R}^n}_{m \text{ times}})$

For all $m \in \mathbf{N}$.

Then, there exists a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $\{X_t\}$ on Ω with $X_t : \Omega \to \mathfrak{R}^n$ s.t.

$$\nu_{t_1,\cdots,t_k}(F_1 \times \cdots \times F_k) = P[X_{t_1} \in F_1, \cdots, X_{t_k} \in F_k]$$

for all $t_i \in T$, $k \in \mathbf{N}$, and all Borel sets F_i.

Construction of Brownian Motion

* Specify a family $\left\{ \nu_{t_1,\cdots,t_k} \right\}$ of probability measures satisfying (K1) and (K2).
* Fix $x \in \Re^n$ and define

$$p(t,x,y) = (2\pi t)^{-n/2} \cdot \exp\left(-\frac{|x-y|^2}{2t} \right) \quad \text{for } y \in \Re^n,\ t > 0$$

* For $0 \le t_1 \le t_2 \le \cdots \le t_k$, define a measure $\nu_{t_1,\cdots,t_k}$ on $\Re^{nk}$ by

$$\nu_{t_1,\cdots,t_k}(F_1 \times \cdots \times F_k) =$$

$$= \int_{F_1 \times \cdots \times F_k} p(t_1,x,x_1) p(t_2 - t_1, x_1, x_2) \cdots p(t_k - t_{k-1}, x_{k-1}, x_k) dx_1 \cdots dx_k$$

Definition: Brownian motion

- By Kolmogorov's theorem, there exists a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $\{B_t\}_{t\geq 0}$ on Ω such that the finite-dimensional distributions of B_t is given by

$$P^x(B_{t_1} \in F_1, \cdots, B_{t_k} \in F_k) =$$

$$= \int_{F_1\times\cdots\times F_k} p(t_1, x, x_1)p(t_2 - t_1, x_1, x_2)\cdots p(t_k - t_{k-1}, x_{k-1}, x_k)dx_1\cdots dx_k$$

- Such a process is called Brownian motion starting at x.
- Observe that $P^x(B_0 = x) = 1$.

Property 1: Brownian motion

* B_t is a Gaussian process.
* For all $0 \le t_1 \le \cdots \le t_k$, the random variable

$$Z = (B_{t_1}, \cdots, B_{t_k}) \in \Re^{nk}$$

has a multi-normal distribution.

* There exists a vector $M \in \Re^{nk}$ and a non-negative definite matrix $C = [c_{jm}] \in \Re^{nk \times nk}$ such that

$$E^x[\exp(i \sum_{j=1}^{nk} u_j Z_j)] = \exp\left(-\frac{1}{2} \sum_{j,m} u_j c_{jm} u_m + i \sum_j u_j M_j \right)$$

for all $u = (u_1, \cdots, u_{nk}) \in \Re^{nk}$.

Property 2: Brownian motion

- B_t has independent increments.
- $B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_k} - B_{t_{k-1}}$ are independent for all $0 \le t_1 \le \cdots \le t_k$
- So, $B_s - B_t$ is independent of $\mathcal{F}_t$ if $s > t$.

Property 3: Brownian motion

- Suppose that $\{X_t\}$ and $\{Y_t\}$ are stochastic processes on $(\Omega, \mathcal{F}, P)$. Then we say that $\{X_t\}$ is a version of (or a modification of) $\{Y_t\}$ if

$$P(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1 \quad \text{for all } t$$

- Kolmogorov's continuity theorem states that if the process $X = \{X_t\}_{t\geq 0}$ satisfies the condition such that for all $T > 0$, there exist positive constants α, β, D such that

$$E[|X_t - X_s|^{\alpha}] \leq D \cdot |t-s|^{1+\beta} \quad \text{for all } s,t \in [0,T],$$

then there exists a continuous version of X.

- If $B_t = (B_t^{(1)}, \cdots, B_t^{(n)})$ is an n-dimensional Brownian motion, then the 1-dimensional processes $\{B_t^{(j)}\}_{t\geq 0}, 1 \leq j \leq n$ are an independent 1-dimensional Brownian motion.

Variation process

* Let $X_t(\cdot):\Omega\to\Re$ be a continuous stochastic process. Then, for $p>0$, the p's variation process of X_t, $\langle X,X\rangle_t^{(p)}$

is defined by (limit in probability)

$$\langle X,X\rangle_t^{(p)}(\omega)=\lim_{\Delta t_k\to 0}\sum_{t_k\le t}\left|X_{t_{k+1}}(\omega)-X_{t_{k+1}}(\omega)\right|^p$$

* If $p=1$, this process is called the total variation process.
* If $p=2$, this process is called the quadratic variation process.

The Ito Integral

Shino Takayama
School of Economics
University of Queensland

Construction of the Ito Integral

* Consider:

$$\frac{dN}{dt} = (r(t) + "noise")N(t)$$

* More generally:

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot "noise"$$

where b and σ are some given functions.

Case of 1-dimensional noise

* Let W_t be a stochastic process to represent the noise term. Then, we obtain:

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot W_t$$

* We will try to rewrite the above equation.
* Let's start with a discrete version.

Discrete Version

- A discrete version is:

$$X_{k+1} - X_k = b(t_k, X_k)\Delta t_k + \sigma(t_k, X_k) W_k \Delta t_k$$

where $X_j = X(t_j)$, $W_k = W_{t_k}$, $\Delta t_k = t_{k+1} - t_k$.

- Replace $W_k \Delta t_k$ by $\Delta B_k = B_{t_{k+1}} - B_{t_k}$

From discrete to continuous time

- Sum up and then we obtain:

$$X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j)\Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta B_j$$

- If we can take $\Delta t_j \to 0$, then we obtain:

$$X_k = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$

Lemma: The Ito isometry

* If $\phi(t,\omega)$ is bounded and elementary, then

$$E\left[\left(\int_s^t \phi(t,\omega)\,dB_t(\omega)\right)^2\right] = E\left[\int_s^t \phi(t,\omega)^2\,dt\right]$$

* Proof – See Note

Definition: The Ito integral

* Let $f \in \mathcal{V}(S,T)$. Then the Ito integral of f from S to T is defined by

$$\int_S^T f(t,\omega)\,dB_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)\,dB_t(\omega)$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$E\left[\int_S^T \left(f(t,\omega) - \phi_n(t,\omega)\right)^2 dt\right] \to 0$$

as $n \to \infty$.

Theorem 3.2.5

- Let $f \in \mathcal{V}(S,T)$. There exists a t-continuous stochastic process J_t on $(\Omega, \mathcal{F}, P)$ such that

$$P\left[J_t = \int_0^t f dB \right] = 1$$

for all t, $0 \le t \le T$

- Proof – See Note

Corollary 3.2.6

* Let $f \in \mathcal{V}(S,T)$ for all T. Then,

$$M_t(\omega) = \int_0^t f(s,\omega)\,dB_s$$

is a martingale w.r.t. $\mathcal{F}_t$ and

$$P\left[\sup_{0\le t\le T} |M_t| \ge \lambda\right] \le \frac{1}{\lambda^2} \cdot E\left[\int_0^t f(s,\omega)^2\,ds\right]$$

for all $\lambda, T > 0$