The Filtering Problem 2

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The 1-Dimensional Linear Filtering Problem

* Consider the 1-dimensional case:

(System)

$$dX_t = F(t)X_t dt + C(t)dU_t; \quad F(t), C(t) \in \Re \ (6.2.3)$$

(Observations)

$$dZ_t = G(t)X_t dt + D(t)dV_t; \quad G(t), D(t) \in \Re \ (6.2.4)$$

* Assume that F, G, C, D are bounded on bounded intervals.
* Assume that $Z_0 = 0$.
* Assume that X_0 is normally distributed and independent of $\{U_t\}, \{V_t\}$.
* Assume that $D(t)$ is bounded away from 0 on bounded intervals.

Theorem 6.2.8 (The 1-dimensional Kalman-Bucy filter)

* The solution $\hat{X}_t = E[X_t \mid \mathcal{G}_t]$ of the 1-dimensional linear filtering problem satisfies the stochastic differential equation

$$d\hat{X}_t = \left(F(t) - \frac{G^2(t)S(t)}{D^2(t)} \right) \hat{X}_t dt + \frac{G(t)S(t)}{D^2(t)} dZ_t;$$

$$\hat{X}_0 = E[X_0] \quad (6.2.28)$$

where $S(t) = E[(X_t - \hat{X}_t)^2]$ satisfies the Riccati equation

$$\frac{dS}{dt} = 2F(t)S(t) - \frac{G^2(t)}{D^2(t)} S^2(t) + C^2(t);$$

$$S(0) = E[(X_0 - E[X_0])^2] \; . \; (6.2.29)$$

Step 1

* Let $\mathcal{L} = \mathcal{L}(Z,t)$ be the closure in $L^2(P)$ of functions which are linear combinations of the form

$$c_0 + c_1 Z_{s_1}(\omega) + \cdots + c_k Z_{s_k}(\omega), \quad \text{with } s_j \le t, c_j \in \Re.$$

* Let

$$\mathcal{P}_{\mathcal{L}} \text{ denote the projection from } L^2(P) \text{ onto } \mathcal{L}.$$

* Then, with $\mathcal{K}$ as in (6.1.9),

$$\hat{X}_t = \mathcal{P}_{\mathcal{K}}(X_t) = E[X_t \mid \mathcal{G}_t] = \mathcal{P}_{\mathcal{L}}(X_t).$$

* Then, the best Z-measurable estimate of X_t coincides with the best Z-linear estimate of X_t .

Step 2: Innovation Process

* Replace Z_t by the innovation process N_t :

$$N_t = Z_t - \int_0^t (GX)_s^{\wedge} ds;$$

$$\text{where } (GX)_s^{\wedge} = \mathcal{P}_{\mathcal{L}(Z,s)}(G(s)X_s) = G(s)\hat{X}_s.$$

* N_t has orthogonal increments, i.e.

$$E[(N_{t_1} - N_{s_1})(N_{t_2} - N_{s_2})] = 0 \text{ for disjoint intervals } [s_1, t_1], [s_2, t_2].$$

* $\mathcal{L}(N,t) = \mathcal{L}(Z,t)$ so $\hat{X}_t = \mathcal{P}_{\mathcal{L}(N,t)}(X_t)$

Lemma 6.2.4

$$\mathcal{L}(Z,T)=\{c_0+\int_0^t f(t)dZ_t;\ f\in L^2[0,T], c_0\in\Re\}$$

* Proof: Denote the right hand side by $\mathcal{N}(Z,T)$. We show:
  1. $\mathcal{N}(Z,T)\subset\mathcal{L}(Z,T)$;
  2. $\mathcal{N}(Z,T)$ contains all linear combinations of the form:
     $$c_0+c_1Z_{t_1}+\cdots+c_kZ_{t_k};\quad 0\le t_i\le T$$
  3. $\mathcal{N}(Z,T)$ is closed in $L^2(P)$.

About 1.

* This follows from the fact that if f is continuous, then

$$\int_0^T f(t)\,dZ_t = \lim_{n\to\infty} \sum_j f(j\cdot 2^{-n})\cdot(Z_{(j+1)2^{-n}} - Z_{j\cdot 2^{-n}}).$$

About 2.

- Suppose $0 \le t_1 < t_2 < \cdots < t_k \le T$.
- We can write:

$$\sum_{i=1}^{k} c_i Z_{t_i} = \sum_{i=0}^{k-1} c_j' \Delta Z_{t_i} = \sum_{i=0}^{k-1} \int_{t_j}^{t_{j+1}} c_j' dZ_t = \int_0^T \left(\sum_{j=0}^{k-1} c_j' \chi_{[t_j, t_{j+1})}(t) \right) dZ_t,$$

where $\Delta Z_j = Z_{t_{j+1}} - Z_{t_j}$.

About 3.

* If $f \in L^2[0,T]$, then

$$E\left[\left(\int_0^T f(t)dZ_t\right)^2\right] =$$

$$= E\left[\left(\int_0^T f(t)G(t)X_t dt\right)^2\right] + E\left[\left(\int_0^T f(t)D(t)dV_t\right)^2\right]$$

$$+2E\left[\left(\int_0^T f(t)G(t)X_t dt\right)\left(\int_0^T f(t)D(t)dV_t\right)\right].$$

About 3 continued

* Since

$$E\left[\left(\int_0^T f(t)G(t)X_t dt\right)^2\right] \le A_1 \cdot \int_0^T f(t)^2 dt;$$

$$E\left[\left(\int_0^T f(t)D(t)dV_t\right)^2\right] = \int_0^T f^2(t)D^2(t)dt.$$

* we conclude that:

$$A_0 \cdot \int_0^T f(t)^2 dt \le E\left[\left(\int_0^T f(t)dZ_t\right)^2\right] \le A_2 \cdot \int_0^T f^2(t)dt. \ (6.2.11)$$

* As $L^2[0,T]$ is complete, this completes the proof.

Innovation Process

- Now we define the Innovation Process N_t as follows:

$$N_t = Z_t - \int_0^t (GX)_s^{\wedge} ds, \text{ where } (GX)_s^{\wedge} = \mathcal{P}_{\mathcal{L}(Z,s)}(G(s)Z_s)G(s)\hat{X}_s$$

i.e. $$dN_t = G(t)(X_t - \hat{X}_t)dt + D(t)dV_t. \qquad (6.2.13)$$

Lemma 6.2.5.

1. N_t has orthogonal increments;
2. $E[N_t^2] = \int_0^t D^2(s)ds$;
3. $\mathcal{L}(N,t) = \mathcal{L}(Z,t) \quad \text{for all } t \geq 0$;
4. N_t is a Gaussian Process.

Proof for 1: Since $X_r - \hat{X}_r \perp \mathcal{L}(Z,r) \subset \mathcal{L}(Z,s) \text{ for } r \geq s$, and V has independents increments,

$$E[(N_t - N_s)Y] = E\left[\left(\int_s^t G(r)(X_r - \hat{X}_r)dr + \int_s^t D(r)dV_r\right)Y\right]$$

$$= \int_s^t G(r)E[(X_r - \hat{X}_r)Y]dr + E\left[\int_s^t D(r)dV_rY\right] = 0.$$

Proof for 2.

- By Ito formula, we have

$$d(N_t^2) = 2N_t dN_t + \frac{1}{2} 2(dN_t)^2 = 2N_t dN_t + D^2 dt.$$

- Thus,

$$E(N_t^2) = E\left[\int_0^t 2N_s dN_s\right] + \int_0^t D^2(s) ds.$$

- Now

$$\int_0^t N_s dN_s = \lim_{\Delta t_j \to \infty} \sum N_{t_j}[N_{t_{j+1}} - N_{t_j}],$$

- Since N_t has orthogonal increments, we have

$$E\left[\int_0^t N_s dN_s\right] = 0.$$

Proof for 3.

* It is clear that $\mathcal{L}(N,t) \subset \mathcal{L}(Z,t)$ for all $t \geq 0$.
* Choose $f \in L^2[0,t]$.
* To prove $\mathcal{L}(Z,t) \subset \mathcal{L}(N,t)$ for all $t \geq 0$, observe:

$$\int_0^t f(s)dN_s = \int_0^t f(s)dZ_s - \int_0^t f(r)D(r)\hat{X}_r dr$$

$$= \int_0^t f(s)dZ_s - \int_0^t f(r)\left[\int_0^t g(r,s)dZ_s\right]dr - \int_0^t f(r)c(r)dr$$

$$= \int_0^t \left[f(s) - \int_0^t f(r)g(r,s)dr \right] dZ_s - \int_0^t f(r)c(r)dr.$$

Proof for 3 continued

- From the theory of Volterra integral equation, there exists for all $h \in L^2[0,t]$ and $f \in L^2[0,t]$ such that

$$f(s) - \int_s^t f(r)g(r,s)dr = h(s).$$

- So by choosing $h = \chi_{[0,t_1]}$ where $0 \le t_1 \le t$, we obtain

$$\int_0^t f(r)c(r)dr + \int_0^t f(s)dN_s = \int_0^t \chi_{[0,t_1]}(s)dZ_s = Z_{t_1},$$

which implies $\mathcal{L}(Z,t) \subset \mathcal{L}(N,t)$ for all $t \ge 0$.

Proof for 4.

* $\hat{X}_t$ is a limit in $L^2(P)$ of linear combinations of the form: $M = c_0 + c_1 Z_{s_1} + \cdots c_k Z_{s_k}$, where $s_k \leq t$.

* Therefore

$$\left(\hat{X}_{t_1}, \cdots, \hat{X}_{t_m}\right)$$

is a limit of m -dimensional random variable

$$\left(M^{(1)}, \cdots, M^{(m)}\right)$$

whose components are linear combinations of this form. $\left(M^{(1)}, \cdots, M^{(m)}\right)$ has a normal distribution since $\{Z_t\}$ is Gaussian and the limit has. So $\{\hat{X}_t\}$is Gaussian. Thus, so is

$$N_t = Z_t - \int_0^t G(S)\hat{X}_s ds.$$

Step 3. The Innovation Process and Brownian Motion

- Let

$$N_t = Z_t - \int_0^t G(s)\hat{X}_s ds$$

be the innovation process defined in Step 2.

We have assumed that $D(t)$ is bounded away from 0 on bounded intervals. Define the process $R_t(\omega)$ by

$$dR_t = \frac{1}{D(t)} dN_t(\omega); \ t \geq 0$$

Lemma 6.2.6 - R_t is a 1-dimensional Brownian motion.

Proof of Lemma 6.2.6.

* Observe that R_t

1. has continuous paths;
2. has orthogonal increments (as N_t is);
3. is Gaussian (as N_t is);
4. and $E[R_t]=0$ and $E[R_t R_s]=\min(s,t)$.

   * because by the Ito formula

$$d(R_t^2)=2R_t dR_t+(dR_t)^2=2R_t dR_t+dt.$$

   * By 2, $E[R_t^2]=E[\int_0^t ds]=t.$
   * For $s<t$,

$$E[R_t R_s]=E[(R_t-R_s)R_s]+E[R_s^2]=E[R_s^2]=s.$$

Lemma 6.2.7

$$\hat{X}_t = E[X_t] + \int_0^t \frac{\partial}{\partial s} E[X_t R_s] dR_s. \ (6.2.15)$$

- Proof: From Lemma 6.2.4, we know that

$$\hat{X}_t = c_0(t) + \int_0^t g(s) dR_s \quad \text{for some } g \in L^2[0,t], c_0(t) \in \Re.$$

- Taking expectations, we see that

$$c_0(t) = E[\hat{X}_t] = E[X_t].$$

- Thus,

$$(X_t - \hat{X}_t) \perp \int_0^t f(s) dR_s \quad \text{for all } f \in L^2[0,t].$$

Proof of Lemma 6.2.7 Continued

- Therefore,

$$E\left[X_t\int_0^t f(s)dR_s\right]=E\left[\hat{X}_t\int_0^t f(s)dR_s\right]=E\left[\int_0^t g(s)dR_s\int_0^t f(s)dR_s\right]$$

$$=E\left[\int_0^t g(s)f(s)ds\right]=\int_0^t g(s)f(s)ds \quad \text{for all } f\in L^2[0,t].$$

- If we choose $f=\chi_{[0,r]}$ for some $r\le t$, we obtain:

$$E[X_tR_\tau]=\int_0^\tau g(s)ds \quad \text{or} \quad g(r)=\frac{\partial}{\partial\tau}E[X_tR_\tau].$$

Step 4: An Explicit Formula

- This is obtained by noting that:

$$X_t = \exp\left(\int_0^t F(s)ds\right)\left[X_0 + \int_0^t \exp\left(-\int_0^t F(u)du\right)C(s)dU_s\right]$$

$$= \exp\left(\int_0^t F(s)ds\right)X_0 + \int_0^t \exp\left(\int_s^t F(u)du\right)C(s)dU_s.$$

- Note that: $E[X_t] = E[X_0]\exp\left(\int_0^t F(s)ds\right).$

Step 4 Continued

* More generally,

for $0 \le r \le t$,

$$X_t = \exp\left(\int_r^t F(s)ds\right)X_0 + \int_r^t \exp\left(\int_s^t F(u)du\right)C(s)dU_s. \ (6.2.16)$$

Step 5: Mean Square Error

- From Lemma 6.2.7,

$$\hat{X}_t = E[X_t] + \int_0^t f(s,t)dR_s$$

$$\text{where} \quad f(s,t) = \frac{\partial}{\partial s} E[X_t R_s] \ (6.2.17)$$

- From (6.2.13) and (6.2.14),

$$R_s = \int_0^s \frac{G(r)}{D(r)}(X_r - \hat{X}_r)dr + V_s$$

Mean Square Error 2

* Then, we obtain:

$$E[X_t R_s] = \int_0^s \frac{G(r)}{D(r)} E[X_t \tilde{X}_r] dr \text{ where } \tilde{X}_r = X_r - \hat{X}_r \text{ (6.2.18).}$$

From (6.2.16), we obtain

$$E[X_t \tilde{X}_r] = \exp\left(\int_r^t F(v)dv\right) E[X_r \tilde{X}_r] = \exp\left(\int_r^t F(v)dv\right) S(r)$$

$$\text{where } S(r) = E[(\tilde{X}_r)^2] \text{ (6.2.19)}$$

Step 5: The Riccati Equation

- We claim that $S(t)$ satisfies:

$$\frac{dS}{dt} = 2F(t)S(t) - \frac{G^2(t)}{D^2(t)} S^2(t) + C^2(t). \quad \text{(The Riccati Equation)}$$

- Proof: Applying the Pythagorean theorem, (6.2.15) and the Ito isometry, we obtain

$$S(t) = E[(X_t - \hat{X}_t)^2] = E[X_t^2] - 2E[X_t \hat{X}_t] + E[\hat{X}_t^2]$$

$$= E[X_t^2] - E[\hat{X}_t^2]$$

$$= T(t) - \int_0^t f(s,t)^2 ds - E[X_t]^2 \quad \text{where } T(t) := E[X_t^2].$$

The Riccati Equation 2

* By (6.2.16) and the Ito isometry, we obtain:

$$T(t) = \exp\left(2\int_0^t F(s)ds\right)E[X_0^2] + \int_0^t \exp\left(2\int_s^t F(u)du\right)C^2(s)ds.$$

* Thus,

$$\frac{dT}{dt} = 2F(t)\exp\left(2\int_0^t F(s)ds\right)E[X_0^2] + C^2(t)$$

$$+\int_0^t 2F(t)\exp\left(2\int_0^t F(u)du\right)C^2(s)ds$$

$$= 2F(t)T(t) + C^2(t). \qquad (6.2.24)$$

The Riccati Equation 3

* By (6.2.22), (6.2.24) and Step 4, we obtain:

$$\frac{dS}{dt} = \frac{dT}{dt} - f(t,t)^2 - \int_0^t 2f(s,t)\frac{\partial}{\partial t} f(s,t)ds - 2F(t)E[X_t]^2$$

$$= 2F(t)T(t) + C^2(t) - \frac{G^2(t)S^2(t)}{D^2(t)} - 2\int_0^t f^2(s,t)F(t)ds - 2F(t)E[X_t]^2$$

$$= 2F(t)T(t) + C^2(t) - \frac{G^2(t)S^2(t)}{D^2(t)}.$$

Step 5: Stochastic Differential Equation

- From the formula:

$$\hat{X}_t = c_0(t) + \int_0^t f(s,t)dR_s \quad \text{where } c_0(t) = E[X_t]$$

- we obtain:

$$d\hat{X}_t = c_0{}'(t)dt + f(t,t)dR_t + \left(\int_0^t \frac{\partial}{\partial t} f(s,t)dR_s \right) dt, \quad (6.2.25)$$

- since

$$\int_0^u \left(\int_0^t \frac{\partial}{\partial t} f(s,t)dR_s \right) dt = \int_0^u \left(\int_s^u \frac{\partial}{\partial t} f(s,t)dt \right) dR_s$$

$$= \int_0^u \left(f(s,u) - f(s,s) \right) dR_s = \hat{X}_u - c_0(u) - \int_0^u f(s,s)dR_s.$$

SDE2

* Thus, since $c_0'(t) = F(t)c_0(t)$ (step 4)

$$d\hat{X}_t = c_0'(t)dt + \frac{G(t)S(t)}{D(t)}dR_t + \left(\int_0^t f(s,t)dR_s\right)F(t)dt$$

$$\text{or}\quad d\hat{X}_t = c_0'(t)dt + \frac{G(t)S(t)}{D(t)}dR_t + F(t)\cdot(\hat{X}_t - c_0(t))dt$$

$$= F(t)\hat{X}_t dt + \frac{G(t)S(t)}{D(t)}dR_t. \quad (6.2.26)$$

Final line

* As $$dR_t = \frac{1}{D(t)}\left[dZ_t - G(t)\hat{X}_t dt\right]$$

we obtain:

$$d\hat{X}_t = \left(F(t) - \frac{G^2(t)S(t)}{D^2(t)}\right)\hat{X}_t dt + \frac{G(t)S(t)}{D^2(t)} dZ_t.$$