Stochastic Differential Equations

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Definition*: 1-dimensional Ito process

* Let B_t be a 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. An Ito process is a stochastic process X_t of the form

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s$$

where $v \in \mathcal{W}_{\mathcal{H}}$, so that

$$P\left[\int_0^t v(s,\omega)^2 ds < \infty \text{ for all } t \geq 0\right] = 1$$

and u is $\mathcal{H}_t$ - adapted and

$$P\left[\int_0^t |u(s,\omega)| ds < \infty \text{ for all } t \geq 0\right] = 1$$

Theorem: 1-dimensional Ito formula

* Let X_t be an Ito process given by $dX_t = udt + vdB_t$. Let $g(t,x) \in C^2\left([0,\infty) \times \Re\right)$. Then, $Y_t = g(t, X_t)$ is again an Ito process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2$$

where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ is computed according to the rules

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \quad dB_t \cdot dB_t = dt$$

Theorem 5.2.1

Let $T > 0$ and $b:[0,T]\times\Re^n \to \Re^n$, $\sigma:[0,T]\times\Re^n \to \Re^{n\times m}$ be measurable functions satisfying

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|); \quad x\in\Re^n, t\in[0,T] \quad (5.2.1)$$

for some constant C, where $|\sigma^2| = \sum|\sigma_{ij}|^2$ and such that

$$|b(t,x)-b(t,y)| + |\sigma(t,x)-\sigma(t,y)| \le D|x-y|; \quad x,y\in\Re^n, t\in[0,T]. \quad (5.2.2)$$

for some constant D. Let Z be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty^{(m)}$ generated by $B_s, s\ge 0$ and such that

$$E[|Z^2|] < \infty.$$

Then, the stochastic differential equation

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t, 0\le t\le T, X_0 = Z. \quad (5.2.3)$$

has a unique t-continuous solution $X_t(\omega)$ with the following property:

Theorem 5.2.1 Continued

$X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and $B_s(\cdot); s \leq t$ and

$$E\left[\int_0^T |X_t|^2 \, dt\right] < \infty.$$

* Conditions (5.2.1) and (5.2.2) are natural in view of the following two simple examples from DDE.

Remark (a)

* The equation

$$\frac{dX_t}{dt} = X_t^2, X_0 = 1$$

has a unique solution

$$X_t = \frac{1}{1-t}; 0 \le t < 1.$$

Remark (b)

* The equation

$$\frac{dX_t}{dt} = 3X_t^{2/3};\, X_0 = 0 \; (5.2.7)$$

has more than one solution. In fact, for any $a > 0$ the function

$$X_t = \begin{cases} 0 & \text{for } t \le a \\ (t-a)^3 & \text{for } t > a \end{cases}$$

solves (5.2.7).

Proof of Theorem 5.2.1

* The uniqueness follows from the Ito isometry and the Lipschitz property.
* For all $\omega \in \Omega$, let

$$X_1(t,\omega) = X_t(\omega) \quad \text{and} \quad X_2(t,\omega) = \hat{X}_t(\omega)$$

with the initial values of

$$X_1(0,\omega) = Z(\omega) \quad \text{and} \quad X_2(0,\omega) = \hat{Z}_t(\omega)$$

Uniqueness Proof of Theorem 5.2.1

Let $a(s,\omega)=b(s,X_s)-b(s,\hat{X}_s)$ and $\gamma(s,\omega)=\sigma(s,X_s)-\sigma(s,\hat{X}_s)$

Then,

$$E[|X_t-\hat{X}_t|^2]=E\left[\left(Z-\hat{Z}+\int_0^t a\,ds+\int_0^t \gamma\,dB_s\right)^2\right]$$

$$\leq 3E[|Z-\hat{Z}|^2]+3E\left[\left(\int_0^t a\,ds\right)^2\right]+3E\left[\left(\int_0^t \gamma\,dB_s\right)^2\right]$$

$$\leq 3E[|Z-\hat{Z}|^2]+3tE\left[\int_0^t a^2\,ds\right]+3E\left[\int_0^t \gamma^2\,ds\right]$$

$$\leq 3E[|Z-\hat{Z}|^2]+3(1+t)D^2\int_0^t E[|X_t-\hat{X}_t|^2]ds.$$

Uniqueness Proof Continued

* So the function

$$v(t) = E[| X_t - \hat{X}_t |^2]; 0 \le t \le T$$

satisfies

$$v(t) \le F + A\int_0^t v(s)ds,$$

where $F = 3E[| Z - \hat{Z} |^2]$ and $A = 3(1+T)D^2.$

* By the Gronwall inequality, we conclude that

$$v(t) \le F \exp(At).$$

* Assume that $Z = \hat{Z}.$ Then, $F = 0$ and so $v(t) = 0$ for all $t \ge 0.$

The Last Step for Uniqueness

* Hence we obtain

$$P\left[\left|X_t - \hat{X}_t\right| = 0\ \forall t \in \mathbf{Q} \cap [0,T]\right] = 1,$$

where $\mathbf{Q}$ denotes the rational numbers.

* By continuity of $t \to |X_t - \hat{X}_t|$, it follows that

$$P\left[\left|X_1(t,\omega) - X_2(t,\omega)\right| = 0\ \forall t \in [0,T]\right] = 1.$$

Existence Proof

* Define $Y_t^{(0)} = X_0$ and $Y_t^{(k)} = Y_t^{(k)}(\omega)$ inductively by
$$Y_t^{(k+1)} = X_0 + \int_0^t b(s, Y_s^{(k)})ds + \int_0^t \sigma(s, Y_s^{(k)})dB_s. \quad (5.2.12)$$
* A similar computation as for the uniqueness proof gives
$$E[|Y_t^{(k+1)} - Y_t^{(k)}|^2] \le (1+T)3D^2 \int_0^t E[|Y_t^{(k+1)} - Y_t^{(k)}|^2]ds \text{ for } k \ge 1, t \le T$$
and
$$E[|Y_t^{(1)} - Y_t^{(0)}|^2] \le 2C^2t^2(1+E[|X_0|^2]) + 2C^2t(1+E[|X_0|^2]) \le A_1 t,$$
where A_1 only depends on C, T and $E[|X_0|^2]$.
* By induction, we obtain
$$E[|Y_t^{(k+1)} - Y_t^{(k)}|^2] \le \frac{A_2^{k+1}t^{k+1}}{(k+1)!}; \; k \ge 0, t \in [0, T]$$
where A_2 only depends on C, D, T and $E[|X_0|^2]$.

Existence Proof Continued

* Let λ denote Lebesgue measure on $[0,T]$ and $m > n \geq 0.$
* Then, we obtain

$$\left\| Y_t^{(m)} - Y_t^{(n)} \right\|_{L^2(\lambda \times P)} = \left\| \sum_{k=n}^{m-1} [Y_t^{(k+1)} - Y_t^{(k)}] \right\|_{L^2(\lambda \times P)}$$

$$\leq \sum_{k=n}^{m-1} \left\| [Y_t^{(k+1)} - Y_t^{(k)}] \right\|_{L^2(\lambda \times P)} = \sum_{k=n}^{m-1} \left(E\left[\int_0^T \left| Y_t^{(k+1)} - Y_t^{(k)} \right|^2 dt \right] \right)^{1/2}$$

$$\leq \sum_{k=n}^{m-1} \left(\int_0^T \frac{A_2^{k+1} t^{k+1}}{(k+1)!} dt \right)^{1/2} = \sum_{k=n}^{m-1} \left(\frac{A_2^{k+1} T^{k+2}}{(k+2)!} \right)^{1/2} \to 0 \quad \text{as} \quad m,n \to 0.$$

* Thus, $\left\{ Y_t^{(n)} \right\}_{n=0}^{\infty}$ is a Cauchy sequence in $L^2(\lambda \times P)$.

Existence Proof Continued

* Define the limit in $L^2(\lambda \times P)$ by $X_t := \lim_{n\to\infty} Y_t^{(n)}$.
* We claim that X_t satisfies (5.2.3).
* For all n and all $t \in [0,T]$, we have

$$Y_t^{(n+1)} = X_0 + \int_0^t b(s, Y_s^{(n)})ds + \int_0^t \sigma(s, Y_s^{(n)})dB_s.$$

* As $n \to \infty$, in $L^2(P)$, by the Holder inequality,

$$\int_0^t b(s, Y_s^{(n)})ds \to \int_0^t b(s, X_s)ds$$

and by the Ito isometry

$$\int_0^t \sigma(s, Y_s^{(n)})dB_s \to \int_0^t \sigma(s, X_s)dB_s.$$

The Last Step in Existence Proof

- It remains to show that X_t can be chosen to be continuous.
- By Theorem 3.2.5, there exists a continuous $\tilde{X}_t$ that for almost all ω satisfies:

$$\tilde{X}_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$

$$= X_0 + \int_0^t b(s, \tilde{X}_s)ds + \int_0^t \sigma(s, \tilde{X}_s)dB_s$$

Weak and Strong Solutions

* Strong solution: The version B_t of Brownian motion is given in advance and the solution X_t constructed from it is $\mathcal{F}_t^{(Z)}$-adapted.
* Weak solution: if we are given only the functions $b(t,x)$ and $\sigma(t,x)$, and ask for a pair of processes $((\tilde{X}_t, \tilde{B}_t), \mathcal{H}_t)$ on a probability space $(\Omega, \mathcal{H}, P)$ such that (5.2.3) holds, then the solution $\tilde{X}_t$ is called a weak solution.
* A strong solution is a weak solution
* But the reverse is not true.
* "Weak uniqueness" means that any two solutions are identical in law.

Lemma 5.3.1

* If b and σ satisfy the conditions of Theorem 5.2.1, then we have

A solution (weak or strong) of (5.2.3) is weakly unique.

* Sketch of Proof: Let

$$\left(\left(\tilde{X}_t, \tilde{B}_t\right), \widetilde{\mathcal{H}}_t\right) \quad \text{and} \quad \left(\left(\hat{X}_t, \hat{B}_t\right), \widehat{\mathcal{H}}_t\right)$$

be two weak solutions. Let

$$X_t \quad \text{and} \quad Y_t$$

be strong solutions constructed from $\tilde{B}_t$ and $\hat{B}_t$.

* The same uniqueness argument applies to show that

$$X_t = \tilde{X}_t \quad \text{and} \quad Y_t = \hat{X}_t \;\; \text{for all } t \;\; \text{a.s.}$$

Sketch of Proof Continued

* It suffices to show that

$$X_t \quad \text{and} \quad Y_t$$

are identical in law.

* We can show this by induction that if

$$X_t^{(k)} \quad \text{and} \quad Y_t^{(k)}$$

are the process in the Picard iteration defined by (5.2.12) with Brownian motions $\tilde{B}_t$ and $\hat{B}_t$, then

$$\left(X_t^{(k)}, \tilde{B}_t\right) \quad \text{and} \quad \left(Y_t^{(k)}, \hat{B}_t\right)$$

have the same law for all k.

The Tanaka equation

- Consider the 1-dimensional stochastic differential equation

$$dX_t = \operatorname{sign}(X_t)dB_t; \ \ X_0 = 0, \ (5.3.1)$$

where

$$\operatorname{sign}(x) = \begin{cases} +1 & \text{if } \ x \geq 0 \\ -1 & \text{if } \ x < 0 \end{cases}.$$

- The Tanaka equation does not have a strong solution.
- Proof: Let $\hat{B}_t$ be a Brownian motion generating the filtration and define

$$Y_t = \int_0^t \operatorname{sign}(\hat{B}_s)d\hat{B}_s.$$

Proof Continued

* By The Tanaka formula, we have

$$Y_t = |\hat{B}_t| - |\hat{B}_0| - \hat{L}_t(\omega)$$

where $\hat{L}_t(\omega)$ is the local time for $\hat{B}_t(\omega)$ at 0.

* It follows that Y_t is measurable w.r.t the σ-algebra $\mathcal{G}_t$ generated by $\left|\hat{B}_S(\cdot)\right|; s \le t$ which is strictly contained in $\widehat{\mathcal{F}}_t$.

* Hence, the σ-algebra $\mathcal{N}_t$ generated by $\left|Y_S(\cdot)\right|; s \le t$ is also strictly contained in $\widehat{\mathcal{F}}_t$.

Theorem 8.4.2

* An Ito process

$$dY_t = v dB_t; \quad Y_0 = 0 \quad \text{with } v(t,\omega) \in \mathcal{V}_{\mathcal{H}}^{n\times m}$$

coincides (in law) with n-dimensional Brownian motion if and only if

$$vv^T(t,\omega) = I_n \quad \text{for a.a. } (t,\omega) \text{ w.r.t. } dt \times dP$$

where I_n is the n-dimensional identity matrix.

Proof Continued

* Suppose that X_t is a strong solution.
* Then, by Theorem 8.4.2, it is a Brownian motion w.r.t. P.
* Let $\mathcal{M}_t$ be the σ -algebra generated by $|X_s(\cdot)|; s \le t$.
* Rewrite (5.3.1) as

$$dB_t = \text{sign}(X_t)dX_t.$$

* This contradicts that X_t is a strong solution because:
  * Let $\hat{B}_t = X_t, Y_t = B_t$ in this argument.
  * Then, $\mathcal{F}_t$ is strictly contained in $\mathcal{M}_t$.

Weak Solution

* Choose X_t to be any Brownian motion $\hat{B}_t$.
* Define $\tilde{B}_t$ by:

$$d\tilde{B}_t = \mathrm{sign}(X_t)dX_t.$$

* Then,

$$dX_t = \mathrm{sign}(X_t)d\tilde{B}_t.$$