Definition: Derivative

The **derivative of** $f(x)$ at $x = a$, denoted $f'(a)$, is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if the limit exists.
When the above limit exists, the function $f(x)$ is said to be **differentiable** at $x = a$.
When the limit does **not** exist, we say $f(x)$ is **not differentiable** at $x = a$.

Theorem: Differentiation Rules

Let $f(x)$ and $g(x)$ be differentiable functions and let $c \in \mathbb{R}$. Then:

$$\frac{d}{dx}[cf(x)] = cf'(x), \qquad \frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x).$$

Product Rule:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Note: the derivative of the quotient exists only at points where $g(x) \neq 0$.

Theorem: The Chain Rule

Let f and g be differentiable functions. Then the composition $y = f(g(x))$ is differentiable, and

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x).$$

Property: Derivative Rules

The following are standard derivative formulas:

$$\frac{d}{dx}[c] = 0,$$
$$\frac{d}{dx}[mx+b] = m,$$
$$\frac{d}{dx}[x^n] = nx^{n-1},$$
$$\frac{d}{dx}[\sin x] = \cos x,$$
$$\frac{d}{dx}[\cos x] = -\sin x,$$
$$\frac{d}{dx}[\tan x] = \frac{1}{\cos^2 x},$$
$$\frac{d}{dx}[e^x] = e^x,$$
$$\frac{d}{dx}[b^x] = b^x \ln b,$$
$$\frac{d}{dx}[\ln x] = \frac{1}{x}.$$

Theorem: Higher-order Derivatives

The **second derivative** of $f(x)$ is

$$f''(x) = \frac{d^2}{dx^2} f(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}.$$

The **n-th derivative** of $f(x)$ is

$$f^{(n)}(x) = \frac{d^n}{dx^n} f(x) = \lim_{h\to 0} \frac{f^{(n-1)}(x+h) - f^{(n-1)}(x)}{h}.$$

Method: Implicit Differentiation

When y is given implicitly as a function of x, differentiate both sides of the equation with respect to x and then solve the resulting equation for $\frac{dy}{dx}$.

Method: Logarithmic Differentiation

1. Take the natural logarithm on both sides of the equation, and use log properties to rewrite any "complicated expression" as a sum of simpler terms.
2. Differentiate both sides with respect to x.
3. Solve the resulting equation for $\frac{dy}{dx}$.

Method: Steps to Find a Tangent Line

To find the tangent line at the point (a, b):

1. Find the first derivative $f'(x)$.
2. Compute $f'(a)$; this is the slope of the tangent line at $x = a$.
3. Plug a, b, and $f'(a)$ into the point-slope form:
$$y - b = f'(a)(x - a).$$
4. Rearrange to slope–intercept form $y = mx + b$ if desired.

Difficulty: Easy ($\star$)

1. Find the derivatives of the following functions:

   (a) $f(x) = (x^3 - 1)(x + 1)$

   (b) $f(x) = \dfrac{\sqrt{x} + 1}{x^2 + 1}$

   (c) $f(x) = (3x^2 - 1)\left(x^2 - \dfrac{1}{x}\right)$

(d) $f(t) = \ln(x^{154})$

(e) $f(t) = \ln(3t + 1)$

(f) $f(t) = \ln(\sqrt{t^2 - 1})$

2. Find an equation of the tangent line to the given curve, at the given point.

   (a) $f(x) = \sqrt{2x - 1}$ at $(5, 3)$

   (b) $f(x) = e^x$ at $(0, 1)$

3. Find $\dfrac{dy}{dx}$ of the following equations.

(a) $x^2y^3 - xy = 10$

(b) $x^3y^5 + 3x = 8y^3 + 1$

(c) $y = \sin x + \cos y$

4. Consider the function $f(x) = 3x^2 + 4x + 30$. Compute the derivative of $f(x)$ when $x = 1$ by using the differentiation rule to first compute the derivative $f'(x)$ and then evaluate it at $x = 1$.

Difficulty: Intermediate ($\star\star$)

1. Consider the function $f(x) = 3x^2 + 4x + 30$. Compute the derivative of $f(x)$ by using the (limit) definition of the derivative and evaluate it at $x = 1$.

   HINT: It is easier to use this definition of derivative: $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

2. Find the derivatives of the following functions:

   (a) $f(x) = \sin(x^5 + 7x + 3)$

   (b) $f(x) = \ln(x - 9)\cos(e^x)$

   (c) $f(x) = \ln\left(\dfrac{\cos(3x + 1)}{x^3 - 2}\right)$

3. Find an equation of the tangent line to the given curve, at the given point.

(a) $f(x) = x\cos(x)$ at $(0, 0)$

(b) $f(x) = \ln(3x^2 - 11) - 5x$ at $(2, -10)$

4. Find an equation of the tangent line to the graph of $f(x) = \sqrt{x^2+3}$ at the point where $x = 1$.

5. Is $f(x) = |x - 10|$ differentiable at $x = 10$?
   If so, find its derivative at $x = 10$ using the definition of the derivative.
   If not, show that it's NOT differentiable at $x = 10$.

6. Find $f^{(n)}(x)$ for given $f(x)$ and n:

(a) $f(x) = 4x^2 - 2x + 1$ for $n = 2$

(b) $f(x) = x^4 - 2x^3 + 6x^2 - 3x + 10$ for $n = 2$

(c) $f(x) = \dfrac{x}{2x+1}$ for $n = 2$

7. Find $\dfrac{dy}{dx}$ of the following equations.

(a) $e^{2x+3y} = x^2 - \ln(xy^3)$

(b) $x^2 + y^2 = 25$ at $(3, -4)$

(c) $x^2 + 3xy + y^2 = -1$ at $(-1, 1)$

8. Use the logarithmic differentiation to find the derivative of $y = x^{x+2}$

9. Consider a manufacturing company that produces a type of product. The average cost function $(\bar{C}(q))$ for producing q units of the product is given by:

$$\bar{C}(q) = q^2 - 8q + 40$$

Use this information to determine the marginal cost function.

10. A small manufacturing company produces a type of widget. The cost (in dollars) to produce q units of the widget is given by the cost function:

$$c(q) = 0.02q^3 - 0.8q^2 + 15q + 500$$

(a) Find the marginal cost function.

(b) Estimate the cost of producing the 101st widget using the marginal cost function.

(c) Estimate the cost of producing the 101st widget using the linear approximation.

11. A tailor is currently producing 80 suits per month and sells them for \$100 per suit. His monthly demand curve is given by $q = 100 - 2\sqrt{p}$. Find the current price elasticity of demand, E_D, and use it to decide whether the price should be raised or lowered to increase the revenue.

12. Find the linear approximation of the function $f(x) = \sqrt{x}$ at $x = 4$ and use it to estimate $\sqrt{4.1}$.

13. Consider the circle $x^2+y^2 = 25$. At the point $(3, 4)$, if $dx = 0.1$, find the corresponding change in y, dy.

Difficulty: Challenging ($\star\star\star$)

1. Find $f^{(n)}(x)$ for given $f(x)$ and n:

   (a) $f(x) = \ln(x)$ for $n = 5$

   (b) $f(t) = \sin(x + 10)$ for $n = 20$

   (c) $f(x) = e^{10x}$ for $n = 101$

2. Use logarithmic differentiation to find the derivative of the function.

   (a) $y = (x-1)^2(x+1)^3(x+3)^4$

   (b) $y = \dfrac{\sqrt{4+3x^2}}{\sqrt[3]{x^2+1}}$

3. Find all value of x where the tangent to $y = 2x^3 + 9x^2 + 5$ has a slope of 24.

4. Find the formula for the kth derivative of $f(x) = x^\alpha$, when $\alpha \notin \mathbb{N}$

5. Consider the demand $q(p) = e^{-p}$, decreases very quickly with price. Determine the price when the marginal revenue is equal to zero.
HINT: Revenue = Price $\cdot$ Quantity

6. Find the quadratic function (i.e. $f(x) = ax^2 + bx + c$) that satisfies the followings:

   (a) $f'(0) = -4$

   (b) $f'(2) = 0$

   (c) $f(0) = 8$

7. Evaluate $\lim_{x \to 1} \dfrac{\sin(x^{154}) - \sin(1)}{x - 1}$

8. Find the coordinates of a point except $(0, 0)$ such that the equation

$$x^3 + y^3 = 6xy$$

have a horizontal tangent line.
HINT: Horizontal tangent line $\iff y' = 0$

9. Very challenging!

   Find the equation of the tangent line to $y = \dfrac{5}{x}$ which passes through $(5, 0)$.

   HINT: The point, $(5, 0)$ is NOT on $y = \dfrac{5}{x}$.