Theorem: Limit

Assume that $f(x)$ exists for all values x close (but not necessarily equal) to a. Then there is a

$$\lim_{x\to a} f(x) = L$$

if and only if

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L.$$

Method: Limit at Infinity

To evaluate the limit at infinity of a rational function we divide the numerator and denominator of the expression by x^n, where n is the highest power present in the denominator of the expression.

Definition: Continuity

A function f is **continuous** at a number a if

1. $f(a)$ is defined,
2. $\lim\limits_{x\to a} f(x)$ exists,
3. $f(a) = \lim\limits_{x\to a} f(x)$

A function f is **continuous on an interval** if it is continuous at every number in the interval.

Theorem: IVT (Existence of zeros)

Assume that

1. f is continuous on a closed interval $[a, b]$,
2. $f(a)$ and $f(b)$ have different signs.

Then there exists at least one solution of the equation $f(x) = 0$ in the interval (a, b).

Method: Evaluating the Limits

When evaluating limits, you may encounter these four scenarios:

1. If substituting the value into the function gives a result of $\dfrac{c}{\infty}$, where c is any real number, then the limit will be 0.
2. If the substitution results in $\dfrac{\pm\infty}{c}$, where c is a non-zero real number, then the limit will be $\pm\infty$.
3. If the substitution results in $\dfrac{c}{0}$, the limit could be either $+\infty$, $-\infty$, or it might not exist (DNE).
4. If the substitution results in an indeterminate form, such as $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, then you cannot directly determine the limit. In such cases, use algebraic techniques, such as factoring or cancelling terms, to simplify the function.

Difficulty: Easy ($\star$)

1. Suppose $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. Which of the following limits can you compute, given the information?

   (a) $\lim_{x\to a} \frac{f(x)}{5}$

   (b) $\lim_{x\to a} \frac{5}{f(x)}$

   (c) $\lim_{x\to a} \frac{f(x)}{g(x)}$

   (d) $\lim_{x\to a} f(x)g(x)$

2. $\lim_{x\to 6} \frac{4(x-6)^2}{x}$

3. $\lim_{x \to 2} \left(\frac{6x - 2}{2x + 1} \right)^3$

4. $\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$

5. $\displaystyle\lim_{r \to -5} \frac{r}{r^2 + 10r + 25}$

6. $\displaystyle\lim_{x \to \infty} \frac{5x^2 + 10}{3x^3 + 2x^2 + x}$

7. $\lim_{x \to -\infty} 2^x$

8. Suppose $f(t)$ is continuous at $t = 5$. Is it true or false that $t = 5$ is in the domain of $f(t)$?

9. Suppose $\lim_{t \to 5} f(t) = 17$, and suppose $f(t)$ is continuous at $t = 5$. Is it true or false that $f(5) = 17$?

10. Find all points for which the function is continuous $f(x) = \dfrac{1}{x^2 - 1}$

Difficulty: Indeterminate ($\star\star$)

1. Find two functions $f(x)$ and $g(x)$ that satisfy $\lim_{x\to 5} f(x) = \lim_{x\to 5} g(x) = 0$ and $\lim_{x\to 5} \frac{f(x)}{g(x)} = 3$.

2. Find two functions $f(x)$ and $g(x)$ that satisfy $\lim_{x\to 5} f(x) = \lim_{x\to 5} g(x) = 0$ and $\lim_{x\to 5} \frac{f(x)}{g(x)} = 0$.

3. $\lim_{h \to 0} \dfrac{(2+h)^2 - 4}{2h}$

4. $\lim_{x \to -1} \dfrac{\sqrt{x^2+8} - 3}{x+1}$

5. $\lim_{t \to 1} \frac{3t-3}{2-\sqrt{5-t}}$

6. $\lim_{t \to \frac{1}{2}} \frac{\frac{1}{3t^2}+\frac{1}{t^2-1}}{2t-1}$

7. Find all values of c such that the following function is continuous at $x = c$

$$f(x) = \begin{cases} 8 - cx, & x \leq c. \\ x^2, & x > c \end{cases}$$

8. Show that equation $x^3 = 20 + \sqrt{x}$ has a solution.

9. Show that equation $x^2 - 4x^3 + 1 = x - 7$ has a solution.

10. Suppose the demand function for a certain product is given by $D(p) = 1000 - 50p$, where D is the quantity demanded and p is the price per unit. The price per unit, p, cannot be less than zero.

    (a) Find the limit of the demand function as the price per unit (p) approaches zero.

    (b) Explain the economic significance of this limit in the context of the demand function.

11. A classmate from your Math 154 course attempted this problem. Identify if they've solved it correctly. If you find any errors, point out the step where the mistake occurred and provide the correct solution.
Problem:
Determine if there exists a root of the equation $f(x) = \frac{1}{x-2}$ in the interval $[1, 3]$ using the Intermediate Value Theorem.

Student's Solution:
Step 1: Note that $f(x) = \frac{1}{x-2}$ is continuous on $[1, 3]$.

Step 2: Evaluate $f(1)$ and $f(3)$:

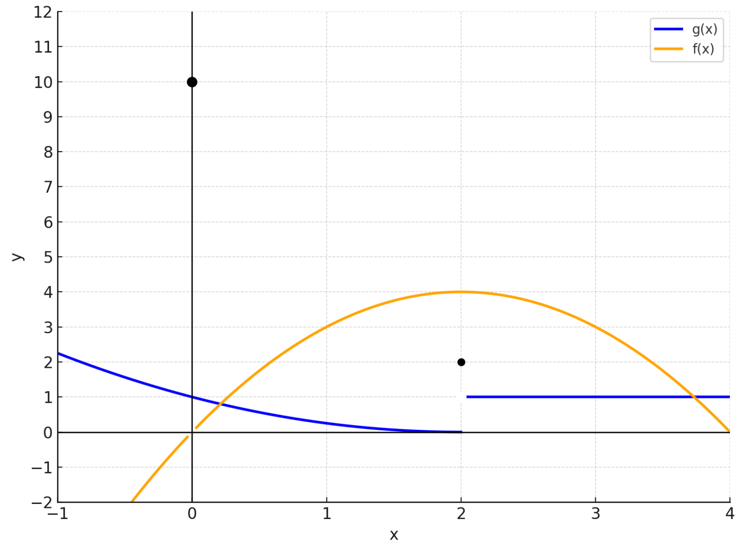
$$f(1) = \frac{1}{1-2} = -1$$

$$f(3) = \frac{1}{3-2} = 1$$

Step 3: Observe that $f(1) < 0$ and $f(3) > 0$.

Step 4: Conclude that, by the Intermediate Value Theorem, there exists a root in the interval $[1, 3]$.

12. Consider the graphs of the two functions f and g shown below.



Using the graph, evaluate the following limits or state that they do not exist.

(a) $\lim_{x \to 0} f(x)$

(b) $\lim_{x \to 2^-} g(x)$

(c) $\lim_{x \to 2^+} g(x)$

(d) $\lim_{x \to 2} g(x)$

(e) $\lim_{x \to 2^+} f\big(g(x)\big)$

Difficulty: Challenging ($\star\star\star$)

1. $\lim_{x \to 0} \left(10 + \frac{|x|}{x} \right)$

2. $\lim_{x \to \infty} \frac{\sqrt{3x^8 + 7x^4} + 10}{x^4 - 2x^2 + 1}$

3. $\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right)$

4. Find all values of c such that the following function is continuous at $x = c$

$$f(x) = \begin{cases} e^{cx} - e^{x}, & x \leq c. \\ xe^{x} - ce^{c}, & x > c \end{cases}$$

5. The total revenue $R(x)$ the company generates from selling the product is given by the function $R(x) = 100x$, where x represents the quantity of the product sold. The total cost $C(x)$ of producing x units of the product is given by the function $C(x) = 500+10x$.

   (a) Find the limit of the average profit per unit as x approaches infinity.

   (b) Explain its economic interpretation.

   Note that

$$\bar{P}(x) = \frac{P(x)}{x}$$

   where:

   – $\bar{P}(x)$is the average profit per unit for x units sold.

   – $P(x)$is the total profit for x units sold

   – x is the number of units sold

6. Let $f(x) = x^3 - 6x^2 + 9x - 3$. Prove that there exist at least two distinct roots of the equation $f(x) = 0$ in the interval $(1, 4)$. Then, provide an interval for each root such that the interval width is no greater than 1.

7. A company produces and sells a product with a price-demand relationship given by $D(p) = 2000 - 50p$, where D is the quantity demanded, and p is the price per unit. The company's production cost per unit is constant and equal to 5. The company wants to know if there exists a price in the quantity interval $[10, 30]$ at which its profit is exactly \$750.

   (a) Find the profit function, $P(q)$ where q is an quantity.

   (b) Show there exists a price in the quantity interval $[10, 30]$ for which the company's profit is exactly \$750.