# MT543 Topics in Algebra

Notes taken by Stephen Nulty and John Brennan

October 13, 2023

**Note:**

Any transcription mistakes and typos are my own.

Lectures by David Wraith. Lie Groups and Lie Algebras.

## Lecture 1 25/09/23

missed this lecture - some intro to do with spheres, transformations and symmetries and other motivational stuff. Definition of an algebra (bilinear product) over a field.

### 0. Introduction

Lie groups have a dual nature: they are groups but also very special topological spaces. The algebraic and topological (spatial) properties are closely aligned. Lie groups and Lie algebras lie at the intersection of algebra, topology, geometry, analysis and more.

**Definition 0.0.1.** *An algebra is a vector space $V$ equipped with a bilinear map $m : V \times V \to V$.*

Note, the "multiplication" map $m$ does not have to be commutative or associative. In general, Lie algebras are neither commutative nor associative. Recall,

$$\text{Commutativity: } m(u, v) = m(v, u),$$
$$\text{Associativity: } m(m(u, v), w) = m(u, m(v, w)).$$

Every Lie group has an associated Lie algebra which encodes many properties of the group. Often this allows problems about Lie groups to be reduced to problems in (fancy!) linear algebra.

## Example of a Lie group

The set of rotations of a ball centred on $O$ in $\mathbb{R}^3$ is a Lie group. It is a group under composition of rotations. To "see" the topology here, notice that it makes sense to talk about two rotations being "close", so there is a sense of space. It makes sense to consider a continuous family of rotations. Continuity implies the existence of topology. We can identify this group with the matrix group $SO(3)$. The map $\mathbb{R}^3 \to \mathbb{R}^3$ given by $x \mapsto Ax$ for $A \in SO(3)$ is a rotation and every rotation occurs in this way. $SO(3)$ is a subset (but not a subgroup) of the set/group of all $(3 \times 3)$-real matrices $M_3(\mathbb{R})$. By listing the elements of any $3 \times 3$ matrix we get a bijection $M_3(\mathbb{R}) \to \mathbb{R}^9$. As $\mathbb{R}^9$ has a natural topology (metric), this gives a natural topology on $M_3(\mathbb{R})$ and by restriction on $SO(3)$.

# Lecture 2 27/09/23

Sorting out tutorial times. Lectures: Monday 2pm MS2, Wednesday 2pm LGH, Thursday 12pm MS2.

Lie Groups, dual nature, Groups but also a topological geometrical character. Can prove things with a mix of both methods - intersection of various areas.

## 1. Groups of matrices

### 1.1. General Linear Groups

Quaternions will have a central role.

Consider groups of $N \times N$ matrices over the fields $\mathbb{R}$ and $\mathbb{C}$ and also over the quaternions.

**Definition 1.1.1.** *The quaternions $\mathbb{H}$ is a 4-dim real vector space with standard basis elements $1, i, j, k$, equipped with an associative linear multiplication operation defined by*

$$i^2 = j^2 = k^2 = -1, \quad ij = k, jk = i, ki = j$$

So a generic quaternion takes the form $a + bi + cj + dk$, $a, b, c, d \in \mathbb{R}$.

Observe, $ji = j(jk) = (jj)k$ (by associativity) $= j^2k = -k$. Similarly $kj = -i$ and $ik = -j$.

e.g. $(2 + i - 3k)(5 + 2i - j + k) = 10 + 4i - 2j + 2k + 5i - 2 - k - j - 15k - 6j - 61 + 3$ etc.

Quaternions is not commutative, so is not a field. However it is a skew field (division algebra).

Terminology - In $a+bi+cj+dk$, $a$ is called the real or scalar part, and the rest $bi+cj+dk$ imaginary or vector part.

In analogy with complex numbers,

**Definition 1.1.2.** *1. The conjugate of* $a+bi+cj+dk$ , *is* $\overline{a+bi+cj+dk} = a-bi-cj-dk$ *2. The norm of* $a+bi+cj+dk$ *is* $|a+bi+cj+dk| = \sqrt{a^2+b^2+c^2+d^2}$

Thus $\mathbb{H}$ is a normed vector space. Next observe that for each $q \in \mathbb{H}$ $q\bar{q} = \bar{q}q = |q|^2$.

therefore (symbol) $q^{-1} = \bar{q}/|q|^2$. So $qq^{-1} = q\bar{q}/|q|^2 = |q|^2 = 1$, similarly for $q^{-1}q = 1$.

This allows division $q_1 \cdot q_2^{-1} = q_1\bar{q_2}/|q_2|^2$. Writing $q_1/q_2$ is ambiguous however. $q_1q_2^{-1} \neq q_2^{-1}q_1$ generically.

Clearly $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$. A classic theorem of Frobenius asserts that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ are the only real associative division algebras. These objects similarly play a distinguished role in Lie group theory.

Convention: Suppose V is a vector space over the quaternions $\mathbb{H}$. We will adopt the convention that whenever we scale a vector $v \in V$ by a scalar $\lambda \in \mathbb{H}$, we multiply on the left, i.e. $\lambda v$

Let $M_n(\mathbb{R}), M_n(\mathbb{C}), M_n(\mathbb{H})$ denote the sets (vector spaces!) of all $n \times n$ matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

**Definition 1.1.3.** *The General Linear Groups* $GL_n(\mathbb{R})$, *resp.* $GL_n(\mathbb{C})$ *is the group of* $n \times n$ *invertible matrices with* $\mathbb{R}$ *resp* $\mathbb{C}$ *coefficients. (Group under multiplication). Equivalently* $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) \neq 0\}$. *Similarly* $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) | \det(A) \neq 0\}$.

(return to the idea of determinants of quaternions later).

Recall that for any matrix $A \in M_n(\mathbb{R})$ we have two associated linear maps $L_A : \mathbb{R}^n \to \mathbb{R}^n, L_a(\vec{x}) = A\vec{x}$, $R_A : \mathbb{R}^n \to \mathbb{R}^n, R_a(\vec{x}) = \vec{x}A$.

It is well know that $A$ is invertible (RC cases) $\iff \det(A) \neq 0 \iff L_A, R_A$ are isomorphisms.

# Lecture 3 02/10/23

Thursday lecture moved to Friday at 10am in MS2.

Reminder:

- Quaternions $\mathbb{H}$, multiplication is associative not commutative. If $V$ is a $\mathbb{H}$- vector space, we scale from the left only, i.e. $\lambda v$ for $\lambda \in \mathbb{H}, v \in V$.

- General linear groups $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$ groups under $*$ of all invertible $\mathbb{R}$ resp. $\mathbb{C}$ $n \times n$ - matrices.

- $A \in M_n(\mathbb{R}), M_n(\mathbb{C})$ is invertible iff $\det A \neq 0$ iff $L_A, R_A$ are both invertible where $L_a(\vec{x}) = A\vec{x}$, $R_a(\vec{x}) = \vec{x}A$.

We now consider $M_n(\mathbb{H})$.

**Definition 1.1.4.** *A function $f : \mathbb{H}^n \to \mathbb{H}^n$ is $\mathbb{H}$ - linear if $f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2)$, $\forall \lambda_1 \lambda_2 \in \mathbb{H}, v_1, v_2 \in \mathbb{H}^n$.*

**Lemma 1.1.5.** *For $A \in M_n(\mathbb{H})$, $R_A : \mathbb{H}^n \to \mathbb{H}^n$ given by $R_a(\vec{x}) = \vec{x}A$ for $v \in \mathbb{H}^n$ a row vector, is $\mathbb{H}$ - linear, however $L_A$ is in general not $\mathbb{H}$ - linear. Proof: exercise*

idea is that associativity makes $\lambda v A$ ok, but not with left multiplication which is interfered by commutativity.

**Lemma 1.1.6.** *For $A \in M_n(\mathbb{H})$, $R_A : \mathbb{H}^n \to \mathbb{H}^n$, is an $\mathbb{H}$ - linear isomorphism iff $A$ is invertible, i.e. $\exists B \in M_n(\mathbb{H})$ such that $AB = BA = I_n$.*

*Proof.* ($\Rightarrow$) If $R_A$ is an iso. then there is a $\mathbb{H}$-linear inverse $(R_A)^{-1} : \mathbb{H}^n \to \mathbb{H}^n$. There is a corresponding matrix $B \in M_n(\mathbb{H})$. Since $R_A \circ (R_A)^{-1} = R_A \circ (R_A)^{-1} = I_n$. we deduce $BA = AB = I_n$ (NB order of matrices here!). Therefore $B = A^{-1}$.

($\Leftarrow$) Similar. $\square$

**Definition 1.1.7.** *The quaternionic general linear group $GL_n(\mathbb{H}) = \{A \in M_n(\mathbb{H}) \,|\, A$ is invertible$\} = \{A \in M_n(\mathbb{H}) \,|\, R_a$ is an iso.$\}$*

NB: There is a problem with the notion of $\mathbb{H}$ - determinant due to non-commutativity we'll return to this later (possible to define determinant and $GL_n(\mathbb{H})$ as ones with non-zero determinant, but defining it requires some thought.)

It turns out that we can view $\mathbb{C}$ and $\mathbb{H}$ - matrices/linear maps in terms of $\mathbb{R}$ - matrices.

**Proposition 1.1.8.** *There is a real linear map $\rho_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ such that the following diagram commutes.*

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "\theta_n"] \arrow[d, "R_A"] & \mathbb{R}^{2n} \arrow[d, "R_{\rho_n(A)}"] \\ \mathbb{C}^n \arrow[r, "\theta_n"] & \mathbb{R}^{2n} \end{tikzcd}$$

*where* $\theta_n : \mathbb{C}^n \to \mathbb{R}^{2n}$ *is given by* $\theta_n(a_1+ib_1,\dots,a_n+ib_n) = (a_1,b_1,\dots,a_n,b_n)$.

(compactly every complex matrix can be viewed as a real matrix of twice the size)

Remark: $\theta_n$ is a real linear isomorphism. This forces $R_{\rho_n(A)} = \theta_n \circ R_A \circ \theta_n^{-1}$.

This is linear and therefore there is a corresponding matrix $\in M_{2n}(\mathbb{R})$.

*Proof.* See moodle. $\square$

**Observation 1.1.9.** $\rho_n$ *is injective. Proof: exercise.*

**Lemma 1.1.10.** $\rho_n$ *satisfies* $\rho_n(AB) = \rho_n(A)\rho_n(B)$. *So* $\rho_n$ *is an injective real-algebra homomorphism.*

*Proof.* We compose commutative squares from 1.1.8 to get

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, "\theta_n"] \arrow[d, "R_A"] & \mathbb{R}^{2n} \arrow[d, "R_{\rho_n(A)}"] \\
\mathbb{C}^n \arrow[r, "\theta_n"] \arrow[d, "R_B"] & \mathbb{R}^{2n} \arrow[d, "R_{\rho_n(B)}"] \\
\mathbb{C}^n \arrow[r, "\theta_n"] & \mathbb{R}^{2n}
\end{tikzcd}$$

On L.H.S. we have $R_B \circ R_A = R_{AB}$. (note order)

On R.H.S we have $R_{\rho_n(B)} \circ R_{\rho_n(A)} = R_{\rho_n(A)\rho_n(B)}$.

But since LHS is $R_{AB}$ this means $R_{\rho_n(AB)}$ = composition on RHS = $R_{\rho_n(A)\rho_n(B)}$. $\square$

It's not surjective however. Q: What exactly is $\rho_n(A)$? Consider $(a+ib) \in M_1(\mathbb{C})$.

$$R_{(a+ib)}(x+iy) = (x+iy)(a+ib) = (ax-by) + i(ay+bx)$$

Now $\theta_1(x+iy) = (x,y) \in \mathbb{R}^2$ etc.

So $\theta_1((ax-by)+i(ay+bx)) = (ax-by, ay+bx)$

The corresponding map from $\mathbb{R}^2 \to \mathbb{R}^2$ is $(x,y) \mapsto (ax-by, ay+bx)$. Observe that

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = (ax-by, ay+bx)$$

So $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M_2(\mathbb{R})$ corresponds under $\rho_1$ to $(a+ib) \in M_1(\mathbb{C})$.

More generally

$$\begin{pmatrix} a_{11}+ib_{11} & \dots & a_{1n}+ib_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1}+ib_{n1} & \dots & a_{nn}+ib_{nn} \end{pmatrix} \in M_n(\mathbb{C})$$

corresponds to

$$\left(\begin{array}{cc|c|cc} a_{11} & b_{11} & \multirow{2}{*}{$\cdots$} & a_{1n} & b_{1n} \\ -b_{11} & a_{11} & & -b_{1n} & a_{1n} \\ \hline \vdots & & \vdots & \vdots & \\ \hline a_{n1} & b_{n1} & \multirow{2}{*}{$\cdots$} & a_{nn} & b_{nn} \\ -b_{n1} & a_{n1} & & -b_{nn} & a_{nn} \end{array}\right) \in M_{2n}(\mathbb{R})$$

is obtained by replacing each $\mathbb{C}$ entry by its corresponding $2\times 2$ real block.

# Lecture 4 04/10/23

Last time:

- $A \in M_n(\mathbb{H})$ then $R_A : \mathbb{H}^n \to \mathbb{H}^n$ given by $R_a(\vec{x}) = \vec{x}A$. is $\mathbb{H}$- linear (assuming coefficients in $\mathbb{H}$ multiply on vectors from the left, x row vector). Left multiplication is not in general $\mathbb{H}$ linear.

- Under the real linear isomorphism $\theta_n : \mathbb{C}^n \to \mathbb{R}^{2n}$, $\theta_n(a_1+ib_1,\dots,a_n+ib_n) = (a_1,b_1,\dots,a_n,b_n)$. Any complex-linear map $\mathbb{C}^n \to \mathbb{C}^n$ corresponds to a real-linear map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and in terms of matrices (an right multiplication) $A \in M_n(\mathbb{C})$ corresponds to some matrix $\rho_n(A) \in M_{2n}(\mathbb{R})$.

- $$\text{If } A = \begin{pmatrix} a_{11}+ib_{11} & \dots & a_{1n}+ib_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1}+ib_{n1} & \dots & a_{nn}+ib_{nn} \end{pmatrix}$$

  then

  $$\rho_n(A) = \left(\begin{array}{cc|c|cc} a_{11} & b_{11} & \multirow{2}{*}{$\cdots$} & a_{1n} & b_{1n} \\ -b_{11} & a_{11} & & -b_{1n} & a_{1n} \\ \hline \vdots & & \vdots & \vdots & \\ \hline a_{n1} & b_{n1} & \multirow{2}{*}{$\cdots$} & a_{nn} & b_{nn} \\ -b_{n1} & a_{n1} & & -b_{nn} & a_{nn} \end{array}\right)$$

Consider the $\mathbb{C}$ linear map $\mathbb{C}^n \to \mathbb{C}^n$ given by $z \to zi$. This is $R_A$ where

$$A = \begin{pmatrix} i & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & i \end{pmatrix} = iI$$

For this matrix we have

$$\rho_n(A) = \left(\begin{array}{c|c|c} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \dots & \mathbf{0} \\ \hline \vdots & \vdots & \vdots \\ \hline \mathbf{0} & \dots & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{array}\right) = \mathcal{I}_n$$

A map $f : \mathbb{C}^n \to \mathbb{C}^n$ is $\mathbb{C}$ linear if it is real linear and $f(zi) = f(z)i$.

Let $bar f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the corresponding $\mathbb{R}$ linear map and suppose this has matrix $B \in M_{2n}(\mathbb{R})$. Then the complex linearity requirement is $R_B \circ R_{\mathcal{I}_n} = R_{\mathcal{I}_n} R_B$.

Since $R_X = R_Y \iff X = Y$ we see this is equivalent to asking $B\mathcal{I}_n = \mathcal{I}_n B$. i.e. $B \in M_{2n}(\mathbb{R})$ corresponds under $\theta_n$ to a complex linear map $\iff$ $B\mathcal{I}_n = \mathcal{I}_n B$.

We'd proved

**Corollary 1.1.11.** *The image of $\rho_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ is the set of all of all matrices in $M_{2n}(\mathbb{R})$ which commute with $\mathcal{I}_n$.*

Remark: This shows that $\rho_n$ is not surjective.

**Lemma 1.1.12.** *There is an injective group homomorphism $\rho_n : GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R})$, given by restricting $\rho_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$.*

*Proof.* We just have to check that if $A \in GL_n(\mathbb{C})$, then $\rho_n(A)$ is invertible. Clearly $\rho_n(AA^{-1}) = \rho_n(A^{-1}A) = \rho_n(I_n)$ so by 1.1.10. $\rho_n(A)\rho_n(A^{-1}) = \rho_n(A^{-1})\rho_n(A) = \rho_n(I_n) = I_{2n}$.

$\therefore$ $\rho_n(A^{-1}) = \rho_n(A)^{-1}$, hence $\rho_n(A) \in GL_{2n}(\mathbb{R})$. So $\rho_n : GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R})$, and by 1.1.10 this is a (multiplicative) group homomorphism $\square$

Now for quaternion matrices.

First observe that there is a $\mathbb{C}$ linear isomorphism $\phi_n : \mathbb{H}^n \to \mathbb{C}^{2n}$ given by $\phi_n(z_1 + w_1 j, \dots, z_n + w_n j) = (z_1, w_1, \dots, z_n, w_n)$.

(exercise to figure out $a + bi + cj + dk$ as $z + wj$, with $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$.)

**Proposition 1.1.13.** *There is an injective $\mathbb{C}$ linear map $\psi_n : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ s.t. the following square commutes:*

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{\phi_n} & \mathbb{C}^{2n} \\ \downarrow{\scriptstyle R_A} & & \downarrow{\scriptstyle R_{\psi_n(A)}} \\ \mathbb{H}^n & \xrightarrow{\phi_n} & \mathbb{C}^{2n} \end{array}$$

*i.e.* $\phi_n \circ R_A = R_{\psi_n(A)} \circ \phi_n$. *Moreover,* $\psi_n$ *satisfies* $\psi_n(AB) = \psi_n(A)\psi_n(B)$.

*Proof.* Analogous to that of prop 1.1.8 and lemma 1.1.10. Exercise! □

Remark: It is easily checked (exercise!) that $\psi_1(z + wj) = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$

More generally, image of $\psi_n$ consists of block matrices with blocks of this form (analogous to $\rho_n$).

By restricting to invertible matrices we obtain:

**Corollary 1.1.14.** *There is an injective group homomorphism* $\psi_n : GL_n(\mathbb{H}) \to GL_{2n}(\mathbb{C})$.

*Proof.* Analogous to 1.1.12 - exercise. □

( you can compose the maps then to get a real $4n$ matrix from a quaternionic one)

Composing $\rho_{2n}$ and $\psi_n$ gives

**Corollary 1.1.15.** *There is an injective* $\mathbb{R}$ *linear map resp. group homomorphism given by* $\rho_{2n} \circ \psi_n : M_n(\mathbb{H}) \to M_{4n}(\mathbb{R})$ *resp.* $\rho_{2n} \circ \psi_n : GL_n(\mathbb{H}) \to GL_{4n}(\mathbb{R})$.

Slogan: all groups of $\mathbb{H}$ or $\mathbb{C}$ matrices can be viewed as groups of real matrices!

**Definition 1.1.16.** *(1.1.16) For* $A \in M_n(\mathbb{H})$, $\det(A) := \det \psi_n(A)$.

# Lecture 5 06/10/23

(Talking about matrices and linear maps, and in $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, there's a standard basis given to go between linear maps and matrices.

$$\begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ \downarrow & & \downarrow \\ \mathbb{R}^{2n} & \longrightarrow & \mathbb{R}^{2n} \end{array}$$

and you can go between $\mathbb{R}$ and $\mathbb{C}$ with a canonical map, where you forget the complex structure going to $\mathbb{R}$ from $\mathbb{C}$ or by pairing up the pairs of reals going to $\mathbb{C}$.)

Last time:

- There is a canonical $\mathbb{C}$ linear isomorphism $\phi_n : \mathbb{H}^n \to \mathbb{C}^{2n}$ given by $\phi_n(z_1 + w_1 j, \ldots, z_n + w_n j) = (z_1, w_1, \ldots, z_n, w_n)$.
- There is an injective homomorphism of complex algebras $\psi_n : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{\phi_n} & \mathbb{C}^{2n} \\ \downarrow{\scriptstyle R_A} & & \downarrow{\scriptstyle R_{\psi_n(A)}} \\ \mathbb{H}^n & \xrightarrow{\phi_n} & \mathbb{C}^{2n} \end{array}$$

- If $A \in M_n(\mathbb{H})$ then $\det(A) := \det \psi_n(A)$.

**Proposition 1.1.17.** *(1.1.17 - need to fix the numbering) For $A \in M_n(\mathbb{H})$, $A$ is invertible $\iff \det A \neq 0$, i.e. $GL_n(\mathbb{H}) = \{A \in M_n(\mathbb{H}) \mid \det A \neq 0\}$.*

*Proof.* We claim that $A$ is invertible $\iff$ $\psi_n(A)$ is invertible.

($\Rightarrow$) This is immediate from the multiplicative properties of $\psi_n$ in 1.1.13.

($\Leftarrow$) In 1.1.14 we noted that the restricted map $\psi_n : GL_n(\mathbb{H}) \to GL_{2n}(\mathbb{C})$ is a group homomorphism. $\therefore$ if $\psi_n(A) \in GL_{2n}(\mathbb{C})$ (i.e. is invertible) for $A \in M_n(\mathbb{H})$, then since $\operatorname{im} \psi_n$ is a subgroup of $GL_{2n}(\mathbb{C})$, $\exists B \in GL_n(\mathbb{H})$ s.t. $\psi_n(B) = [\psi_n(A)]^{-1}$. (Want to show now that $B$ is the inverse of $A$.)

We have $\psi_n(AB) = \psi_n(A)\psi_n(B) = \psi_n(A)[\psi_n(A)]^{-1} = I_{2n}$. But $\psi_n$ is injective, so we must have $AB = I_n$. Similarly $BA = I$. $\therefore B = A^{-1}$ i.e. $A$ is invertible.

So the claim is true.

$$\therefore A \text{ is invertible} \underset{\text{claim}}{\iff} \psi_n \text{ is invertible}$$
$$\underset{\text{elementary linear algebra}}{\iff} \det \psi_n(A) \neq 0 \underset{\text{defn 1.1.16}}{\iff} \det A \neq 0$$

$\square$

### 1.2. Orthogonal Groups

From now on we will only consider (skew) fields $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ ($=\mathbb{F}$).

Recall that an inner product on a vector space $V$ over $\mathbb{F}$ is a map $\langle \, , \rangle : V \times V \to \mathbb{F}$ which is bilinear (i.e. linear in each entry) and is positive definite, i.e. $\langle v, v\rangle \geq 0 \, \forall v \in V$. If $v \neq 0, \langle v, v\rangle > 0$. e.g. dot product in $\mathbb{R}^n$.

**Definition 1.2.1.**

a *The $\underline{\text{standard inner product}}$ in $\mathbb{F}^n$ is given by* $\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = x_1\bar{y_1} + \ldots + x_n\bar{y_n}$, *where $\bar{y_i}$ is the conjugate of $y_i$. (If $y_i \in \mathbb{R}$ then $\bar{y_i} = y_i$). This is the dot product if $\mathbb{F} = \mathbb{R}$, and is called a $\underline{\text{Hermitian}}$ inner product if $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$.*

b *The standard basis for $\mathbb{F}^n$ is* $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, 1)$.

Remarks: As $x\bar{x} = |x|^2$ for all $x \in \mathbb{F}^n$ we see that $\langle x, x \rangle \geq 0 \forall x \in \mathbb{F}^n$ for the standard inner products.

For $\lambda \in \mathbb{F}$,

- $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$
- $\langle x, \lambda y \rangle = \langle x, y \rangle \bar{\lambda}$ (NB for $q_1, q_2 \in \mathbb{H}$ $\overline{q_1 q_2} = \bar{q_2}\bar{q_1}$.)
- $\overline{\langle x, y \rangle} = \langle y, x \rangle$.

In the real case, vectors $x, y$ are orthogonal if $\langle x, y \rangle = 0$. A basis $\{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$ is orthonormal if $|v_i| = 1 \forall i$ and $\langle v_i, v_j \rangle = 0$ for $i \neq j$.

Exactly the same language is used if $\mathbb{F} = \mathbb{C}, \mathbb{H}$.

**Lemma 1.2.2.** $\{v_1, \ldots, v_n\} \in \mathbb{C}^n$ *is a (Hermitian) orthonormal basis* $\iff$ $\{\theta_n(v_1), \theta_n(iv_1), \ldots, \theta_n(v_n), \theta_n(iv_n)\}$ *is an orthonormal basis for $\mathbb{R}^n$.*

*($\theta_n : \mathbb{C}^n \cong \mathbb{R}^{2n}$).*

*Proof.* An easy computation sows that

$$\underbrace{\langle x, y \rangle_{\mathbb{C}}}_{\text{Hermitian I.P. on } \mathbb{C}^n} = \underbrace{\langle \theta_n(x), \theta_n(y) \rangle_{\mathbb{R}}}_{\text{dot product on } \mathbb{R}^n} + i \langle \theta_n(x), \theta_n(iy) \rangle_{\mathbb{R}}$$

.

Thus $\langle x, y \rangle = 0 \iff \langle \theta_n(x), \theta_n(y) \rangle_{\mathbb{R}}$ and $\langle \theta_n(x), \theta_n(iy) \rangle_{\mathbb{R}}$. The result now follows easily. (Exercise: complete the argument) $\square$

# Lecture 6 09/10/23

$v \cdot v = |v|^2$ $v \cdot w = |v||w| \cos\theta$ Dot product tells you about lengths and angles.
Last time:

- standard inner product in $\mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n$ is given by $\langle x, y \rangle = \sum_{i=1}^{n} x_i \bar{y_i}$. ($\bar{y_i} = y_i$ if $y_i \in \mathbb{R}$)

- $\langle x,\, y\rangle_{\mathbb{C}^n} = \langle \theta_n(x),\, \theta_n(y)\rangle_{\mathbb{R}^{2n}} + i\,\langle \theta_n(x),\, \theta_n(iy)\rangle_{\mathbb{R}^{2n}}$
- (1.2.2) If $\{z_1, \ldots, z_n\}$ is an orthonormal basis for $\mathbb{C}^n$, then

  $$\{\theta_n(z_1), \theta_n(iz_1), \ldots, \theta_n(z_n), \theta_n(iz_n)\}$$

  is orthonormal for $\mathbb{R}^n$

**Lemma 1.2.3.** $\{q_1, \ldots, q_n)\}$ *is an orthonormal basis for* $\mathbb{H}^n \iff$

$$\{\theta_{2n} \circ \phi_n(q_1), \theta_{2n} \circ \phi_n(iq_1), \theta_{2n} \circ \phi_n(jq_1), \theta_{2n} \circ \phi_n(kq_1), \ldots,\\ \theta_{2n} \circ \phi_n(q_n), \theta_{2n} \circ \phi_n(iq_n), \theta_{2n} \circ \phi_n(jq_n), \theta_{2n} \circ \phi_n(kq_n)\}$$

*is orthonormal for* $\mathbb{R}^{4n}$*.*
*(*$\phi_n : \mathbb{H}^n \to \mathbb{C}^{2n},\ \theta_{2n} : \mathbb{C}^{2n} \to \mathbb{R}^{4n}$*)*

*Proof.* This follows in the manner of 1.2.2 from the easily established formula

$$\langle x,\, y\rangle_{\mathbb{H}^n} = \langle \theta_{2n} \circ \phi_n(x),\, \theta_{2n} \circ \phi_n(y)\rangle_{\mathbb{R}^{4n}} + i\,\langle \theta_{2n} \circ \phi_n(x),\, \theta_{2n} \circ \phi_n(iy)\rangle_{\mathbb{R}^{4n}}\\ + j\,\langle \theta_{2n} \circ \phi_n(x),\, \theta_{2n} \circ \phi_n(jy)\rangle_{\mathbb{R}^{4n}} + k\,\langle \theta_{2n} \circ \phi_n(x),\, \theta_{2n} \circ \phi_n(ky)\rangle_{\mathbb{R}^{4n}}$$

□

**Definition 1.2.4.** *1) The orthogonal group $O(n)$ is*

$$O(n) = \{A \in GL_n(\mathbb{R}) \mid \langle xA,\, yA\rangle_{\mathbb{R}^n} = \langle x,\, y\rangle_{\mathbb{R}^n}, \forall x, y \in \mathbb{R}^n\}$$

*2) The unitary group $U(n)$ is*

$$U(n) = \{A \in GL_n(\mathbb{C}) \mid \langle xA,\, yA\rangle_{\mathbb{C}^n} = \langle x,\, y\rangle_{\mathbb{C}^n}, \forall x, y \in \mathbb{C}^n\}$$

*3) The symplectic group $Sp(n)$ is*

$$Sp(n) = \{A \in GL_n(\mathbb{H}) \mid \langle xA,\, yA\rangle_{\mathbb{H}^n} = \langle x,\, y\rangle_{\mathbb{H}^n}, \forall x, y \in \mathbb{H}^n\}$$

Exercise: Show $O(n)\,, U(n)\,, Sp(n)$ are groups under multiplication.
Remark: We could also define

$$O(n) = \{A \in GL_n(\mathbb{R}) \mid \langle Ax,\, Ay\rangle_{\mathbb{R}^n} = \langle x,\, y\rangle_{\mathbb{R}^n}, \forall x, y \in \mathbb{R}^n\}$$
$$U(n) = \{A \in GL_n(\mathbb{C}) \mid \langle Ax,\, Ay\rangle_{\mathbb{C}^n} = \langle x,\, y\rangle_{\mathbb{C}^n}, \forall x, y \in \mathbb{C}^n\}$$

Exercise: show this agrees with 1.2.4

**Lemma 1.2.5.** *For* $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ *and* $A \in GL_n(\mathbb{F}^n)$. *The following are equivalent (tfae),*

*a)* $A \in O(n), U(n), Sp(n)$ *(as appropriate),*

*b)* $R_A$ *maps* $\mathbb{F}^n$*- orthonormal bases to* $\mathbb{F}^n$*-orthonormal bases,*

*c) The rows of* $A$ *from an orthonormal for* $\mathbb{F}^n$*,*

*d)* $AA^* = I_n$ *where* $A^* = \bar{A}^T$*.*

*Proof.* (a)$\Rightarrow$(b) This is clear since by definition, multiplication by $A$ preserves the inner product and in particular preserves orthogonality and lengths.

(b)$\Rightarrow$(c) The standard basis for $\mathbb{F}^n$ is orthonormal with respect to the standard inner product $\langle\ ,\ \rangle_{\mathbb{F}^n}$. By part (b) $R_A$ maps this orthonormal basis to another orthonormal basis. But $R_A(e_i) = i^{\text{th}}$ row of $A$. Hence rows of $A$ form an o.n. basis. (o.n. stands for orthonormal).

(c) $\iff$ (d) Observe that $(AA^*)_{ij} = \sum\limits_k a_{ik}a^*_{kj} = \sum\limits_k a_{ik}\bar{a}_{jk}$. The rows of $A$ being o.n. means

$$\left\langle i^{\text{th}} \text{ row of } A,\ j^{\text{th}} \text{ row of } A \right\rangle = \delta_{ij}$$

($\delta_{ii} = 1$, $\delta_{ij} = 0$ if $i \neq j$)
i.e. $\sum\limits_k^n a_{ik}\bar{a}_{jk} = \delta_{ij}$ (maybe color this to show it matches previous?)
i.e. $(AA^*)_{ij} = \delta_{ij} \iff AA^* = I_n$

(c)$\Rightarrow$(a) The rows of $A$ are o.n., i.e. $\sum\limits_k^n a_{ik}\bar{a}_{jk} = \delta_{ij}$.

$$\begin{aligned}
\langle xA,\ yA\rangle_{\mathbb{F}^n} &= \left\langle \left(\sum_l x_l a_{l1}, \ldots, \sum_l x_l a_{ln}\right), \left(\sum_m y_m a_{m1}, \ldots, \sum_m y_m a_{mn}\right)\right\rangle \\
&= \sum_{k,l,m} x_l a_{lk} \overline{y_m a_{mk}} \\
&= \sum_{k,l,m} x_l a_{lk} \overline{a_{mk} y_m} \\
&= \sum_{l,m} x_l \delta_{lm} \overline{y_m} \\
&= \sum_l x_l \bar{y}_l \\
&= \langle x,\ y\rangle_{\mathbb{F}^n}
\end{aligned}$$

$\square$

# Lecture 7 11/10/23

Previously:

$$O(n) = \{A \in M_n(\mathbb{R}) \,|\, \langle xA,\, yA\rangle_{\mathbb{R}^n} = \langle x,\, y\rangle_{\mathbb{R}^n}\,, \forall x, y \in \mathbb{R}^n\}$$
$$U(n) = \{A \in M_n(\mathbb{C}) \,|\, \langle xA,\, yA\rangle_{\mathbb{C}^n} = \langle x,\, y\rangle_{\mathbb{C}^n}\,, \forall x, y \in \mathbb{C}^n\}$$
$$Sp(n) = \{A \in M_n(\mathbb{H}) \,|\, \langle xA,\, yA\rangle_{\mathbb{H}^n} = \langle x,\, y\rangle_{\mathbb{H}^n}\,, \forall x, y \in \mathbb{H}^n\}$$

Note: other people might use $Sp$ and symplectic group for something related but not exactly the same (something like $Sp(n, \mathbb{F})$).

1.2.5 For $A \in GL_n(\mathbb{F}^n)$ t.f.a.e.

a) $A \in O(n)\,, U(n)\,, Sp(n)$

b) $R_A$ maps $\mathbb{F}$ orthonormal bases to $\mathbb{F}$ orthonormal bases.

c) Rows of $A$ are an o.n. basis for $\mathbb{F}^n$.

d) $AA^* = I$ where $A^* = \bar{A}^T$.

Remark: It follows from this that

$$A \in O(n) \iff A^{-1} = A^T$$
$$A \in U(n) \iff A^{-1} = \bar{A}^T$$
$$A \in Sp(n) \iff A^{-1} = \bar{A}^T$$

i.e. we could define $O(n) = \{A \in M_n(\mathbb{R}) \,|A^{-1} = A^T\}$.

(This follows from 1.2.5 (d), provided we can also show $A^*A = I$. The latter follows from the fact that if $AB = 1$, $(A, B \in M_n(\mathbb{F}))$ then $\Rightarrow BA = I$. Why is this?)

By definition a matrix $\in O(n)$ if the corresponding linear map preserves the dot product, i.e. preserves lengths and angles. What about $U(n)$ and $Sp(n)$?

**Proposition 1.2.6.**

*1)* $\rho_n(U(n)) = O(2n) \cap \rho_n(GL_n(\mathbb{C}))$.

*2)* $\psi_n(Sp(n)) = U(2n) \cap \psi_n(GL_n(\mathbb{H}))$.

*3)* $\rho_{2n}\circ\psi_n(Sp(n)) = O(4n)\cap\rho_{2n}\circ\psi_n(GL_n(\mathbb{H}))$.

*Proof.* See Moodle $\square$

This says that the real matrices corresponding to matrices in $U(n)$ and $Sp(n)$ are precisely the dot-product (i.e. length and angle) preserving transformation which have the form of the real version a complex/quaterionic matrix.

We now give an alternative description of $O(n)$, $U(n)$, $Sp(n)$.

Starting point: In $\mathbb{R}^n$, the norm and dot product determine each other via a "polarization" formula.

$$\langle x+y,\, x+y\rangle = \langle x,\, x\rangle + 2\langle x,\, y\rangle + \langle y,\, y\rangle$$

i.e. $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x,\, y\rangle$.

$\therefore \langle x,\, y\rangle = \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right)$.

This shows that we could define orthogonal transformation to be those preserving all norms.

For $U(n)$, $Sp(n)$, we have the following fact:

**Lemma 1.2.7.** *$\|x\|_{\mathbb{C}^n} = \langle x,\, x\rangle_{\mathbb{C}^n}$ is equal to $\|\theta_n(x)\|_{\mathbb{R}^{2n}}$. Similarly $\|x\|_{\mathbb{H}^n} = \|\theta_{2n}\circ\phi_n(x)\|_{\mathbb{R}^{4n}}$.*

*Proof.* Exercise. (Hint: $\langle x,\, y\rangle_{\mathbb{C}^n} = \langle\theta_n(x),\, \theta_n(y)\rangle_{\mathbb{R}^{2n}} + i\,\langle\theta_n(x),\, \theta_n(iy)\rangle_{\mathbb{R}^{2n}}$ etc.) $\square$

**Proposition 1.2.8.**

*1)* $O(n) = \{A\in GL_n(\mathbb{R})\,|\,\|xA\|_{\mathbb{R}^n} = \|x\|_{\mathbb{R}^n}, \forall x\in\mathbb{R}^n\}$.

*2)* $U(n) = \{A\in GL_n(\mathbb{C})\,|\,\|xA\|_{\mathbb{C}^n} = \|x\|_{\mathbb{C}^n}, \forall x\in\mathbb{C}^n\}$.

*3)* $Sp(n) = \{A\in GL_n(\mathbb{H})\,|\,\|xA\|_{\mathbb{H}^n} = \|x\|_{\mathbb{H}^n}, \forall x\in\mathbb{H}^n\}$.

*Proof.* (1) is established by the polarization formula.

(2) We need to show that $A\in GL_n(\mathbb{C})$ preserves all norms then $A\in U(n)$. (if it preserves inner products it automatically preserves norms). But if $\|xA\|_{\mathbb{C}^n} = \|x\|_{\mathbb{C}^n}\forall x\in\mathbb{C}^n$ we have

$$\begin{aligned}&\|\theta_n(xA)\|_{\mathbb{R}^{2n}} = \|\theta_n(x)\|_{\mathbb{R}^{2n}}\\ &\implies \|\theta_n(x)\rho_n(A)\|_{\mathbb{R}^{2n}} = \|\theta_n(x)\|_{\mathbb{R}^{2n}}\end{aligned}$$

$\therefore \rho_n(A)\in O(2n)$ by (1). By 1.2.6(1), $\rho_n(U(n)) = O(2n)\cap\rho_n(GL_n(\mathbb{C}))$ and since $\rho_n$ is injective, we conclude that $A\in U(n)$ as required.

$\square$

# Lecture 8 13/10/23

coming up: semi-direct products of groups, group actions (on sets), basic topology.

**Proposition 1.2.9.** *For $A \in O(n)\,, U(n)\,, Sp(n)$ we have* $|\det A| = 1$.

*Proof.* For $A \in O(n)$, $AA^T = I$ by 1.2.5(d). So $\det\left(AA^T\right) = \det(A)\det\left(A^T\right) = \det(A)^2 = \det I = 1$.

For $A \in U(n)$, $A\bar{A}^T = I$ by 1.2.5(d). So $\det\left(A\bar{A}^T\right) = \det(A)\det\left(\bar{A}^T\right) = \det(A)\det\left(\bar{A}\right)$. But $\det\left(\bar{A}\right) = \overline{\det(A)}$. To see this, note that $\det(A)$ is a sum of products of entries of $A$, and for complex conjugation, $\overline{a+b} = \bar{a} + \bar{b}$ and $\overline{ab} = \bar{a}\bar{b}$. So then $\det\left(A\bar{A}^T\right)\det(A)\overline{\det(A)} = |\det(A)|^2 = \det I = 1$. As $|z| \geq 0$, we see $|\det(A)| = 1$ implies $|\det(A)| = 1$.

For $A \in Sp(n)$, $\det A := \det_{\mathbb{C}} \psi_n(A)$. By 1.2.6(2) we know that $\psi_n(A) \in U(2n)$. $\therefore$ $\det \psi_n(A)\overline{\det \psi_n(A)} = \det\Big(\psi_n(A)\overline{\psi_n(A^T)}\Big) = \det I = 1$. $\therefore$ $|\det(\psi_n(A))|^2 = 1$, so $|\det(\psi_n(A))| = 1$ and $\therefore$ $|\det(A)| = 1$ $\square$

Remark For $A \in gon$, $\det A = \pm 1$.
For $A \in gun$, $\det A \in \{e^{i\theta} \mid \theta \in [0, 2\pi]\}$. (circle in complex plane).
For $A \in gspn$, it turns out that $\det A = 1$. This is not obvious!

**Definition 1.2.10.** *"Special Groups".*

*1) The special orthogonal group* $SO(n) = \{A \in O(n) \mid \det A = 1\}$,

*2) The special unitary group* $SU(n) = \{A \in U(n) \mid \det A = 1\}$,

*3) The special linear group* $SO(n) = \{A \in M_n(\mathbb{R})$ ( *or* $GL_n(\mathbb{R})$ ) $\mid \det A = 1\}$

Remark

1. $SL(n)$ could be also written $SL(n, \mathbb{R})$ or $SL_n(\mathbb{R})$. Also have complex version $SL(n, \mathbb{C})$ with the obvious definition.

2. The significance of $SL(n)$ is that this is the group of corresponding linear maps are precisely the volume preserving linear transformations. This follows from:

3. FACT: $A \in M_n(\mathbb{R})$. Then the volume of the parallelepiped determined by the vectors $R_A(e_1), \ldots, R_A(e_n)$ is $|\det A|$. Exercise: prove this fact! (vol $= |\det A|$.

   (insert vector picture).

4. $Sp(n)$ is already special!

Question What is the relationship between $SO(n)$ and $O(n)$, $SU(n)$ and $U(n)$?

**Theorem 1.2.11.** $U(n) = SU(n) \rtimes U(1)$.

$\rtimes$ is a semidirect product. Quick definition: $G$ is a semidirect product of subgroups $N, H$ if

1) $N \lhd G$,

2) $G = NH$ i.e. $G = \{nh \mid n \in N, h \in H\}$,

1) $N \cap H = \{e\}$.

Remark $U(1) = \{(e^{i\theta}) \mid \theta \in [0, 2\pi]\}$ which we identify with $\{e^{i\theta} \mid \theta \in [0, 2\pi]\}$. For the purposes of the theorem, we will identify $U(1)$ with $\begin{pmatrix} e^{i\theta} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$.

*Proof.* We observe that $SU(n) \lhd U(n)$ since $SU(n)$ is the kernel of $\det : U(n) \to \{e^{i\theta} \mid \theta \in [0, 2\pi]\}$. (det is a homomorphism and kernel of any homomorphism is normal.)

Next observe that any $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \in U(n)$ can be expressed as a product

(insert matrices)

As $A$ and (matrix 3) both $\in U(n)$, so is left hand matrix above.

Moreover, det of L.H. matrix is $\frac{1}{\det A}$ is $\frac{1}{\det A} \det A = 1$. $\therefore$ L.H. matrix $\in SU(n)$. It remains to show $SU(n) \cap U(1) = \{e\}$. □

This is clear since any matrix of the form (matrix 4) with $\det = 1$ must be $I_n$. Analogous arguments show

**Theorem 1.2.12.** $O(n) = SO(n) \rtimes \mathbb{Z}_2$.

*Proof.* exercise. □

# Bonus Lecture 1: Semidirect products of groups

Direct product: $G, H$ can form $G \times H = \{(g,h) \mid g \in G, h \in H\}$ with multiplication given by $(g,h)\cdot(g',h') = (gg', hh')$. (External direct product).

Now suppose that $G$ has subgroups $H, K$ such that $G = HK$, i.e. $G = \{hk \mid h \in H, k \in K\}$. What about multiplication in $G$?

$(hk)(h'k') =$? In the nicest case we might have $kh' = h'k$, so $(hk)(h'k') = hh'kk' \in HK$. This will work precisely when $H$ and $K$ commute with each other. In that case $G$ is an (internal) direct product of $H$ and $K$.

Definition If $H, K \subset G$, then G is a direct product of $H$ and $K$ if

1) $G = HK$
2) $H \cap K = \{e\}$ ( this means any expression $g = hk$ is unique)
3) $H, K$ commute with each other, i.e. $hk = kh$, $\forall h \in H, k \in K$,

   (3) is often replaced by

3') $H \lhd G$, $K \lhd G$. (not equivalent by itself with 1 and 2 it is)

Definition $G$ is a semidirect product of $K \subset G$ and $N \lhd G$ if (1) and (2) hold.

What's going on?

(Aside) $N \lhd G$ means $N$ is "normal" in $G$ i.e. $N \subset G$ with the following property $gNg^{-1} \subset N \forall g \in G$.

$\therefore$ for any $n \in N$ and any $g \in G$, $gng^{-1} = n'$ for some $n' \in N$, i.e. $gn = n'g$ (*).

If $G = NH$ then consider the product

$$\begin{aligned} &(n_1h_1)(n_2h_2) \\ =&n_1(h_1n_2)h_2 \quad \text{associativity} \\ \overset{*}{=}&n_1n_2'h_1h_2 \quad \in NH. \end{aligned}$$

Remark

(1) As a set $N \rtimes H$ is just $N \times H$. But as a group it is in general a "twisted" product.

(2) Example dihedral groups $D_n$, where $D_n$ is the group of symmetries of a regular $n$-gon. $|D_n| = 2n$. There are two types of symmetries here:

   i) rotations about centre: subgroup isomorphic to $\cong \mathbb{Z}_n$
   ii) flips : subgroup $\cong \mathbb{Z}_2$.

$D_n \cong Z_n \rtimes Z_2$ not a product since rotations and flips do not commute.

(3) There are several different looking ways to define a semidirect product. (e.g. short exact sequences, group extensions)

# Bonus Lecture 1: Group Actions

Let $X$ be a set, and let $\mathrm{Bij}(X)$ be the set of bijections $X \to X$. This a group under composition.

Definition An action of a group G on the set $X$ is a homomorphism $\alpha : G \to \mathrm{Bij}(X)$.

If X has some extra structure e.g. $X$ is a topological space, then we often replace $\mathrm{Bij}(X)$ with a group of bijections which preserve the extra structure.

For a topological space $X$ we consider $\alpha : G \to \underbrace{\mathrm{Homeo}(X)}$ group of "homeomorphisms".

Examples

i) $GL_n(\mathbb{R})$ acts on $\mathbb{R}^n$ by right or left multiplication. $\underbrace{R}$ right multiplication : $GL_n(\mathbb{R}) \to \underbrace{\mathrm{Isom}(\mathbb{R}^n)}$ group of linear isomorphisms of $\mathbb{R}^n$

ii) $\underbrace{\mathbb{Z}_2} = \{\pm 1\}$ acts on $\mathbb{R}$ by flips about 0. (picture of real line arrows indicating flips)

iii) $SO(2)$ acts on the circle $S^1$, $\alpha : SO(2) \to \mathrm{Homeo}(S^1)$. If $S^1 = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ then action is by matrix multiplication

$$\underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}} generic element of SO(2) \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} \cdot \\ \cdot \end{pmatrix}} again element of S^1$$

iv) $guk1 = \{e^{i\theta}\}$ =(circle) acts on 2-sphere $S^2$ by rotation (picture of sphere rotation through line connection poles say).

# Bonus Lecture 1: Topology

Everything topological in module can be interpreted in terms of metric spaces (but doing so might be needlessly cumbersome).

Recall a metric space consists of a set $X$ and a "metric" $d : X \times X \to [0, \infty)$ such that

- $d(x, y) = d(y, x)$

- $d(x, y) = 0 \iff x = y$
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangle).

$d$ is a distance function.

An open ball in $(X, d)$: $B(p, r) = \{x \in X \mid d(p, x) < r\}$, $p \in X$, $r > 0$.

Unions of open balls are called open sets. Metric allows us to define continuity and convergence.

It turns out that these notions can be describe using only open sets, e.g. $f : (X, d) \to (Y, d')$ is continuous $\iff$ for every open set $U \subset Y$, preimage $f^{-1}(U)$ is open in $X$. (*)

Open sets satisfy the following properties.

1. $X, \emptyset$ are open
2. unions of open sets are open
3. intersections of finitely many open sets are open.

Definition A topological space $(X, T)$ is a set $X$ together with a collection of subsets $T$ ("topology") satisfying (1), (2), (3).

Remark Every metric space is a topological space. Many different metrics on $X$ will generate the same topology (collection of open sets). Not every topology $T$ will arise from a metric (more general in a sense).

Definition A homeomorphism $f : (X, T) \to (Y, T')$ such that

1. $f$ is a bijection &
2. $f, f^{-1}$ are both continuous as in (*)

Other terms

1. A subset $C$ of $(X, T)$ is closed if $X \setminus C \in T$, i.e. $X \setminus C$ is open.
2. A subset $S \subset (X, T)$ is compact if every collection of $S$ by open sets in $X$ has a finite subcovering (i.e. still covering $S$). idea Compact is small and neat (highly non technical)
3. A topological space is path-connected if you can join any two points by a continuous path ($p : \underset{[0,1]}{\text{standard metric topology}} \to (X, T)$)
4. A space is simply connected if any continuous loop in space can be contracted through the space to a point. Write this as $\pi_1(\underset{X}{\text{space}}) = 0$.

   For any space $\pi_1(X)$ is the "fundamental group" of $X$ and measures the failure of loops to be contractible. (e.g. Circle has fundamental group isomorphic to $\mathbb{Z}$ ).