

# MT543 Topics in Algebra

Notes taken by Stephen Nulty

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## **Note:**

Any transcription mistakes and typos are my own.

Lectures by David Wraith. Lie Groups and Lie Algebras.

## **Lecture 1 25/09/23**

missed this lecture - some intro to do with spheres, transformations and symmetries and other motivational stuff. Definition of an algebra (bilinear product) over a field.

## **0. Something**

## **Lecture 2 27/09/23**

Sorting out tutorial times. Lectures: Monday 2pm MS2, Wednesday 2pm LGH, Thursday 12pm MS2.

Lie Groups, dual nature, Groups but also a topological geometrical character. Can prove things with a mix of both methods - intersection of various areas.

## **1. Groups of matrices**

### **1.1. General Linear Groups**

Quaternions will have a central role.

Consider groups of  $N \times N$  matrices over the fields  $\mathbb{R}$  and  $\mathbb{C}$  and also over the quaternions.

**Definition 1.1.1.** *The quaternions  $\mathbb{H}$  is a 4-dim real vector space with standard basis elements  $1, i, j, k$ , equipped with an associative linear multiplication operation defined by*

$$i^2 = j^2 = k^2 = -1, \quad ij = k, jk = i, ki = j$$

So a generic quaternion takes the form  $a + bi + cj + dk$ ,  $a, b, c, d \in \mathbb{R}$ .

Observe,  $ji = j(jk) = (jj)k$  (by associativity)  $= j^2k = -k$ . Similarly  $kj = -i$  and  $ik = -j$ .

e.g.  $(2 + i - 3k)(5 + 2i - j + k) = 10 + 4i - 2j + 2k + 5i - 2 - k - j - 15k - 6j - 6i + 3$  etc.

Quaternions is not commutative, so is not a field. However it is a skew field (division algebra).

Terminology - In  $a + bi + cj + dk$ ,  $a$  is called the real or scalar part, and the rest  $bi + cj + dk$  imaginary or vector part.

In analogy with complex numbers,

**Definition 1.1.2.** 1. *The conjugate of  $a + bi + cj + dk$ , is  $\overline{a + bi + cj + dk} = a - bi - cj - dk$*  2. *The norm of  $a + bi + cj + dk$  is  $|a + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2}$*

Thus  $\mathbb{H}$  is a normed vector space. Next observe that for each  $q \in \mathbb{H}$   $q\bar{q} = \bar{q}q = |q|^2$ .

therefore (symbol)  $q^{-1} = \bar{q}/|q|^2$ . So  $qq^{-1} = q\bar{q}/|q|^2 = |q|^2/|q|^2 = 1$ , similarly for  $q^{-1}q = 1$ .

This allows division  $q_1 \cdot q_2^{-1} = q_1\bar{q}_2/|q_2|^2$ . Writing  $q_1/q_2$  is ambiguous however.  $q_1q_2^{-1} \neq q_2^{-1}q_1$  generically.

Clearly  $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$ . A classic theorem of Frobenius asserts that  $\mathbb{R}, \mathbb{C}, \mathbb{H}$  are the only real associative division algebras. These objects similarly play a distinguished role in Lie group theory.

Convention: Suppose  $V$  is a vector space over the quaternions  $\mathbb{H}$ . We will adopt the convention that whenever we scale a vector  $v \in V$  by a scalar  $\lambda \in \mathbb{H}$ , we multiply on the left, i.e.  $\lambda v$

Let  $M_n(\mathbb{R}), M_n(\mathbb{C}), M_n(\mathbb{H})$  denote the sets (vector spaces!) of all  $n \times n$  matrices over  $\mathbb{R}, \mathbb{C}, \mathbb{H}$ .

**Definition 1.1.3.** *The General Linear Groups  $GL_n(\mathbb{R})$ , resp.  $GL_n(\mathbb{C})$  is the group of  $n \times n$  invertible matrices with  $\mathbb{R}$  resp  $\mathbb{C}$  coefficients. (Group under multiplication). Equivalently  $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) \neq 0\}$ . Similarly  $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) | \det(A) \neq 0\}$ .*

(return to the idea of determinants of quaternions later).

Recall that for any matrix  $A \in M_n(\mathbb{R})$  we have two associated linear maps  $L_A : \mathbb{R}^n \rightarrow \mathbb{R}^n, L_A(\vec{x}) = A\vec{x}, R_A : \mathbb{R}^n \rightarrow \mathbb{R}^n, R_A(\vec{x}) = \vec{x}A$ .

It is well known that  $A$  is invertible (RC cases)  $\iff \det(A) \neq 0$  iff  $L_A, R_A$  are isomorphisms.

## Lecture 3 02/10/23

Thursday lecture moved to Friday at 10am in MS2.

Reminder:

- Quaternions  $\mathbb{H}$ , multiplication is associative not commutative. If  $V$  is a  $\mathbb{H}$ -vector space, we scale from the left only, i.e.  $\lambda v$  for  $\lambda \in \mathbb{H}, v \in V$ .
- General linear groups  $GL_n(\mathbb{R}), GL_n(\mathbb{C})$  groups under  $*$  of all invertible  $\mathbb{R}$  resp.  $\mathbb{C} n \times n$ -matrices.
- $A \in M_n(\mathbb{R}), M_n(\mathbb{C})$  is invertible iff  $\det A \neq 0$  iff  $L_A, R_A$  are both invertible where  $L_A(\vec{x}) = A\vec{x}, R_A(\vec{x}) = \vec{x}A$ .

We now consider  $M_n(\mathbb{H})$ .

**Definition 1.1.4.** A function  $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$  is  $\mathbb{H}$ -linear if  $f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2), \forall \lambda_1, \lambda_2 \in \mathbb{H}, v_1, v_2 \in \mathbb{H}^n$ .

**Lemma 1.1.5.** For  $A \in M_n(\mathbb{H}), R_A : \mathbb{H}^n \rightarrow \mathbb{H}^n$  given by  $R_A(\vec{x}) = \vec{x}A$  for  $v \in \mathbb{H}^n$  a row vector, is  $\mathbb{H}$ -linear, however  $L_A$  is in general not  $\mathbb{H}$ -linear.  
Proof: exercise

idea is that associativity makes  $\lambda v A$  ok, but not with left multiplication which is interfered by commutativity.

**Lemma 1.1.6.** For  $A \in M_n(\mathbb{H}), R_A : \mathbb{H}^n \rightarrow \mathbb{H}^n$ , is an  $\mathbb{H}$ -linear isomorphism iff  $A$  is invertible, i.e.  $\exists B \in M_n(\mathbb{H})$  such that  $AB = BA = I_n$ .

*Proof.* ( $\Rightarrow$ ) If  $R_A$  is an iso. then there is a  $\mathbb{H}$ -linear inverse  $(R_A)^{-1} : \mathbb{H}^n \rightarrow \mathbb{H}^n$ . There is a corresponding matrix  $B \in M_n(\mathbb{H})$ . Since  $R_A \circ (R_A)^{-1} = R_A \circ (R_A)^{-1} = I_n$ . we deduce  $BA = AB = I_n$  (NB order of matrices here!). Therefore  $B = A^{-1}$ .

( $\Leftarrow$ ) Similar. □

**Definition 1.1.7.** The quaternionic general linear group  $GL_n(\mathbb{H}) = \{A \in M_n(\mathbb{H}) \mid A \text{ is invertible}\} = \{A \in M_n(\mathbb{H}) \mid R_A \text{ is an iso.}\}$

NB: There is a problem with the notion of  $\mathbb{H}$ -determinant due to non-commutativity we'll return to this later (possible to define determinant and gl as ones with non-zero determinant, but defining it requires some thought.)

It turns out that we can view  $\mathbb{C}$  and  $\mathbb{H}$ -matrices/linear maps in terms of  $\mathbb{R}$ -matrices.

**Proposition 1.1.8.** *There is a real linear map  $\rho_n : M_n(\mathbb{C}) \rightarrow M_{2n}(\mathbb{R})$  such that the following diagram commutes.*

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\theta_n} & \mathbb{R}^{2n} \\ \downarrow R_A & & \downarrow R_{\rho_n(A)} \\ \mathbb{C}^n & \xrightarrow{\theta_n} & \mathbb{R}^{2n} \end{array}$$

where  $\theta_n : \mathbb{C}^n \rightarrow \mathbb{R}^{2n}$  is given by  $\theta_n(a_1+ib_1, \dots, a_n+ib_n) = (a_1, b_1, \dots, a_n, b_n)$ .

(compactly every complex matrix can be viewed as a real matrix of twice the size)

Remark:  $\theta_n$  is a real linear isomorphism. This forces  $R_{\rho_n(A)} = \theta_n \circ R_A \circ \theta_n^{-1}$ .

This is linear and therefore there is a corresponding matrix  $\in M_{2n}(\mathbb{R})$ .

*Proof.* See moodle. □

**Observation 1.1.9.**  $\rho_n$  is injective. *Proof:* exercise.

**Lemma 1.1.10.**  $\rho_n$  satisfies  $\rho_n(AB) = \rho_n(A)\rho_n(B)$ . So  $\rho_n$  is an injective real-algebra homomorphism.

*Proof.* We compose commutative squares from 1.1.8 to get ... (insert diagram)

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\theta_n} & \mathbb{R}^{2n} \\ \downarrow R_A & & \downarrow R_{\rho_n(A)} \\ \mathbb{C}^n & \xrightarrow{\theta_n} & \mathbb{R}^{2n} \\ \downarrow R_B & & \downarrow R_{\rho_n(B)} \\ \mathbb{C}^n & \xrightarrow{\theta_n} & \mathbb{R}^{2n} \end{array}$$

On L.H.S. we have  $R_B \circ R_A = R_{AB}$ . (note order)

On R.H.S we have  $R_{\rho_n(B)} \circ R_{\rho_n(A)} = R_{\rho_n(A)\rho_n(B)}$ .

But since LHS is  $R_{AB}$  this means  $R_{\rho_n(AB)} =$  composition on RHS =  $R_{\rho_n(A)\rho_n(B)}$ . □

It's not surjective however. Q: What exactly is  $\rho_n(A)$ ? Consider  $(a+ib) \in M_1(\mathbb{C})$ .

$$R_{(a+ib)}(x+iy) = (x+iy)(a+ib) = (ax-by) + i(ay+bx)$$

Now  $\theta_1(x+iy) = (x, y) \in \mathbb{R}^2$  etc.

So  $\theta_1((ax-by) + i(ay+bx)) = (ax-by, ay+bx)$

The corresponding map from  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  is  $(x, y) \mapsto (ax-by, ay+bx)$ .

Observe that

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = (ax-by, ay+bx)$$

So  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M_2(\mathbb{R})$  corresponds under  $\rho_1$  to  $(a+ib) \in M_1(\mathbb{C})$ .

More generally

$$\begin{pmatrix} a_{11}+ib_{11} & \dots & a_{1n}+ib_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1}+ib_{n1} & \dots & a_{nn}+ib_{nn} \end{pmatrix} \in M_n(\mathbb{C})$$

corresponds to

$$\left( \begin{array}{cc|ccc|cc} a_{11} & b_{11} & & & & a_{1n} & b_{1n} \\ -b_{11} & a_{11} & & & & -b_{1n} & a_{1n} \\ \hline & \vdots & & & & & \vdots \\ a_{n1} & b_{n1} & & & & a_{nn} & b_{nn} \\ -b_{n1} & a_{n1} & & & & -b_{nn} & a_{nn} \end{array} \right) \in M_{2n}(\mathbb{R})$$

is obtained by replacing each  $\mathbb{C}$  entry by its corresponding  $2 \times 2$  real block.

## Lecture 4 04/10/23

Last time:

- $A \in M_n(\mathbb{H})$  then  $R_A : \mathbb{H}^n \rightarrow \mathbb{H}^n$  given by  $R_A(\vec{x}) = \vec{x}A$  is  $\mathbb{H}$ -linear (assuming coefficients in  $\mathbb{H}$  multiply on vectors from the left, c row vector). Left multiplication is not in general  $\mathbb{H}$  linear.
- Under the real linear isomorphism  $\theta_n : \mathbb{C}^n \rightarrow \mathbb{R}^{2n}$ ,  $\theta_n(a_1+ib_1, \dots, a_n+ib_n) = (a_1, b_1, \dots, a_n, b_n)$ . Any complex-linear map  $\mathbb{C}^n \rightarrow \mathbb{C}^n$  corresponds to a real-linear map  $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  and in terms of matrices (an right multiplication)  $A \in M_n(\mathbb{C})$  corresponds to some matrix  $\rho_n(A) \in M_{2n}(\mathbb{R})$ .

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$$\text{If } A = \begin{pmatrix} a_{11} + ib_{11} & \dots & a_{1n} + ib_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} + ib_{n1} & \dots & a_{nn} + ib_{nn} \end{pmatrix}$$

then

$$\rho_n(A) = \left( \begin{array}{cc|ccc|cc} a_{11} & b_{11} & & & & a_{1n} & b_{1n} \\ -b_{11} & a_{11} & & & & -b_{1n} & a_{1n} \\ \hline & \vdots & & & & & \vdots \\ a_{n1} & b_{n1} & & & & a_{nn} & b_{nn} \\ -b_{n1} & a_{n1} & & & & -b_{nn} & a_{nn} \end{array} \right)$$

Consider the  $\mathbb{C}$  linear map  $\mathbb{C}^n \rightarrow \mathbb{C}^n$  given by  $z \rightarrow zi$ . This is  $R_A$  where

$$A = \begin{pmatrix} i & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & i \end{pmatrix} = iI$$

For this matrix we have

$$\rho_n(A) = \left( \begin{array}{cc|ccc|cc} 0 & 1 & & & & \mathbf{0} & \\ -1 & 0 & & & & & \\ \hline & \vdots & & & & & \vdots \\ \mathbf{0} & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{array} \right) = \mathcal{I}_n$$

A map  $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is  $\mathbb{C}$  linear if it is real linear and  $f(zi) = f(z)i$ .

Let  $\text{bar } f : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  be the corresponding  $\mathbb{R}$  linear map and suppose this has matrix  $B \in M_{2n}(\mathbb{R})$ . Then the complex linearity requirement is  $R_B \circ R_{\mathcal{I}_n} = R_{\mathcal{I}_n} R_B$ .

Since  $R_X = R_Y \iff X = Y$  we see this is equivalent to asking  $B\mathcal{I}_n = \mathcal{I}_n B$ . i.e.  $B \in M_{2n}(\mathbb{R})$  corresponds under  $\theta_n$  to a complex linear map  $\iff B\mathcal{I}_n = \mathcal{I}_n B$ .

We'd proved

**Corollary 1.1.11.** (1.1.11) The image of  $\rho_n : M_n(\mathbb{C}) \rightarrow M_{2n}(\mathbb{R})$  is the set of all of all matrices in  $M_{2n}(\mathbb{R})$  which commute with  $\mathcal{I}_n$ .

Remark: This shows that  $\rho_n$  is not surjective.

**Lemma 1.1.12.** There is an injective group homomorphism  $\rho_n : GL_n(\mathbb{C}) \rightarrow GL_{2n}(\mathbb{R})$ , given by restricting  $\rho_n : M_n(\mathbb{C}) \rightarrow M_{2n}(\mathbb{R})$ .

*Proof.* We just have to check that if  $A \in GL_n(\mathbb{C})$ , then  $\rho_n(A)$  is invertible. Clearly  $\rho_n(AA^{-1}) = \rho_n(A^{-1}A) = \rho_n(I_n)$  so by 1.1.10.  $\rho_n(A)\rho_n(A^{-1}) = \rho_n(A^{-1})\rho_n(A) = \rho_n(I_n) = I_{2n}$ .

$\therefore \rho_n(A^{-1}) = \rho_n(A)^{-1}$ , hence  $\rho_n(A) \in GL_{2n}(\mathbb{R})$ . So  $\rho_n : GL_n(\mathbb{C}) \rightarrow GL_{2n}(\mathbb{R})$ , and by 1.1.10 this is a (multiplicative) group homomorphism  $\square$

Now for quaternion matrices.

First observe that there is a  $\mathbb{C}$  linear isomorphism  $\phi_n : \mathbb{H}^n \rightarrow \mathbb{C}^{2n}$  given by  $\phi_n(z_1 + w_1j, \dots, z_n + w_nj) = (z_1, w_1, \dots, z_n, w_n)$ .

(exercise to figure out  $a + bi + cj + dk$  as  $z + wj$ , with  $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$ .)

**Proposition 1.1.13.** (1.1.13) *There is an injective  $\mathbb{C}$  linear map  $\psi_n : M_n(\mathbb{H}) \rightarrow M_{2n}(\mathbb{C})$  s.t. the following square commutes:*

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{\phi_n} & \mathbb{C}^{2n} \\ \downarrow R_A & & \downarrow R_{\psi_n(A)} \\ \mathbb{H}^n & \xrightarrow{\phi_n} & \mathbb{C}^{2n} \end{array}$$

i.e.  $\phi_n \circ R_A = R_{\psi_n(A)} \circ \phi_n$ . Moreover,  $\psi_n$  satisfies  $\psi_n(AB) = \psi_n(A)\psi_n(B)$ .

*Proof.* Analogous to that of prop 1.1.8 and lemma 1.1.10. Exercise!  $\square$

Remark: It is easily checked (exercise!) that  $\psi_1(z + wj) = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$

More generally, image of  $\psi_n$  consists of block matrices with blocks of this form (analogous to  $\rho_n$ ).

By restricting to invertible matrices we obtain:

**Corollary 1.1.14.** 1.1.14 *There is an injective group homomorphism  $\psi_n : GL_n(\mathbb{H}) \rightarrow GL_{2n}(\mathbb{C})$ .*

*Proof.* Analogous to 1.1.12 - exercise.  $\square$

( you can compose the maps then to get a real  $4n$  matrix from a quaternionic one)

Composing  $\rho_{2n}$  and  $\psi_n$  gives

**Corollary 1.1.15.** 1.1.15 *There is an injective  $\mathbb{R}$  linear map resp. group homomorphism given by  $\rho_{2n} \circ \psi_n : M_n(\mathbb{H}) \rightarrow M_{4n}(\mathbb{R})$  resp.  $\rho_{2n} \circ \psi_n : GL_n(\mathbb{H}) \rightarrow GL_{4n}(\mathbb{R})$ .*

Slogan: all groups of  $\mathbb{H}$  or  $\mathbb{C}$  matrices can be viewed as groups of real matrices!

**Definition 1.1.16.** 1.1.16 *For  $A \in M_n(\mathbb{H})$ ,  $\det(A) := \det \psi_n(A)$ .*