MT543 Topics in Algebra

Notes taken by Stephen Nulty

October 4, 2023

Note:

Any transcription mistakes and typos are my own.

Lectures by David Wraith. Lie Groups and Lie Algebras.

Lecture 1 25/09/23

missed this lecture - some intro to do with spheres, transformations and symmetries and other motivational stuff. Definition of an algebra (bilinear product) over a field.

0. Something

Lecture 2 27/09/23

Sorting out tutorial times. Lectures: Monday 2pm MS2, Wednesday 2pm LGH, Thursday 12pm MS2.

Lie Groups, dual nature, Groups but also a topological geometrical character. Can prove things with a mix of both methods - intersection of various areas.

1. Groups of matrices

1.1. General Linear Groups

Quaternions will have a central role.

Consider groups of $N \times N$ matrices over the fields $\mathbb{R}$ and $\mathbb{C}$ and also over the quaternions.

Definition 1.1.1. *The quaternions* $\mathbb{H}$ *is a 4-dim real vector space with standard basis elements* $1, i, j, k$*, equipped with an associative linear multiplication operation defined by*

$$i^2 = j^2 = k^2 = -1, \quad ij = k, jk = i, ki = j$$

So a generic quaternion takes the form $a + bi + cj + dk$, $a, b, c, d \in \mathbb{R}$.

Observe, $ji = j(jk) = (jj)k$ (by associativity) $= j^2k = -k$. Similarly $kj = -i$ and $ik = -j$.

e.g. $(2 + i - 3k)(5 + 2i - j + k) = 10 + 4i - 2j + 2k + 5i - 2 - k - j - 15k - 6j - 61 + 3$ etc.

Quaternions is not commutative, so is not a field. However it is a skew field (division algebra).

Terminology - In $a + bi + cj + dk$, a is called the real or scalar part, and the rest $bi + cj + dk$ imaginary or vector part.

In analogy with complex numbers,

Definition 1.1.2. *1. The conjugate of* $a+bi+cj+dk$ *, is* $\overline{a + bi + cj + dk} = a - bi - cj - dk$ *2. The norm of* $a + bi + cj + dk$ *is* $|a + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2}$

Thus $\mathbb{H}$ is a normed vector space. Next observe that for each $q \in \mathbb{H}$ $q\bar{q} = \bar{q}q = |q|^2$.

therefore (symbol) $q^{-1} = \bar{q}/|q|^2$. So $qq^{-1} = q\bar{q}/|q|^2 = |q|^2 = 1$, similarly for $q^{-1}q = 1$.

This allows division $q_1 \cdot q_2^{-1} = q_1\bar{q_2}/|q_2|^2$. Writing q_1/q_2 is ambiguous however. $q_1q_2^{-1} \neq q_2^{-1}q_1$ generically.

Clearly $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$. A classic theorem of Frobenius asserts that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ are the only real associative division algebras. These objects similarly play a distinguished role in Lie group theory.

Convention: Suppose V is a vector space over the quaternions $\mathbb{H}$. We will adopt the convention that whenever we scale a vector $v \in V$ by a scalar $\lambda \in \mathbb{H}$, we multiply on the left, i.e. λv

Let $M_n(\mathbb{R}), M_n(\mathbb{C}), M_n(\mathbb{H})$ denote the sets (vector spaces!) of all $n \times n$ matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

Definition 1.1.3. *The General Linear Groups* $GL_n(\mathbb{R})$*, resp.* $GL_n(\mathbb{C})$ *is the group of* $n \times n$ *invertible matrices with* $\mathbb{R}$ *resp* $\mathbb{C}$ *coefficients. (Group under multiplication). Equivalently* $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) \neq 0\}$*. Similarly* $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) | \det(A) \neq 0\}$.

(return to the idea of determinants of quaternions later).

Recall that for any matrix $A \in M_n(\mathbb{R})$ we have two associated linear maps $L_A : \mathbb{R}^n \to \mathbb{R}^n, L_a(\vec{x}) = A\vec{x}$, $R_A : \mathbb{R}^n \to \mathbb{R}^n, R_a(\vec{x}) = \vec{x}A$.

It is well know that A is invertible (RC cases) $\iff$ $\det(A) \neq 0$ $\iff$ L_A, R_A are isomorphisms.

Lecture 3 02/10/23

Thursday lecture moved to Friday at 10am in MS2.

Reminder:

- Quaternions $\mathbb{H}$, multiplication is associative not commutative. If V is a $\mathbb{H}$- vector space, we scale from the left only, i.e. λv for $\lambda \in \mathbb{H}, v \in V$.
- General linear groups $GL_n(\mathbb{R}), GL_n(\mathbb{C})$ groups under $*$ of all invertible $\mathbb{R}$ resp. $\mathbb{C}$ $n \times n$- matrices.
- $A \in M_n(\mathbb{R}), M_n(\mathbb{C})$ is invertible iff $\det A \neq 0$ iff L_A, R_A are both invertible where $L_a(\vec{x}) = A\vec{x}$, $R_a(\vec{x}) = \vec{x}A$.

We now consider $M_n(\mathbb{H})$.

Definition 1.1.4. *A function $f : \mathbb{H}^n \to \mathbb{H}^n$ is $\mathbb{H}$ - linear if $f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2)$, $\forall \lambda_1 \lambda_2 \in \mathbb{H}, v_1, v_2 \in \mathbb{H}^n$.*

Lemma 1.1.5. *For $A \in M_n(\mathbb{H})$, $R_A : \mathbb{H}^n \to \mathbb{H}^n$ given by $R_a(\vec{x}) = \vec{x}A$ for $v \in \mathbb{H}^n$ a row vector, is $\mathbb{H}$ - linear, however L_A is in general not $\mathbb{H}$ - linear. Proof: exercise*

idea is that associativity makes $\lambda v A$ ok, but not with left multiplication which is interfered by commutativity.

Lemma 1.1.6. *For $A \in M_n(\mathbb{H})$, $R_A : \mathbb{H}^n \to \mathbb{H}^n$, is an $\mathbb{H}$ - linear isomorphism iff A is invertible, i.e. $\exists B \in M_n(\mathbb{H})$ such that $AB = BA = I_n$.*

Proof. ($\Rightarrow$) If R_A is an iso. then there is a $\mathbb{H}$-linear inverse $(R_A)^{-1} : \mathbb{H}^n \to \mathbb{H}^n$. There is a corresponding matrix $B \in M_n(\mathbb{H})$. Since $R_A \circ (R_A)^{-1} = R_A \circ (R_A)^{-1} = I_n$. we deduce $BA = AB = I_n$ (NB order of matrices here!). Therefore $B = A^{-1}$.

($\Leftarrow$) Similar. $\square$

Definition 1.1.7. *The quaternionic general linear group $GL_n(\mathbb{H}) = \{A \in M_n(\mathbb{H}) | A$ is invertible$\} = \{A \in M_n(\mathbb{H}) | R_a$ is an iso.$\}$*

NB: There is a problem with the notion of $\mathbb{H}$- determinant due to non-commutativity we'll return to this later (possible to define determinant and gl as ones with non-zero determinant, but defining it requires some thought.)

It turns out that we can view $\mathbb{C}$ and $\mathbb{H}$- matrices/linear maps in terms of $\mathbb{R}$- matrices.

Proposition 1.1.8. *There is a real linear map $\rho_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ such that the following diagram commutes.*

$$\begin{array}{ccc}
\mathbb{C}^n & \xrightarrow{\theta_n} & \mathbb{R}^{2n} \\
\downarrow{\scriptstyle R_A} & & \downarrow{\scriptstyle R_{\rho_n(A)}} \\
\mathbb{C}^n & \xrightarrow{\theta_n} & \mathbb{R}^{2n}
\end{array}$$

where $\theta_n : \mathbb{C}^n \to \mathbb{R}^{2n}$ is given by $\theta_n(a_1+ib_1, \ldots, a_n+ib_n) = (a_1, b_1, \ldots, a_n, b_n)$.

(compactly every complex matrix can be viewed as a real matrix of twice the size)

Remark: θ_n is a real linear isomorphism. This forces $R_{\rho_n(A)} = \theta_n \circ R_A \circ \theta_n^{-1}$.

This is linear and therefore there is a corresponding matrix $\in M_{2n}(\mathbb{R})$.

Proof. See moodle. □

Observation 1.1.9. *ρ_n is injective. Proof: exercise.*

Lemma 1.1.10. *ρ_n satisfies $\rho_n(AB) = \rho_n(A)\rho_n(B)$. So ρ_n is an injective real-algebra homomorphism.*

Proof. We compose commutative squares from 1.1.8 to get

$$\begin{array}{ccc}
\mathbb{C}^n & \xrightarrow{\theta_n} & \mathbb{R}^{2n} \\
\downarrow{\scriptstyle R_A} & & \downarrow{\scriptstyle R_{\rho_n(A)}} \\
\mathbb{C}^n & \xrightarrow{\theta_n} & \mathbb{R}^{2n} \\
\downarrow{\scriptstyle R_B} & & \downarrow{\scriptstyle R_{\rho_n(B)}} \\
\mathbb{C}^n & \xrightarrow{\theta_n} & \mathbb{R}^{2n}
\end{array}$$

On L.H.S. we have $R_B \circ R_A = R_{AB}$. (note order)

On R.H.S we have $R_{\rho_n(B)} \circ R_{\rho_n(A)} = R_{\rho_n(A)\rho_n(B)}$.

But since LHS is R_{AB} this means $R_{\rho_n(AB)} =$ composition on RHS $= R_{\rho_n(A)\rho_n(B)}$. □

It's not surjective however. Q: What exactly is $\rho_n(A)$? Consider $(a+ib) \in M_1(\mathbb{C})$.

$$R_{(a+ib)}(x+iy) = (x+iy)(a+ib) = (ax-by) + i(ay+bx)$$

Now $\theta_1(x+iy) = (x,y) \in \mathbb{R}^2$ etc.

So $\theta_1((ax-by)+i(ay+bx)) = (ax-by, ay+bx)$

The corresponding map from $\mathbb{R}^2 \to \mathbb{R}^2$ is $(x,y) \mapsto (ax-by, ay+bx)$. Observe that

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = (ax-by, ay+bx)$$

So $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M_2(\mathbb{R})$ corresponds under ρ_1 to $(a+ib) \in M_1(\mathbb{C})$.

More generally

$$\begin{pmatrix} a_{11}+ib_{11} & \dots & a_{1n}+ib_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1}+ib_{n1} & \dots & a_{nn}+ib_{nn} \end{pmatrix} \in M_n(\mathbb{C})$$

corresponds to

$$\left(\begin{array}{cc|c|cc} a_{11} & b_{11} & \multirow{2}{*}{$\dots$} & a_{1n} & b_{1n} \\ -b_{11} & a_{11} & & -b_{1n} & a_{1n} \\ \hline \vdots & & \vdots & \vdots & \\ \hline a_{n1} & b_{n1} & \multirow{2}{*}{$\dots$} & a_{nn} & b_{nn} \\ -b_{n1} & a_{n1} & & -b_{nn} & a_{nn} \end{array}\right) \in M_{2n}(\mathbb{R})$$

is obtained by replacing each $\mathbb{C}$ entry by its corresponding 2×2 real block.

Lecture 4 04/10/23

Last time:

- $A \in M_n(\mathbb{H})$ then $R_A : \mathbb{H}^n \to \mathbb{H}^n$ given by $R_a(\vec{x}) = \vec{x}A$. is $\mathbb{H}$- linear (assuming coefficients in $\mathbb{H}$ multiply on vectors from the left, x row vector). Left multiplication is not in general $\mathbb{H}$ linear.
- Under the real linear isomorphism $\theta_n : \mathbb{C}^n \to \mathbb{R}^{2n}$, $\theta_n(a_1+ib_1, \dots, a_n+ib_n) = (a_1, b_1, \dots, a_n, b_n)$. Any complex-linear map $\mathbb{C}^n \to \mathbb{C}^n$ corresponds to a real-linear map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and in terms of matrices (an right multiplication) $A \in M_n(\mathbb{C})$ corresponds to some matrix $\rho_n(A) \in M_{2n}(\mathbb{R})$.

- $$\text{If } A = \begin{pmatrix} a_{11} + ib_{11} & \dots & a_{1n} + ib_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} + ib_{n1} & \dots & a_{nn} + ib_{nn} \end{pmatrix}$$

  then

$$\rho_n(A) = \left(\begin{array}{cc|c|cc} a_{11} & b_{11} & & a_{1n} & b_{1n} \\ -b_{11} & a_{11} & \dots & -b_{1n} & a_{1n} \\ \hline & \vdots & \vdots & & \vdots \\ \hline a_{n1} & b_{n1} & & a_{nn} & b_{nn} \\ -b_{n1} & a_{n1} & \dots & -b_{nn} & a_{nn} \end{array}\right)$$

Consider the $\mathbb{C}$ linear map $\mathbb{C}^n \to \mathbb{C}^n$ given by $z \to zi$. This is R_A where

$$A = \begin{pmatrix} i & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & i \end{pmatrix} = iI$$

For this matrix we have

$$\rho_n(A) = \left(\begin{array}{c|c|c} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \dots & \mathbf{0} \\ \hline \vdots & \vdots & \vdots \\ \hline \mathbf{0} & \dots & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{array}\right) = \mathcal{I}_n$$

A map $f : \mathbb{C}^n \to \mathbb{C}^n$ is $\mathbb{C}$ linear if it is real linear and $f(zi) = f(z)i$.

Let $barf : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the corresponding $\mathbb{R}$ linear map and suppose this has matrix $B \in M_{2n}(\mathbb{R})$. Then the complex linearity requirement is $R_B \circ R_{\mathcal{I}_n} = R_{\mathcal{I}_n} R_B$.

Since $R_X = R_Y \iff X = Y$ we see this is equivalent to asking $B\mathcal{I}_n = \mathcal{I}_n B$. i.e. $B \in M_{2n}(\mathbb{R})$ corresponds under θ_n to a complex linear map $\iff$ $B\mathcal{I}_n = \mathcal{I}_n B$.

We'd proved

Corollary 1.1.11. *The image of $\rho_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ is the set of all of all matrices in $M_{2n}(\mathbb{R})$ which commute with $\mathcal{I}_n$.*

Remark: This shows that ρ_n is not surjective.

Lemma 1.1.12. *There is an injective group homomorphism $\rho_n : GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R})$, given by restricting $\rho_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$.*

Proof. We just have to check that if $A \in GL_n(\mathbb{C})$, then $\rho_n(A)$ is invertible. Clearly $\rho_n(AA^{-1}) = \rho_n(A^{-1}A) = \rho_n(I_n)$ so by 1.1.10. $\rho_n(A)\rho_n(A^{-1}) = \rho_n(A^{-1})\rho_n(A) = \rho_n(I_n) = I_{2n}$.

$\therefore \rho_n(A^{-1}) = \rho_n(A)^{-1}$, hence $\rho_n(A) \in GL_{2n}(\mathbb{R})$. So $\rho_n : GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R})$, and by 1.1.10 this is a (multiplicative) group homomorphism □

Now for quaternion matrices.

First observe that there is a $\mathbb{C}$ linear isomorphism $\phi_n : \mathbb{H}^n \to \mathbb{C}^{2n}$ given by $\phi_n(z_1 + w_1 j, \dots, z_n + w_n j) = (z_1, w_1, \dots, z_n, w_n)$.

(exercise to figure out $a + bi + cj + dk$ as $z + wj$, with $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$.)

Proposition 1.1.13. *There is an injective $\mathbb{C}$ linear map $\psi_n : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ s.t. the following square commutes:*

$$\begin{tikzcd}
\mathbb{H}^n \arrow[r, "\phi_n"] \arrow[d, "R_A"] & \mathbb{C}^{2n} \arrow[d, "R_{\psi_n(A)}"] \\
\mathbb{H}^n \arrow[r, "\phi_n"] & \mathbb{C}^{2n}
\end{tikzcd}$$

i.e. $\phi_n \circ R_A = R_{\psi_n(A)} \circ \phi_n$. *Moreover, ψ_n satisfies* $\psi_n(AB) = \psi_n(A)\psi_n(B)$.

Proof. Analogous to that of prop 1.1.8 and lemma 1.1.10. Exercise! □

Remark: It is easily checked (exercise!) that $\psi_1(z + wj) = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$

More generally, image of ψ_n consists of block matrices with blocks of this form (analogous to ρ_n).

By restricting to invertible matrices we obtain:

Corollary 1.1.14. *There is an injective group homomorphism $\psi_n : GL_n(\mathbb{H}) \to GL_{2n}(\mathbb{C})$.*

Proof. Analogous to 1.1.12 - exercise. □

(you can compose the maps then to get a real $4n$ matrix from a quaternionic one)

Composing ρ_{2n} and ψ_n gives

Corollary 1.1.15. *There is an injective $\mathbb{R}$ linear map resp. group homomorphism given by $\rho_{2n} \circ \psi_n : M_n(\mathbb{H}) \to M_{4n}(\mathbb{R})$ resp. $\rho_{2n} \circ \psi_n : GL_n(\mathbb{H}) \to GL_{4n}(\mathbb{R})$.*

Slogan: all groups of $\mathbb{H}$ or $\mathbb{C}$ matrices can be viewed as groups of real matrices!

Definition 1.1.16. *For $A \in M_n(\mathbb{H})$,* $\det(A) := \det \psi_n(A)$.