MT543 Topics in Algebra

Notes taken by Stephen Nulty and John Brennan

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Note:

Any transcription mistakes and typos are my own.

Lectures by David Wraith. Lie Groups and Lie Algebras.

Lecture 1 25/09/23

missed this lecture - some intro to do with spheres, transformations and symmetries and other motivational stuff. Definition of an algebra (bilinear product) over a field.

0. Introduction

Lie groups have a dual nature: they are groups but also very special topological spaces. The algebraic and topological (spatial) properties are closely aligned. Lie groups and Lie algebras lie at the intersection of algebra, topology, geometry, analysis and more.

Definition 0.0.1. *An algebra is a vector space V equipped with a bilinear map $m : V \times V \to V$.*

Note, the "multiplication" map m does not have to be commutative or associative. In general, Lie algebras are neither commutative nor associative. Recall,

$$\text{Commutativity: } m(u, v) = m(v, u),$$
$$\text{Associativity: } m(m(u, v), w) = m(u, m(v, w)).$$

Every Lie group has an associated Lie algebra which encodes many properties of the group. Often this allows problems about Lie groups to be reduced to problems in (fancy!) linear algebra.

Example of a Lie group

The set of rotations of a ball centred on O in $\mathbb{R}^3$ is a Lie group. It is a group under composition of rotations. To "see" the topology here, notice that it makes sense to talk about two rotations being "close", so there is a sense of space. It makes sense to consider a continuous family of rotations. Continuity implies the existence of topology. We can identify this group with the matrix group $SO(3)$. The map $\mathbb{R}^3 \to \mathbb{R}^3$ given by $x \mapsto Ax$ for $A \in SO(3)$ is a rotation and every rotation occurs in this way. $SO(3)$ is a subset (but not a subgroup) of the set/group of all (3×3)-real matrices $M_3(\mathbb{R})$. By listing the elements of any 3×3 matrix we get a bijection $M_3(\mathbb{R}) \to \mathbb{R}^9$. As $\mathbb{R}^9$ has a natural topology (metric), this gives a natural topology on $M_3(\mathbb{R})$ and by restriction on $SO(3)$.

Lecture 2 27/09/23

Sorting out tutorial times. Lectures: Monday 2pm MS2, Wednesday 2pm LGH, Thursday 12pm MS2.

Lie Groups, dual nature, Groups but also a topological geometrical character. Can prove things with a mix of both methods - intersection of various areas.

1. Groups of matrices

1.1. General Linear Groups

Quaternions will have a central role.

Consider groups of $N \times N$ matrices over the fields $\mathbb{R}$ and $\mathbb{C}$ and also over the quaternions.

Definition 1.1.1. *The quaternions $\mathbb{H}$ is a 4-dim real vector space with standard basis elements $1, i, j, k$, equipped with an associative linear multiplication operation defined by*

$$i^2 = j^2 = k^2 = -1, \quad ij = k, jk = i, ki = j$$

So a generic quaternion takes the form $a + bi + cj + dk$, $a, b, c, d \in \mathbb{R}$.

Observe, $ji = j(jk) = (jj)k$ (by associativity) $= j^2k = -k$. Similarly $kj = -i$ and $ik = -j$.

e.g. $(2 + i - 3k)(5 + 2i - j + k) = 10 + 4i - 2j + 2k + 5i - 2 - k - j - 15k - 6j - 61 + 3$ etc.

Quaternions is not commutative, so is not a field. However it is a skew field (division algebra).

Terminology - In $a+bi+cj+dk$, a is called the real or scalar part, and the rest $bi+cj+dk$ imaginary or vector part.

In analogy with complex numbers,

Definition 1.1.2. *1. The conjugate of* $a+bi+cj+dk$ *, is* $\overline{a+bi+cj+dk} = a-bi-cj-dk$ *2. The norm of* $a+bi+cj+dk$ *is* $|a+bi+cj+dk| = \sqrt{a^2+b^2+c^2+d^2}$

Thus $\mathbb{H}$ is a normed vector space. Next observe that for each $q \in \mathbb{H}$ $q\bar{q} = \bar{q}q = |q|^2$.

therefore (symbol) $q^{-1} = \bar{q}/|q|^2$. So $qq^{-1} = q\bar{q}/|q|^2 = |q|^2 = 1$, similarly for $q^{-1}q = 1$.

This allows division $q_1 \cdot q_2^{-1} = q_1\bar{q_2}/|q_2|^2$. Writing q_1/q_2 is ambiguous however. $q_1q_2^{-1} \neq q_2^{-1}q_1$ generically.

Clearly $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$. A classic theorem of Frobenius asserts that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ are the only real associative division algebras. These objects similarly play a distinguished role in Lie group theory.

Convention: Suppose V is a vector space over the quaternions $\mathbb{H}$. We will adopt the convention that whenever we scale a vector $v \in V$ by a scalar $\lambda \in \mathbb{H}$, we multiply on the left, i.e. λv

Let $M_n(\mathbb{R}), M_n(\mathbb{C}), M_n(\mathbb{H})$ denote the sets (vector spaces!) of all $n \times n$ matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

Definition 1.1.3. *The General Linear Groups* $GL_n(\mathbb{R})$*, resp.* $GL_n(\mathbb{C})$ *is the group of* $n \times n$ *invertible matrices with* $\mathbb{R}$ *resp* $\mathbb{C}$ *coefficients. (Group under multiplication). Equivalently* $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) \neq 0\}$*. Similarly* $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) | \det(A) \neq 0\}$.

(return to the idea of determinants of quaternions later).

Recall that for any matrix $A \in M_n(\mathbb{R})$ we have two associated linear maps $L_A : \mathbb{R}^n \to \mathbb{R}^n, L_a(\vec{x}) = A\vec{x}$, $R_A : \mathbb{R}^n \to \mathbb{R}^n, R_a(\vec{x}) = \vec{x}A$.

It is well know that A is invertible (RC cases) $\iff \det(A) \neq 0 \iff L_A, R_A$ are isomorphisms.

Lecture 3 02/10/23

Thursday lecture moved to Friday at 10am in MS2.

Reminder:

- Quaternions $\mathbb{H}$, multiplication is associative not commutative. If V is a $\mathbb{H}$- vector space, we scale from the left only, i.e. λv for $\lambda \in \mathbb{H}, v \in V$.

- General linear groups $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$ groups under $*$ of all invertible $\mathbb{R}$ resp. $\mathbb{C}$ $n \times n$ - matrices.

- $A \in M_n(\mathbb{R}), M_n(\mathbb{C})$ is invertible iff $\det A \neq 0$ iff L_A, R_A are both invertible where $L_a(\vec{x}) = A\vec{x}$, $R_a(\vec{x}) = \vec{x}A$.

We now consider $M_n(\mathbb{H})$.

Definition 1.1.4. *A function $f : \mathbb{H}^n \to \mathbb{H}^n$ is $\mathbb{H}$ - linear if $f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2)$, $\forall \lambda_1 \lambda_2 \in \mathbb{H}, v_1, v_2 \in \mathbb{H}^n$.*

Lemma 1.1.5. *For $A \in M_n(\mathbb{H})$, $R_A : \mathbb{H}^n \to \mathbb{H}^n$ given by $R_a(\vec{x}) = \vec{x}A$ for $v \in \mathbb{H}^n$ a row vector, is $\mathbb{H}$ - linear, however L_A is in general not $\mathbb{H}$ - linear. Proof: exercise*

idea is that associativity makes $\lambda v A$ ok, but not with left multiplication which is interfered by commutativity.

Lemma 1.1.6. *For $A \in M_n(\mathbb{H})$, $R_A : \mathbb{H}^n \to \mathbb{H}^n$, is an $\mathbb{H}$ - linear isomorphism iff A is invertible, i.e. $\exists B \in M_n(\mathbb{H})$ such that $AB = BA = I_n$.*

Proof. ($\Rightarrow$) If R_A is an iso. then there is a $\mathbb{H}$-linear inverse $(R_A)^{-1} : \mathbb{H}^n \to \mathbb{H}^n$. There is a corresponding matrix $B \in M_n(\mathbb{H})$. Since $R_A \circ (R_A)^{-1} = R_A \circ (R_A)^{-1} = I_n$. we deduce $BA = AB = I_n$ (NB order of matrices here!). Therefore $B = A^{-1}$.

($\Leftarrow$) Similar. □

Definition 1.1.7. *The quaternionic general linear group $GL_n(\mathbb{H}) = \{A \in M_n(\mathbb{H}) \,|\, A \text{ is invertible}\} = \{A \in M_n(\mathbb{H}) \,|\, R_a \text{ is an iso.}\}$*

NB: There is a problem with the notion of $\mathbb{H}$ - determinant due to non-commutativity we'll return to this later (possible to define determinant and $GL_n(\mathbb{H})$ as ones with non-zero determinant, but defining it requires some thought.)

It turns out that we can view $\mathbb{C}$ and $\mathbb{H}$ - matrices/linear maps in terms of $\mathbb{R}$ - matrices.

Proposition 1.1.8. *There is a real linear map $\rho_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ such that the following diagram commutes.*

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, "\theta_n"] \arrow[d, "R_A"] & \mathbb{R}^{2n} \arrow[d, "R_{\rho_n(A)}"] \\
\mathbb{C}^n \arrow[r, "\theta_n"] & \mathbb{R}^{2n}
\end{tikzcd}$$

where $\theta_n : \mathbb{C}^n \to \mathbb{R}^{2n}$ *is given by* $\theta_n(a_1+ib_1, \ldots, a_n+ib_n) = (a_1, b_1, \ldots, a_n, b_n)$.

(compactly every complex matrix can be viewed as a real matrix of twice the size)

Remark: θ_n is a real linear isomorphism. This forces $R_{\rho_n(A)} = \theta_n \circ R_A \circ \theta_n^{-1}$.

This is linear and therefore there is a corresponding matrix $\in M_{2n}(\mathbb{R})$.

Proof. Given that θ_n is a real-linear isomorphism, consider the map $\theta_n \circ R_A \circ \theta_n^{-1} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. This is clearly real-linear, and hence corresponds to some $(2n \times 2n)$-real matrix which depends on A. Call this matrix $\rho_n(A)$. Thus we obtain a map $\rho_n(A) : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$, and the requirement that $\theta_n \circ R_A = R_{\rho_n(A)} \circ \theta_n$ is automatrically satisfied by construction.

It remains to show that ρ_n is a real-linear map. To this end we compute for $A_1, A_2 \in M_n(\mathbb{C})$ and $\lambda, \mu \in \mathbb{R}$:

$$\begin{aligned}
R_{\rho_n(\lambda A_1+\mu A_2)} &= \theta_n \circ R_{\lambda A_1+\mu A_2} \circ \theta_n^{-1} \\
&= \theta_n \circ (\lambda R_{A_1} + \mu R_{A_2}) \circ \theta_n^{-1} \\
&= \lambda \theta_n \circ R_{A_1} \circ \theta_n^{-1} + \mu \theta_n \circ R_{A_2} \circ \theta_n^{-1} \\
&= \lambda R_{\rho_n(A_1)} + \mu R_{\rho_n(A_2)} \\
&= R_{\lambda \rho_n(A_1)+\mu \rho_n(A_2)}.
\end{aligned}$$

As for any $X, Y \in M_{2n}(\mathbb{R})$ we have $R_X = R_Y$ if and only if $X = Y$, we deduce that $\rho_n(\lambda A_1 + \mu A_2) = \lambda \rho_n(A_1) + \mu \rho_n(A_2)$ as required. □

Observation 1.1.9. ρ_n *is injective. Proof: exercise.*

Lemma 1.1.10. ρ_n *satisfies* $\rho_n(AB) = \rho_n(A)\rho_n(B)$.

Remark: When we take 1.1.10 together with 1.1.9 and 1.1.8, we see that ρ_n is an injective real-algebra homomorphism.

Proof. We compose commutative squares from 1.1.8 to get

$$\begin{array}{ccc}
\mathbb{C}^n & \xrightarrow{\theta_n} & \mathbb{R}^{2n} \\
\downarrow R_A & & \downarrow R_{\rho_n(A)} \\
\mathbb{C}^n & \xrightarrow{\theta_n} & \mathbb{R}^{2n} \\
\downarrow R_B & & \downarrow R_{\rho_n(B)} \\
\mathbb{C}^n & \xrightarrow{\theta_n} & \mathbb{R}^{2n}
\end{array}$$

On L.H.S. we have $R_B \circ R_A = R_{AB}$. (note order)

On R.H.S we have $R_{\rho_n(B)} \circ R_{\rho_n(A)} = R_{\rho_n(A)\rho_n(B)}$.

But since LHS is R_{AB} this means $R_{\rho_n(AB)}$ = composition on RHS = $R_{\rho_n(A)\rho_n(B)}$. □

It's not surjective however. Q: What exactly is $\rho_n(A)$? Consider $(a+ib) \in M_1(\mathbb{C})$.

$$R_{(a+ib)}(x+iy) = (x+iy)(a+ib) = (ax-by) + i(ay+bx)$$

Now $\theta_1(x+iy) = (x,y) \in \mathbb{R}^2$ etc.

So $\theta_1((ax-by)+i(ay+bx)) = (ax-by, ay+bx)$

The corresponding map from $\mathbb{R}^2 \to \mathbb{R}^2$ is $(x,y) \mapsto (ax-by, ay+bx)$. Observe that

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = (ax-by, ay+bx)$$

So $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M_2(\mathbb{R})$ corresponds under ρ_1 to $(a+ib) \in M_1(\mathbb{C})$.

More generally

$$\begin{pmatrix} a_{11}+ib_{11} & \dots & a_{1n}+ib_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1}+ib_{n1} & \dots & a_{nn}+ib_{nn} \end{pmatrix} \in M_n(\mathbb{C})$$

corresponds to

$$\left(\begin{array}{cc|c|cc} a_{11} & b_{11} & \multirow{2}{*}{$\cdots$} & a_{1n} & b_{1n} \\ -b_{11} & a_{11} & & -b_{1n} & a_{1n} \\ \hline \vdots & & \vdots & \vdots & \\ \hline a_{n1} & b_{n1} & \multirow{2}{*}{$\cdots$} & a_{nn} & b_{nn} \\ -b_{n1} & a_{n1} & & -b_{nn} & a_{nn} \end{array}\right) \in M_{2n}(\mathbb{R})$$

is obtained by replacing each $\mathbb{C}$ entry by its corresponding 2×2 real block.

Lecture 4 04/10/23

Last time:

- $A \in M_n(\mathbb{H})$ then $R_A : \mathbb{H}^n \to \mathbb{H}^n$ given by $R_a(\vec{x}) = \vec{x}A$. is $\mathbb{H}$- linear (assuming coefficients in $\mathbb{H}$ multiply on vectors from the left, x row vector). Left multiplication is not in general $\mathbb{H}$ linear.

- Under the real linear isomorphism $\theta_n : \mathbb{C}^n \to \mathbb{R}^{2n}$, $\theta_n(a_1+ib_1,\dots,a_n+ib_n) = (a_1,b_1,\dots,a_n,b_n)$. Any complex-linear map $\mathbb{C}^n \to \mathbb{C}^n$ corresponds to a real-linear map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and in terms of matrices (an right multiplication) $A \in M_n(\mathbb{C})$ corresponds to some matrix $\rho_n(A) \in M_{2n}(\mathbb{R})$.

- $$\text{If } A = \begin{pmatrix} a_{11}+ib_{11} & \dots & a_{1n}+ib_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1}+ib_{n1} & \dots & a_{nn}+ib_{nn} \end{pmatrix}$$

  then

  $$\rho_n(A) = \left(\begin{array}{cc|c|cc} a_{11} & b_{11} & \multirow{2}{*}{$\cdots$} & a_{1n} & b_{1n} \\ -b_{11} & a_{11} & & -b_{1n} & a_{1n} \\ \hline \vdots & & \vdots & \vdots & \\ \hline a_{n1} & b_{n1} & \multirow{2}{*}{$\cdots$} & a_{nn} & b_{nn} \\ -b_{n1} & a_{n1} & & -b_{nn} & a_{nn} \end{array}\right)$$

Consider the $\mathbb{C}$ linear map $\mathbb{C}^n \to \mathbb{C}^n$ given by $z \to zi$. This is R_A where

$$A = \begin{pmatrix} i & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & i \end{pmatrix} = iI$$

For this matrix we have

$$\rho_n(A) = \left(\begin{array}{cc|c|cc} 0 & 1 & \multirow{2}{*}{$\cdots$} & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}} \\ -1 & 0 & & & \\ \hline \vdots & & \vdots & \vdots & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}$}} & \multirow{2}{*}{$\cdots$} & 0 & 1 \\ & & & -1 & 0 \end{array}\right) = \mathcal{I}_n$$

A map $f : \mathbb{C}^n \to \mathbb{C}^n$ is $\mathbb{C}$ linear if it is real linear and $f(zi) = f(z)i$.

Let $barf : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the corresponding $\mathbb{R}$ linear map and suppose this has matrix $B \in M_{2n}(\mathbb{R})$. Then the complex linearity requirement is $R_B \circ R_{\mathcal{I}_n} = R_{\mathcal{I}_n} R_B$.

Since $R_X = R_Y \iff X = Y$ we see this is equivalent to asking $B\mathcal{I}_n = \mathcal{I}_n B$. i.e. $B \in M_{2n}(\mathbb{R})$ corresponds under θ_n to a complex linear map $\iff B\mathcal{I}_n = \mathcal{I}_n B$.

We'd proved

Corollary 1.1.11. *The image of $\rho_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ is the set of all of all matrices in $M_{2n}(\mathbb{R})$ which commute with $\mathcal{I}_n$.*

Remark: This shows that ρ_n is not surjective.

Lemma 1.1.12. *There is an injective group homomorphism* $\rho_n : GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R})$*, given by restricting* $\rho_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$*.*

Proof. We just have to check that if $A \in GL_n(\mathbb{C})$, then $\rho_n(A)$ is invertible. Clearly $\rho_n(AA^{-1}) = \rho_n(A^{-1}A) = \rho_n(I_n)$ so by 1.1.10. $\rho_n(A)\rho_n(A^{-1}) = \rho_n(A^{-1})\rho_n(A) = \rho_n(I_n) = I_{2n}$.

$\therefore$ $\rho_n(A^{-1}) = \rho_n(A)^{-1}$, hence $\rho_n(A) \in GL_{2n}(\mathbb{R})$. So $\rho_n : GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R})$, and by 1.1.10 this is a (multiplicative) group homomorphism $\square$

Now for quaternion matrices.

First observe that there is a $\mathbb{C}$ linear isomorphism $\phi_n : \mathbb{H}^n \to \mathbb{C}^{2n}$ given by $\phi_n(z_1 + w_1 j, \ldots, z_n + w_n j) = (z_1, w_1, \ldots, z_n, w_n)$.

(exercise to figure out $a + bi + cj + dk$ as $z + wj$, with $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$.)

Proposition 1.1.13. *There is an injective* $\mathbb{C}$ *linear map* $\psi_n : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ *s.t. the following square commutes:*

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{\phi_n} & \mathbb{C}^{2n} \\ \downarrow{\scriptstyle R_A} & & \downarrow{\scriptstyle R_{\psi_n(A)}} \\ \mathbb{H}^n & \xrightarrow{\phi_n} & \mathbb{C}^{2n} \end{array}$$

i.e. $\phi_n \circ R_A = R_{\psi_n(A)} \circ \phi_n$*. Moreover,* ψ_n *satisfies* $\psi_n(AB) = \psi_n(A)\psi_n(B)$*.*

Proof. Analogous to that of prop 1.1.8 and lemma 1.1.10. Exercise! $\square$

Remark: It is easily checked (exercise!) that $\psi_1(z + wj) = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$

More generally, image of ψ_n consists of block matrices with blocks of this form (analogous to ρ_n).

By restricting to invertible matrices we obtain:

Corollary 1.1.14. *There is an injective group homomorphism* $\psi_n : GL_n(\mathbb{H}) \to GL_{2n}(\mathbb{C})$*.*

Proof. Analogous to 1.1.12 - exercise. $\square$

(you can compose the maps then to get a real $4n$ matrix from a quaternionic one)

Composing ρ_{2n} and ψ_n gives

Corollary 1.1.15. *There is an injective* $\mathbb{R}$ *linear map resp. group homomorphism given by* $\rho_{2n} \circ \psi_n : M_n(\mathbb{H}) \to M_{4n}(\mathbb{R})$ *resp.* $\rho_{2n} \circ \psi_n : GL_n(\mathbb{H}) \to GL_{4n}(\mathbb{R})$*.*

Slogan: all groups of $\mathbb{H}$ or $\mathbb{C}$ matrices can be viewed as groups of real matrices!

Definition 1.1.16. *(1.1.16) For $A \in M_n(\mathbb{H})$, $\det(A) := \det \psi_n(A)$.*

Lecture 5 06/10/23

(Talking about matrices and linear maps, and in $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, there's a standard basis given to go between linear maps and matrices.

$$\begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ \downarrow & & \downarrow \\ \mathbb{R}^{2n} & \longrightarrow & \mathbb{R}^{2n} \end{array}$$

and you can go between $\mathbb{R}$ and $\mathbb{C}$ with a canonical map, where you forget the complex structure going to $\mathbb{R}$ from $\mathbb{C}$ or by pairing up the pairs of reals going to $\mathbb{C}$.)

Last time:

- There is a canonical $\mathbb{C}$ linear isomorphism $\phi_n : \mathbb{H}^n \to \mathbb{C}^{2n}$ given by $\phi_n(z_1 + w_1 j, \ldots, z_n + w_n j) = (z_1, w_1, \ldots, z_n, w_n)$.
- There is an injective homomorphism of complex algebras $\psi_n : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{\phi_n} & \mathbb{C}^{2n} \\ \Big\downarrow{\scriptstyle R_A} & & \Big\downarrow{\scriptstyle R_{\psi_n(A)}} \\ \mathbb{H}^n & \xrightarrow{\phi_n} & \mathbb{C}^{2n} \end{array}$$

- If $A \in M_n(\mathbb{H})$ then $\det(A) := \det \psi_n(A)$.

Proposition 1.1.17. *(1.1.17 - need to fix the numbering) For $A \in M_n(\mathbb{H})$, A is invertible $\iff$ $\det A \neq 0$, i.e. $GL_n(\mathbb{H}) = \{A \in M_n(\mathbb{H}) \mid \det A \neq 0\}$.*

Proof. We claim that A is invertible $\iff$ $\psi_n(A)$ is invertible.

($\Rightarrow$) This is immediate from the multiplicative properties of ψ_n in 1.1.13.

($\Leftarrow$) In 1.1.14 we noted that the restricted map $\psi_n : GL_n(\mathbb{H}) \to GL_{2n}(\mathbb{C})$ is a group homomorphism. $\therefore$ if $\psi_n(A) \in GL_{2n}(\mathbb{C})$ (i.e. is invertible) for $A \in M_n(\mathbb{H})$, then since $\operatorname{im} \psi_n$ is a subgroup of $GL_{2n}(\mathbb{C})$, $\exists B \in GL_n(\mathbb{H})$ s.t. $\psi_n(B) = [\psi_n(A)]^{-1}$. (Want to show now that B is the inverse of A.)

We have $\psi_n(AB) = \psi_n(A)\psi_n(B) = \psi_n(A)[\psi_n(A)]^{-1} = I_{2n}$. But ψ_n is injective, so we must have $AB = I_n$. Similarly $BA = I$. $\therefore B = A^{-1}$ i.e. A is invertible.

So the claim is true.

$$\therefore A \text{ is invertible} \underset{\text{claim}}{\iff} \psi_n \text{ is invertible}$$
$$\underset{\text{elementary linear algebra}}{\iff} \det \psi_n(A) \neq 0 \underset{\text{defn 1.1.16}}{\iff} \det A \neq 0$$

□

1.2. Orthogonal Groups

From now on we will only consider (skew) fields $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (=$\mathbb{F}$).

Recall that an inner product on a vector space V over $\mathbb{F}$ is a map $\langle\ ,\ \rangle : V \times V \to \mathbb{F}$ which is bilinear (i.e. linear in each entry) and is positive definite, i.e. $\langle v, v\rangle \geq 0\ \forall v \in V$. If $v \neq 0, \langle v, v\rangle > 0$. e.g. dot product in $\mathbb{R}^n$.

Definition 1.2.1.

a *The standard inner product in $\mathbb{F}^n$ is given by $\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle = x_1\bar{y_1} + \ldots + x_n\bar{y_n}$, where $\bar{y_i}$ is the conjugate of y_i. (If $y_i \in \mathbb{R}$ then $\bar{y_i} = y_i$). This is the dot product if $\mathbb{F} = \mathbb{R}$, and is called a Hermitian inner product if $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$.*

b *The standard basis for $\mathbb{F}^n$ is $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, …, $(0, 0, \ldots, 1)$.*

Remarks: As $x\bar{x} = |x|^2$ for all $x \in \mathbb{F}^n$ we see that $\langle x, x\rangle \geq 0 \forall x \in \mathbb{F}^n$ for the standard inner products.

For $\lambda \in \mathbb{F}$,

- $\langle \lambda x, y\rangle = \lambda \langle x, y\rangle$
- $\langle x, \lambda y\rangle = \langle x, y\rangle \bar{\lambda}$ (NB for $q_1, q_2 \in \mathbb{H}$ $\overline{q_1 q_2} = \bar{q_2}\bar{q_1}$.)
- $\overline{\langle x, y\rangle} = \langle y, x\rangle$.

In the real case, vectors x, y are orthogonal if $\langle x, y\rangle = 0$. A basis $\{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$ is orthonormal if $|v_i| = 1 \forall i$ and $\langle v_i, v_j\rangle = 0$ for $i \neq j$.

Exactly the same language is used if $\mathbb{F} = \mathbb{C}, \mathbb{H}$.

Lemma 1.2.2. *$\{v_1, \ldots, v_n\} \in \mathbb{C}^n$ is a (Hermitian) orthonormal basis $\iff$ $\{\theta_n(v_1), \theta_n(iv_1), \ldots, \theta_n(v_n), \theta_n(iv_n)\}$ is an orthonormal basis for $\mathbb{R}^n$.*

($\theta_n : \mathbb{C}^n \cong \mathbb{R}^{2n}$).

Proof. An easy computation sows that

$$\underbrace{\langle x,\, y\rangle_{\mathbb{C}}}_{\text{Hermitian I.P. on } \mathbb{C}^n} = \underbrace{\langle \theta_n(x),\, \theta_n(y)\rangle_{\mathbb{R}}}_{\text{dot product on } \mathbb{R}^n} + i\,\langle \theta_n(x),\, \theta_n(iy)\rangle_{\mathbb{R}}$$

.

Thus $\langle x,\, y\rangle = 0 \iff \langle \theta_n(x),\, \theta_n(y)\rangle_{\mathbb{R}}$ and $\langle \theta_n(x),\, \theta_n(iy)\rangle_{\mathbb{R}}$. The result now follows easily. (Exercise: complete the argument) □

Lecture 6 09/10/23

$v \cdot v = |v|^2$ $v \cdot w = |v||w|\cos\theta$ Dot product tells you about lengths and angles. Last time:

- standard inner product in $\mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n$ is given by $\langle x,\, y\rangle = \sum_{i=1}^{n} x_i \bar{y}_i$. ($\bar{y}_i = y_i$ if $y_i \in \mathbb{R}$)
- $\langle x,\, y\rangle_{\mathbb{C}^n} = \langle \theta_n(x),\, \theta_n(y)\rangle_{\mathbb{R}^{2n}} + i\,\langle \theta_n(x),\, \theta_n(iy)\rangle_{\mathbb{R}^{2n}}$
- (1.2.2) If $\{z_1, \ldots, z_n\}$ is an orthonormal basis for $\mathbb{C}^n$, then

  $$\{\theta_n(z_1), \theta_n(iz_1), \ldots, \theta_n(z_n), \theta_n(iz_n)\}$$

  is orthonormal for $\mathbb{R}^n$

Lemma 1.2.3. *$\{q_1, \ldots, q_n)\}$ is an orthonormal basis for $\mathbb{H}^n$ $\iff$*

$$\{\theta_{2n} \circ \phi_n(q_1), \theta_{2n} \circ \phi_n(iq_1), \theta_{2n} \circ \phi_n(jq_1), \theta_{2n} \circ \phi_n(kq_1), \ldots,$$
$$\theta_{2n} \circ \phi_n(q_n), \theta_{2n} \circ \phi_n(iq_n), \theta_{2n} \circ \phi_n(jq_n), \theta_{2n} \circ \phi_n(kq_n)\}$$

is orthonormal for $\mathbb{R}^{4n}$.

($\phi_n : \mathbb{H}^n \to \mathbb{C}^{2n}$, $\theta_{2n} : \mathbb{C}^{2n} \to \mathbb{R}^{4n}$)

Proof. This follows in the manner of 1.2.2 from the easily established formula

$$\langle x,\, y\rangle_{\mathbb{H}^n} = \langle \theta_{2n} \circ \phi_n(x),\, \theta_{2n} \circ \phi_n(y)\rangle_{\mathbb{R}^{4n}} + i\,\langle \theta_{2n} \circ \phi_n(x),\, \theta_{2n} \circ \phi_n(iy)\rangle_{\mathbb{R}^{4n}}$$
$$+ j\,\langle \theta_{2n} \circ \phi_n(x),\, \theta_{2n} \circ \phi_n(jy)\rangle_{\mathbb{R}^{4n}} + k\,\langle \theta_{2n} \circ \phi_n(x),\, \theta_{2n} \circ \phi_n(ky)\rangle_{\mathbb{R}^{4n}}$$

□

Definition 1.2.4. *1) The <u>orthogonal group</u> $O(n)$ is*

$$O(n) = \{A \in GL_n(\mathbb{R}) \mid \langle xA,\, yA\rangle_{\mathbb{R}^n} = \langle x,\, y\rangle_{\mathbb{R}^n}, \forall x, y \in \mathbb{R}^n\}$$

2) The *unitary group* $U(n)$ *is*

$$U(n) = \{A \in GL_n(\mathbb{C}) \,|\, \langle xA,\, yA\rangle_{\mathbb{C}^n} = \langle x,\, y\rangle_{\mathbb{C}^n}\,, \forall x, y \in \mathbb{C}^n\}$$

3) The *symplectic group* $Sp(n)$ *is*

$$Sp(n) = \{A \in GL_n(\mathbb{H}) \,|\, \langle xA,\, yA\rangle_{\mathbb{H}^n} = \langle x,\, y\rangle_{\mathbb{H}^n}\,, \forall x, y \in \mathbb{H}^n\}$$

Exercise: Show $O(n)\,, U(n)\,, Sp(n)$ are groups under multiplication.
Remark: We could also define

$$O(n) = \{A \in GL_n(\mathbb{R}) \,|\, \langle Ax,\, Ay\rangle_{\mathbb{R}^n} = \langle x,\, y\rangle_{\mathbb{R}^n}\,, \forall x, y \in \mathbb{R}^n\}$$
$$U(n) = \{A \in GL_n(\mathbb{C}) \,|\, \langle Ax,\, Ay\rangle_{\mathbb{C}^n} = \langle x,\, y\rangle_{\mathbb{C}^n}\,, \forall x, y \in \mathbb{C}^n\}$$

Exercise: show this agrees with 1.2.4

Lemma 1.2.5. *For* $\mathbb{F} = \mathbb{R}\,, \mathbb{C}\,, \mathbb{H}$ *and* $A \in GL_n(\mathbb{F}^n)$ *. The following are equivalent (tfae),*

a) $A \in O(n)\,, U(n)\,, Sp(n)$ *(as appropriate),*

b) R_A *maps* $\mathbb{F}^n$*- orthonormal bases to* $\mathbb{F}^n$*-orthonormal bases,*

c) The rows of A *from an orthonormal for* $\mathbb{F}^n$*,*

d) $AA^* = I_n$ *where* $A^* = \bar{A}^T$*.*

Proof. (a)$\Rightarrow$(b) This is clear since by definition, multiplication by A preserves the inner product and in particular preserves orthogonality and lengths.

(b)$\Rightarrow$(c) The standard basis for $\mathbb{F}^n$ is orthonormal with respect to the standard inner product $\langle\ ,\ \rangle_{\mathbb{F}^n}$. By part (b) R_A maps this orthonormal basis to another orthonormal basis. But $R_A(e_i) = i^{\text{th}}$ row of A. Hence rows of A form an o.n. basis. (o.n. stands for orthonormal).

(c) $\iff$ (d) Observe that $(AA^*)_{ij} = \sum\limits_k a_{ik}a^*_{kj} = \sum\limits_k a_{ik}\bar{a}_{jk}$. The rows of A being o.n. means

$$\left\langle i^{\text{th}} \text{ row of } A,\, j^{\text{th}} \text{ row of } A\right\rangle = \delta_{ij}$$

($\delta_{ii} = 1,\ \delta_{ij} = 0$ if $i \neq j$)
i.e. $\sum\limits_k^n a_{ik}\bar{a}_{jk} = \delta_{ij}$ (maybe color this to show it matches previous?)
i.e. $(AA^*)_{ij} = \delta_{ij} \iff AA^* = I_n$

(c)⇒(a) The rows of A are o.n., i.e. $\sum_k^n a_{ik}\bar{a}_{jk} = \delta_{ij}$.

$$\begin{aligned}
\langle xA,\, yA\rangle_{\mathbb{F}^n} &= \left\langle \left(\sum_l x_l a_{l1}, \dots, \sum_l x_l a_{ln}\right),\, \left(\sum_m y_m a_{m1}, \dots, \sum_m y_m a_{mn}\right)\right\rangle \\
&= \sum_{k,l,m} x_l a_{lk} \overline{y_m a_{mk}} \\
&= \sum_{k,l,m} x_l a_{lk} \overline{a_{mk} y_m} \\
&= \sum_{l,m} x_l \delta_{lm} \overline{y_m} \\
&= \sum_l x_l \bar{y}_l \\
&= \langle x,\, y\rangle_{\mathbb{F}^n}
\end{aligned}$$

□

Lecture 7 11/10/23

Previously:

$$\begin{aligned}
O(n) &= \{A \in M_n(\mathbb{R}) \mid \langle xA,\, yA\rangle_{\mathbb{R}^n} = \langle x,\, y\rangle_{\mathbb{R}^n}, \forall x, y \in \mathbb{R}^n\} \\
U(n) &= \{A \in M_n(\mathbb{C}) \mid \langle xA,\, yA\rangle_{\mathbb{C}^n} = \langle x,\, y\rangle_{\mathbb{C}^n}, \forall x, y \in \mathbb{C}^n\} \\
Sp(n) &= \{A \in M_n(\mathbb{H}) \mid \langle xA,\, yA\rangle_{\mathbb{H}^n} = \langle x,\, y\rangle_{\mathbb{H}^n}, \forall x, y \in \mathbb{H}^n\}
\end{aligned}$$

Note: other people might use Sp and symplectic group for something related but not exactly the same (something like $Sp(n, \mathbb{F})$).

1.2.5 For $A \in GL_n(\mathbb{F}^n)$ t.f.a.e.

a) $A \in O(n), U(n), Sp(n)$

b) R_A maps $\mathbb{F}$ orthonormal bases to $\mathbb{F}$ orthonormal bases.

c) Rows of A are an o.n. basis for $\mathbb{F}^n$.

d) $AA^* = I$ where $A^* = \bar{A}^T$.

Remark: It follows from this that

$$A \in O(n) \iff A^{-1} = A^T$$
$$A \in U(n) \iff A^{-1} = \bar{A}^T$$
$$A \in Sp(n) \iff A^{-1} = \bar{A}^T$$

i.e. we could define $O(n) = \{A \in M_n(\mathbb{R}) \,|\, A^{-1} = A^T\}$.

(This follows from 1.2.5 (d), provided we can also show $A^*A = I$. The latter follows from the fact that if $AB = 1$, $(A, B \in M_n(\mathbb{F}))$ then $\Rightarrow BA = I$. Why is this?)

By definition a matrix $\in O(n)$ if the corresponding linear map preserves the dot product, i.e. preserves lengths and angles. What about $U(n)$ and $Sp(n)$?

Proposition 1.2.6.

1) $\rho_n(U(n)) = O(2n) \cap \rho_n(GL_n(\mathbb{C}))$.

2) $\psi_n(Sp(n)) = U(2n) \cap \psi_n(GL_n(\mathbb{H}))$.

3) $\rho_{2n} \circ \psi_n(Sp(n)) = O(4n) \cap \rho_{2n} \circ \psi_n(GL_n(\mathbb{H}))$.

Proof. See Moodle □

This says that the real matrices corresponding to matrices in $U(n)$ and $Sp(n)$ are precisely the dot-product (i.e. length and angle) preserving transformation which have the form of the real version a complex/quaterionic matrix.

We now give an alternative description of $O(n)$, $U(n)$, $Sp(n)$.

Starting point: In $\mathbb{R}^n$, the norm and dot product determine each other via a "polarization" formula.

$$\langle x + y,\, x + y\rangle = \langle x,\, x\rangle + 2\,\langle x,\, y\rangle + \langle y,\, y\rangle$$

i.e. $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\langle x,\, y\rangle$.

$\therefore \langle x,\, y\rangle = \frac{1}{2}\left(\|x + y\|^2 - \|x\|^2 - \|y\|^2\right)$.

This shows that we could define orthogonal transformation to be those preserving all norms.

For $U(n)$, $Sp(n)$, we have the following fact:

Lemma 1.2.7. $\|x\|_{\mathbb{C}^n} = \langle x,\, x\rangle_{\mathbb{C}^n}$ *is equal to* $\|\theta_n(x)\|_{\mathbb{R}^{2n}}$. *Similarly* $\|x\|_{\mathbb{H}^n} = \|\theta_{2n} \circ \phi_n(x)\|_{\mathbb{R}^{4n}}$.

Proof. Exercise. (Hint: $\langle x, y\rangle_{\mathbb{C}^n} = \langle \theta_n(x), \theta_n(y)\rangle_{\mathbb{R}^{2n}} + i\,\langle \theta_n(x), \theta_n(iy)\rangle_{\mathbb{R}^{2n}}$ etc.) $\square$

Proposition 1.2.8.

1) $O(n) = \{A \in GL_n(\mathbb{R}) \,|\, \|xA\|_{\mathbb{R}^n} = \|x\|_{\mathbb{R}^n}, \forall x \in \mathbb{R}^n\}$.
2) $U(n) = \{A \in GL_n(\mathbb{C}) \,|\, \|xA\|_{\mathbb{C}^n} = \|x\|_{\mathbb{C}^n}, \forall x \in \mathbb{C}^n\}$.
3) $Sp(n) = \{A \in GL_n(\mathbb{H}) \,|\, \|xA\|_{\mathbb{H}^n} = \|x\|_{\mathbb{H}^n}, \forall x \in \mathbb{H}^n\}$.

Proof. (1) is established by the polarization formula.

(2) We need to show that $A \in GL_n(\mathbb{C})$ preserves all norms then $A \in U(n)$. (if it preserves inner products it automatically preserves norms). But if $\|xA\|_{\mathbb{C}^n} = \|x\|_{\mathbb{C}^n} \forall x \in \mathbb{C}^n$ we have

$$\begin{aligned}&\|\theta_n(xA)\|_{\mathbb{R}^{2n}} = \|\theta_n(x)\|_{\mathbb{R}^{2n}}\\ \implies &\|\theta_n(x)\rho_n(A)\|_{\mathbb{R}^{2n}} = \|\theta_n(x)\|_{\mathbb{R}^{2n}}\end{aligned}$$

$\therefore \rho_n(A) \in O(2n)$ by (1). By 1.2.6(1), $\rho_n(U(n)) = O(2n) \cap \rho_n(GL_n(\mathbb{C}))$ and since ρ_n is injective, we conclude that $A \in U(n)$ as required.

$\square$

Lecture 8 13/10/23

coming up: semi-direct products of groups, group actions (on sets), basic topology.

Proposition 1.2.9. *For $A \in O(n), U(n), Sp(n)$ we have* $|\det A| = 1$.

Proof. For $A \in O(n)$, $AA^T = I$ by 1.2.5(d). So $\det\big(AA^T\big) = \det(A)\det\big(A^T\big) = \det(A)^2 = \det I = 1$.

For $A \in U(n)$, $A\bar{A}^T = I$ by 1.2.5(d). So $\det\big(A\bar{A}^T\big) = \det(A)\det\big(\bar{A}^T\big) = \det(A)\det\big(\bar{A}\big)$. But $\det\big(\bar{A}\big) = \overline{\det(A)}$. To see this, note that $\det(A)$ is a sum of products of entries of A, and for complex conjugation, $\overline{a+b} = \bar{a} + \bar{b}$ and $\overline{ab} = \bar{a}\bar{b}$. So then $\det\big(A\bar{A}^T\big)\det(A)\overline{\det(A)} = |\det(A)|^2 = \det I = 1$. As $|z| \geq 0$, we see $|\det(A)| = 1$ implies $|\det(A)| = 1$.

For $A \in Sp(n)$, $\det A := \det_{\mathbb{C}} \psi_n(A)$. By 1.2.6(2) we know that $\psi_n(A) \in U(2n)$. $\therefore$ $\det \psi_n(A)\overline{\det \psi_n(A)} = \det\Big(\psi_n(A)\overline{\psi_n(A^T)}\Big) = \det I = 1$. $\therefore$ $|\det(\psi_n(A))|^2 = 1$, so $|\det(\psi_n(A))| = 1$ and $\therefore |\det(A)| = 1$ $\square$

Remark For $A \in O(n)$, $\det A = \pm 1$.
For $A \in U(n)$, $\det A \in \{e^{i\theta} \mid \theta \in [0, 2\pi]\}$. (circle in complex plane).
For $A \in Sp(n)$, it turns out that $\det A = 1$. This is not obvious!

Definition 1.2.10. *"Special Groups".*

1) *The special orthogonal group* $SO(n) = \{A \in O(n) \mid \det A = 1\}$,
2) *The special unitary group* $SU(n) = \{A \in U(n) \mid \det A = 1\}$,
3) *The special linear group* $SO(n) = \{A \in M_n(\mathbb{R})$ (*or* $GL_n(\mathbb{R})$) $\mid \det A = 1\}$

Remark

1. $SL(n)$ could be also written $SL(n, \mathbb{R})$ or $SL_n(\mathbb{R})$. Also have complex version $SL(n, \mathbb{C})$ with the obvious definition.
2. The significance of $SL(n)$ is that this is the group of corresponding linear maps are precisely the volume preserving linear transformations. This follows from:
3. FACT: $A \in M_n(\mathbb{R})$. Then the volume of the parallelepiped determined by the vectors $R_A(e_1), \ldots, R_A(e_n)$ is $|\det A|$. Exercise: prove this fact! (vol $= |\det A|$.

   (insert vector picture).
4. $Sp(n)$ is already special!

Question What is the relationship between $SO(n)$ and $O(n)$, $SU(n)$ and $U(n)$?

Theorem 1.2.11. $U(n) = SU(n) \rtimes U(1)$.

$\rtimes$ is a semidirect product. Quick definition: G is a semidirect product of subgroups N, H if

1) $N \lhd G$,
2) $G = NH$ i.e. $G = \{nh \mid n \in N, h \in H\}$,
1) $N \cap H = \{e\}$.

Remark $U(1) = \{(e^{i\theta}) \mid \theta \in [0, 2\pi]\}$ which we identify with $\{e^{i\theta} \mid \theta \in [0, 2\pi]\}$. For the purposes of the theorem, we will identify $U(1)$ with $\begin{pmatrix} e^{i\theta} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$.

Proof. We observe that $SU(n) \lhd U(n)$ since $SU(n)$ is the kernel of det : $U(n) \to \{e^{i\theta} \mid \theta \in [0, 2\pi]\}$. (det is a homomorphism and kernel of any homomorphism is normal.)

Next observe that any $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \in U(n)$ can be expressed as a product

$$A = \begin{pmatrix} \frac{1}{\det A} a_{11} & a_{12} & \dots & a_{1n} \\ \frac{1}{\det A} a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{1}{\det A} a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} \det A & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

As A and $\begin{pmatrix} \det A & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ are both $\in U(n)$, so is left hand matrix above.

Moreover, det of L.H. matrix is $\frac{1}{\det A}$ is $\frac{1}{\det A} \det A = 1$. $\therefore$ L.H. matrix $\in SU(n)$. It remains to show $SU(n) \cap U(1) = \{e\}$. $\square$

This is clear since any matrix of the form $\begin{pmatrix} e^{i\theta} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ with det $= 1$ must be I_n. Analogous arguments show

Theorem 1.2.12. $O(n) = SO(n) \rtimes \mathbb{Z}_2$.

Proof. exercise. $\square$

Bonus Lecture 1: Semidirect products of groups

Direct product: G, H can form $G \times H = \{(g,h) \mid g \in G, h \in H\}$ with multiplication given by $(g,h) \cdot (g',h') = (gg', hh')$. (External direct product).

Now suppose that G has subgroups H, K such that $G = HK$, i.e. $G = \{hk \mid h \in H, k \in K\}$. What about multiplication in G?

$(hk)(h'k') =$? In the nicest case we might have $kh' = h'k$, so $(hk)(h'k') = hh'kk' \in HK$. This will work precisely when H and K commute with each other. In that case G is an (internal) direct product of H and K.

<u>Definition</u> If $H, K \subset G$, then G is a direct product of H and K if

1) $G = HK$
2) $H \cap K = \{e\}$ (this means any expression $g = hk$ is unique)
3) H, K commute with each other, i.e. $hk = kh$, $\forall h \in H, k \in K$,

   (3) is often replaced by

3') $H \lhd G$, $K \lhd G$. (not equivalent by itself with 1 and 2 it is)

<u>Definition</u> G is a semidirect product of $K \subset G$ and $N \lhd G$ if (1) and (2) hold.

What's going on?

(Aside) $N \lhd G$ means N is "normal" in G i.e. $N \subset G$ with the following property $gNg^{-1} \subset N \forall g \in G$.

$\therefore$ for any $n \in N$ and any $g \in G$, $gng^{-1} = n'$ for some $n' \in N$, i.e. $gn = n'g$ (*).

If $G = NH$ then consider the product

$$\begin{aligned}
&(n_1h_1)(n_2h_2) \\
=&n_1(h_1n_2)h_2 \quad \text{associativity} \\
\underset{(*)}{=}&n_1n_2'h_1h_2 \quad \in NH.
\end{aligned}$$

<u>Remark</u>

(1) As a <u>set</u> $N \rtimes H$ is just $N \times H$. But as a group it is in general a "twisted" product.

(2) Example dihedral groups D_n, where D_n is the group of symmetries of a regular n-gon. $|D_n| = 2n$. There are two types of symmetries here:

   i) rotations about centre: subgroup isomorphic to $\cong \mathbb{Z}_n$
   ii) flips : subgroup $\cong \mathbb{Z}_2$.

$D_n \cong Z_n \rtimes Z_2$ not a product since rotations and flips do not commute.

(3) There are several different looking ways to define a semidirect product. (e.g. short exact sequences, group extensions)

Bonus Lecture 1: Group Actions

Let X be a set, and let $\text{Bij}(X)$ be the set of bijections $X \to X$. This a group under composition.

Definition An action of a group G on the set X is a homomorphism $\alpha : G \to \text{Bij}(X)$.

If X has some extra structure e.g. X is a topological space, then we often replace $\text{Bij}(X)$ with a group of bijections which preserve the extra structure.

For a topological space X we consider $\alpha : G \to \underbrace{\text{Homeo}(X)}_{\text{group of "homeomorphisms"}}$.

Examples

i) $GL_n(\mathbb{R})$ acts on $\mathbb{R}^n$ by right or left multiplication. $\underbrace{R}_{\text{right multiplication}}$: $GL_n(\mathbb{R}) \to \underbrace{\text{Isom}(\mathbb{R}^n)}_{\text{group of linear isomorphisms of } \mathbb{R}^n}$

ii) $\underbrace{\mathbb{Z}_2}_{=\{\pm 1\}}$ acts on $\mathbb{R}$ by flips about 0. (picture of real line arrows indicating flips)

iii) $SO(2)$ acts on the circle S^1, $\alpha : SO(2) \to \text{Homeo}(S^1)$. If $S^1 = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ then action is by matrix multiplication

$$\underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{\text{generic element of } SO(2)} \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} \cdot \\ \cdot \end{pmatrix}}_{\text{again element of } S^1}$$

iv) $U(1) = \{e^{i\theta}\}$ =(circle) acts on 2-sphere S^2 by rotation (picture of sphere rotation through line connection poles say).

Bonus Lecture 1: Topology

Everything topological in module can be interpreted in terms of metric spaces (but doing so might be needlessly cumbersome).

Recall a metric space consists of a set X and a "metric" $d : X \times X \to [0, \infty)$ such that

- $d(x, y) = d(y, x)$
- $d(x, y) = 0 \iff x = y$
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangle).

d is a distance function.

An open ball in (X, d): $B(p, r) = \{x \in X \mid d(p, x) < r\}$, $p \in X$, $r > 0$.

Unions of open balls are called open sets. Metric allows us to define continuity and convergence.

It turns out that these notions can be describe using only open sets, e.g. $f : (X, d) \to (Y, d')$ is continuous $\iff$ for every open set $U \subset Y$, preimage $f^{-1}(U)$ is open in X. (*)

Open sets satisfy the following properties.

1. $X, \emptyset$ are open
2. unions of open sets are open
3. intersections of finitely many open sets are open.

Definition A topological space (X, T) is a set X together with a collection of subsets T ("topology") satisfying (1), (2), (3).

Remark Every metric space is a topological space. Many different metrics on X will generate the same topology (collection of open sets). Not every topology T will arise from a metric (more general in a sense).

Definition A homeomorphism $f : (X, T) \to (Y, T')$ such that

1. f is a bijection &
2. f, f^{-1} are both continuous as in (*)

Other terms

1. A subset C of (X, T) is closed if $X \setminus C \in T$, i.e. $X \setminus C$ is open.
2. A subset $S \subset (X, T)$ is compact if every collection of S by open sets in X has a finite subcovering (i.e. still covering S). idea Compact is small and neat (highly non technical)
3. A topological space is path-connected if you can join any two points by a continuous path ($p : \underbrace{[0, 1]}_{\text{standard metric topology}} \to (X, T)$)

4. A space is simply connected if any continuous loop in space can be contracted through the space to a point. Write this as $\underset{X}{\pi_1(\text{space})} = 0$.

   For any space $\pi_1(X)$ is the "fundamental group" of X and measures the failure of loops to be contractible. (e.g. Circle has fundamental group isomorphic to $\mathbb{Z}$).

Lecture 9 16/10/23

Last time

- $U(n) \cong SU(n) \rtimes U(1)$, here $U(1)$ is identified with $\left\{\begin{pmatrix} e^{i\theta} & & & \\ & 1 & & \\ & & \vdots & \\ & & & 1 \end{pmatrix}\right\} \cong U(1)$.
- $O(n) \cong SO(n) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2 = \left\{I, \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \vdots & \\ & & & 1 \end{pmatrix}\right\}$

Definition 1.2.13. *Metric spaces* (X, d) *and* (Y, d') *are* <u>*isometric*</u> *if* $\exists$ *bijection* $f : X \to Y$ *s.t.* $d'(f(x_1), f(x_2)) = d(x_1, x_2)$, $x_1, x_2 \in X$. f *is an* <u>*isometry*</u>. *(distance preserving maps)*

In $\mathbb{R}^n$ we have a standard distance d_{st} and a norm related by $d_{st} = \|x_1 - x_2\| (= \langle x_1 - x_2, x_1 - x_2 \rangle^{1/2})$.

By 1.2.8 ($O(n) = \{A \in M_n(\mathbb{R}) \mid \underbrace{\|xA\|}_{=\|R_A(x)\|} = \|x\|, \forall x \in \mathbb{R}^n\}$), we see that if $A \in O(n)$ then $R_A : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry.

Sort of conversely:

Proposition 1.2.14. *Suppose* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is an isometry (for* d_{st}*) which fixes* $0 \in \mathbb{R}^n$*, (i.e.* $f(0) = 0$*) then* $f = R_A$ *for some* $A \in O(n)$. *In particular* f *is linear.*

Proof. see moodle. (reasonably straightforward but takes time to write out). □

<u>Remark</u> The above establishes the fact that $O(n)$ is precisely the group of origin fixing isometries of $\mathbb{R}^n$. ($O(n)$ is a group of matrices, but every matrix corresponds to a linear transformation in the standard basis)

What about isometries of $\mathbb{R}^n$ that do not fix the origin?

Observe that for any $v \in \mathbb{R}^n$, the translation map $T_v : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x + v$ is an isometry.

Theorem 1.2.15. $\mathrm{Isom}(\mathbb{R}^n)$, *the group of isometries of* $\mathbb{R}^n$ *(under composition) is* $\mathrm{Isom}(\mathbb{R}^n) \cong \mathbb{R}^n \rtimes O(n)$, *where* $\mathbb{R}^n \cong \{T_v \mid v \in \mathbb{R}^n\}$. *In particular, any isometry is a composition of an orthogonal transformation and a translation.*

Proof. Exercise. (think about what happens to the origin - say to a point v, translate by $-v$ and then a origin fixing transformation is orthogonal.) □

1.3. Basic topology of groups of matrices

All groups of matrices seen so far are subsets of $M_n(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\therefore$ bijective with subsets of $M_n(\mathbb{R}), M_{2n}(\mathbb{R}), M_{4n}(\mathbb{R})$ respectively.

Observe that there is a bijection $b : M_n(\mathbb{R}) \to \mathbb{R}^{n^2}$, given by

$$b\left(\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}\right) = (a_{11}, a_{12}, \dots, a_{1n}, a_{21}, \dots, a_{nn}).$$

The standard topology on $\mathbb{R}^{n^2}$ can be "pulled-back" to give a natural topology on $M_n(\mathbb{R})$ which makes b a homeomorphism (continuous bijection with continuous inverse).

$\therefore$ all matrix groups seen so far inherit a (subspace) topology from some $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

Observation 1.3.1. *With respect to these natural topologies, the functions* det *and* tr *(trace) are continuous.*

Definition 1.3.2.

1) *A* <u>*matrix group*</u> *(as opposed to a group of matrices!) is a subgroup of* $GL_n(\mathbb{F})$ *which is closed.*

2) *A* <u>*topological group*</u> *is a group G which is equipped with a topology with respect to which the operations of multiplication and inverse are continuous.*

<u>Remark</u> The point of the closed condition in (1) is that it keeps the topology 'nice'.

Proposition 1.3.3. *All groups seen so far are matrix groups*

Proof. View det as a map det : $GL_n(\mathbb{R}) \to \mathbb{R}$. This is continuous. $SL(n) := \det^{-1}\{1\}$. As $\{1\} \subset \mathbb{R}$ is closed, so is its pre-image $\therefore SL(n)$ is a matrix group. Ditto $SL(n, \mathbb{C})$ etc.

(pre-image)

($\mathbb{R} \setminus \{1\} = (-\infty, 1) \cup (1, \infty)$ is open).

For $O(n), U(n), Sp(n)$ consider the map $f : GL_n(\mathbb{F}) \to GL_n(\mathbb{F})$ given by $f(A) = AA^*$. $f^{-1}(I) = O(n), U(n), Sp(n)$ depending on $\mathbb{F}$. As f is continuous and $\{I\}$ is closed in $GL_n(\mathbb{F})$, we see $O(n), U(n), Sp(n)$ are closed in $GL_n(\mathbb{F})$.

Finally $SU(n) = U(n) \cap SL(n, \mathbb{C})$. $SO(n) = O(n) \cap SL(n)$, and intersection of closed sets is closed. Hence these are matrix groups. □

Lecture 10 18/10/23

Previously

- $\mathbb{R}^n$ has a standard (metric) topology.

  Bijection $b : M_n(\mathbb{R}) \to \mathbb{R}^{n^2}$.

  "Pull-back" metric/topology to $M_n(\mathbb{R})$.

  $M_n(\mathbb{F})$ bijective to a subset of $M_n(\mathbb{R})$, $M_{2n}(\mathbb{R})$, $M_{4n}(\mathbb{R})$

  $\to M_n(\mathbb{F})$ has a natural topology. By restriction, so do all groups seen.

- Matrix group := closed subgroup of $GL_n(\mathbb{F})$

(Aside: compactness $[0,1]$ is compact, but $(0,1)$ isn't as $(0,1) \cong \mathbb{R}$, $\sim$ compact is like small but has to make sense topologically, small caveat to intuitiveness)

Proposition 1.3.4. *$O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$ are compact*

Proof. We prove $O(n)$ is compact. Arguments for other groups analogous.

Consider a map $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $f(A) = AA^T$. $O(n) = f^{-1}(\{I\})$. As $\{I\}$ is a closed subset of $M_n(\mathbb{R})$ and f is continuous, we see $O(n)$ is a closed subset of $M_n(\mathbb{R})$.

We have a homeomorphism $b : M_n(\mathbb{R}) \to \mathbb{R}^{n^2}$. $\therefore b(O(n))$ is closed in $\mathbb{R}^{n^2}$.

Recall the Heine-Borel Thm: a subset of Euclidean space is compact $\iff$ it is closed and bounded.

It remains to show that $b(O(n))$ is bounded. (so then $b(O(n))$ is compact in $\mathbb{R}^{n^2}$, and since b is a homeomorphism preserving topologies, so $O(n)$ is compact in $M_n(\mathbb{R})$.)

Recall 1.2.5 the rows of any $A \in O(n)$ form an orthonormal set. So each row is a unit vector in $\mathbb{R}^n$.

$$\|b(A)\| = (\underbrace{a_{11}^2 + a_{12}^2 + \ldots + a_{1n}^2}_{=1} + \underbrace{a_{21}^2 + \ldots + a_{2n}^2}_{=1} + \ldots + \underbrace{a_{n1}^2 + \ldots + a_{nn}^2}_{=1})^{1/2}$$

$\therefore \|b(A)\| = \sqrt{n}$.
$\therefore b(O(n)) \subset$ sphere in $\mathbb{R}^{n^2}$ of radius $\sqrt{n}$.
$\therefore b(O(n))$ is bounded as required.

$\square$

Proposition 1.3.5. *$GL_n(\mathbb{F})$, $Sp(n, \mathbb{F})$ are non-compact.*

Proof. exercise $\square$

Low dimensional examples
$O(1) = \{(1), (-1)\} \cong \mathbb{Z}_2$, $SO(1) = \{(1)\}$ trivial.
$SO(2) = \{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi)\} \underset{\text{isomorphic \& homeomorphic}}{\cong} \{e^{i\theta} \mid \theta \in [0, 2\pi)\} \underset{\text{homeo}}{\cong} S^1$.
$U(1) = \{(e^{i\theta}) \mid \theta \in [0, 2\pi)\}$, $\therefore SO(2) \underset{\text{homeo \& isomorphic}}{\cong} U(1)$.
$SU(1)$ is trivial.
$SU(2) = \{\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \mid z, w \in \mathbb{C}, |z|^2 + |w|^2 = 1\}$. Now $\{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \underset{\text{homeo}}{\cong} \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \sum_{i=1}^{4} |x_i|^2 = 1\} = S^3$, 3-sphere $\subset \mathbb{R}^4$.
$\therefore$ as a topological space $SU(2) \cong S^3$.
$Sp(1) = \{(q) \mid q\bar{q} = 1\}$, group of unit length quaternions.
As $\mathbb{H} \cong \mathbb{C}^2 \cong \mathbb{R}^4$ topologically/as real vector spaces, we see that $Sp(1) \cong S^3$.
In fact $Sp(1)$ and $SU(2)$ are isomorphic as groups, with isomorphism given by ψ_1.
Observe that all the above groups except $O(1)$ are path-connected.
Recall that an action of a group G on a set X is a map $\alpha : X \times G \to X$ (this is a right action!) such that $\alpha(x, 1) = x, \forall x \in X$ and $\alpha(\alpha(x, g), h) = \alpha(x, gh)$.
For fixed $g \in G$ we obtain a map $\alpha_g : X \to X$ given by $\alpha(\cdot, g)$. This is a bijection with inverse $\alpha_{g^{-1}}$.
$\therefore$ an action can be viewed as a homomorphism $\alpha : G \to \mathrm{Bij}(X)$.
e.g. a group can act on itself from left or right. i.e. $R_g : G \to G$, $R_g(g') = g'g$ etc.

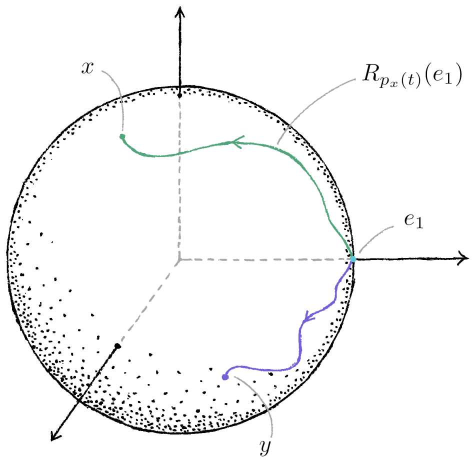

Figure 1: Here we depict the trajectories of e_1 in S^2 as it is evolved under paths in $SO(3)$, as decribed in 1.3.6, corresponding to points $x, y \in S^2$. Clearly, we can map x to y via $R^{-1}_{p_x(1)} \circ R_{p_y(1)}$.

If X is a topological space then we replace $\mathrm{Bij}(X)$ by $\mathrm{Homeo}(X)$, so actions respect topology.

Moreover, if G has a topology we will assume action depends continuously on $g \in G$.

An action of G on X is transitive if for any points $x, y \in X$ $\exists g \in G$ s.t. $\alpha_g(x) = y$.

Theorem 1.3.6.

1. *$SO(n)$ acts transitively by right (or left) multiplication on $S^{n-1} \subset \mathbb{R}^n$.*
2. *Given any point $x \in S^{n-1}$ there is a continuous path $p : [0,1] \to SO(n)$ s.t. $p(0) = I_n$ and $R_{p(1)}(e_1) = x$ $(n \geq 2)$.*

Remarks $SO(n)$ acts by right/left mult. on $\mathbb{R}^n$. As this action preserves distances it restricts to given an action on any sphere (say unit radius sphere) about 0, S^{n-1}, i.e. $R : S^{n-1} \times SO(n) \to S^{n-1}$.

Lecture 11 20/10/23

Proof. The first statement follows from the second (by composing paths as in Fig 1).

For the second statement we proceed by induction. For $n = 2$ this is easy since

$$SO(2) = \{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi)\}.$$

Clearly, for $x = (\cos\theta, \sin\theta) \in S^1$ we have a path

$$p(t) = \begin{pmatrix} \cos(t\theta) & \sin(t\theta) \\ -\sin(t\theta) & \cos(t\theta) \end{pmatrix},$$

such that $(1\ 0)p(t) = (\cos(t\theta)\ \sin(t\theta)$ which starts at $(1\ 0)$ and ends at x. (insert figure of circle here)

Now assume the result is true for $n = k$ and consider $SO(k+1)$ acting on $\mathbb{R}^{k+1}$. We can write any $x \in S^k \subset \mathbb{R}^{k+1}$ as $x = \cos\theta e_1 + \sin\theta y$ where $y \in \text{Span}\{e_2, \ldots, e_{k+1}\}$ is a unit vector. Choose a path $p_1(t)$ in $SO(n)$ taking the form

$$\left(\begin{array}{cc|c} \cos(t\theta) & \sin(t\theta) & \multirow{2}{*}{$\mathbf{0}$} \\ -\sin(t\theta) & \cos(t\theta) & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathcal{I}_{k-1} \end{array}\right)$$

Observe that $R_{p_1(1)}(e_1) = \cos\theta e_1 + \sin\theta e_2$. Now, identify $\mathbb{R}^k$ with $\text{Span}\{e_2, \ldots, e_{k+1}\}$. We know there exists a path $\bar{p}_2 : [0,1] \to SO(k)$ such that $\bar{p}_2(0) = \mathcal{I}_k$ and $R_{\bar{p}_2(1)}(e_2) = y$ by the inductive hypothesis. By linearity $R_{\bar{p}_2(1)}(\sin(\theta)y) = \sin(\theta)y$. Now, we can define $p_2 : [0,1] \to SO(k+1)$ by

$$p_2(t) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \bar{p}_2(t) \end{array}\right).$$

Finally, define $p : [0,1] \to SO(k+1)$ by

$$p(t) = \begin{cases} p_1(2t) & t \in [0, \frac{1}{2}) \\ p_2(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

By construction, $p(0) = \mathcal{I}_{k+1}$ and $R_{p(1)}(e_1) = \cos(\theta)e_1 + \sin(\theta)y = x$. $\square$

Remark Similar arguments apply to $SU(n)$ and $Sp(n)$ with very little change.

Corollary 1.3.7. *$SO(n)$ is path-connected.*

Proof. For any $A \in SO(n)$ we construct a path from A to $\mathcal{I}_n$. By Lemma 1.2.5 the rows of A form an orthonormal basis for $\mathbb{R}^n$. Call these $r_1, \ldots, r_n$. By theorem 1.3.6 there exisits a path p in $SO(n)$ starting with $\mathcal{I}_n$ and ending with $p(1)$ which satisfies $R_{p_1(1)}(r_1) = e_1$. As matrices in $SO(n)$ preserve orthonormality (by lemma 1.2.5) we see that $R_{p_1(1)}(r_i) \in \text{Span}\{e_2, \dots, e_n\}$ for $i \geq 2$

Now let $s_2 = R_{p_1(1)}(r_2)$. By theorem 1.3.6, there exists a path $\bar{p}_2$ in $SO(n-1)$ such that the corresponding path,

$$p_2(t) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \bar{p}_2(t) \end{array}\right),$$

moves s_2 to e_2, ie $R_{p_2(1)}(s_2) = e_2$ (and $p_2(0) = \mathcal{I}_n$). We can continue in this way to construct paths p_i moving r_i to e_i but fixing $e_1, \dots, e_{i-1}$. We can also compose these paths to obtain a path $p(t)$ such that $p(0) = \mathcal{I}_n$ and $R_{p(1)}(r_i) = e_i$ for all i.

Finally, consider the $n \times n$ matrix obtained by stacking the images of $r_1, \dots, r_n$ under $R_{p(t)}$:

$$\begin{pmatrix} — & R_{p(t)}(r_1) & — \\ — & R_{p(t)}(r_2) & — \\ & \vdots & \\ — & R_{p(t)}(r_n) & — \end{pmatrix}$$

Notice that, for eact t, the rows are orthonormal, so this matrix is an element of $SO(n)$ for all $t \in [0, 1]$. Moreover, at $t = 0$ this matrix is

$$\begin{pmatrix} — & r_1 & — \\ — & r_2 & — \\ & \vdots & \\ — & r_n & — \end{pmatrix} = A$$

and at $t = 1$

$$\begin{pmatrix} — & e_1 & — \\ — & e_2 & — \\ & \vdots & \\ — & e_n & — \end{pmatrix} = \mathcal{I}_n$$

Hence, we have constructe a path in $SO(n)$ from A to $\mathcal{I}_n$. $\square$

Lecture 12 23/10/23

Previously:

- 1.3.6 $SO(n)$ acts transitively on S^{n-1}

- 1.3.7 $SO(n)$ is path-connected.

<u>Remark</u> Essentially the same arguments used in 1.3.6 & 1.3.7 show

- $SU(n)$, $Sp(n)$ act transitively on S^{2n-1} and S^{4n-1} respectively.
- $SU(n)$, $Sp(n)$ are path-connected. $SL(n)$ is also path connected, but the argument is different.

Corollary 1.3.8. *$U(n)$ is path connected by $O(n)$ has two path components (i.e. <u>not</u> path connected).*

Proof. By 1.2.11, $U(n) = SU(n) \rtimes U(1)$. Topologically, $U(n) \cong SU(n) \times S^1$ (forgetting the group structure), and is $\therefore$ path connected as $SU(n)$ and S^1 both are.

By 1.2.12, $O(n) = SO(n) \rtimes \mathbb{Z}_2$. Topologically this means $O(n) \cong SO(n) \underset{\text{disjoint union}}{\amalg} SO(n)$. Hence $O(n)$ has two path components, each $\cong SO(n)$. $\square$

<u>Remark</u> $\mathbb{Z}_2$ here corresponds to $\{I_n, \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\}$. The matrix $\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ corresponds to a reflection in the hyperplane $\operatorname{span}\{e_2, \dots, e_n\}$ (flipping the e_1 coordinate over). Notice that we can't have a continuous path from I_n to $\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ in $O(n)$ as the determinant (which is continuous) would have to jump from 1 to -1.

If we define a rotation of $\mathbb{R}^n$ to be an origin fixing distance preserving map which can be linked via a continuous path of such maps to I_n, then we can deduce

- $SO(n)$ is precisely the group of rotations of $\mathbb{R}^n$,
- $O(n)$ is precisely the group of rotations and reflections of $\mathbb{R}^n$.

Final remark: "Homogeneous spaces".

If G is a topological group and $H \subset G$ a subgroup, then the set of cosets ${}^{G}/_{H}$ (or ${}_{H}\backslash^{G}$) is a topological space with the quotient topology. ${}^{G}/_{H}$ is called a "Homogeneous space", as, when equipped with an appropriate geometry ${}^{G}/_{H}$ "looks" the same at all points.

Examples $S^{n-1} \cong {}^{SO(n)}/_{SO(n-1)} \cong {}^{O(n)}/_{O(n-1)}$, where $SO(n-1)$ is identified with $\begin{pmatrix} 1 & \\ & SO(n-1) \end{pmatrix} \subset SO(n)$ etc.

$S^{2n-1} \cong {}^{SU(n)}/_{SU(n-1)} \cong {}^{U(n)}/_{U(n-1)}$
$S^{4n-1} \cong {}^{Sp(n)}/_{Sp(n-1)}$.

2. Lie groups & Lie Algebras

2.1. Manifolds

Roughly speaking a manifold is a "nice" topological space such that a neighbourhood of each point "looks like" euclidean space of a fixed dimension. We can use the local euclidean property to transfer calculus (differential and with a bit more work integral) from $\mathbb{R}^n$ to manifolds.

Definition 2.1.1. *A topological n-manifold M is a Hausdoff topological space with a countable basis for its topology, satisfying the following locally euclidean property: For any $x \in M$ there is an open set $x \in U \subset M$, and an open set $V \subset \mathbb{R}^n$, and a homeomorphism $\phi : U \to V$.*

(nice picture of torus, open sets, and arrow between it and V in $\mathbb{R}^n$).

Remarks

1) n is the "dimension" of the manifold.

2) The map ϕ is called a "chart".

3) A locally Euclidean space which is a subset of some $\mathbb{R}^m$ $m \geq n$ is automatically Hausdorff and has a countable basis for its topology.

Lecture 13 25/10/23

Previously 2.1.1 A topological n-manifold M is a Hausdoff topological space with a countable basis for its topology,which is locally euclidean, i.e. for each $x \in M \exists U \subset M$ open with $x \in U$ and a homeomorphism $\phi : U \to V$, where $V \subset \mathbb{R}^n$ is an open set. ϕ is a "chart".

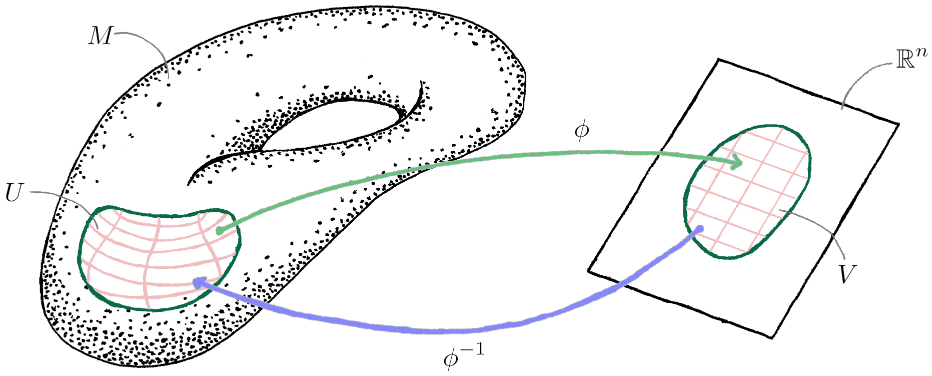

Figure 2: The inverse of a chart ϕ provides M with a local coordinate system. Coordinate lines in $V \subset \mathbb{R}^n$ are mapped by ϕ^{-1} to continuous curves in M which trace the local coordinate system on M.

Definition 2.1.2. *A family of charts $\{\phi_\alpha \mid \alpha \in \Lambda\}$ (Λ an indexing set) on a manifold M is called an atlas if for each $x \in M$ $\exists \alpha \in \Lambda$ s.t. $x \in$ domain of ϕ_α.*

Remark The inverse of any chart provides M with a local coordinate system. See figure 2.

(today upgrade definition of topological manifold to a smooth one)

We next describe what it means for a manifold to be smooth. We use charts to interpret smoothness.

Given charts $\phi : U \to V$, $\theta : U' \to V'$ on M, assume $U \cap U' \neq \emptyset$. Then there $\exists h : \phi(U \cap U') \to \theta(U \cap U')$ such that the following diagram commutes

$$\begin{tikzcd} & U \cap U' \arrow[dl, "\phi|_{U\cap U'}"'] \arrow[dr, "\theta|_{U\cap U'}"] & \\ \underset{\subset V \subset \mathbb{R}^n}{\phi(U \cap U')} \arrow[rr, "h"'] & & \underset{\subset V' \subset \mathbb{R}^n}{\theta(U \cap U')} \end{tikzcd}$$

$h := \theta|_{U\cap U'} \circ \phi|^{-1}_{U\cap U'}$. Observe that h is a homeomorphism between open sets in $\mathbb{R}^n$. As a map of Euclidean spaces, it makes sense to ask: is h smooth, i.e. infinitely differentiable? h is called a "transition function".

Definition 2.1.3. *An atlas $\{\phi_\alpha\}$ is said to be a smooth atlas if all transition functions are smooth (C^∞) with smooth inverses.*

(note smooth functions like x^3 don't necessarily have smooth inverses everywhere).

Two overlapping charts are said to be "smoothly compatible" if the corresponding transition function is smooth.

Given a smooth atlas, we could expand the atlas by including all possible charts which are smoothly compatible with each other. So any smooth atlas can be expanded to a "maximal C^∞ atlas".

Definition 2.1.4. *A smooth manifold is a topological manifold equipped with a maximal smooth atlas. Such a maximal atlas is called a "C^∞ structure"*

(question about uniqueness of maximal atlases - 28 different ones for S^7, Milnor came up with an example. Exotic differentiable structures)

Remark One of the benefits of a smooth structure is that it allows us to define an ambiguous way whether 'objects' defined on the manifold (e.g. functions) are smooth or not.

Examples

- Dim 0: one-point spaces (or disjoint unions of one point spaces). $S^0 = \{1, -1\}$.
- Dim 1: $\mathbb{R}$ or S^1. (other manifolds with boundary but not defining that).
- Dim 2 (theorem) Two families
  - the "orientable" family, S^2, $T^2 = S^1 \times S^1$, $T^2 \underset{\text{connected sum}}{\#} T^2$, $T^2 \# T^2 \# T^2$, ... (connected sum is removing a disk from each and joining with a cylinder).
  - the "non-orientable" family: $\underbrace{\mathbb{RP}^2 \text{ (real projective space)}, K \text{(Klein bottle)}}_{\text{these don't live in } \mathbb{R}^3}$, S^2 with n discs removed with n mobius bands glued in.
- In any dimension $\mathbb{R}^n$ is a manifold with atlas consisting of a single chart $\{\mathrm{id}_{\mathbb{R}^n}\}$. Ditto $M_n(\mathbb{F}) \, (\cong \mathbb{F}^{n^2})$.
- We also have an n-sphere $S^n \subset \mathbb{R}^{n+1}$. $S^n = \{(x_1, \ldots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$.

Lecture 14 27/10/23

To see that S^n is a manifold we describe an atlas involving $2n + 2$ charts. Take open hemispheres about the points $\pm e_1, \pm e_2, \ldots, \pm e_{n+1}$. We define a chart for each of these half spheres to $\mathbb{R}^n$ = space orthogonal to $\pm e_i$ by projection (picture of projection). This is a smooth atlas as the transition functions, which are compositions of projections are clearly smooth maps. (There's also an atlas with just two charts, using stereographic projection from the poles. Messy exercise).

Exercise If M and N are manifolds, show that $M \times N$ (with the product topology) is a also a manifold with dimension $\dim M + \dim N$.

Example 2-torus $T^2 = S^1 \times S^1$. 3-torus $T^3 = S^1 \times S^1 \times S^1$, n-torus $T^n = S^1 \times \ldots \times S^1$. Tori are extremely important in the theory of Lie groups. (they're abelian groups).

Definition 2.1.5. *If M^m (notation for dimension of a manifold) is a C^∞-manifold and $N \subset M$ is a subset equipped with the subspace topology (intersect open sets with subset) then N is a dimension $n(<= m)$ submanifold if for every $x \in N$ there is a chart $\phi : U \to V$ of M with $x \in U$, such that $\phi(U \cap N) = \phi(U) \cap \left(\underbrace{\mathbb{R}^n \times \{0\}}_{\subset \mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n}}\right)$. The restriction of charts taking this form to N gives an atlas for N.*

Examples

i) S^n is a submanifold of $\mathbb{R}^{n+1}$.

ii) The equator $S^{n-1} \subset S^n$ is a submanifold of S^n.

iii) $S^1 \times \{p\} \subset S^1 \times S^1 = T^2$ is a circular submanifold of T^2.

Remark There are ways of detecting submanifolds which are easier than finding atlases.

Theorem 2.1.6. *(Whitney embedding theorem) Any smooth n-manifold can be $\underbrace{\text{realised}}_{\text{(diffeomorphic)}}$ as a submanifold of $\mathbb{R}^{2n}$. (No proof included)*

Maps between manifolds

Definition 2.1.7. *A map of smooth manifolds $f : M \to N$ is smooth if for every $x \in M$ and every chart $\theta : U' \to V'$ of N with $f(x) \in U'$ and every chart $\phi : U \to V$ for M with $x \in U$ and $f(U) \subset U'$, there is a smooth map $\bar{f}$ making the following square commute:*

$$\begin{array}{ccc} U \subset M & \xrightarrow{f|_U} & f(U) \subset N \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \theta|_{f(U)}} \\ V & \xrightarrow{\bar{f}} & \theta(f(U)) \end{array}$$

Idea $\bar{f}$ is the 'same' as f just viewed through charts. Need to use charts here as smooth only makes sense in Euclidean spaces.

Definition 2.1.8. *A bijection of smooth manifolds $f : M \to N$ is a diffeomorphism if it is smooth with smooth inverse.*

Definition 2.1.9. *A Lie group is a group G which is also a smooth manifold for which the multiplication map $G \times G \to G$ and the inverse map $G \to G$ are both smooth.*

(Aside: Gromoll Meyer sphere ${}_{Sp(1)}/{}^{Sp(2)}\backslash_{Sp(1)}$ biquotient. Exotic sphere with non-negative curved)

Lecture 15 06/11/23

Previously

- Manifold M^n is a topological space locally $\cong \mathbb{R}^n$. Charts give a way of doing calculus locally on a manifold. Transition functions from a smooth atlas give us a way of making the local calculus global: they guarantee that calculus in one chart agrees with that in any overlapping chart.
- Smooth functions $f : M \to N$ are those which are smooth when viewed through charts.

$$\begin{array}{ccc} U \subset M & \xrightarrow{f|_U} & f(U) \subset N \\ \downarrow \phi & & \downarrow \theta|_{f(U)} \\ V & \xrightarrow{\bar{f}} & W \end{array}$$

  i.e. want $\bar{f}$ to be smooth for any suitable choice of charts in M, N.

- Lie groups: this is a group G which is also a C^∞ manifold s.t. multiplication $G \times G \to G$ & inverse $G \to G$ are both C^∞.

Consider a differentiable map $g : \mathbb{R}^n \to \mathbb{R}^m$. The derivative of g is described by the Jacobian matrix. Writing $g = (g_1, g_2, \ldots, g_m)$,

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix}$$

At some point $p \in \mathbb{R}^n$ the Jacobian has "a rank" = max no of linearly independent rows/columns. (A result of linear algebra says that the row and column rank are the same.) For a differentiable map $f : M \to N$ the rank of f at $p \in M$ is the rank of the Jacobian matrix $\bar{f}$ at the point corresponding to p in $\mathbb{R}^n$. NB This concept of rank for maps between manifolds is well defined, i.e. is independent of the charts used to define $\bar{f}$. This is a consequence of the smooth atlas definition (smooth transition functions).

Definition 2.1.10. *$f : M^m \to N^n$ a C^∞ map, $m \geq n$.*

a) A point $p \in M$ at which (less than maximal) $\mathrm{rank}_p f < n$ is called a critical value of f, and $f(p)$ is a critical value.

b) A point $y \in N$ is a regular value of f if $\mathrm{rank}_p f = n$ (i.e. maximal) $\forall p \in f^{-1}(\{y\})$ (preimage of y).

Important convention Every point of N which is not in image of f is declared to be regular.

Theorem 2.1.11. *(The implicit Function Theorem - manifolds version).*

If $y \in N$ is a regular value of $f : M^m \to N^n$ $(m \geq n)$ a smooth map, then the pre-image $f^{-1}(\{y\})$ is a smooth submanifold of M with dimension $m - n$.

(No proof)

This is particularly useful when $M = \mathbb{R}^n$ or $M = M_n(\mathbb{F}) \; (\cong \mathbb{F}^{n^2})$.

e.g.

$f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i^2$. Then $f^{-1}(\{1\}) = S^{n-1}$. We conclude from IFT (2.1.11) that S^{n-1} is a manifold of dimension $n-1$ provided that 1 is a regular value of f. $\mathrm{Jac}\, f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}) = (2x_1, 2x_2, \ldots, 2x_n)$. This has rank 1 unless $(2x_1, 2x_2, \ldots, 2x_n) = 0$, but $0 \notin f^{-1}(\{1\})$, so this does not occur. So 1 is a regular value of f.

Applications to groups of matrices

Firstly, since $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and $\mathbb{R}^{n^2}$ is trivially a manifold, $M_n(\mathbb{R}^n)$ is a manifold. Next, observe that any open subset of a manifold is again a manifold under restriction of the topology and charts.

$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}$. This is an open subset of $M_n(\mathbb{R})$ $(GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$. (preimage of an open set under a continuous function is open, $\mathbb{R} \setminus \{0\}$ is open). $\therefore GL_n(\mathbb{R})$ is a manifold of dim n^2. (Similar comments for $GL_n(\mathbb{C})$ and $GL_n(\mathbb{H})$).

Lecture 16 08/11/23

Last time:

- $GL_n(\mathbb{R})$ is a manifold of dimension n^2. (because it's an open subset of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Similar comments for $GL_n(\mathbb{C})$, $GL_n(\mathbb{H})$.
- Differentiable map $f : M^n \to N^n$, with $m \geq n$, then the rank of f at $p \in M \underset{\text{defn}}{=}$ rank of Jacobian map of $\bar{f}$ (f viewed through chart, i.e. as a map between open subsets $\mathbb{R}^m$, resp. $\mathbb{R}^n$.
- $y \in N$ is a regular value of $\operatorname{rank}_p f = n$ (maximal!) $\forall p \in f^{-1}(\{y\})$.
- Implicit function theorem: if y is a regular point, then pre-image $f^{-1}(\{y\})$ is a smooth submanifold of M with dimension $m - n$.

Examples of Lie Groups

(i) $SL(n) = \{A \in M_n(\mathbb{R}) \mid \det A = 1\}$. Consider $\det : GL_n(\mathbb{R}) \to \mathbb{R}$. So $SL(n) = \det^{-1}(\{1\})$. By the IFT we will see that $SL(n)$ is a submanifold of $GL_n(\mathbb{R})$ with dimension $n^2 - 1$ and hence a Lie group since the multiplication and inversion in $SL(n)$ are the restrictions of the smooth multiplication and inversion in $GL_n(\mathbb{R})$. We $\therefore$ need to show that $1 \in \mathbb{R}$ is a regular value for det, i.e. we need to show that the rank of det $\neq 0$ at each $A \in GL_n(\mathbb{R})$ with $\det A = 1$, i.e. the derivative at A (in some direction through $GL_n(\mathbb{R})$ is non-zero). Consider the path $p(t) = (1-t)A$. $\det p(t) = (1-t)^n \det A = (1-t)^n$. $p(t) \in GL_n(\mathbb{R})$ for t small. Differentiating $\frac{\mathrm{d}}{\mathrm{d}t} \det p(t)\big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(1-t)^n\big|_{t=0} = n(1-t)^{n-1}\big|_{t=0} = n \neq 0$. Thus the rank of the det is maximal (i.e.$= 1$), at every $A \in SL(n)$ as required.

(ii) $SO(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = I_n\}$. Observe that AA^T is symmetric, i.e. $(AA^T)^T = AA^T$. Let $S_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A \text{ is symmetric i.e. } A^T = A\}$. Notice that $S_n(\mathbb{R}) \cong \mathbb{R}^{\frac{1}{2}n(n+1)}$ as any element of $S_n(\mathbb{R})$ is determined by its upper/lower triangle of entries. So $S_n(\mathbb{R})$ is a manifold!

Define $f : \underset{\cong \mathbb{R}^{n^2}}{M_n(\mathbb{R})} \to \underset{\cong \mathbb{R}^{\frac{1}{2}n(n+1)}}{S_n(\mathbb{R})}$, by $f(A) = AA^T$. (f " $=$ " $\bar{f}$)) Apply IFT to f.

Claim every element of $S_n(\mathbb{R})$ in $f^{-1}(\{I_n\})$ is a regular value of f. Choose $A, B \in M_n(\mathbb{R})$ and consider the path $A + tB \in M_n(\mathbb{R})$. Differentiating f along this path at $t = 0$: $df_A(B)$ (Deriv of f at A (i.e.

$t = 0$) in direction of B).

$$\begin{aligned} df_A(B) &= \frac{\mathrm{d}}{\mathrm{d}t} f(A + tB)\Big|_{t=0} \\ &= \lim_{t=0} \frac{f(A+tB) - f(A)}{t} \\ &= BA^T + AB^T \end{aligned}$$

Calculation: exercise. Notice that $BA^T + AB^T$ is also symmetric! Thus $df_A : M_n(\mathbb{R}) \to S_n(\mathbb{R})$. The claim follows if we can show df_A is surjective (since df_A is linear) for each $A \in f^{-1}(I_n)$. To see this: for any $C \in S_n(\mathbb{R})$ we can easily check that by setting $B = \frac{1}{2}CA$, we obtain $df_A(B) = C$. $\therefore$ by IFT $SO(n) = f^{-1}(\{I_n\})$ is a manifold of dimension $n^2 - \frac{1}{2}n^2 - \frac{1}{2}n = \frac{1}{2}n(n-1)$.

Lecture 17 10/11/23

Last time:

Showed $SO(n)$ is a submanifold of $M_n(\mathbb{R})$ of dimension $\frac{1}{2}n(n-1)$ using Implicit Function Theorem.

To show that $SO(n)$ is a Lie group we simply observe that the multiplication and inverse operations agree with those (where defined) in $M_n(\mathbb{R})$, and hence are smooth.

2.2. Tangent Spaces

Consider a submanifold $M^m \subset \mathbb{R}^n$. Let $p \in M$ and consider a smooth path $\gamma : (-\epsilon, \epsilon) \to M \subset \mathbb{R}^n$ with $\gamma(0) = p$. (small picture of gamma in manifold, with velocity vector).

The velocity vector $\gamma'(0)$ is a vector in $\mathbb{R}^n$. Now consider the set of <u>all</u> such paths γ. From this we get a set of velocity vectors $\subset \mathbb{R}^n$. Intuitively it is clear that this set is an affine vector space, i.e. a vector space translated to the point p.

Definition 2.2.1. *(a) The <u>tangent space</u> to $M \subset \mathbb{R}^n$ to p, T_pM, is the set of velocity vectors to M at p.*

<u>Aim</u>: formulate an intrinsic notion of tangent space, i.e. not depending on a choice of embedding into some $\mathbb{R}^n$.

<u>Observation</u>: Given a path γ through p in M <u>as before</u>, and a differentiable function $f : M \to \mathbb{R}$, consider the composition $f \circ \gamma : (-\epsilon, \epsilon) \to \mathbb{R}$.

This is differentiable. We can think of this derivative as a directional derivative of f in the direction of $\gamma'(0)$. We can therefore identify velocity vectors $\gamma'(0)$ at p with directional derivative operations, i.e. with differential operators. The point is that this makes as much sense for abstract manifolds as for submanifolds of $\mathbb{R}^n$.

Denote the set of all C^∞ real-valued functions on M by $C^\infty(M)$.

Definition 2.2.2. *Given a smooth curve $\gamma(t)$ on M with $\gamma(0) = p$, the tangent vector to γ at p is a map $\gamma'(0) : C^\infty(M) \to \mathbb{R}$ given by $\gamma'(0)(f) = \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\big|_{t=0}$.*

Given a chart $\phi : \underset{\subset M}{U} \to \underset{\subset \mathbb{R}^m}{V}$, we can use the standard coordinates $(x_1, x_2, \ldots, x_m)$ in V, $x_i : V \to \mathbb{R}$, to obtain a local coordinate system in M, given by $(x_1 \circ \phi, \ldots, x_m \circ \phi)$. For simplicity, we'll call these coordinates on M, $(x_1, \ldots, x_m)$.

Assume that $p \in U$ and $\phi(p) = 0 \in V \subset \mathbb{R}^m$. In these coordinates we have $\gamma(t) = (\gamma_1(t), \ldots, \gamma_m(t))$, $(\gamma_i : (-\epsilon, \epsilon) \to \mathbb{R})$ and $\therefore$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\bigg|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_1(t), \ldots, \gamma_m(t))\bigg|_{t=0} \\ &\underset{\text{chain rule}}{=} \sum_{i=1}^{m} \frac{\partial f}{\partial x_i} \gamma_i'(0) \\ &= \left(\sum_{i=1}^{m} \gamma_i'(0) \frac{\partial}{\partial x_i}(p) \right)(f) \end{aligned}$$

where $\frac{\partial}{\partial x_i}(p)$ is the tangent vector to the x_i coordinate line at p, i.e. $(0, \ldots, 0, \underset{i^{th} \text{ place}}{t}, 0, \ldots, 0)$ in coords of U.

$\therefore \gamma'(0) = \sum_{i=1}^{m} \gamma_i'(0) \frac{\partial}{\partial x_i}(p)$ $(*)$.

Observe that this only depends on the derivative of γ at p.

Definition 2.2.1. *(b) The tangent space T_pM for any manifold M is the set of velocity vectors (as defined above) to M at p. The following is evident from $(*)$.*

Lemma 2.2.3. *T_pM is a vector space of dimension $= \dim M$, and given local coordinates system $(x_1, x_2, \ldots, x_m)$ around p, a basis for T_pM is given by*

$$\{\frac{\partial}{\partial x_1}(p), \ldots, \frac{\partial}{\partial x_m}(p)\}$$

Notice that T_pM is independent of local coord systems, since tangent vectors themselves are indep. of local coords.

Lecture 18 13/11/23

Last time:

- tangent vector to a manifold M at $p \in M$ is a directional differentiation operator at p. (Acting on smooth functions $M \to \mathbb{R}$).
- Tangent space $T_pM$is the set of all tangent vectors at p.
- $T_pM = \text{span}\{\frac{\partial}{\partial x_1}(p), \dots, \frac{\partial}{\partial x_n}(p)\}$, where $(x_1, x_2, \dots, x_n)$ is any local coordinate system on M about p, and $\frac{\partial}{\partial x_i}$ are the usual partial differentiation operations.

The collection of all tangent spaces $\bigcup\limits_{p \in M} T_pM^n$ has the structure of a manifold (non-compact!) of dimension $2n$. This is denoted TM and is called the "tangent bundle".

Examples

(i) $T_p\mathbb{R}^n \cong \mathbb{R}^n$. Consider a path $\gamma(t) = p + vt$ for some $v \in \mathbb{R}^n$, $\gamma'(0) = v$. Thus the corresponding tangent (velocity) vector is v.

Alternatively natively consider $\frac{\mathrm{d}}{\mathrm{d}t}\, f(p+vt)\big|_{t=0}$. If $v = \sum \lambda_i e_i$ then this is

$$\begin{aligned} &\sum \frac{\partial f}{\partial x_i}\lambda_i \quad \text{by the chain rule.} \\ &= \left(\sum \lambda_i \frac{\partial}{\partial x_i}\right)(f) \end{aligned}$$

So $\sum \lambda_i \frac{\partial}{\partial x_i}$ is the tangent vector corresp to $v = \sum \lambda_i e_i$. The correspondence $T_p\mathbb{R}^n \cong \mathbb{R}^n$ is precisely $\sum \lambda_i \frac{\partial}{\partial x_i} \leftrightarrow \sum \lambda_i e_i$.

(ii) $T_A M_n(\mathbb{R}) \cong M_n(\mathbb{R})$. (Essentially a special case of (i)). Consider a path $\gamma(t) = A + tB$ for some $B \in M_n(\mathbb{R})$. Clearly $\gamma'(0) = B$. Alternatively, consider $\frac{\mathrm{d}}{\mathrm{d}t}\, f(A+tB)\big|_{t=0}$. Let $v_{ij} : M_n(\mathbb{R}) \to \mathbb{R}$ be the function which picks out the $(i,j)^{th}$ entry. Standard coordinates on $M_n(\mathbb{R})$.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f(A+tB) &\underset{\text{chain rule}}{=} \sum_{i,j} \frac{\partial f}{\partial v_{ij}} b_{ij} \\ &= \left(\sum_{i,j} \frac{\partial f}{\partial v_{ij}} b_{ij}\right)(f) \end{aligned}$$

So $\sum_{i,j} \frac{\partial f}{\partial v_{ij}} b_{ij}$ is our tangent vector which we can identify with matrix $(b_{ij}) = B$.

(iii) $S_n(\mathbb{R}) = \{(n \times n)$ symmetric matrices $\} \cong \mathbb{R}^{\frac{1}{2}n(n+1)}$. Same argument as in (ii) shows that $T_A S_n(\mathbb{R}) \cong \S_n(\mathbb{R})$

(iv) $T_A GL_n(\mathbb{R}) \cong M_n(\mathbb{R})$. We can see this two ways:

  (a) Since $GL_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$

  (b) using same argument as (ii) after noting that for $A \in GL_n(\mathbb{R}), B \in M_n(\mathbb{R})$, the path $A + tB$ lies in $GL_n(\mathbb{R})$ for $tcloseto0$.

Definition 2.2.4. *Given a differentiable map $f : M^n \to N^n$, the derivative of f at $p \in M$, df_p is a map $df_p : T_pM \to T_{f(p)}N$ given by $df_p(X) = X(g \circ f) \forall g \in C^\infty(N), \forall X \in T_pM$.*

Idea Given a differentiable path $\gamma(t)$ through $p \in M$, $f(\gamma(t))$ is a differentiable path through $f(p)$ in N. If $X \in T_pM$ is the tangent vector $\gamma'(0)$ then $df_p(X)$ is the tangent vector corresp. to $f(\gamma(t))$ at $f(p)$. (picture of path and image of path, tangent vectors etc)

Lemma 2.2.5. *df_p is a linear map.*

Proof. Trivial, from definition of df_p. □

The individual derivatives df_p can be assembled into a 'total' derivative $df : TM \to TN$ between tangent bundles.

Remarks

1. Given coordinate systems about p and $f(p)$ we obtain bases for T_pM and $T_{f(p)}N$ and with respect to these bases, df_p is given by a matrix. This is just the usual Jacobian matrix.

2. We can now re-interpret the notions of regular/critical points in terms of df_p being surjective/ not surjective.

3. Previously we saw a map $f : M_n(\mathbb{R}) \to S_n(\mathbb{R})$, $f(A) = AA^T$ we decided that df_A is a map $M_n(\mathbb{R}) \to S_n(\mathbb{R})$. In reality $df_A : \underbrace{T_A M_n(\mathbb{R})}_{\cong M_n(\mathbb{R})} \to \underbrace{T_{AA^T} S_n(\mathbb{R})}_{\cong S_n(\mathbb{R})}$

Vector Fields

Definition 2.2.6. *A (tangent) vector field on a smooth manifold M is a choice of tangent vector at each $p \in M$, i.e. a rule which assigns $X(p) \in T_pM$ to each $p \in M$.*

Given a coord system $(x_1, x_2, \ldots, x_n)$ locally on $U \subset M$. We can express (locally) a vector field on U as $\sum \lambda_i(p)\frac{\partial}{\partial x_i}(p)$. Here $\lambda_i : U \to \mathbb{R}$. We say the vector field is smooth if the λ_i are all smooth. The vector field is globally smooth (which we will assume from now on) if it is smooth in every chart. This is a well-defined concept, as the notion of smoothness is the same viewed from any charts in a smooth atlas.

Lecture 19 15/11/23

Last time:

- A tangent vector field on manifold M is a choice of tangent vector in $T_pM$for each $p \in M$. This choice is assumed to be smoothly varying with p.

From now on we will use the term vector field to mean tangent vector field.

Remark: A flow on M is a one-parameter, smoothly varying, family of diffeomorphisms $M \to M$. (for example, flow of air around the earth) (TODO: insert figure of flow here). Differentiating a flow gives a vector field. Conversely, vector field can be integrated to give a flow.

2.3. Lie Algebras

Let X, Y be vector fields on M.

Definition 2.3.1. *The Lie bracket of X and Y, denoted $[X, Y]$ is*[1]

$$[X, Y] = XY - YX.$$

Lemma 2.3.2. *$[X, Y]$ is a vector field on M. (ie it is a first order differential operator, not second order)*

Proof. Let $f \in C^\infty(M)$. So $XY(f) = X(Yf)$ etc. Note, $Yf \in C^\infty(M)$. With respect to a local coordinate system $(x_1, x_2, \ldots, x_n)$ we have

$$X = \sum_i \lambda_i \frac{\partial}{\partial x_i}, \quad Y = \sum_i \mu_i \frac{\partial}{\partial x_i},$$

[1] multiplication on the R.H.S. makes sense if vector fields are interpreted as differential operators.

for some $\lambda_i, \mu_i \in C^\infty(M)$.

Computing, we have

$$XYf - YXf = \sum_{i,j} \left(\lambda_i \frac{\partial \mu_j}{\partial x_i} \frac{\partial f}{\partial x_j} - \mu_j \frac{\partial \lambda_i}{\partial x_j} \frac{\partial f}{\partial x_i} \right) + \sum_{i,j} \lambda_j \mu_j \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i} \right).$$

But the second partial derivatives commute.

$$\therefore [X,Y](f) = \sum_{i,j} \left(\lambda_i \frac{\partial \mu_j}{\partial x_i} \frac{\partial f}{\partial x_j} - \mu_j \frac{\partial \lambda_i}{\partial x_j} \frac{\partial f}{\partial x_i} \right).$$

$$\therefore [X,Y] = \sum_{i,j} \left(\lambda_i \frac{\partial \mu_j}{\partial x_i} \frac{\partial}{\partial x_j} - \mu_j \frac{\partial \lambda_i}{\partial x_j} \frac{\partial}{\partial x_i} \right).$$

□

Remark: This has a nice interpretation in terms of flows and is sometimes called a Lie derivative. One can define a way to push/pull Y along the flow of X and differentiate it, this gives a notion of differentiating Y with respect to X. We won't explore this further here.

Lemma 2.3.3. *The Lie bracket has the following properties:*

1. *It is bilinear.*
2. *It is anti-symmetric (ie $[X,Y] = -[Y,X]$).*
3. *It satisfies the "Jacobi identity":*

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0.$$

Proof. Properties 1 and 2 are trivial. For 3 we can simple write out each term and add. ie $[[X,Y],Z] = [XY - YX, Z] = XYZ - YXZ - ZXY + ZYX$ etc. (exercise!) □

We now focus on vector fields on Lie groups.

For $g \in G$ (G a Lie group) we have diffeomorphisms $L_g : G \to G$ given by $L_g(h) = gh$ (ie by left multiplication) and $R_g : G \to G$ given by $R_g(h) = hg$. We will concentrate on L_g but everything will hold true for R_g also. We have $dL_g : T_hG \to T_{gh}G$ for any $h \in G$.

Definition 2.3.4. *A vector field X on G is left-invariant if $dL_g(X) = X$ $\forall g \in G$. ie $dL_g(X_h) = X_{gh}$ $\forall g, h \in G$.*

A similar definition can be made for right-invariance.

Lemma 2.3.5. *There is a bijection between the set of left invariant vector fields (LIVF) and T_eG.*

Remark: Since the set of LIVFs is a vector space (if X, Y are LIVFs then trivially so is $\lambda X + \mu Y$) this bijection is actually a linear ismorphism.

Proof. For any $v \in T_eG$ we can construct a LIVF by "left propagation". ie let $X_g = dL_g(v)$ $\forall g \in G$. The resulting field X is left invariant:

$$\begin{aligned} dL_g(X_h) &= dL_g \circ dL_h(v) \\ &= d(L_g \circ L_h)(v) && \text{(by the chain rule)} \\ &= dL_{gh}(v) \\ &= X_{gh}. \end{aligned}$$

This gives a map $T_eG \to \{LIVFs\}$. This map is trivally injective. Observe that there is an inverse map $X \mapsto X_e \in T_eG$. Therefore the map is a bijection. $\square$

Lemma 2.3.6. *If X, Y are LIVFs, then $[X, Y]$ is a LIVF.*

Proof. We begin by making the following claim: Let $f \in C^\infty(G)$. X is left invariant $\iff (Xf) \circ L_g = X(f \circ L_g)$.

$$\begin{tikzcd} G \arrow[r, "L_g"] \arrow[dr, "f \circ L_g"'] & G \arrow[d, "Xf"] \\ & \mathbb{R} \end{tikzcd}$$

To see this, suppose $p(t)$ is a C^∞ path in G with $p(0) = e, p'(0) = X_e$. Then X is left invariant $\iff X_h = \frac{d}{dt} hp(t)|_{t=0}$ $\forall h \in G$.

$$\begin{aligned} \implies (Xf) \circ L_g(h) &= (Xf)(gh) \\ &= X_{gh}(f) && \text{(as } X \text{ is left invariant)} \\ &= \frac{d}{dt} f(ghp(t))|_{t=0}. \end{aligned}$$

Compare this with:

$$\begin{aligned} X(f \circ L_g)(h) &= (X_h)(f \circ L_g) \\ &= \frac{d}{dt}(f \circ L_g)(hp(t))|_{t=0} \qquad \text{(as } X \text{ is left invariant)} \\ &= \frac{d}{dt} f(ghp(t))|_{t=0}. \end{aligned}$$

as before. So $(Xf) \circ L_g = X(f \circ L_g)$ (To be continued next lecture...) $\square$

Lecture 20 17/11/23

Last time:

- A X is a (tangent) vector field on Lie group G, then X if left-invariant (LI) if, for all $g \in G$, $dL_g(X) = X$ where $L_g : G \to G$, $L_g(h) = gh$ for all $h \in G$ and $dL_g(X) = X$ means $dL_g(X_h) = X_{gh}$.
- (2.3.5) There exists a bijection $T_eG \to${LIVFs on G} given by "left propagation", ie for $v \in T_eG$ $X_h := dL_h(v)$ $\forall h \in G$.
- (2.3.6) If X, Y are LI then so is $[X, Y]$.

Proof. (2.3.6 continued) Previously, we had made the following claim: For $f \in C^\infty(G)$, X is left invariant if and only iff the following holds:

$$(Xf) \circ L_g = X(f \circ L_g). \tag{$\circledast$}$$

($\Longrightarrow$) was prove lasted time.
($\Longleftarrow$) Assume $(Xf) \circ L_g = X(f \circ L_g)$ holds. Then for any $h \in G$,

$$(Xf) \circ L_g(h) = (Xf)(gh) = X_{gh}f.$$

Therefore, $X_{gh}f = X_h(f \circ L_g)$. By definition of the derivative map (2.2.4) we have

$$X_h(f \circ L_g) = dL_g(X_h)(f).$$

Therefore,

$$X_{gh}f = dL_g(X_h)(f), \quad \forall f \in C^\infty(G).$$

Hence, $X_{gh} = dL_g(X_h)$, ie X is LI. This proves the claim.

Now to prove 2.3.6, consider the following:

$$
\begin{aligned}
([X,Y]f)\circ L_g &= (X(Yf) - Y(Xf))\circ L_g, && \\
&= X(Yf)\circ L_g - Y(Xf)\circ L_g, && \text{(trivially)}\\
&= X((Yf)\circ L_g) - Y((Xf)\circ L_g) && \text{(by } \circledast)\\
&= X(Y(f\circ L_g)) - Y(X(f\circ L_g)) && \text{(by } \circledast)\\
&= (XY)(f\circ L_g) - (YX)(f\circ L_g) && \\
&= [X,Y](f\circ L_g) &&
\end{aligned}
$$

Hence, $([X,Y]f)\circ L_g = [X,Y](f\circ L_g)$ and so, by the claim made at the start of the proof, we conclude that $[X,Y]$ is LI. $\square$

Combining (2.3.5) nd (2.3.6) we immediately deduce:

Corollary 2.3.7. *Restricting the Lie bracket operation from $\{$LIVFs on $G\}$ to T_eG gives a map*

$$[\cdot,\cdot] : T_eG \times T_eG \to T_eG,$$

which is bilinear, anti-symmetric and satisfies the Jacobi identity.

Definition 2.3.8.

1. *A Lie algebra consists of a vector space V over a field $\mathbb{F}$, together with an operation $[\cdot,\cdot] : V\times V \to V$ which is bilinear, anti-symmetric and satisfies the Jacobi identity.*

2. *The Lie algebra $\mathfrak{g}$ of a Lie group G is T_eG equipped with the operation $[\cdot,\cdot]$ from 2.3.7.*

Remark: The Lie bracket captures, in an infinitesimal and nonobvious way, the group structure of G.

Observation 2.3.9. *In general, the "multiplication" map $[\cdot,\cdot]$ in a Lie algebra is not associative.*

If $[\cdot,\cdot]$ was associative then $[[X,Y],Z] = [X,[Y,Z]]$. Then, by anti-symmetry we have $[[X,Y],Z] + [[Y,Z],X] = 0$. But the Jacobi identity tells us that

$$[[X,Y],Z] + [[Y,Z],X] = -[[Z,X],Y]$$

which is non-zero in general.

Examples of a Lie algebras

1. Let V be any vector space and set $[v, w] = 0$ for all $v, w \in V$. Then $(V, [\cdot,\cdot])$ is a Lie algebra. Note, in this special case $[\cdot,\cdot]$ is associative.

2. On $M_n(\mathbb{F})$ define the commutator $[A, B] = AB - BA$. Then $(M_n(\mathbb{F}), [\cdot,\cdot])$ is a Lie algebra.

3. Let M be a smooth manifold and let $\Gamma(M)$ be the (real) vector space of all (tangent) vector fields on M. The Lie bracket $[X, Y] = XY - YX$ makes $\Gamma(M)$ into a Lie algebra. Note that this object is infinite dimensional.

We have seen that $T_e GL_n(\mathbb{R}) \cong M_n(\mathbb{R})$. Therefore, as a vector space, $\mathfrak{gl}_n(\mathbb{R}) \cong M_n(\mathbb{R})$. What is the Lie bracket in this case?

Theorem 2.3.10. *The Lie bracket operation on $\mathfrak{gl}_n(\mathbb{R})$ is just $[A, B] = AB - BA \in M_n(\mathbb{R})$.*

Proof. See moodle! Here is the main idea:

Corresponding to A and B we have paths through $\mathcal{I}_n$ in $GL_n(\mathbb{R})$, parameterised by s and t respectively. By left translation (L_g) by elements close to $\mathcal{I}_n$, we can create a "mesh" of paths surrounding "$\mathcal{I}_n$" by translating one of these paths along the other. This gives us a local 2-dimensional "slice" around $\mathcal{I}_n$ parameterised by (s, t). We can compute second derivatives of $f \in C^\infty(GL_n(\mathbb{R}))$ with respect to the parameters. This allows us to identify the Lie bracket operation. (TODO: insert figure of coordinate patch construction around identity) $\square$

Lecture 21 20/11/23

Last time:

- Lie algebra is a vector space V equipped with an operation $[\cdot,\cdot]$, $[\cdot,\cdot] : V \times V \to V$ s.t. $[\cdot,\cdot]$ is

  i) bilinear
  ii) antisymmetric
  iii) satisfies the Jacobi identity

  (In general $[\cdot,\cdot]$ is not associative)

  e.g. $\{C^\infty$ vector fields on a manifold $\}$ is a Lie algebra under standard Lie bracket operation.

e.g. G a Lie group, then $\{LIVF\}$ is a Lie bracket under standard Lie bracket. This restricts to give a map $[\cdot,\cdot] : T_eG \times T_eG \to T_eG$, $(T_eG, [\cdot,\cdot])$ is the Lie algebra of G.

- The general linear group $GL_n(\mathbb{R})$ has Lie algebra $(\underbrace{M_n(\mathbb{R})}_{=T_eGL_n(\mathbb{R})}, [\cdot,\cdot])$ where $[A,B] = \underbrace{AB - BA}_{\text{difference of matrix products}}$ for any $A, B \in M_n(\mathbb{R})$.

  Same argument for $GL_n(\mathbb{C})$ or $GL_n(\mathbb{H})$.

  (an aside - if we think of $GL_n(\mathbb{H})$ as an abstract object, not as linear maps, we don't have to worry about the left/right multiplication issue, we can do left or right multiplication of matrices)

Corollary 2.3.11. *The Lie bracket operation for the other matrix groups (which are all Lie subgroups of $GL_n(\mathbb{F})$) is just the restriction of that for $GL_n(\mathbb{R})$ to the appropriate space of matrices, i.e. the Lie bracket of matrix groups is just the matrix commutator.*

Theorem 2.3.12. *The Lie algebra of $O(n)$ is*

$$\left(\underbrace{\{A \in M_n(\mathbb{R}) \mid \underset{\textit{skew-symmetric matrices}}{A^T = -A}\}}_{=T_eO(n)}, [\cdot,\cdot]\}\right)$$

Proof. Consider a C^∞ path $A(t)$ in $O(n)$ with $A(0) = I_n$. We have $A(t)A^T(t) = I_n$ (by orthogonality).

$\therefore \frac{\mathrm{d}}{\mathrm{d}t}AA^T \underset{\text{product rule}}{=} A'A^T + AA'^T = 0.$

At $t = 0$ we obtain $A' + A'^T = 0$, i.e. $A'^T = -A'$. So $A' \in \underset{\text{skew-symmetric matrices}}{\mathrm{SSym}_n(\mathbb{R})}$, i.e. $\mathfrak{o}_n = T_eO(n) \subset \mathrm{SSym}_n(\mathbb{R})$.

Conversely, E_{ij} be the $(n \times n)-$matrix with 1 in position (i,j) and 0s elsewhere.

Observe that $\{E_{ij} - E_{ji}\}_{i \neq j}$ is a basis for $\mathrm{SSym}_n(\mathbb{R})$.

It suffices to find a path for each (i,j), $i \neq j$ in $O(n)$ through I_n with derivative $E_{ij} - E_{ji}$ at I_n.

Set $\gamma_{ij} = I_n + \sin(t)(E_{ij} - E_{ji}) + (-1 + \cos(t))(E_{ii} - E_{jj})$. Notice that $\gamma'_{ij}(0) = E_{ij} - E_{ji}$.

To see that $\gamma_{ij}(t) \in O(n)$ (actually $\in SO(n)$ since $O(n)$ has two components, and we're in the identity component), we re-write in matrix form

$$
\begin{array}{cc}
\begin{matrix} \\ \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \\ \end{matrix} &
\begin{pmatrix}
1 & & & & & & & & & & & & \\
& 1 & & & & & & & & & & & \\
& & \ddots & & & & & & & & & & \\
& & & 1 & & & & & & & & & \\
& & & & \cos(t) & & & & \sin(t) & & & & \\
& & & & & 1 & & & & & & & \\
& & & & & & \ddots & & & & & & \\
& & & & & & & 1 & & & & & \\
& & & & -\sin(t) & & & & \cos(t) & & & & \\
& & & & & & & & & 1 & & & \\
& & & & & & & & & & 1 & & \\
& & & & & & & & & & & \ddots & \\
& & & & & & & & & & & & 1
\end{pmatrix} \\
& \begin{matrix} & & & & i & & & & j & & & & \end{matrix}
\end{array}.
$$

Since the rows of this matrix form an orthonormal set, we see by 1.2.5 that this matrix is orthogonal $\forall t$. (Actually this matrix represents a rotation in the plane $\mathrm{Span}\{e_i, e_j\}$ through angle t.) $\square$

Similar arguments show:

Theorem 2.3.13. *The Lie algebra of $U(n)$ is*

$$\mathfrak{u}_n = \{A \in M_n(\mathbb{C}) \mid \underbrace{\bar{A}^T = -A}_{\text{"skew Hermitian"}}\},$$

and the Lie algebra of $Sp(n)$ is

$$\mathfrak{sp}_n = \{A \in M_n(\mathbb{H}) \mid \bar{A}^T = -A\},$$

both with matrix commutator as Lie bracket.

Corollary 2.3.14.

$$\begin{aligned}
\dim O(n) &= \frac{1}{2}n(n-1) \\
\dim U(n) &= n^2 \\
\dim Sp(n) &= 2n^2 + n
\end{aligned}$$

Proof. $\dim O(n) = \dim \mathfrak{o}_n = \dim \mathrm{SSym}_n(\mathbb{R})$.

A skew symmetric matrix is determined by the entries above the diagonal. There are
$$\frac{1}{2}\left(\underset{\text{total no. of entries}}{n^2} - \underset{\text{no. of entries on diagonal}}{n^2}\right)$$
(all diag entries $= 0$)

For $\mathfrak{u}_n$, the diagonal entries must be pure imaginary. Off the diagonal, each entry determines two real nos. Overall we have
$$\frac{1}{2}\left(\underset{\text{if diagonal entries }=0}{2n^2 - 2n}\right) + \underset{\text{add back } n \text{for imaginary entries on diagonal}}{n}$$
$= n^2 - n + n = n^2$.

In symplectic case, we have four real numbers for each entry, but diagonal entries must be pure imaginary (determined by three real nos) $\therefore \dim \mathfrak{sp}_n = \frac{4n^2-4n}{2} + \underset{\text{diagonal entries}}{3n} = 2n^2 - 2n + 3n = 2n^2 + n$ $\square$

Lecture 22 22/11/23

Previously:

- Lie bracket of matrix groups is the matrix commutator. $[A, B] = AB - BA$. So identifying the Lie algebra just requires identifying the tangent space at the identity T_eG.
- $GL_n(\mathbb{R})$ has Lie algebra $\mathfrak{gl}_n(\mathbb{R}) = (M_n(\mathbb{R}), [\cdot,\cdot])$, $\mathfrak{o}_n = \{\text{skew symmetric matrices}, [\cdot,\cdot]\}$, $\mathfrak{u}_n = (\{A \in M_n(\mathbb{C}) \mid \bar{A}^T = -A\}, [\cdot,\cdot])$, $\mathfrak{sp}_n = \ldots$. This leaves $SL(n)$ and $SU(n)$. (As $O(n) \underset{\text{topologically}}{\cong} SO(n) \amalg SO(n)$, we get $\mathfrak{so}_n = \mathfrak{o}_n$

For $SL(n)$ and $SU(n)$ we need a lemma:

Lemma 2.3.15. *Suppose $A(t)$ is a C^∞ path of matrices in $M_n(\mathbb{F})$ such that $A(0) = I_n$. Then $\frac{\mathrm{d}}{\mathrm{d}t} \det A(t)\big|_{t=0} = \operatorname{tr} A'(0)$.*

Proof. Recall that
$$\det A = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$
(S_n = symmetric group, i.e. permutations of $\{1, \ldots, n\}$, $A = (a_{ij})$)

Each term of sum, when differentiated gives an expression (product rule)
$$a'_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} + a_{1\sigma(1)} a'_{2\sigma(2)} \cdots a_{n\sigma(n)} + \ldots + a_{1\sigma(1)} a_{2\sigma(2)} \cdots a'_{n\sigma(n)}$$

At I_n, $a_{ij} = \delta_{ij}$, so this gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \det A(t)\bigg|_{t=0} = a'_{11} + a'_{22} + \ldots + a'_{nn} = \operatorname{tr} A'(0)$$

as all terms a_{ij} with $i \neq j$ vanish. $\square$

Theorem 2.3.16. *As vectors spaces*

$$\begin{aligned} \mathfrak{sl}_n(\mathbb{R}) &= \{A \in M_n(\mathbb{R}) \mid \operatorname{tr} A = 0\} \\ \mathfrak{su}_n &= \{A \in \mathfrak{u}_n \mid \operatorname{tr} A = 0\} = \{A \in M_n(\mathbb{C}) \mid \bar{A}^T = -A, \operatorname{tr} A = 0\}. \end{aligned}$$

Proof. The condition $\det A = 1$ in the definitions of both $SL(n)$ and $SU(n)$ differentiate at I_n to give $\operatorname{tr} A = 0$ (by 2.3.15). $\square$

A linear map of Lie algebras $f : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism if $f([u,v]_{\mathfrak{g}}) = [f(u), f(v)]_{\mathfrak{h}}$, $\forall u, v \in \mathfrak{g}$.

Theorem 2.3.17. *A smooth homomorphism of Lie Groups $\theta : G \to H$ induces a Lie algebra homomorphism $d\theta_e : \mathfrak{g} \to \mathfrak{h}$.*

Proof. Consider $u, v \in \mathfrak{g}$ and corresponding paths $\theta(t)$, $\psi(s)$ s.t. $\phi'(0) = u$, $\psi'(0) = v$. (picture).

Let $f \in C^\infty(H)$, so $f \circ \theta \in C^\infty(G)$. If U and V extend u, v to LIVFs we have

$$d\theta_e(V_g)(f) = \frac{\mathrm{d}}{\mathrm{d}s} f(\theta \circ Ł_g \circ \psi(s))\bigg|_{s=0}$$

by the definition of derivatives (2.2.4).

Then

$$\begin{aligned} d\theta(U)d\theta(V)(f)\bigg|_{e_H} &= \frac{\mathrm{d}}{\mathrm{d}t} d\theta_{\phi(t)}(V_{\phi(t)})(f)\bigg|_{t=0} \\ &= \frac{\partial^2}{\partial t \partial s} f(\theta \circ L_{\phi(t)} \circ \psi(s))\bigg|_{s=t=0} \\ &= \frac{\partial^2}{\partial t \partial s} f(\theta(\phi(t)\psi(s)))\bigg|_{s=t=0} \end{aligned}$$

So

$$[d\theta(U), d\theta(V)]_{e_H}(f) = \frac{\partial^2}{\partial t \partial s} \left[f \circ \theta(\phi(t)\psi(s)) - f \circ \theta(\psi(s)\phi(t))\right]\bigg|_{s=t=0}$$

Compare

$$V_g(f \circ \theta) = \frac{\mathrm{d}}{\mathrm{d}s} f \circ \theta(g\psi(s))\bigg|_{s=0}$$

to

$$UV(f\circ\theta)\Big|_{e_G} = \frac{\partial^2}{\partial t\partial s}\, f\circ\theta(\phi(t)\psi(s))\Big|_{s=t=0}$$

$$\begin{aligned}
\therefore\ d\theta([U,V])(f)\Big|_{e_H} &= [U,V](f\circ\theta)\Big|_{e_H} \\
&= UV(f\circ\theta)\Big|_{e_G} - VU(f\circ\theta)\Big|_{e_G} \\
&= \frac{\partial^2}{\partial t\partial s}\left[f\circ\theta(\phi(t)\psi(s)) - f\circ\theta(\psi(s)\phi(t))\right]\Big|_{s=t=0} \\
&= [d\theta(U), d\theta(V)]_{e_H}(f) \quad \text{from above}
\end{aligned}$$

$\square$

A Lie group isomorphism is an isomorphism of Lie Groups which is a C^∞ diffeomorphism.

Corollary 2.3.18. *A Lie Group isomorphism induces an isomorphism of Lie algebras.*

Question: To what extent is the reverse true, i.e. If G and H have isomorphic Lie algebras, is $G \cong H$? (If yes, this would mean we could classify Lie groups via linear algebra) Answer No!

Simple counter examples

- $O(n) \ncong SO(n)$ but $\mathfrak{o}_n = \mathfrak{so}_n$.
- $SU(2) \ncong SO(3)$ but $\mathfrak{su}(2) \cong \mathfrak{so}(3) = \mathfrak{o}(3)$.

But

Theorem 2.3.19. *(Lie - very hard!) Simply - connected Lie groups are $\cong \iff$ their Lie algebras are $\cong$, (Simply - connected: loops can be contracted to a point).*