# MT543 Topics in Algebra

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**Note:**

Any transcription mistakes and typos are my own.

Lectures by David Wraith. Lie Groups and Lie Algebras.

# 1 Lecture 1 25/09/23

missed this lecture - some intro to do with spheres, transformations and symmetries and other motivational stuff. Definition of an algebra (bilinear product) over a field.

# 2 Lecture 2 27/09/23

Sorting out tutorial times. Lectures: Monday 2pm MS2, Wednesday 2pm LGH, Thursday 12pm MS2.

Lie Groups, dual nature, Groups but also a topological geometrical character. Can prove things with a mix of both methods - intersection of various areas.

## 2.1 Groups of matrices

### 2.1.1 General Linear Groups

Quaternions will have a central role.

Consider groups of $N \times N$ matrices over the fields $\mathbb{R}$ and $\mathbb{C}$ and also over the quaternions.

**Definition 1.** *The quaternions $\mathbb{H}$ is a 4-dim real vector space with standard basis elements $1, i, j, k$, equipped with an associative linear multiplication operation defined by*

$$i^2 = j^2 = k^2 = -1, \quad ij = k, jk = i, ki = j$$

So a generic quaternion takes the form $a + bi + cj + dk$, $a, b, c, d \in \mathbb{R}$.

Observe, $ji = j(jk) = (jj)k$ (by associativity) $= j^2k = -k$. Similarly $kj = -i$ and $ik = -j$.

e.g. $(2 + i - 3k)(5 + 2i - j + k) = 10 + 4i - 2j + 2k + 5i - 2 - k - j - 15k - 6j - 61 + 3$ etc.

Quaternions is not commutative, so is not a field. However it is a skew field (division algebra).

Terminology - In $a + bi + cj + dk$, $a$ is called the real or scalar part, and the rest $bi + cj + dk$ imaginary or vector part.

In analogy with complex numbers,

**Definition 2.** *1. The conjugate of $a + bi + cj + dk$, is $\overline{a + bi + cj + dk} = a - bi - cj - dk$ 2. The norm of $a + bi + cj + dk$ is $|a + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2}$*

Thus $\mathbb{H}$ is a normed vector space. Next observe that for each $q \in \mathbb{H}$ $q\bar{q} = \bar{q}q = |q|^2$.

therefore (symbol) $q^{-1} = \bar{q}/|q|^2$. So $qq^{-1} = q\bar{q}/|q|^2 = |q|^2 = 1$, similarly for $q^{-1}q = 1$.

This allows division $q_1 \cdot q_2^{-1} = q_1\bar{q}_2/|q_2|^2$. Writing $q_1/q_2$ is ambiguous however. $q_1q_2^{-1} \neq q_2^{-1}q_1$ generically.

Clearly $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$. A classic theorem of Frobenius asserts that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ are the only real associative division algebras. These objects similarly play a distinguished role in Lie group theory.

Convention: Suppose V is a vector space over the quaternions $\mathbb{H}$. We will adopt the convention that whenever we scale a vector $v \in V$ by a scalar $\lambda \in \mathbb{H}$, we multiply on the left, i.e. $\lambda v$

Let $M_n(\mathbb{R}), M_n(\mathbb{C}), M_n(\mathbb{H})$ denote the sets (vector spaces!) of all $n \times n$ matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

**Definition 3.** *The General Linear Groups $GL_n(\mathbb{R})$, resp. $GL_n(\mathbb{C})$ is the group of $n \times n$ invertible matrices with $\mathbb{R}$ resp $\mathbb{C}$ coefficients. (Group under multiplication). Equivalently $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) \neq 0\}$. Similarly $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) | \det(A) \neq 0\}$.*

(return to the idea of determinants of quaternions later).

Recall that for any matrix $A \in M_n(\mathbb{R})$ we have two associated linear maps $L_A : \mathbb{R}^n \to \mathbb{R}^n, L_a(\vec{x}) = A\vec{x}$, $R_A : \mathbb{R}^n \to \mathbb{R}^n, R_a(\vec{x}) = \vec{x}A$.

It is well know that $A$ is invertible (RC cases) iff $\det(A) \neq 0$ iff $L_A, R_A$ are isomorphisms.

# 3 Lecture 3 02/10/23

Thursday lecture moved to Friday at 10am in MS2.

Reminder:

- Quaternions $\mathbb{H}$, multiplication is associative not commutative. If $V$ is a $\mathbb{H}$- vector space, we scale from the left only, i.e. $\lambda v$ for $\lambda \in \mathbb{H}, v \in V$.
- General linear groups $GL_n(\mathbb{R}), GL_n(\mathbb{C})$ groups under $*$ of all invertible $\mathbb{R}$ resp. $\mathbb{C}$ $n \times n$- matrices.
- $A \in M_n(\mathbb{R}), M_n(\mathbb{C})$ is invertible iff $\det A \neq 0$ iff $L_A, R_A$ are both invertible where $L_a(\vec{x}) = A\vec{x}$, $R_a(\vec{x}) = \vec{x}A$.

We now consider $M_n(\mathbb{H})$.

**Definition 4.** *A function $f : \mathbb{H}^n \to \mathbb{H}^n$ is $\mathbb{H}$- linear if $f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2)$, $\forall \lambda_1 \lambda_2 \in \mathbb{H}, v_1, v_2 \in \mathbb{H}^n$.*

**Lemma 1.** *For $A \in M_n(\mathbb{H})$, $R_A : \mathbb{H}^n \to \mathbb{H}^n$ given by $R_a(\vec{x}) = \vec{x}A$ for $v \in \mathbb{H}^n$ a row vector, is $\mathbb{H}$- linear, however $L_A$ is in general not $\mathbb{H}$- linear. Proof: exercise*

idea is that associativity makes $\lambda v A$ ok, but not with left multiplication which is interfered by commutativity.

**Lemma 2.** *For $A \in M_n(\mathbb{H})$, $R_A : \mathbb{H}^n \to \mathbb{H}^n$, is an $\mathbb{H}$- linear isomorphism iff $A$ is invertible, i.e. $\exists B \in M_n(\mathbb{H})$ such that $AB = BA = I_n$.*

*Proof.* ($\Rightarrow$) If $R_A$ is an iso. then there is a $\mathbb{H}$-linear inverse $(R_A)^{-1} : \mathbb{H}^n \to \mathbb{H}^n$. There is a corresponding matrix $B \in M_n(\mathbb{H})$. Since $R_A \circ (R_A)^{-1} = R_A \circ (R_A)^{-1} = I_n$. we deduce $BA = AB = I_n$ (NB order of matrices here!). Therefore $B = A^{-1}$.

($\Leftarrow$) Similar. $\square$

**Definition 5.** *The quaternionic general linear group $GL_n(\mathbb{H}) = \{A \in M_n(\mathbb{H}) \,|A$ is invertible$\} = \{A \in M_n(\mathbb{H}) \,|R_a$ is an iso.$\}$*

NB: There is a problem with the notion of $\mathbb{H}$- determinant due to non-commutativity we'll return to this later (possible to define determinant and gl as ones with non-zero determinant, but defining it requires some thought.)

It turns out that we can view $\mathbb{C}$ and $\mathbb{H}$- matrices/linear maps in terms of $\mathbb{R}$- matrices.

**Proposition 1.** *There is a real linear map $\rho_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ such that the following diagram commutes.*

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "\theta_n"] \arrow[d, "R_A"] & \mathbb{R}^{2n} \arrow[d, "R_{\rho_n(A)}"] \\ \mathbb{C}^n \arrow[r, "\theta_n"] & \mathbb{R}^{2n} \end{tikzcd}$$

*where* $\theta_n : \mathbb{C}^n \to \mathbb{R}^{2n}$ *is given by* $\theta_n(a_1+ib_1, \ldots, a_n+ib_n) = (a_1, b_1, \ldots, a_n, b_n)$.

(compactly every complex matrix can be viewed as a real matrix of twice the size)

Remark: $\theta_n$ is a real linear isomorphism. This forces $R_{\rho_n(A)} = \theta_n \circ R_A \circ \theta_n^{-1}$.

This is linear and therefore there is a corresponding matrix $\in M_{2n}(\mathbb{R})$.

*Proof.* See moodle. □

**Observation 1.** *$\rho_n$ is injective. Proof: exercise.*

**Lemma 3.** *$\rho_n$ satisfies $\rho_n(AB) = \rho_n(A)\rho_n(B)$. So $\rho_n$ is an injective real-algebra homomorphism.*

*Proof.* We compose commutative squares from 1.1.8 to get ... (insert diagram)

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "\theta_n"] \arrow[d, "R_A"] & \mathbb{R}^{2n} \arrow[d, "R_{\rho_n(A)}"] \\ \mathbb{C}^n \arrow[r, "\theta_n"] \arrow[d, "R_B"] & \mathbb{R}^{2n} \arrow[d, "R_{\rho_n(B)}"] \\ \mathbb{C}^n \arrow[r, "\theta_n"] & \mathbb{R}^{2n} \end{tikzcd}$$

On L.H.S. we have $R_B \circ R_A = R_{AB}$. (note order)

On R.H.S we have $R_{\rho_n(B)} \circ R_{\rho_n(A)} = R_{\rho_n(A)\rho_n(B)}$.

But since LHS is $R_{AB}$ this means $R_{\rho_n(AB)}$ = composition on RHS = $R_{\rho_n(A)\rho_n(B)}$. □

It's not surjective however. Q: What exactly is $\rho_n(A)$? Consider $(a+ib) \in M_1(\mathbb{C})$.

$$R_{(a+ib)}(x+iy) = (x+iy)(a+ib) = (ax-by) + i(ay+bx)$$

Now $\theta_1(x+iy) = (x, y) \in \mathbb{R}^2$ etc.

So $\theta_1((ax-by) + i(ay+bx)) = (ax-by, ay+bx)$

The corresponding map from $\mathbb{R}^2 \to \mathbb{R}^2$ is $(x, y) \mapsto (ax-by, ay+bx)$. Observe that

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = (ax - by, ay + bx)$$

So $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M_2(\mathbb{R})$ corresponds under $\rho_1$ to $(a + ib) \in M_1(\mathbb{C})$.
More generally

$$\begin{pmatrix} a_{11} + ib_{11} & \dots & a_{1n} + ib_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} + ib_{n1} & \dots & a_{nn} + ib_{nn} \end{pmatrix} \in M_n(\mathbb{C})$$

corresponds to

$$\left(\begin{array}{cc|c|cc} a_{11} & b_{11} & \multirow{2}{*}{$\cdots$} & a_{1n} & b_{1n} \\ -b_{11} & a_{11} & & -b_{1n} & a_{1n} \\ \hline \multicolumn{2}{c|}{\vdots} & \vdots & \multicolumn{2}{c}{\vdots} \\ \hline a_{n1} & b_{n1} & \multirow{2}{*}{$\cdots$} & a_{nn} & b_{nn} \\ -b_{n1} & a_{n1} & & -b_{nn} & a_{nn} \end{array}\right) \in M_{2n}(\mathbb{R})$$

is obtained by replacing each $\mathbb{C}$ entry by its correspinding $2 \times 2$ real block.

# 4 Lecture 4 04/10/23