MT543 Topics in Algebra

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Note:

Any transcription mistakes and typos are my own.

Lectures by David Wraith. Lie Groups and Lie Algebras.

Lecture 1 25/09/23

missed this lecture - some intro to do with spheres, transformations and symmetries and other motivational stuff. Definition of an algebra (bilinear product) over a field.

0. Introduction

Lie groups have a dual nature: they are groups but also very special topological spaces. The algebraic and topological (spatial) properties are closely aligned. Lie groups and Lie algebras lie at the intersection of algebra, topology, geometry, analysis and more.

Definition 0.0.1. *An algebra is a vector space V equipped with a bilinear map $m : V \times V \to V$.*

Note, the "multiplication" map m does not have to be commutative or associative. In general, Lie algebras are neither commutative nor associative. Recall,

$$\text{Commutativity: } m(u, v) = m(v, u),$$
$$\text{Associativity: } m(m(u, v), w) = m(u, m(v, w)).$$

Every Lie group has an associated Lie algebra which encodes many properties of the group. Often this allows problems about Lie groups to be reduced to problems in (fancy!) linear algebra.

Example of a Lie group

The set of rotations of a ball centred on O in $\mathbb{R}^3$ is a Lie group. It is a group under composition of rotations. To "see" the topology here, notice that it makes sense to talk about two rotations being "close", so there is a sense of space. It makes sense to consider a continuous family of rotations. Continuity implies the existence of topology. We can identify this group with the matrix group $SO(3)$. The map $\mathbb{R}^3 \to \mathbb{R}^3$ given by $x \mapsto Ax$ for $A \in SO(3)$ is a rotation and every rotation occurs in this way. $SO(3)$ is a subset (but not a subgroup) of the set/group of all (3×3)-real matrices $M_3(\mathbb{R})$. By listing the elements of any 3×3 matrix we get a bijection $M_3(\mathbb{R}) \to \mathbb{R}^9$. As $\mathbb{R}^9$ has a natural topology (metric), this gives a natural topology on $M_3(\mathbb{R})$ and by restriction on $SO(3)$.

Lecture 2 27/09/23

Sorting out tutorial times. Lectures: Monday 2pm MS2, Wednesday 2pm LGH, Thursday 12pm MS2.

Lie Groups, dual nature, Groups but also a topological geometrical character. Can prove things with a mix of both methods - intersection of various areas.

1. Groups of matrices

1.1. General Linear Groups

Quaternions will have a central role.

Consider groups of $N \times N$ matrices over the fields $\mathbb{R}$ and $\mathbb{C}$ and also over the quaternions.

Definition 1.1.1. *The quaternions $\mathbb{H}$ is a 4-dim real vector space with standard basis elements $1, i, j, k$, equipped with an associative linear multiplication operation defined by*

$$i^2 = j^2 = k^2 = -1, \quad ij = k, jk = i, ki = j$$

So a generic quaternion takes the form $a + bi + cj + dk$, $a, b, c, d \in \mathbb{R}$.

Observe, $ji = j(jk) = (jj)k$ (by associativity) $= j^2k = -k$. Similarly $kj = -i$ and $ik = -j$.

e.g. $(2 + i - 3k)(5 + 2i - j + k) = 10 + 4i - 2j + 2k + 5i - 2 - k - j - 15k - 6j - 61 + 3$ etc.

Quaternions is not commutative, so is not a field. However it is a skew field (division algebra).

Terminology - In $a+bi+cj+dk$, a is called the real or scalar part, and the rest $bi+cj+dk$ imaginary or vector part.

In analogy with complex numbers,

Definition 1.1.2. *1. The conjugate of* $a+bi+cj+dk$ *, is* $\overline{a+bi+cj+dk} = a-bi-cj-dk$ *2. The norm of* $a+bi+cj+dk$ *is* $|a+bi+cj+dk| = \sqrt{a^2+b^2+c^2+d^2}$

Thus $\mathbb{H}$ is a normed vector space. Next observe that for each $q \in \mathbb{H}$ $q\bar{q} = \bar{q}q = |q|^2$.

therefore (symbol) $q^{-1} = \bar{q}/|q|^2$. So $qq^{-1} = q\bar{q}/|q|^2 = |q|^2 = 1$, similarly for $q^{-1}q = 1$.

This allows division $q_1 \cdot q_2^{-1} = q_1\bar{q_2}/|q_2|^2$. Writing q_1/q_2 is ambiguous however. $q_1q_2^{-1} \neq q_2^{-1}q_1$ generically.

Clearly $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$. A classic theorem of Frobenius asserts that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ are the only real associative division algebras. These objects similarly play a distinguished role in Lie group theory.

Convention: Suppose V is a vector space over the quaternions $\mathbb{H}$. We will adopt the convention that whenever we scale a vector $v \in V$ by a scalar $\lambda \in \mathbb{H}$, we multiply on the left, i.e. λv

Let $M_n(\mathbb{R}), M_n(\mathbb{C}), M_n(\mathbb{H})$ denote the sets (vector spaces!) of all $n \times n$ matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

Definition 1.1.3. *The General Linear Groups* $GL_n(\mathbb{R})$*, resp.* $GL_n(\mathbb{C})$ *is the group of* $n \times n$ *invertible matrices with* $\mathbb{R}$ *resp* $\mathbb{C}$ *coefficients. (Group under multiplication). Equivalently* $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) \neq 0\}$. *Similarly* $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) | \det(A) \neq 0\}$.

(return to the idea of determinants of quaternions later).

Recall that for any matrix $A \in M_n(\mathbb{R})$ we have two associated linear maps $L_A : \mathbb{R}^n \to \mathbb{R}^n, L_a(\vec{x}) = A\vec{x}$, $R_A : \mathbb{R}^n \to \mathbb{R}^n, R_a(\vec{x}) = \vec{x}A$.

It is well know that A is invertible (RC cases) $\iff$ $\det(A) \neq 0$ $\iff$ L_A, R_A are isomorphisms.

Lecture 3 02/10/23

Thursday lecture moved to Friday at 10am in MS2.

Reminder:

- Quaternions $\mathbb{H}$, multiplication is associative not commutative. If V is a $\mathbb{H}$- vector space, we scale from the left only, i.e. λv for $\lambda \in \mathbb{H}, v \in V$.

- General linear groups $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$ groups under $*$ of all invertible $\mathbb{R}$ resp. $\mathbb{C}$ $n \times n$- matrices.

- $A \in M_n(\mathbb{R}), M_n(\mathbb{C})$ is invertible iff $\det A \neq 0$ iff L_A, R_A are both invertible where $L_a(\vec{x}) = A\vec{x}$, $R_a(\vec{x}) = \vec{x}A$.

We now consider $M_n(\mathbb{H})$.

Definition 1.1.4. *A function $f : \mathbb{H}^n \to \mathbb{H}^n$ is $\mathbb{H}$ - linear if $f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2)$, $\forall \lambda_1 \lambda_2 \in \mathbb{H}, v_1, v_2 \in \mathbb{H}^n$.*

Lemma 1.1.5. *For $A \in M_n(\mathbb{H})$, $R_A : \mathbb{H}^n \to \mathbb{H}^n$ given by $R_a(\vec{x}) = \vec{x}A$ for $v \in \mathbb{H}^n$ a row vector, is $\mathbb{H}$ - linear, however L_A is in general not $\mathbb{H}$ - linear. Proof: exercise*

idea is that associativity makes $\lambda v A$ ok, but not with left multiplication which is interfered by commutativity.

Lemma 1.1.6. *For $A \in M_n(\mathbb{H})$, $R_A : \mathbb{H}^n \to \mathbb{H}^n$, is an $\mathbb{H}$ - linear isomorphism iff A is invertible, i.e. $\exists B \in M_n(\mathbb{H})$ such that $AB = BA = I_n$.*

Proof. ($\Rightarrow$) If R_A is an iso. then there is a $\mathbb{H}$-linear inverse $(R_A)^{-1} : \mathbb{H}^n \to \mathbb{H}^n$. There is a corresponding matrix $B \in M_n(\mathbb{H})$. Since $R_A \circ (R_A)^{-1} = R_A \circ (R_A)^{-1} = I_n$. we deduce $BA = AB = I_n$ (NB order of matrices here!). Therefore $B = A^{-1}$.

($\Leftarrow$) Similar. $\square$

Definition 1.1.7. *The quaternionic general linear group $GL_n(\mathbb{H}) = \{A \in M_n(\mathbb{H}) \,|\, A$ is invertible$\} = \{A \in M_n(\mathbb{H}) \,|\, R_a$ is an iso.$\}$*

NB: There is a problem with the notion of $\mathbb{H}$ - determinant due to non-commutativity we'll return to this later (possible to define determinant and $GL_n(\mathbb{H})$ as ones with non-zero determinant, but defining it requires some thought.)

It turns out that we can view $\mathbb{C}$ and $\mathbb{H}$ - matrices/linear maps in terms of $\mathbb{R}$ - matrices.

Proposition 1.1.8. *There is a real linear map $\rho_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ such that the following diagram commutes.*

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, "\theta_n"] \arrow[d, "R_A"] & \mathbb{R}^{2n} \arrow[d, "R_{\rho_n(A)}"] \\
\mathbb{C}^n \arrow[r, "\theta_n"] & \mathbb{R}^{2n}
\end{tikzcd}$$

where $\theta_n : \mathbb{C}^n \to \mathbb{R}^{2n}$ *is given by* $\theta_n(a_1+ib_1, \ldots, a_n+ib_n) = (a_1, b_1, \ldots, a_n, b_n)$.

(compactly every complex matrix can be viewed as a real matrix of twice the size)

Remark: θ_n is a real linear isomorphism. This forces $R_{\rho_n(A)} = \theta_n \circ R_A \circ \theta_n^{-1}$.

This is linear and therefore there is a corresponding matrix $\in M_{2n}(\mathbb{R})$.

Proof. Given that θ_n is a real-linear isomorphism, consider the map $\theta_n \circ R_A \circ \theta_n^{-1} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. This is clearly real-linear, and hence corresponds to some $(2n \times 2n)$-real matrix which depends on A. Call this matrix $\rho_n(A)$. Thus we obtain a map $\rho_n(A) : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$, and the requirement that $\theta_n \circ R_A = R_{\rho_n(A)} \circ \theta_n$ is automatrically satisfied by construction.

It remains to show that ρ_n is a real-linear map. To this end we compute for $A_1, A_2 \in M_n(\mathbb{C})$ and $\lambda, \mu \in \mathbb{R}$:

$$\begin{aligned}
R_{\rho_n(\lambda A_1 + \mu A_2)} &= \theta_n \circ R_{\lambda A_1 + \mu A_2} \circ \theta_n^{-1} \\
&= \theta_n \circ (\lambda R_{A_1} + \mu R_{A_2}) \circ \theta_n^{-1} \\
&= \lambda \theta_n \circ R_{A_1} \circ \theta_n^{-1} + \mu \theta_n \circ R_{A_2} \circ \theta_n^{-1} \\
&= \lambda R_{\rho_n(A_1)} + \mu R_{\rho_n(A_2)} \\
&= R_{\lambda \rho_n(A_1) + \mu \rho_n(A_2)}.
\end{aligned}$$

As for any $X, Y \in M_{2n}(\mathbb{R})$ we have $R_X = R_Y$ if and only if $X = Y$, we deduce that $\rho_n(\lambda A_1 + \mu A_2) = \lambda \rho_n(A_1) + \mu \rho_n(A_2)$ as required. $\square$

Observation 1.1.9. ρ_n *is injective. Proof: exercise.*

Lemma 1.1.10. ρ_n *satisfies* $\rho_n(AB) = \rho_n(A)\rho_n(B)$.

Remark: When we take 1.1.10 together with 1.1.9 and 1.1.8, we see that ρ_n is an injective real-algebra homomorphism.

Proof. We compose commutative squares from 1.1.8 to get

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, "\theta_n"] \arrow[d, "R_A"] & \mathbb{R}^{2n} \arrow[d, "R_{\rho_n(A)}"] \\
\mathbb{C}^n \arrow[r, "\theta_n"] \arrow[d, "R_B"] & \mathbb{R}^{2n} \arrow[d, "R_{\rho_n(B)}"] \\
\mathbb{C}^n \arrow[r, "\theta_n"] & \mathbb{R}^{2n}
\end{tikzcd}$$

On L.H.S. we have $R_B \circ R_A = R_{AB}$. (note order)

On R.H.S we have $R_{\rho_n(B)} \circ R_{\rho_n(A)} = R_{\rho_n(A)\rho_n(B)}$.

But since LHS is R_{AB} this means $R_{\rho_n(AB)}$ = composition on RHS = $R_{\rho_n(A)\rho_n(B)}$. $\square$

It's not surjective however. Q: What exactly is $\rho_n(A)$? Consider $(a+ib) \in M_1(\mathbb{C})$.

$$R_{(a+ib)}(x+iy) = (x+iy)(a+ib) = (ax-by) + i(ay+bx)$$

Now $\theta_1(x+iy) = (x,y) \in \mathbb{R}^2$ etc.

So $\theta_1((ax-by)+i(ay+bx)) = (ax-by, ay+bx)$

The corresponding map from $\mathbb{R}^2 \to \mathbb{R}^2$ is $(x,y) \mapsto (ax-by, ay+bx)$. Observe that

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = (ax-by, ay+bx)$$

So $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M_2(\mathbb{R})$ corresponds under ρ_1 to $(a+ib) \in M_1(\mathbb{C})$.

More generally

$$\begin{pmatrix} a_{11}+ib_{11} & \dots & a_{1n}+ib_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1}+ib_{n1} & \dots & a_{nn}+ib_{nn} \end{pmatrix} \in M_n(\mathbb{C})$$

corresponds to

$$\left(\begin{array}{cc|c|cc} a_{11} & b_{11} & \multirow{2}{*}{$\dots$} & a_{1n} & b_{1n} \\ -b_{11} & a_{11} & & -b_{1n} & a_{1n} \\ \hline \vdots & & \vdots & \vdots & \\ \hline a_{n1} & b_{n1} & \multirow{2}{*}{$\dots$} & a_{nn} & b_{nn} \\ -b_{n1} & a_{n1} & & -b_{nn} & a_{nn} \end{array}\right) \in M_{2n}(\mathbb{R})$$

is obtained by replacing each $\mathbb{C}$ entry by its corresponding 2×2 real block.

Lecture 4 04/10/23

Last time:

- $A \in M_n(\mathbb{H})$ then $R_A : \mathbb{H}^n \to \mathbb{H}^n$ given by $R_a(\vec{x}) = \vec{x}A$. is $\mathbb{H}$- linear (assuming coefficients in $\mathbb{H}$ multiply on vectors from the left, x row vector). Left multiplication is not in general $\mathbb{H}$ linear.

- Under the real linear isomorphism $\theta_n : \mathbb{C}^n \to \mathbb{R}^{2n}$, $\theta_n(a_1+ib_1, \ldots, a_n+ib_n) = (a_1, b_1, \ldots, a_n, b_n)$. Any complex-linear map $\mathbb{C}^n \to \mathbb{C}^n$ corresponds to a real-linear map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and in terms of matrices (an right multiplication) $A \in M_n(\mathbb{C})$ corresponds to some matrix $\rho_n(A) \in M_{2n}(\mathbb{R})$.

- $$\text{If } A = \begin{pmatrix} a_{11}+ib_{11} & \ldots & a_{1n}+ib_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1}+ib_{n1} & \ldots & a_{nn}+ib_{nn} \end{pmatrix}$$

  then

  $$\rho_n(A) = \left(\begin{array}{cc|c|cc} a_{11} & b_{11} & \multirow{2}{*}{$\cdots$} & a_{1n} & b_{1n} \\ -b_{11} & a_{11} & & -b_{1n} & a_{1n} \\ \hline \multicolumn{2}{c|}{\vdots} & \vdots & \multicolumn{2}{c}{\vdots} \\ \hline a_{n1} & b_{n1} & \multirow{2}{*}{$\cdots$} & a_{nn} & b_{nn} \\ -b_{n1} & a_{n1} & & -b_{nn} & a_{nn} \end{array}\right)$$

Consider the $\mathbb{C}$ linear map $\mathbb{C}^n \to \mathbb{C}^n$ given by $z \to zi$. This is R_A where

$$A = \begin{pmatrix} i & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & i \end{pmatrix} = iI$$

For this matrix we have

$$\rho_n(A) = \left(\begin{array}{cc|c|cc} 0 & 1 & \multirow{2}{*}{$\cdots$} & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}} \\ -1 & 0 & & & \\ \hline \multicolumn{2}{c|}{\vdots} & \vdots & \multicolumn{2}{c}{\vdots} \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}$}} & \multirow{2}{*}{$\cdots$} & 0 & 1 \\ & & & -1 & 0 \end{array}\right) = \mathcal{I}_n$$

A map $f : \mathbb{C}^n \to \mathbb{C}^n$ is $\mathbb{C}$ linear if it is real linear and $f(zi) = f(z)i$.

Let $barf : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the corresponding $\mathbb{R}$ linear map and suppose this has matrix $B \in M_{2n}(\mathbb{R})$. Then the complex linearity requirement is $R_B \circ R_{\mathcal{I}_n} = R_{\mathcal{I}_n} R_B$.

Since $R_X = R_Y \iff X = Y$ we see this is equivalent to asking $B\mathcal{I}_n = \mathcal{I}_n B$. i.e. $B \in M_{2n}(\mathbb{R})$ corresponds under θ_n to a complex linear map $\iff B\mathcal{I}_n = \mathcal{I}_n B$.

We'd proved

Corollary 1.1.11. *The image of $\rho_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ is the set of all of all matrices in $M_{2n}(\mathbb{R})$ which commute with $\mathcal{I}_n$.*

Remark: This shows that ρ_n is not surjective.

Lemma 1.1.12. *There is an injective group homomorphism $\rho_n : GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R})$, given by restricting $\rho_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$.*

Proof. We just have to check that if $A \in GL_n(\mathbb{C})$, then $\rho_n(A)$ is invertible. Clearly $\rho_n(AA^{-1}) = \rho_n(A^{-1}A) = \rho_n(I_n)$ so by 1.1.10. $\rho_n(A)\rho_n(A^{-1}) = \rho_n(A^{-1})\rho_n(A) = \rho_n(I_n) = I_{2n}$.

$\therefore$ $\rho_n(A^{-1}) = \rho_n(A)^{-1}$, hence $\rho_n(A) \in GL_{2n}(\mathbb{R})$. So $\rho_n : GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R})$, and by 1.1.10 this is a (multiplicative) group homomorphism $\square$

Now for quaternion matrices.

First observe that there is a $\mathbb{C}$ linear isomorphism $\phi_n : \mathbb{H}^n \to \mathbb{C}^{2n}$ given by $\phi_n(z_1 + w_1 j, \ldots, z_n + w_n j) = (z_1, w_1, \ldots, z_n, w_n)$.

(exercise to figure out $a + bi + cj + dk$ as $z + wj$, with $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$.)

Proposition 1.1.13. *There is an injective $\mathbb{C}$ linear map $\psi_n : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ s.t. the following square commutes:*

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{\phi_n} & \mathbb{C}^{2n} \\ \downarrow{\scriptstyle R_A} & & \downarrow{\scriptstyle R_{\psi_n(A)}} \\ \mathbb{H}^n & \xrightarrow{\phi_n} & \mathbb{C}^{2n} \end{array}$$

i.e. $\phi_n \circ R_A = R_{\psi_n(A)} \circ \phi_n$. Moreover, ψ_n satisfies $\psi_n(AB) = \psi_n(A)\psi_n(B)$.

Proof. Analogous to that of prop 1.1.8 and lemma 1.1.10. Exercise! $\square$

Remark: It is easily checked (exercise!) that $\psi_1(z + wj) = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$

More generally, image of ψ_n consists of block matrices with blocks of this form (analogous to ρ_n).

By restricting to invertible matrices we obtain:

Corollary 1.1.14. *There is an injective group homomorphism $\psi_n : GL_n(\mathbb{H}) \to GL_{2n}(\mathbb{C})$.*

Proof. Analogous to 1.1.12 - exercise. $\square$

(you can compose the maps then to get a real $4n$ matrix from a quaternionic one)

Composing ρ_{2n} and ψ_n gives

Corollary 1.1.15. *There is an injective $\mathbb{R}$ linear map resp. group homomorphism given by $\rho_{2n} \circ \psi_n : M_n(\mathbb{H}) \to M_{4n}(\mathbb{R})$ resp. $\rho_{2n} \circ \psi_n : GL_n(\mathbb{H}) \to GL_{4n}(\mathbb{R})$.*

Slogan: all groups of $\mathbb{H}$ or $\mathbb{C}$ matrices can be viewed as groups of real matrices!

Definition 1.1.16. *(1.1.16) For $A \in M_n(\mathbb{H})$, $\det(A) := \det \psi_n(A)$.*

Lecture 5 06/10/23

(Talking about matrices and linear maps, and in $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, there's a standard basis given to go between linear maps and matrices.

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r] \arrow[d] & \mathbb{C}^n \arrow[d] \\
\mathbb{R}^{2n} \arrow[r] & \mathbb{R}^{2n}
\end{tikzcd}$$

and you can go between $\mathbb{R}$ and $\mathbb{C}$ with a canonical map, where you forget the complex structure going to $\mathbb{R}$ from $\mathbb{C}$ or by pairing up the pairs of reals going to $\mathbb{C}$.)

Last time:

- There is a canonical $\mathbb{C}$ linear isomorphism $\phi_n : \mathbb{H}^n \to \mathbb{C}^{2n}$ given by $\phi_n(z_1 + w_1 j, \ldots, z_n + w_n j) = (z_1, w_1, \ldots, z_n, w_n)$.
- There is an injective homomorphism of complex algebras $\psi_n : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ such that the following diagram commutes

$$\begin{tikzcd}
\mathbb{H}^n \arrow[r, "\phi_n"] \arrow[d, "R_A"] & \mathbb{C}^{2n} \arrow[d, "R_{\psi_n(A)}"] \\
\mathbb{H}^n \arrow[r, "\phi_n"] & \mathbb{C}^{2n}
\end{tikzcd}$$

- If $A \in M_n(\mathbb{H})$ then $\det(A) := \det \psi_n(A)$.

Proposition 1.1.17. *(1.1.17 - need to fix the numbering) For $A \in M_n(\mathbb{H})$, A is invertible $\iff \det A \neq 0$, i.e. $GL_n(\mathbb{H}) = \{A \in M_n(\mathbb{H}) \mid \det A \neq 0\}$.*

Proof. We claim that A is invertible $\iff \psi_n(A)$ is invertible.

($\Rightarrow$) This is immediate from the multiplicative properties of ψ_n in 1.1.13.

($\Leftarrow$) In 1.1.14 we noted that the restricted map $\psi_n : GL_n(\mathbb{H}) \to GL_{2n}(\mathbb{C})$ is a group homomorphism. $\therefore$ if $\psi_n(A) \in GL_{2n}(\mathbb{C})$ (i.e. is invertible) for $A \in M_n(\mathbb{H})$, then since $\operatorname{im} \psi_n$ is a subgroup of $GL_{2n}(\mathbb{C})$, $\exists B \in GL_n(\mathbb{H})$ s.t. $\psi_n(B) = [\psi_n(A)]^{-1}$. (Want to show now that B is the inverse of A.)

We have $\psi_n(AB) = \psi_n(A)\psi_n(B) = \psi_n(A)[\psi_n(A)]^{-1} = I_{2n}$. But ψ_n is injective, so we must have $AB = I_n$. Similarly $BA = I$. $\therefore B = A^{-1}$ i.e. A is invertible.

So the claim is true.

$$\therefore A \text{ is invertible } \underset{\text{claim}}{\iff} \psi_n \text{ is invertible}$$
$$\underset{\text{elementary linear algebra}}{\iff} \det \psi_n(A) \neq 0 \underset{\text{defn 1.1.16}}{\iff} \det A \neq 0$$

□

1.2. Orthogonal Groups

From now on we will only consider (skew) fields $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (=$\mathbb{F}$).

Recall that an inner product on a vector space V over $\mathbb{F}$ is a map $\langle\ ,\ \rangle : V \times V \to \mathbb{F}$ which is bilinear (i.e. linear in each entry) and is positive definite, i.e. $\langle v, v\rangle \geq 0\ \forall v \in V$. If $v \neq 0, \langle v, v\rangle > 0$. e.g. dot product in $\mathbb{R}^n$.

Definition 1.2.1.

a *The standard inner product in $\mathbb{F}^n$ is given by $\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle = x_1\bar{y_1} + \ldots + x_n\bar{y_n}$, where $\bar{y_i}$ is the conjugate of y_i. (If $y_i \in \mathbb{R}$ then $\bar{y_i} = y_i$). This is the dot product if $\mathbb{F} = \mathbb{R}$, and is called a Hermitian inner product if $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$.*

b *The standard basis for $\mathbb{F}^n$ is $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, ..., $(0, 0, \ldots, 1)$.*

Remarks: As $x\bar{x} = |x|^2$ for all $x \in \mathbb{F}^n$ we see that $\langle x, x\rangle \geq 0 \forall x \in \mathbb{F}^n$ for the standard inner products.

For $\lambda \in \mathbb{F}$,

- $\langle \lambda x, y\rangle = \lambda \langle x, y\rangle$
- $\langle x, \lambda y\rangle = \langle x, y\rangle \bar{\lambda}$ (NB for $q_1, q_2 \in \mathbb{H}$ $\overline{q_1 q_2} = \bar{q_2}\bar{q_1}$.)
- $\overline{\langle x, y\rangle} = \langle y, x\rangle$.

In the real case, vectors x, y are orthogonal if $\langle x, y\rangle = 0$. A basis $\{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$ is orthonormal if $|v_i| = 1 \forall i$ and $\langle v_i, v_j\rangle = 0$ for $i \neq j$.

Exactly the same language is used if $\mathbb{F} = \mathbb{C}, \mathbb{H}$.

Lemma 1.2.2. *$\{v_1, \ldots, v_n\} \in \mathbb{C}^n$ is a (Hermitian) orthonormal basis $\iff$ $\{\theta_n(v_1), \theta_n(iv_1), \ldots, \theta_n(v_n), \theta_n(iv_n)\}$ is an orthonormal basis for $\mathbb{R}^n$.*

($\theta_n : \mathbb{C}^n \cong \mathbb{R}^{2n}$).

Proof. An easy computation sows that

$$\underbrace{\langle x,\, y\rangle_{\mathbb{C}}}_{\text{Hermitian I.P. on } \mathbb{C}^n} = \underbrace{\langle \theta_n(x),\, \theta_n(y)\rangle_{\mathbb{R}}}_{\text{dot product on } \mathbb{R}^n} + i\,\langle \theta_n(x),\, \theta_n(iy)\rangle_{\mathbb{R}}$$

.

Thus $\langle x,\, y\rangle = 0 \iff \langle \theta_n(x),\, \theta_n(y)\rangle_{\mathbb{R}}$ and $\langle \theta_n(x),\, \theta_n(iy)\rangle_{\mathbb{R}}$. The result now follows easily. (Exercise: complete the argument) $\square$

Lecture 6 09/10/23

$v \cdot v = |v|^2$ $v \cdot w = |v||w|\cos\theta$ Dot product tells you about lengths and angles. Last time:

- standard inner product in $\mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n$ is given by $\langle x,\, y\rangle = \sum_{i=1}^{n} x_i \bar{y}_i$. ($\bar{y}_i = y_i$ if $y_i \in \mathbb{R}$)
- $\langle x,\, y\rangle_{\mathbb{C}^n} = \langle \theta_n(x),\, \theta_n(y)\rangle_{\mathbb{R}^{2n}} + i\,\langle \theta_n(x),\, \theta_n(iy)\rangle_{\mathbb{R}^{2n}}$
- (1.2.2) If $\{z_1, \ldots, z_n\}$ is an orthonormal basis for $\mathbb{C}^n$, then

  $$\{\theta_n(z_1), \theta_n(iz_1), \ldots, \theta_n(z_n), \theta_n(iz_n)\}$$

  is orthonormal for $\mathbb{R}^n$

Lemma 1.2.3. *$\{q_1, \ldots, q_n)\}$ is an orthonormal basis for $\mathbb{H}^n$ $\iff$*

$$\{\theta_{2n}\circ\phi_n(q_1), \theta_{2n}\circ\phi_n(iq_1), \theta_{2n}\circ\phi_n(jq_1), \theta_{2n}\circ\phi_n(kq_1), \ldots,$$
$$\theta_{2n}\circ\phi_n(q_n), \theta_{2n}\circ\phi_n(iq_n), \theta_{2n}\circ\phi_n(jq_n), \theta_{2n}\circ\phi_n(kq_n)\}$$

is orthonormal for $\mathbb{R}^{4n}$.

($\phi_n : \mathbb{H}^n \to \mathbb{C}^{2n}$, $\theta_{2n} : \mathbb{C}^{2n} \to \mathbb{R}^{4n}$)

Proof. This follows in the manner of 1.2.2 from the easily established formula

$$\langle x,\, y\rangle_{\mathbb{H}^n} = \langle \theta_{2n}\circ\phi_n(x),\, \theta_{2n}\circ\phi_n(y)\rangle_{\mathbb{R}^{4n}} + i\,\langle \theta_{2n}\circ\phi_n(x),\, \theta_{2n}\circ\phi_n(iy)\rangle_{\mathbb{R}^{4n}}$$
$$+ j\,\langle \theta_{2n}\circ\phi_n(x),\, \theta_{2n}\circ\phi_n(jy)\rangle_{\mathbb{R}^{4n}} + k\,\langle \theta_{2n}\circ\phi_n(x),\, \theta_{2n}\circ\phi_n(ky)\rangle_{\mathbb{R}^{4n}}$$

$\square$

Definition 1.2.4. *1) The orthogonal group $O(n)$ is*

$$O(n) = \{A \in GL_n(\mathbb{R}) \mid \langle xA,\, yA\rangle_{\mathbb{R}^n} = \langle x,\, y\rangle_{\mathbb{R}^n}, \forall x, y \in \mathbb{R}^n\}$$

2) The <u>unitary group</u> $U(n)$ is

$$U(n) = \{A \in GL_n(\mathbb{C}) \mid \langle xA, yA\rangle_{\mathbb{C}^n} = \langle x, y\rangle_{\mathbb{C}^n}, \forall x, y \in \mathbb{C}^n\}$$

3) The <u>symplectic group</u> $Sp(n)$ is

$$Sp(n) = \{A \in GL_n(\mathbb{H}) \mid \langle xA, yA\rangle_{\mathbb{H}^n} = \langle x, y\rangle_{\mathbb{H}^n}, \forall x, y \in \mathbb{H}^n\}$$

<u>Exercise</u>: Show $O(n), U(n), Sp(n)$ are groups under multiplication.
<u>Remark</u>: We could also define

$$O(n) = \{A \in GL_n(\mathbb{R}) \mid \langle Ax, Ay\rangle_{\mathbb{R}^n} = \langle x, y\rangle_{\mathbb{R}^n}, \forall x, y \in \mathbb{R}^n\}$$
$$U(n) = \{A \in GL_n(\mathbb{C}) \mid \langle Ax, Ay\rangle_{\mathbb{C}^n} = \langle x, y\rangle_{\mathbb{C}^n}, \forall x, y \in \mathbb{C}^n\}$$

Exercise: show this agrees with 1.2.4

Lemma 1.2.5. *For $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $A \in GL_n(\mathbb{F}^n)$. The following are equivalent (tfae),*

a) $A \in O(n), U(n), Sp(n)$ (as appropriate),

b) R_A maps $\mathbb{F}^n$- orthonormal bases to $\mathbb{F}^n$-orthonormal bases,

c) The rows of A from an orthonormal for $\mathbb{F}^n$,

d) $AA^* = I_n$ where $A^* = \bar{A}^T$.

Proof. (a)$\Rightarrow$(b) This is clear since by definition, multiplication by A preserves the inner product and in particular preserves orthogonality and lengths.

(b)$\Rightarrow$(c) The standard basis for $\mathbb{F}^n$ is orthonormal with respect to the standard inner product $\langle\ ,\ \rangle_{\mathbb{F}^n}$. By part (b) R_A maps this orthonormal basis to another orthonormal basis. But $R_A(e_i) = i^{\text{th}}$ row of A. Hence rows of A form an o.n. basis. (o.n. stands for orthonormal).

(c) $\iff$ (d) Observe that $(AA^*)_{ij} = \sum_k a_{ik}a^*_{kj} = \sum_k a_{ik}\bar{a}_{jk}$. The rows of A being o.n. means

$$\left\langle i^{\text{th}} \text{ row of } A,\ j^{\text{th}} \text{ row of } A\right\rangle = \delta_{ij}$$

$(\delta_{ii} = 1,\ \delta_{ij} = 0 \text{ if } i \neq j)$
i.e. $\sum_k^n a_{ik}\bar{a}_{jk} = \delta_{ij}$ (maybe color this to show it matches previous?)
i.e. $(AA^*)_{ij} = \delta_{ij} \iff AA^* = I_n$

(c)⇒(a) The rows of A are o.n., i.e. $\sum\limits_k^n a_{ik}\bar{a}_{jk} = \delta_{ij}$.

$$\begin{aligned}
\langle xA,\, yA\rangle_{\mathbb{F}^n} &= \left\langle \left(\sum_l x_l a_{l1}, \ldots, \sum_l x_l a_{ln}\right),\, \left(\sum_m y_m a_{m1}, \ldots, \sum_m y_m a_{mn}\right)\right\rangle \\
&= \sum_{k,l,m} x_l a_{lk} \overline{y_m a_{mk}} \\
&= \sum_{k,l,m} x_l a_{lk} \overline{a_{mk} y_m} \\
&= \sum_{l,m} x_l \delta_{lm} \overline{y_m} \\
&= \sum_l x_l \bar{y}_l \\
&= \langle x,\, y\rangle_{\mathbb{F}^n}
\end{aligned}$$

$\square$

Lecture 7 11/10/23

Previously:

$$\begin{aligned}
O(n) &= \{A \in M_n(\mathbb{R}) \,|\, \langle xA,\, yA\rangle_{\mathbb{R}^n} = \langle x,\, y\rangle_{\mathbb{R}^n}, \forall x, y \in \mathbb{R}^n\} \\
U(n) &= \{A \in M_n(\mathbb{C}) \,|\, \langle xA,\, yA\rangle_{\mathbb{C}^n} = \langle x,\, y\rangle_{\mathbb{C}^n}, \forall x, y \in \mathbb{C}^n\} \\
Sp(n) &= \{A \in M_n(\mathbb{H}) \,|\, \langle xA,\, yA\rangle_{\mathbb{H}^n} = \langle x,\, y\rangle_{\mathbb{H}^n}, \forall x, y \in \mathbb{H}^n\}
\end{aligned}$$

Note: other people might use Sp and symplectic group for something related but not exactly the same (something like $Sp(n, \mathbb{F})$).

1.2.5 For $A \in GL_n(\mathbb{F}^n)$ t.f.a.e.

a) $A \in O(n), U(n), Sp(n)$

b) R_A maps $\mathbb{F}$ orthonormal bases to $\mathbb{F}$ orthonormal bases.

c) Rows of A are an o.n. basis for $\mathbb{F}^n$.

d) $AA^* = I$ where $A^* = \bar{A}^T$.

Remark: It follows from this that

$$\begin{aligned} A \in O(n) &\iff A^{-1} = A^T \\ A \in U(n) &\iff A^{-1} = \bar{A}^T \\ A \in Sp(n) &\iff A^{-1} = \bar{A}^T \end{aligned}$$

i.e. we could define $O(n) = \{A \in M_n(\mathbb{R}) \,|A^{-1} = A^T\}$.

(This follows from 1.2.5 (d), provided we can also show $A^*A = I$. The latter follows from the fact that if $AB = 1$, $(A, B \in M_n(\mathbb{F}))$ then $\Rightarrow BA = I$. Why is this?)

By definition a matrix $\in O(n)$ if the corresponding linear map preserves the dot product, i.e. preserves lengths and angles. What about $U(n)$ and $Sp(n)$?

Proposition 1.2.6.

1) $\rho_n(U(n)) = O(2n) \cap \rho_n(GL_n(\mathbb{C}))$.

2) $\psi_n(Sp(n)) = U(2n) \cap \psi_n(GL_n(\mathbb{H}))$.

3) $\rho_{2n} \circ \psi_n(Sp(n)) = O(4n) \cap \rho_{2n} \circ \psi_n(GL_n(\mathbb{H}))$.

Proof. See Moodle □

This says that the real matrices corresponding to matrices in $U(n)$ and $Sp(n)$ are precisely the dot-product (i.e. length and angle) preserving transformation which have the form of the real version a complex/quaterionic matrix.

We now give an alternative description of $O(n)$, $U(n)$, $Sp(n)$.

Starting point: In $\mathbb{R}^n$, the norm and dot product determine each other via a "polarization" formula.

$$\langle x + y,\, x + y\rangle = \langle x,\, x\rangle + 2\,\langle x,\, y\rangle + \langle y,\, y\rangle$$

i.e. $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\langle x,\, y\rangle$.

$\therefore \langle x,\, y\rangle = \frac{1}{2}\left(\|x + y\|^2 - \|x\|^2 - \|y\|^2\right)$.

This shows that we could define orthogonal transformation to be those preserving all norms.

For $U(n)$, $Sp(n)$, we have the following fact:

Lemma 1.2.7. $\|x\|_{\mathbb{C}^n} = \langle x,\, x\rangle_{\mathbb{C}^n}$ *is equal to* $\|\theta_n(x)\|_{\mathbb{R}^{2n}}$. *Similarly* $\|x\|_{\mathbb{H}^n} = \|\theta_{2n} \circ \phi_n(x)\|_{\mathbb{R}^{4n}}$.

Proof. Exercise. (Hint: $\langle x,\, y\rangle_{\mathbb{C}^n} = \langle \theta_n(x),\, \theta_n(y)\rangle_{\mathbb{R}^{2n}} + i\,\langle \theta_n(x),\, \theta_n(iy)\rangle_{\mathbb{R}^{2n}}$ etc.) $\square$

Proposition 1.2.8.

1) $O(n) = \{A \in GL_n(\mathbb{R}) \mid \|xA\|_{\mathbb{R}^n} = \|x\|_{\mathbb{R}^n}, \forall x \in \mathbb{R}^n\}$.

2) $U(n) = \{A \in GL_n(\mathbb{C}) \mid \|xA\|_{\mathbb{C}^n} = \|x\|_{\mathbb{C}^n}, \forall x \in \mathbb{C}^n\}$.

3) $Sp(n) = \{A \in GL_n(\mathbb{H}) \mid \|xA\|_{\mathbb{H}^n} = \|x\|_{\mathbb{H}^n}, \forall x \in \mathbb{H}^n\}$.

Proof. (1) is established by the polarization formula.

(2) We need to show that $A \in GL_n(\mathbb{C})$ preserves all norms then $A \in U(n)$. (if it preserves inner products it automatically preserves norms). But if $\|xA\|_{\mathbb{C}^n} = \|x\|_{\mathbb{C}^n} \forall x \in \mathbb{C}^n$ we have

$$\begin{aligned} &\|\theta_n(xA)\|_{\mathbb{R}^{2n}} = \|\theta_n(x)\|_{\mathbb{R}^{2n}} \\ &\implies \|\theta_n(x)\rho_n(A)\|_{\mathbb{R}^{2n}} = \|\theta_n(x)\|_{\mathbb{R}^{2n}} \end{aligned}$$

$\therefore \rho_n(A) \in O(2n)$ by (1). By 1.2.6(1), $\rho_n(U(n)) = O(2n) \cap \rho_n(GL_n(\mathbb{C}))$ and since ρ_n is injective, we conclude that $A \in U(n)$ as required.

$\square$

Lecture 8 13/10/23

coming up: semi-direct products of groups, group actions (on sets), basic topology.

Proposition 1.2.9. *For $A \in O(n), U(n), Sp(n)$ we have* $|\det A| = 1$.

Proof. For $A \in O(n)$, $AA^T = I$ by 1.2.5(d). So $\det\big(AA^T\big) = \det(A)\det\big(A^T\big) = \det(A)^2 = \det I = 1$.

For $A \in U(n)$, $A\bar{A}^T = I$ by 1.2.5(d). So $\det\big(A\bar{A}^T\big) = \det(A)\det\big(\bar{A}^T\big) = \det(A)\det\big(\bar{A}\big)$. But $\det\big(\bar{A}\big) = \overline{\det(A)}$. To see this, note that $\det(A)$ is a sum of products of entries of A, and for complex conjugation, $\overline{a+b} = \bar{a} + \bar{b}$ and $\overline{ab} = \bar{a}\bar{b}$. So then $\det\big(A\bar{A}^T\big)\det(A)\overline{\det(A)} = |\det(A)|^2 = \det I = 1$. As $|z| \geq 0$, we see $|\det(A)| = 1$ implies $|\det(A)| = 1$.

For $A \in Sp(n)$, $\det A := \det_{\mathbb{C}} \psi_n(A)$. By 1.2.6(2) we know that $\psi_n(A) \in U(2n)$. $\therefore$ $\det\psi_n(A)\overline{\det\psi_n(A)} = \det\Big(\psi_n(A)\overline{\psi_n(A^T)}\Big) = \det I = 1$. $\therefore$ $|\det(\psi_n(A))|^2 = 1$, so $|\det(\psi_n(A))| = 1$ and $\therefore$ $|\det(A)| = 1$ $\square$

Remark For $A \in gon$, $\det A = \pm 1$.
For $A \in gun$, $\det A \in \{e^{i\theta} \mid \theta \in [0, 2\pi]\}$. (circle in complex plane).
For $A \in gspn$, it turns out that $\det A = 1$. This is not obvious!

Definition 1.2.10. *"Special Groups".*

1) *The special orthogonal group* $SO(n) = \{A \in O(n) \mid \det A = 1\}$,
2) *The special unitary group* $SU(n) = \{A \in U(n) \mid \det A = 1\}$,
3) *The special linear group* $SO(n) = \{A \in M_n(\mathbb{R}) (\text{ or } GL_n(\mathbb{R})) \mid \det A = 1\}$

Remark

1. $SL(n)$ could be also written $SL(n, \mathbb{R})$ or $SL_n(\mathbb{R})$. Also have complex version $SL(n, \mathbb{C})$ with the obvious definition.
2. The significance of $SL(n)$ is that this is the group of corresponding linear maps are precisely the volume preserving linear transformations. This follows from:
3. FACT: $A \in M_n(\mathbb{R})$. Then the volume of the parallelepiped determined by the vectors $R_A(e_1), \ldots, R_A(e_n)$ is $|\det A|$. Exercise: prove this fact! (vol $= |\det A|$.

   (insert vector picture).
4. $Sp(n)$ is already special!

Question What is the relationship between $SO(n)$ and $O(n)$, $SU(n)$ and $U(n)$?

Theorem 1.2.11. $U(n) = SU(n) \rtimes U(1)$.

$\rtimes$ is a semidirect product. Quick definition: G is a semidirect product of subgroups N, H if

1) $N \triangleleft G$,
2) $G = NH$ i.e. $G = \{nh \mid n \in N, h \in H\}$,
1) $N \cap H = \{e\}$.

Remark $U(1) = \{(e^{i\theta}) \mid \theta \in [0,2\pi]\}$ which we identify with $\{e^{i\theta} \mid \theta \in [0,2\pi]\}$. For the purposes of the theorem, we will identify $U(1)$ with $\begin{pmatrix} e^{i\theta} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$.

Proof. We observe that $SU(n) \lhd U(n)$ since $SU(n)$ is the kernel of $\det : U(n) \to \{e^{i\theta} \mid \theta \in [0,2\pi]\}$. (det is a homomorphism and kernel of any homomorphism is normal.)

Next observe that any $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \in U(n)$ can be expressed as a product

$$A = \begin{pmatrix} \frac{1}{\det A}a_{11} & a_{12} & \dots & a_{1n} \\ \frac{1}{\det A}a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{1}{\det A}a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} \det A & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

As A and $\begin{pmatrix} \det A & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ are both $\in U(n)$, so is left hand matrix above.

Moreover, det of L.H. matrix is $\frac{1}{\det A}$ is $\frac{1}{\det A} \det A = 1$. $\therefore$ L.H. matrix $\in SU(n)$. It remains to show $SU(n) \cap U(1) = \{e\}$. $\square$

This is clear since any matrix of the form $\begin{pmatrix} e^{i\theta} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ with $\det = 1$ must be I_n. Analogous arguments show

Theorem 1.2.12. $O(n) = SO(n) \rtimes \mathbb{Z}_2$.

Proof. exercise. $\square$

Bonus Lecture 1: Semidirect products of groups

Direct product: G, H can form $G \times H = \{(g,h) \mid g \in G, h \in H\}$ with multiplication given by $(g,h)\cdot(g',h') = (gg', hh')$. (External direct product).

Now suppose that G has subgroups H, K such that $G = HK$, i.e. $G = \{hk \mid h \in H, k \in K\}$. What about multiplication in G?

$(hk)(h'k') =?$ In the nicest case we might have $kh' = h'k$, so $(hk)(h'k') = hh'kk' \in HK$. This will work precisely when H and K commute with each other. In that case G is an (internal) direct product of H and K.

Definition If $H, K \subset G$, then G is a direct product of H and K if

1) $G = HK$
2) $H \cap K = \{e\}$ (this means any expression $g = hk$ is unique)
3) H, K commute with each other, i.e. $hk = kh$, $\forall h \in H, k \in K$,

   (3) is often replaced by

3') $H \lhd G$, $K \lhd G$. (not equivalent by itself with 1 and 2 it is)

Definition G is a semidirect product of $K \subset G$ and $N \lhd G$ if (1) and (2) hold.

What's going on?

(Aside) $N \lhd G$ means N is "normal" in G i.e. $N \subset G$ with the following property $gNg^{-1} \subset N \forall g \in G$.

$\therefore$ for any $n \in N$ and any $g \in G$, $gng^{-1} = n'$ for some $n' \in N$, i.e. $gn = n'g$ (*).

If $G = NH$ then consider the product

$$\begin{aligned} &(n_1h_1)(n_2h_2) \\ =&n_1(h_1n_2)h_2 \quad \text{associativity} \\ \underset{(*)}{=}&n_1n_2'h_1h_2 \quad \in NH. \end{aligned}$$

Remark

(1) As a set $N \rtimes H$ is just $N \times H$. But as a group it is in general a "twisted" product.

(2) Example dihedral groups D_n, where D_n is the group of symmetries of a regular n-gon. $|D_n| = 2n$. There are two types of symmetries here:

   i) rotations about centre: subgroup isomorphic to $\cong \mathbb{Z}_n$
   ii) flips : subgroup $\cong \mathbb{Z}_2$.

$D_n \cong Z_n \rtimes Z_2$ not a product since rotations and flips do not commute.

(3) There are several different looking ways to define a semidirect product. (e.g. short exact sequences, group extensions)

Bonus Lecture 1: Group Actions

Let X be a set, and let $\mathrm{Bij}(X)$ be the set of bijections $X \to X$. This a group under composition.

Definition An action of a group G on the set X is a homomorphism $\alpha : G \to \mathrm{Bij}(X)$.

If X has some extra structure e.g. X is a topological space, then we often replace $\mathrm{Bij}(X)$ with a group of bijections which preserve the extra structure.

For a topological space X we consider $\alpha : G \to \underbrace{\mathrm{Homeo}(X)}_{\text{group of "homeomorphisms"}}$.

Examples

i) $GL_n(\mathbb{R})$ acts on $\mathbb{R}^n$ by right or left multiplication. $\underbrace{R}_{\text{right multiplication}}$: $GL_n(\mathbb{R}) \to \underbrace{\mathrm{Isom}(\mathbb{R}^n)}_{\text{group of linear isomorphisms of } \mathbb{R}^n}$

ii) $\underbrace{\mathbb{Z}_2}_{=\{\pm 1\}}$ acts on $\mathbb{R}$ by flips about 0. (picture of real line arrows indicating flips)

iii) $SO(2)$ acts on the circle S^1, $\alpha : SO(2) \to \mathrm{Homeo}(S^1)$. If $S^1 = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ then action is by matrix multiplication

$$\underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{\text{generic element of } SO(2)} \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} \cdot \\ \cdot \end{pmatrix}}_{\text{again element of } S^1}$$

iv) $U(1) = \{e^{i\theta}\}$ =(circle) acts on 2-sphere S^2 by rotation (picture of sphere rotation through line connection poles say).

Bonus Lecture 1: Topology

Everything topological in module can be interpreted in terms of metric spaces (but doing so might be needlessly cumbersome).

Recall a metric space consists of a set X and a "metric" $d : X \times X \to [0, \infty)$ such that

- $d(x, y) = d(y, x)$
- $d(x, y) = 0 \iff x = y$
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangle).

d is a <u>distance function</u>.

An open ball in (X, d): $B(p, r) = \{x \in X \mid d(p, x) < r\}$, $p \in X$, $r > 0$.

Unions of open balls are called open sets. Metric allows us to define continuity and convergence.

It turns out that these notions can be describe using only open sets, e.g. $f : (X, d) \to (Y, d')$ is continuous $\iff$ for every open set $U \subset Y$, preimage $f^{-1}(U)$ is open in X. (*)

Open sets satisfy the following properties.

1. $X, \emptyset$ are open
2. unions of open sets are open
3. intersections of finitely many open sets are open.

<u>Definition</u> A topological space (X, T) is a set X together with a collection of subsets T ("topology") satisfying (1), (2), (3).

<u>Remark</u> Every metric space is a topological space. Many different metrics on X will generate the same topology (collection of open sets). Not every topology T will arise from a metric (more general in a sense).

<u>Definition</u> A homeomorphism $f : (X, T) \to (Y, T')$ such that

1. f is a bijection &
2. f, f^{-1} are both continuous as in (*)

<u>Other terms</u>

1. A subset C of (X, T) is <u>closed</u> if $X \setminus C \in T$, i.e. $X \setminus C$ is open.
2. A subset $S \subset (X, T)$ is <u>compact</u> if every collection of S by open sets in X has a finite subcovering (i.e. still covering S). <u>idea</u> Compact is small and neat (highly non technical)
3. A topological space is path-connected if you can join any two points by a continuous path ($p : \underbrace{[0, 1]}_{\text{standard metric topology}} \to (X, T)$)

4. A space is simply connected if any continuous loop in space can be contracted through the space to a point. Write this as $\pi_1(\underset{X}{\text{space}}) = 0$.

   For any space $\pi_1(X)$ is the "fundamental group" of X and measures the failure of loops to be contractible. (e.g. Circle has fundamental group isomorphic to $\mathbb{Z}$).

Lecture 9 16/10/23

Last time

- $U(n) \cong SU(n) \rtimes U(1)$, here $U(1)$ is identified with $\left\{ \begin{pmatrix} e^{i\theta} & & & \\ & 1 & & \\ & & \vdots & \\ & & & 1 \end{pmatrix} \right\} \cong$ $U(1)$.
- $O(n) \cong SO(n) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2 = \left\{ I, \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \vdots & \\ & & & 1 \end{pmatrix} \right\}$

Definition 1.2.13. *Metric spaces (X, d) and (Y, d') are* <u>*isometric*</u> *if $\exists$ bijection $f : X \to Y$ s.t. $d'(f(x_1), f(x_2)) = d(x_1, x_2)$, $x_1, x_2 \in X$. f is an* <u>*isometry*</u>. *(distance preserving maps)*

In $\mathbb{R}^n$ we have a standard distance d_{st} and a norm related by $d_{st} = \|x_1 - x_2\| (= \langle x_1 - x_2,\, x_1 - x_2 \rangle^{1/2})$.

By 1.2.8 ($O(n) = \{A \in M_n(\mathbb{R}) \mid \underbrace{\|xA\|}_{=\|R_A(x)\|} = \|x\|, \forall x \in \mathbb{R}^n\}$), we see that if $A \in O(n)$ then $R_A : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry.

Sort of conversely:

Proposition 1.2.14. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry (for d_{st}) which fixes $0 \in \mathbb{R}^n$, (i.e. $f(0) = 0$) then $f = R_A$ for some $A \in O(n)$. In particular f is linear.*

Proof. see moodle. (reasonably straightforward but takes time to write out). $\square$

<u>Remark</u> The above establishes the fact that $O(n)$ is precisely the group of origin fixing isometries of $\mathbb{R}^n$. ($O(n)$ is a group of matrices, but every matrix corresponds to a linear transformation in the standard basis)

What about isometries of $\mathbb{R}^n$ that do not fix the origin?

Observe that for any $v \in \mathbb{R}^n$, the translation map $T_v : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x + v$ is an isometry.

Theorem 1.2.15. $\mathrm{Isom}(\mathbb{R}^n)$*, the group of isometries of* $\mathbb{R}^n$ *(under composition) is* $\mathrm{Isom}(\mathbb{R}^n) \cong \mathbb{R}^n \rtimes O(n)$*, where* $\mathbb{R}^n \cong \{T_v \mid v \in \mathbb{R}^n\}$*. In particular, any isometry is a composition of an orthogonal transformation and a translation.*

Proof. Exercise. (think about what happens to the origin - say to a point v, translate by $-v$ and then a origin fixing transformation is orthogonal.) □

1.3. Basic topology of groups of matrices

All groups of matrices seen so far are subsets of $M_n(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\therefore$ bijective with subsets of $M_n(\mathbb{R}), M_{2n}(\mathbb{R}), M_{4n}(\mathbb{R})$ respectively.

Observe that there is a bijection $b : M_n(\mathbb{R}) \to \mathbb{R}^{n^2}$, given by

$$b\left(\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}\right) = (a_{11}, a_{12}, \dots, a_{1n}, a_{21}, \dots, a_{nn}).$$

The standard topology on $\mathbb{R}^{n^2}$ can be "pulled-back" to give a natural topology on $M_n(\mathbb{R})$ which makes b a homeomorphism (continuous bijection with continuous inverse).

$\therefore$ all matrix groups seen so far inherit a (subspace) topology from some $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

Observation 1.3.1. *With respect to these natural topologies, the functions* det *and* tr *(trace) are continuous.*

Definition 1.3.2.

1) *A matrix group (as opposed to a group of matrices!) is a subgroup of* $GL_n(\mathbb{F})$ *which is closed.*

2) *A topological group is a group G which is equipped with a topology with respect to which the operations of multiplication and inverse are continuous.*

Remark The point of the closed condition in (1) is that it keeps the topology 'nice'.

Proposition 1.3.3. *All groups seen so far are matrix groups*

Proof. View det as a map det : $GL_n(\mathbb{R}) \to \mathbb{R}$. This is continuous. $SL(n) := \det^{-1}\{1\}$. As $\{1\} \subset \mathbb{R}$ is closed, so is its pre-image $\therefore SL(n)$ is a matrix group. Ditto $SL(n, \mathbb{C})$ etc.
(pre-image)

($\mathbb{R} \setminus \{1\} = (-\infty, 1) \cup (1, \infty)$ is open).

For $O(n), U(n), Sp(n)$ consider the map $f : GL_n(\mathbb{F}) \to GL_n(\mathbb{F})$ given by $f(A) = AA^*$. $f^{-1}(I) = O(n), U(n), Sp(n)$ depending on $\mathbb{F}$. As f is continuous and $\{I\}$ is closed in $GL_n(\mathbb{F})$, we see $O(n), U(n), Sp(n)$ are closed in $GL_n(\mathbb{F})$.

Finally $SU(n) = U(n) \cap SL(n, \mathbb{C})$. $SO(n) = O(n) \cap SL(n)$, and intersection of closed sets is closed. Hence these are matrix groups. □

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Previously

- $\mathbb{R}^n$ has a standard (metric) topology.

  Bijection $b : M_n(\mathbb{R}) \to \mathbb{R}^{n^2}$.

  "Pull-back" metric/topology to $M_n(\mathbb{R})$.

  $M_n(\mathbb{F})$ bijective to a subset of $M_n(\mathbb{R})$, $M_{2n}(\mathbb{R})$, $M_{4n}(\mathbb{R})$

  $\to M_n(\mathbb{F})$ has a natural topology. By restriction, so do all groups seen.

- Matrix group := closed subgroup of $GL_n(\mathbb{F})$

(Aside: compactness $[0,1]$ is compact, but $(0,1)$ isn't as $(0,1) \cong \mathbb{R}$, $\sim$ compact is like small but has to make sense topologically, small caveat to intuitiveness)

Proposition 1.3.4. *$O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$ are compact*

Proof. We prove $O(n)$ is compact. Arguments for other groups analogous.

Consider a map $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $f(A) = AA^T$. $O(n) = f^{-1}(\{I\})$. As $\{I\}$ is a closed subset of $M_n(\mathbb{R})$ and f is continuous, we see $O(n)$ is a closed subset of $M_n(\mathbb{R})$.

We have a homeomorphism $b : M_n(\mathbb{R}) \to \mathbb{R}^{n^2}$. $\therefore b(O(n))$ is closed in $\mathbb{R}^{n^2}$.

Recall the Heine-Borel Thm: a subset of Euclidean space is compact $\iff$ it is closed and bounded.

It remains to show that $b(O(n))$ is bounded. (so then $b(O(n))$ is compact in $\mathbb{R}^{n^2}$, and since b is a homeomorphism preserving topologies, so $O(n)$ is compact in $M_n(\mathbb{R})$.)

Recall 1.2.5 the rows of any $A \in O(n)$ form an orthonormal set. So each row is a unit vector in $\mathbb{R}^n$.

$$\|b(A)\| = (\underbrace{a_{11}^2 + a_{12}^2 + \ldots + a_{1n}^2}_{=1} + \underbrace{a_{21}^2 + \ldots + a_{2n}^2}_{=1} + \ldots + \underbrace{a_{n1}^2 + \ldots + a_{nn}^2}_{=1})^{1/2}$$

$\therefore \|b(A)\| = \sqrt{n}$.
$\therefore b(O(n)) \subset$ sphere in $\mathbb{R}^{n^2}$ of radius $\sqrt{n}$.
$\therefore b(O(n))$ is bounded as required.

□

Proposition 1.3.5. *$GL_n(\mathbb{F})$, $Sp(n, \mathbb{F})$ are non-compact.*

Proof. exercise □

Low dimensional examples

$O(1) = \{(1), (-1)\} \cong \mathbb{Z}_2$, $SO(1) = \{(1)\}$ trivial.

$SO(2) = \{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi)\} \underset{\text{isomorphic \& homeomorphic}}{\cong} \{e^{i\theta} \mid \theta \in [0, 2\pi)\} \underset{\text{homeo}}{\cong} S^1$.

$U(1) = \{(e^{i\theta}) \mid \theta \in [0, 2\pi)\}$, $\therefore SO(2) \underset{\text{homeo \& isomorphic}}{\cong} U(1)$.

$SU(1)$ is trivial.

$SU(2) = \{\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \mid z, w \in \mathbb{C}, |z|^2 + |w|^2 = 1\}$. Now $\{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \underset{\text{homeo}}{\cong} \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \sum_{i=1}^{4} |x_i|^2 = 1\} = S^3$, 3-sphere $\subset \mathbb{R}^4$.

$\therefore$ as a topological space $SU(2) \cong S^3$.

$Sp(1) = \{(q) \mid q\bar{q} = 1\}$, group of unit length quaternions.

As $\mathbb{H} \cong \mathbb{C}^2 \cong \mathbb{R}^4$ topologically/as real vector spaces, we see that $Sp(1) \cong S^3$.

In fact $Sp(1)$ and $SU(2)$ are isomorphic as groups, with isomorphism given by ψ_1.

Observe that all the above groups except $O(1)$ are path-connected.

Recall that an action of a group G on a set X is a map $\alpha : X \times G \to X$ (this is a right action!) such that $\alpha(x, 1) = x, \forall x \in X$ and $\alpha(\alpha(x, g), h) = \alpha(x, gh)$.

For fixed $g \in G$ we obtain a map $\alpha_g : X \to X$ given by $\alpha(\cdot, g)$. This is a bijection with inverse $\alpha_{g^{-1}}$.

$\therefore$ an action can be viewed as a homomorphism $\alpha : G \to \text{Bij}(X)$.

e.g. a group can act on itself from left or right. i.e. $R_g : G \to G$, $R_g(g') = g'g$ etc.

If X is a topological space then we replace $\mathrm{Bij}(X)$ by $\mathrm{Homeo}(X)$, so actions respect topology.

Moreover, if G has a topology we will assume action depends continuously on $g \in G$.

An action of G on X is transitive if for any points $x, y \in X$ $\exists g \in G$ s.t. $\alpha_g(x) = y$.

Theorem 1.3.6.

1. *$SO(n)$ acts transitively by right (or left) multiplication on $S^{n-1} \subset \mathbb{R}^n$.*
2. *Given any point $x \in S^{n-1}$ there is a continuous path $p : [0,1] \to SO(n)$ s.t. $p(0) = I_n$ and $R_{p(1)}(e_1) = x$ ($n \geq 2$).*

Remarks $SO(n)$ acts by right/left mult. on $\mathbb{R}^n$. As this action preserves distances it restricts to given an action on any sphere (say unit radius sphere) about 0, S^{n-1}, i.e. $R : S^{n-1} \times SO(n) \to S^{n-1}$.

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Proof. The first statement follows from the second (by composing paths) (TODO: insert figure of sphere and path here)

For the second statement we proceed by induction. For $n = 2$ this is easy since

$$SO(2) = \{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi)\}.$$

Clearly, for $x = (\cos\theta, \sin\theta) \in S^1$ we have a path

$$p(t) = \begin{pmatrix} \cos(t\theta) & \sin(t\theta) \\ -\sin(t\theta) & \cos(t\theta) \end{pmatrix},$$

such that $(1\ 0)p(t) = (\cos(t\theta)\ \ \sin(t\theta)$ which starts at $(1\ 0)$ and ends at x. (insert figure of circle here)

Now assume the result is true for $n = k$ and consider $SO(k+1)$ acting on $\mathbb{R}^{k+1}$. We can write any $x \in S^k \subset \mathbb{R}^{k+1}$ as $x = \cos\theta e_1 + \sin\theta y$ where $y \in \mathrm{Span}\{e_2, \ldots, e_{k+1}\}$ is a unit vector. Choose a path $p_1(t)$ in $SO(n)$ taking the form

$$\left(\begin{array}{cc|c} \cos(t\theta) & \sin(t\theta) & \multirow{2}{*}{$\mathbf{0}$} \\ -\sin(t\theta) & \cos(t\theta) & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathcal{I}_{k-1} \end{array}\right)$$

Observe that $R_{p_1(1)}(e_1) = \cos\theta e_1 + \sin\theta e_2$. Now, identify $\mathbb{R}^k$ with $\text{Span}\{e_2, \dots, e_{k+1}\}$. We know there exists a path $\bar{p}_2 : [0,1] \to SO(k)$ such that $\bar{p}_2(0) = \mathcal{I}_k$ and $R_{\bar{p}_2(1)}(e_2) = y$ by the inductive hypothesis. By linearity $R_{\bar{p}_2(1)}(\sin(\theta)y) = \sin(\theta)y$. Now, we can define $p_2 : [0,1] \to SO(k+1)$ by

$$p_2(t) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \bar{p}_2(t) \end{array}\right).$$

Finally, define $p : [0,1] \to SO(k+1)$ by

$$p(t) = \begin{cases} p_1(2t) & t \in [0, \frac{1}{2}) \\ p_2(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

By construction, $p(0) = \mathcal{I}_{k+1}$ and $R_{p(1)}(e_1) = \cos(\theta)e_1 + \sin(\theta)y = x$. $\square$

<u>Remark</u> Similar arguments apply to $SU(n)$ and $Sp(n)$ with very little change.

Corollary 1.3.7. *$SO(n)$ is path-connected.*

Proof. For any $A \in SO(n)$ we construct a path from A to $\mathcal{I}_n$. By Lemma 1.2.5 the rows of A form an orthonormal basis for $\mathbb{R}^n$. Call these $r_1, \dots, r_n$. By theorem 1.3.6 there exisits a path p in $SO(n)$ starting with $\mathcal{I}_n$ and ending with $p(1)$ which satisfies $R_{p_1(1)}(r_1) = e_1$. As matrices in $SO(n)$ preserve orthonormality (by lemma 1.2.5) we see that $R_{p_1(1)}(r_i) \in \text{Span}\{e_2, \dots, e_n\}$ for $i \geq 2$

Now let $s_2 = R_{p_1(1)}(r_2)$. By theorem 1.3.6, there exists a path $\bar{p}_2$ in $SO(n-1)$ such that the corresponding path,

$$p_2(t) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \bar{p}_2(t) \end{array}\right),$$

moves s_2 to e_2, ie $R_{p_2(1)}(s_2) = e_2$ (and $p_2(0) = \mathcal{I}_n$). We can continue in this way to construct paths p_i moving r_i to e_i but fixing $e_1, \dots, e_{i-1}$. We can also compose these paths to obtain a path $p(t)$ such that $p(0) = \mathcal{I}_n$ and $R_{p(1)}(r_i) = e_i$ for all i.

Finally, consider the $n \times n$ matrix obtained by stacking the images of $r_1, \dots, r_n$ under $R_{p(t)}$:

$$\begin{pmatrix} — & R_{p(t)}(r_1) & — \\ — & R_{p(t)}(r_2) & — \\ & \vdots & \\ — & R_{p(t)}(r_n) & — \end{pmatrix}$$

Notice that, for eact t, the rows are orthonormal, so this matrix is an element of $SO(n)$ for all $t \in [0, 1]$. Moreover, at $t = 0$ this matrix is

$$\begin{pmatrix} — & r_1 & — \\ — & r_2 & — \\ & \vdots & \\ — & r_n & — \end{pmatrix} = A$$

and at $t = 1$

$$\begin{pmatrix} — & e_1 & — \\ — & e_2 & — \\ & \vdots & \\ — & e_n & — \end{pmatrix} = \mathcal{I}_n$$

Hence, we have constructe a path in $SO(n)$ from A to $\mathcal{I}_n$. $\square$

Lecture 12 23/10/23

Previously:

- 1.3.6 $SO(n)$ acts transitively on S^{n-1}
- 1.3.7 $SO(n)$ is path-connected.

<u>Remark</u> Essentially the same arguments used in 1.3.6 & 1.3.7 show

- $SU(n)$, $Sp(n)$ act transitively on S^{2n-1} and S^{4n-1} respectively.
- $SU(n)$, $Sp(n)$ are path-connected. $SL(n)$ is also path connected, but the argument is different.

Corollary 1.3.8. *$U(n)$ is path connected by $O(n)$ has two path components (i.e. <u>not</u> path connected).*

Proof. By 1.2.11, $U(n) = SU(n) \rtimes U(1)$. Topologically, $U(n) \cong SU(n) \times S^1$ (forgetting the group structure), and is $\therefore$ path connected as $SU(n)$ and S^1 both are.

By 1.2.12, $O(n) = SO(n) \rtimes \mathbb{Z}_2$. Topologically this means $O(n) \cong SO(n) \underset{\text{disjoint union}}{\amalg} SO(n)$. Hence $O(n)$ has two path components, each $\cong SO(n)$. $\square$

Remark $\mathbb{Z}_2$ here corresponds to $\{I_n, \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\}$. The matrix $\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ corresponds to a reflection in the hyperplane span$\{e_2, \ldots, e_n\}$ (flipping the e_1 coordinate over). Notice that we can't have a continuous path from I_n to $\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ in $O(n)$ as the determinant (which is continuous) would have to jump from 1 to -1.

If we define a rotation of $\mathbb{R}^n$ to be an origin fixing distance preserving map which can be linked via a continuous path of such maps to I_n, then we can deduce

- $SO(n)$ is precisely the group of rotations of $\mathbb{R}^n$,
- $O(n)$ is precisely the group of rotations and reflections of $\mathbb{R}^n$.

Final remark: "Homogeneous spaces".

If G is a topological group and $H \subset G$ a subgroup, then the set of cosets ${}^{G}/_{H}$ (or ${}_{H}\backslash^{G}$) is a topological space with the quotient topology. ${}^{G}/_{H}$ is called a "Homogeneous space", as, when equipped with an appropriate geometry ${}^{G}/_{H}$ "looks" the same at all points.

Examples $S^{n-1} \cong {}^{SO(n)}/_{SO(n-1)} \cong {}^{O(n)}/_{O(n-1)}$, where $SO(n-1)$ is identified with $\begin{pmatrix} 1 & \\ & SO(n-1) \end{pmatrix} \subset SO(n)$ etc.

$S^{2n-1} \cong {}^{SU(n)}/_{SU(n-1)} \cong {}^{U(n)}/_{U(n-1)}$

$S^{4n-1} \cong {}^{Sp(n)}/_{Sp(n-1)}$.

2. Lie groups & Lie Algebras

2.1. Manifolds

Roughly speaking a manifold is a "nice" topological space such that a neighbourhood of each point "looks like" euclidean space of a fixed dimension. We can use the local euclidean property to transfer calculus (differential and with a bit more work integral) from $\mathbb{R}^n$ to manifolds.

Definition 2.1.1. *A topological n-manifold M is a Hausdoff topological space with a countable basis for its topology, satisfying the following locally euclidean property: For any $x \in M$ there is an open set $x \in U \subset M$, and an open set $V \subset \mathbb{R}^n$, and a homeomorphism $\phi : U \to V$.*

(nice picture of torus, open sets, and arrow between it and V in $\mathbb{R}^n$).

Remarks

1) n is the "dimension" of the manifold.

2) The map ϕ is called a "chart".

3) A locally Euclidean space which is a subset of some $\mathbb{R}^m$ $m \geq n$ is automatically Hausdorff and has a countable basis for its topology.