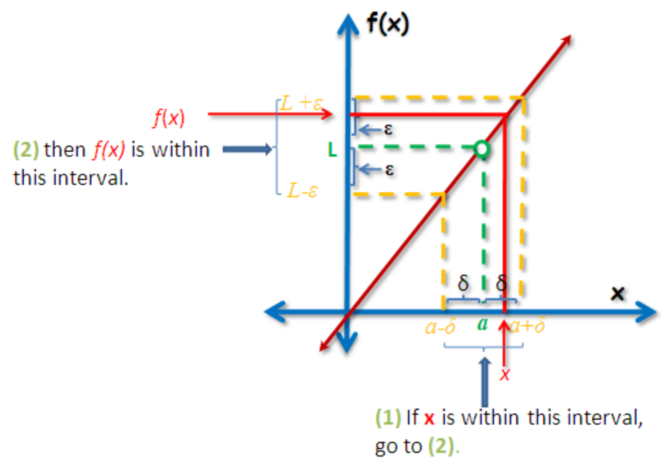
Harold's Calculus Notes
Cheat Sheet
26 April 2016

AP Calculus

Limits	
Definition of Limit Let f be a function defined on an open interval containing c and let L be a real number. The statement: $\lim_{x \to a} f(x) = L$ means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < \lvert x - a \rvert < \delta$, then $\lvert f(x) - L \rvert < \epsilon$ **Tip :** Direct substitution: Plug in $f(a)$ and see if it provides a legal answer. If so then $L = f(a)$.	
The Existence of a Limit The limit of $f(x)$ as x approaches a is L if and only if:	$\lim_{x \to a^-} f(x) = L$ $\lim_{x \to a^+} f(x) = L$
Definition of Continuity A function **f is continuous** at c if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\lvert x - c \rvert < \delta$ and $\lvert f(x) - f(c) \rvert < \varepsilon$. Tip: Rearrange $\lvert f(x) - f(c) \rvert$ to have $\lvert (x - c) \rvert$ as a factor. Since $\lvert x - c \rvert < \delta$ we can find an equation that relates both δ and ε together.	**<u>Prove</u> that $f(x) = x^2 - 1$ is a continuous function.** $\lvert f(x) - f(c) \rvert$ $= \lvert (x^2 - 1) - (c^2 - 1) \rvert$ $= \lvert x^2 - 1 - c^2 + 1 \rvert$ $= \lvert x^2 - c^2 \rvert$ $= \lvert (x + c)(x - c) \rvert$ $= \lvert (x + c) \rvert \lvert (x - c) \rvert$ Since $\lvert (x + c) \rvert \le \lvert 2c \rvert$ $\lvert f(x) - f(c) \rvert \le \lvert 2c \rvert \lvert (x - c) \rvert < \varepsilon$ So given $\varepsilon > 0$, we can **choose** $\boldsymbol{\delta} = \left\lvert \frac{1}{2c} \right\rvert \boldsymbol{\varepsilon} > \mathbf{0}$ in the Definition of Continuity. So substituting the chosen δ for $\lvert (x - c) \rvert$ we get: $\lvert f(x) - f(c) \rvert \le \lvert 2c \rvert \left(\left\lvert \frac{1}{2c} \right\rvert \varepsilon \right) = \varepsilon$ Since both conditions are met, $f(x)$ is continuous.
Two Special Trig Limits	$\lim_{x \to 0} \frac{\sin x}{x} = 1$ $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

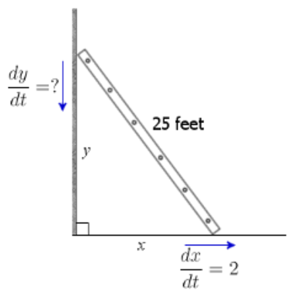
Derivatives	(See Larson's 1-pager of common derivatives)
Definition of a Derivative of a Function Slope Function	$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$
Notation for Derivatives	$f'(x),\ f^{(n)}(x), \frac{dy}{dx}, y', \frac{d}{dx}[f(x)],\ D_x[y]$
0. The Chain Rule	$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$ $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
1. The Constant Multiple Rule	$\frac{d}{dx}[cf(x)] = cf'(x)$
2. The Sum and Difference Rule	$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
3. The Product Rule	$\frac{d}{dx}[fg] = fg' + g\,f'$
4. The Quotient Rule	$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$
5. The Constant Rule	$\frac{d}{dx}[c] = 0$
6a. The Power Rule	$\frac{d}{dx}[x^n] = nx^{n-1}$
6b. The General Power Rule	$\frac{d}{dx}[u^n] = nu^{n-1}\,u'\ \ where\ u = u(x)$
7. The Power Rule for x	$\frac{d}{dx}[x] = 1\ \ (think\ \ x = x^1\ and\ x^0 = 1)$
8. Absolute Value	$\frac{d}{dx}[\lvert x \rvert] = \frac{x}{\lvert x \rvert}$
9. Natural Logorithm	$\frac{d}{dx}[\ln x] = \frac{1}{x}$
10. Natural Exponential	$\frac{d}{dx}[e^x] = e^x$
11. Logorithm	$\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)\,x}$
12. Exponential	$\frac{d}{dx}[a^x] = (\ln a)\,a^x$
13. Sine	$\frac{d}{dx}[sin(x)] = \cos(x)$
14. Cosine	$\frac{d}{dx}[cos(x)] = -sin(x)$
15. Tangent	$\frac{d}{dx}[tan(x)] = sec^2(x)$
16. Cotangent	$\frac{d}{dx}[cot(x)] = -csc^2(x)$
17. Secant	$\frac{d}{dx}[sec(x)] = sec(x)\,tan(x)$

Derivatives	(See Larson's 1-pager of common derivatives)
18. Cosecant	$\frac{d}{dx}[csc(x)] = -csc(x)\,cot(x)$
19. Arcsine	$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$
20. Arccosine	$\frac{d}{dx}[\cos^{-1}(x)] = \frac{-1}{\sqrt{1-x^2}}$
21. Arctangent	$\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2}$
22. Arccotangent	$\frac{d}{dx}[\cot^{-1}(x)] = \frac{-1}{1+x^2}$
23. Arcsecant	$\frac{d}{dx}[\sec^{-1}(x)] = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
24. Arccosecant	$\frac{d}{dx}[\csc^{-1}(x)] = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
25. Hyperbolic Sine	$\frac{d}{dx}[sinh(x)] = \cosh(x)$
26. Hyperbolic Cosine	$\frac{d}{dx}[cosh(x)] = sinh(x)$
27. Hyperbolic Tangent	$\frac{d}{dx}[tanh(x)] = sech^2(x)$
28. Hyperbolic Cotangent	$\frac{d}{dx}[coth(x)] = -csch^2(x)$
29. Hyperbolic Secant	$\frac{d}{dx}[sech(x)] = -sech(x)\,tanh(x)$
30. Hyperbolic Cosecant	$\frac{d}{dx}[csch(x)] = -csch(x)\,coth(x)$
31. Hyperbolic Arcsine	$\frac{d}{dx}[\sinh^{-1}(x)] = \frac{1}{\sqrt{x^2+1}}$
32. Hyperbolic Arccosine	$\frac{d}{dx}[\cosh^{-1}(x)] = \frac{1}{\sqrt{x^2-1}}$
33. Hyperbolic Arctangent	$\frac{d}{dx}[\tanh^{-1}(x)] = \frac{1}{1-x^2}$
34. Hyperbolic Arccotangent	$\frac{d}{dx}[\coth^{-1}(x)] = \frac{1}{1-x^2}$
35. Hyperbolic Arcsecant	$\frac{d}{dx}[\mathrm{sech}^{-1}(x)] = \frac{-1}{x\sqrt{1-x^2}}$
36. Hyperbolic Arccosecant	$\frac{d}{dx}[\mathrm{csch}^{-1}(x)] = \frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$
Position Function	$s(t) = \frac{1}{2}gt^2 + v_0t + s_0$
Velocity Function	$v(t) = s'(t) = gt + v_0$
Acceleration Function	$a(t) = v'(t) = s''(t)$
Jerk Function	$j(t) = a'(t) = v''(t) = s^{(3)}(t)$

Applications of Differentiation	
Rolle's Theorem *f* is continuous on the closed interval [a,b], and *f* is differentiable on the open interval (a,b).	If $f(a) = f(b)$, then there exists at least one number c in (a,b) such that $f'(c) = 0$.
Mean Value Theorem If *f* meets the conditions of Rolle's Theorem, then	$f'(c) = \frac{f(b) - f(a)}{b - a}$ $f(b) = f(a) + (b - a)f'(c)$ Find '*c*'.
L'Hôpital's Rule	$If \ \lim_{x \to c} f(x) = \lim_{x \to c} \frac{P(x)}{Q(x)} =$ $\left\{\frac{0}{0}, \frac{\infty}{\infty}, 0 \bullet \infty, 1^{\infty}, 0^0, \infty^0, \infty - \infty\right\}, but \ not \ \{0^{\infty}\},$ $then \ \lim_{x \to c} \frac{P(x)}{Q(x)} = \lim_{x \to c} \frac{P'(x)}{Q'(x)} = \lim_{x \to c} \frac{P''(x)}{Q''(x)} = \cdots$

Graphing with Derivatives	
Test for Increasing and Decreasing Functions	1. If $f'(x) > 0$, then f is increasing (slope up) ↗ 2. If $f'(x) < 0$, then f is decreasing (slope down) ↘ 3. If $f'(x) = 0$, then f is constant (zero slope) →
The First Derivative Test	1. If $f'(x)$ changes from – to + at c, then f has a *relative minimum* at $(c, f(c))$ 2. If $f'(x)$ changes from + to - at c, then f has a *relative maximum* at $(c, f(c))$ 3. If $f'(x)$, is + c + or - c -, then $f(c)$ is neither
The Second Deriviative Test Let $f'(c)=0$, and $f''(x)$ exists, then	1. If $f''(x) > 0$, then f has a relative minimum at $(c, f(c))$ 2. If $f''(x) < 0$, then f has a relative maximum at $(c, f(c))$ 3. If $f'(x) = 0$, then the test fails (See 1^{st} derivative test)
Test for Concavity	1. If $f''(x) > 0$ for all x, then the graph is concave up ∪ 2. If $f''(x) < 0$ for all x, then the graph is concave down ∩
Points of Inflection Change in concavity	If $(c, f(c))$ is a point of inflection of f, then either 1. $f''(c) = 0$ or 2. f'' does not exist at $x = c$.

Analyzing the Graph of a Function	(See Harold's Illegals and Graphing Rationals Cheat Sheet)
x-Intercepts (Zeros or Roots)	$f(x) = 0$
y-Intercept	$f(0) = y$
Domain	Valid x values
Range	Valid y values
Continuity	No division by 0, no negative square roots or logs
Vertical Asymptotes (VA)	x = division by 0 or undefined
Horizontal Asymptotes (HA)	$\lim_{x \to \infty^-} f(x) \to y$ and $\lim_{x \to \infty^+} f(x) \to y$
Infinite Limits at Infinity	$\lim_{x \to \infty^-} f(x) \to \infty$ and $\lim_{x \to \infty^+} f(x) \to \infty$
Differentiability	Limit from both directions arrives at the same slope
Relative Extrema	Create a table with *domains, f(x), f'(x), and f''(x)*
Concavity	If $f''(x) \to +$, then cup up ∪ If $f''(x) \to -$, then cup down ∩
Points of Inflection	$f''(x) = 0$ (concavity changes)

Approximating with Differentials	
Newton's Method Finds zeros of *f*, or finds *c* if *f(c) = 0*.	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
Tangent Line Approximations	$y = mx + b$ $y = f'(c)(x - c) + f(c)$
Function Approximations with Differentials	$f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)\,dx$
Related Rates 	Steps to solve: 1. Identify the known variables and rates of change. $\left(x = 15\ m; y = 20\ m;\ x' = 2\frac{m}{s};\ y' = ?\right)$ 2. Construct an equation relating these quantities. $(x^2 + y^2 = r^2)$ 3. Differentiate both sides of the equation. $(2xx' + 2yy' = 0)$ 4. Solve for the desired rate of change. $\left(y' = -\frac{x}{y}\ x'\right)$ 5. Substitute the known rates of change and quantities into the equation. $\left(y' = -\frac{15}{20} \bullet 2 = \frac{3}{2}\ \frac{m}{s}\right)$

Summation Formulas

Stop point

Summation symbol $$\sum_{i=1}^{n} x_i$$ Typical element

Index Start point

$$\sum_{i=1}^{4} x_i = x_1 + x_2 + x_3 + x_4$$

Sum of Powers

$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2 = \frac{n^2(n+1)^2}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

$$\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12} = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}$$

$$\sum_{i=1}^{n} i^6 = \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}$$

$$\sum_{i=1}^{n} i^7 = \frac{n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)}{24}$$

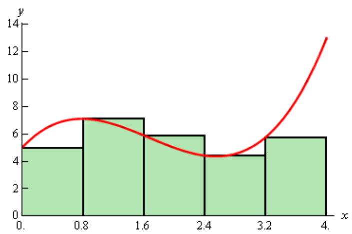
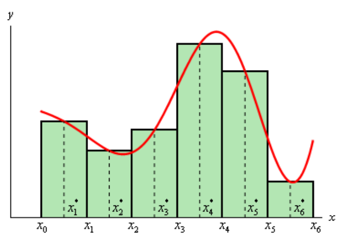
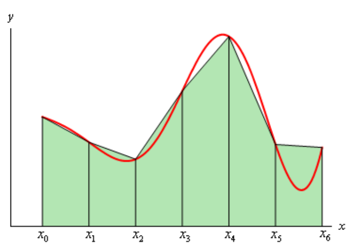
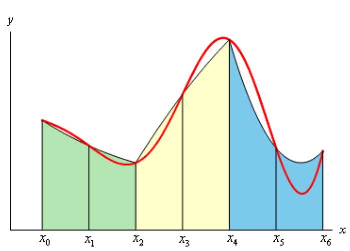
$$S_k(n) = \sum_{i=1}^{n} i^k = \frac{(n+1)^{k+1}}{k+1} - \frac{1}{k+1}\sum_{r=0}^{k-1}\binom{k+1}{r} S_r(n)$$

Misc. Summation Formulas

$$\sum_{i=1}^{n} i(i+1) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = \frac{n(n+1)(n+2)}{3}$$

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

$$\sum_{i=1}^{n} \frac{1}{i(i+1)(i+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

Numerical Methods	
Riemann Sum 	$P_0(x) = \int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and $\Delta x_i = x_i - x_{i-1}$ and $\|P\| = max\{\Delta x_i\}$ Types: • Left Sum (LHS) • Middle Sum (MHS) • Right Sum (RHS)
Midpoint Rule 	$P_0(x) = \int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x =$ $\Delta x[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + \cdots + f(\bar{x}_n)]$ where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = midpoint\ of\ [x_{i-1}, x_i]$ Error Bounds: $\|E_M\| \le \frac{K(b-a)^3}{24n^2}$
Trapezoidal Rule 	$P_1(x) = \int_a^b f(x)\,dx \approx$ $\frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_3) + \cdots + 2f(x_{n-1})$ $+ f(x_n)]$ where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ Error Bounds: $\|E_T\| \le \frac{K(b-a)^3}{12n^2}$
Simpson's Rule 	$P_2(x) = \int_a^b f(x)dx \approx$ $\frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots$ $+ 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$ Where n is even and $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ Error Bounds: $\|E_S\| \le \frac{K(b-a)^5}{180n^4}$
TI-84 Plus 	[MATH] fnInt(f(x),x,a,b), [MATH] [1] [ENTER] Example: [MATH] fnInt(x^2,x,0,1) $\int_0^1 x^2\,dx = \frac{1}{3}$
TI-Nspire CAS 	[MENU] [4] Calculus [3] Integral [TAB] [TAB] [X] [^] [2] [TAB] [TAB] [X] [ENTER]

Integration	(See Harold's Fundamental Theorem of Calculus Cheat Sheet)
Basic Integration Rules Integration is the "inverse" of differentiation, and vice versa.	$\int f'(x)\,dx = f(x) + C$ $\frac{d}{dx}\int f(x)\,dx = f(x)$
$f(x) = 0$	$\int 0\,dx = C$
$f(x) = k = kx^0$	$\int k\,dx = kx + C$
The Constant Multiple Rule	$\int k\,f(x)\,dx = k\int f(x)\,dx$
The Sum and Difference Rule	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
The Power Rule $f(x) = kx^n$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, where\ n \neq -1$ $If\ n = -1, then \int x^{-1}\,dx = \ln\lvert x\rvert + C$
The General Power Rule	If $u = g(x), and\ u' = \frac{d}{dx}g(x)$ then $\int u^n\,u'dx = \frac{u^{n+1}}{n+1} + C, where\ n \neq -1$
Reimann Sum	$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \quad where\ \ x_{i-1} \leq c_i \leq x_i$ $\lVert\Delta\rVert = \Delta x = \frac{b-a}{n}$
Definition of a Definite Integral Area under curve	$\lim_{\lVert\Delta\rVert\to 0}\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx$
Swap Bounds	$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
Additive Interval Property	$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
The Fundamental Theorem of Calculus	$\int_a^b f(x)\,dx = F(b) - F(a)$
The Second Fundamental Theorem of Calculus	$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f\big(g(x)\big)g'(x)$ $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f\big(h(x)\big)h'(x) - f\big(g(x)\big)g'(x)$
Mean Value Theorem for Integrals	$\int_a^b f(x)\,dx = f(c)(b-a)$ Find 'c'.
The Average Value for a Function	$\frac{1}{b-a}\int_a^b f(x)\,dx$

Integration Methods	
1. Memorized	See Larson's 1-pager of common integrals
2. U-Substitution	$\int f(g(x))g'(x)dx = F(g(x)) + C$ Set $u = g(x)$, then $du = g'(x)\ dx$ $\int f(u)\ du = F(u) + C$ $u =$ ____ $du =$ ____ dx
3. Integration by Parts	$\int u\ dv = uv - \int v\ du$ $u =$ ____ $v =$ ____ $du =$ ____ $dv =$ ____ Pick 'u' using the **LIATED** Rule: **L – Logarithmic** : $\ln x, \log_b x, etc.$ **I – Inverse Trig.**: $\tan^{-1} x, \sec^{-1} x, etc.$ **A – Algebraic**: $x^2, 3x^{60}, etc.$ **T – Trigonometric**: $\sin x, \tan x, etc.$ **E – Exponential**: $e^x, 19^x, etc.$ **D – Derivative of**: ${}^{dy}/_{dx}$
4. Partial Fractions	$\int \frac{P(x)}{Q(x)}\ dx$ where $P(x)$ *and* $Q(x)$ are polynomials **Case 1:** If degree of $P(x) \geq Q(x)$ then do long division first **Case 2:** If degree of $P(x) < Q(x)$ then do partial fraction expansion
5a. Trig Substitution for $\sqrt{a^2 - x^2}$	$\int \sqrt{a^2 - x^2}\ dx$ Substutution: $x = a \sin\theta$ Identity: $1 - sin^2\,\theta = cos^2\,\theta$
5b. Trig Substitution for $\sqrt{x^2 - a^2}$	$\int \sqrt{x^2 - a^2}\ dx$ Substutution: $x = a \sec\theta$ Identity: $sec^2\,\theta - 1 = tan^2\,\theta$
5c. Trig Substitution for $\sqrt{x^2 + a^2}$	$\int \sqrt{x^2 + a^2}\ dx$ Substutution: $x = a \tan\theta$ Identity: $tan^2\,\theta + 1 = sec^2\,\theta$
6. Table of Integrals	CRC Standard Mathematical Tables book
7. Computer Algebra Systems (CAS)	TI-Nspire CX CAS Graphing Calculator TI –Nspire CAS iPad app
8. Numerical Methods	Riemann Sum, Midpoint Rule, Trapezoidal Rule, Simpson's Rule, TI-84
9. WolframAlpha	Google of mathematics. Shows steps. Free. www.wolframalpha.com

Partial Fractions	(See Harold's Partial Fractions Cheat Sheet)
Condition	$f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ *and* $Q(x)$ are polynomials and degree of $P(x) < Q(x)$ If degree of $P(x) \geq Q(x)$ *then do long division first*
Example Expansion	$\frac{P(x)}{(ax+b)(cx+d)^2(ex^2+fx+g)}$ $= \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2} + \frac{Dx+E}{(ex^2+fx+g)}$
Typical Solution	$\int \frac{a}{x+b}\, dx = a\ ln\lvert x+b\rvert + C$

Sequences & Series	(See Harold's Series Cheat Sheet)
Sequence	$\lim_{n\to\infty} a_n = L$ (Limit) Example: $(a_n, a_{n+1}, a_{n+2}, \ldots)$
Geometric Series	$S = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$ only if $\lvert r\rvert < 1$ where r is the radius of convergence and $(-r, r)$ is the interval of convergence

Convergence Tests	(See Harold's Series Convergence Tests Cheat Sheet)
Series Convergence Tests	1. Divergence or n^{th} Term 2. Geometric Series 3. p-Series 4. Alternating Series 5. Integral 6. Ratio 7. Root 8. Direct Comparison 9. Limit Comparison 10. Telescoping

Taylor Series	(See Harold's Taylor Series Cheat Sheet)
Taylor Series	$f(x) = P_n(x) + R_n(x)$ $= \sum_{n=0}^{+\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(x^*)}{(n+1)!}(x-c)^{n+1}$ where $x \leq x^* \leq c$ (worst case scenario x^*) and $\lim_{x\to+\infty} R_n(x) = 0$