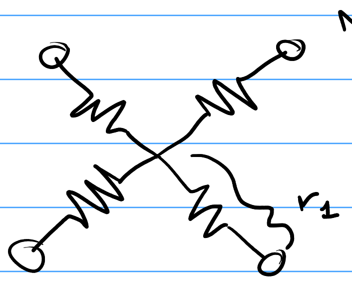
$$\vec{q} = \langle \theta, r_1, \cdots, r_N \rangle$$



ALL SPRINGS HAVE
REST LENGTH l_0,
STIFFNESS k.

$$T = \sum_{n=0}^{N-1} \frac{1}{2} m (\dot{r}_n^2 + r_n^2 \dot{\theta}^2)$$

$$U = \sum_{n=0}^{N-1} \frac{1}{2} k (R - r_n)^2 + \sum_{n=0}^{N-1} m g r_n \sin\left(\theta + \frac{n}{N} 2\pi\right)$$

$$\mathcal{L} = T - U$$

$$\mathcal{L} = \sum_{n=0}^{N-1} \frac{1}{2} m (\dot{r}_n^2 + r_n^2 \dot{\theta}_n^2) - \frac{1}{2} k (R - r_n)^2 - m g r_n \sin\left(\theta + \frac{n}{N} 2\pi\right)$$

$$S = N + 1$$

E.L:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}_n}\right) = \frac{d}{dt}\left(m \dot{r}_n\right) = \qquad m\ddot{r}_n = \frac{\partial \mathcal{L}}{\partial r_n}$$

$$= \qquad m r_n \dot{\theta}^2 + k(R - r_n) - m g \sin\left(\theta + \frac{n}{N} 2\pi\right)$$

$$\hookrightarrow \qquad m\ddot{r}_n = m r_n \dot{\theta}^2 + k(R - r_n) - m g \sin\left(\theta + \frac{n}{N} 2\pi\right)$$

$$\sum_{n=0}^{N-1} \frac{d}{dt}\left[m \dot{r}_n^2 \dot{\theta} \right] = \sum_{n=0}^{N-1} -mgr_n \cos\left(\theta + \frac{n}{N} 2\pi\right)$$

$$\sum_n \cancel{m} \dot{r}_n^2 \ddot{\theta} + 2\cancel{m} \dot{r}_n \ddot{r}_n \dot{\theta} = -\sum_n mgr_n \cos\left(\theta + 2\pi \frac{n}{N}\right)$$

$N = 2$

EL IN r_n

$$\boxed{0} \quad m\ddot{r}_0 = mr_0\dot{\theta}^2 + k(R - r_0) - mg\sin(\theta)$$

$$\boxed{1} \quad m\ddot{r}_1 = mr_1\dot{\theta}^2 + k(R - r_1) - mg\sin(\theta + \pi)$$

EL IN θ

$$\boxed{2} \quad (\dot{r}_0^2 + \dot{r}_1^2)\ddot{\theta} + 2(\dot{r}_0\ddot{r}_0 + \dot{r}_1\ddot{r}_1)\dot{\theta} = -g\left[r_0\cos(\theta) + r_1\cos(\theta + \pi)\right]$$

$$\ddot{f}(x) \approx \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2}$$

$$\ddot{f}(i\Delta t) \doteq \frac{f(i\Delta t) - 2f(i\Delta t - \Delta t) + f(i\Delta t - 2\Delta t)}{\Delta t^2}$$

$$f_i \equiv f(i\Delta t)$$

$$\nabla^2_{\Delta t} f_i = \frac{f_i - 2f_{i-1} + f_{i-2}}{\Delta t^2}$$

$$\nabla_{\Delta t} f_i = \frac{f_i - f_{i-1}}{\Delta t}$$

$$\boxed{0} \quad m\ddot{r}_0 = mr_0\dot{\theta}^2 + k(R - r_0) - mg\sin\left(\theta + \frac{n}{N}2\pi\right)$$

$$\Rightarrow \quad m\,\frac{r_{n,i} - 2r_{n,i-1} + r_{n,i-2}}{\Delta t^2}$$

$$= m r_{n,t}\left(\frac{\theta_i - \theta_{i-1}}{\Delta t}\right)^2 + k\left(R - r_{o,i}\right) - mg\sin\left(\theta_i\right)$$

SOLVE FOR $r_{n,i}$

$$r_n^i - 2r_n^{i-1} + r_n^{i-2} = r_n^t\left(\theta_i - \theta_{i-1}\right)^2 + \frac{k}{m}\Delta t^2\left(R - r_n^i\right) - \cancel{m}g\sin\left(\theta_i\right)\Delta t^2$$

$$= r_n^i\left(\theta_i - \theta_{i-1}\right)^2 + \frac{k}{m}\Delta t^2 R - \frac{k}{m}\Delta t^2 r_n^i - \cancel{mg}\sin\left(\theta_i\right)\Delta t^2$$

$$r_n^i\left(1 + \frac{k}{m}\Delta t^2 - \left(\theta_i - \theta_{i-1}\right)^2\right) = \frac{k}{m}\Delta t^2 R - \overset{\Delta t^2}{\cancel{m}}g\sin\left(\theta_i\right) + 2r_n^{i-1} - r_n^{i-2}$$

$$r_n^i = \frac{\frac{k}{m}R\Delta t^2 - \overset{\Delta t^2}{\cancel{m}}g\sin\left(\theta_i\right) + 2r_n^{i-1} - r_n^{i-2}}{1 + \frac{k}{m}\Delta t^2 - \left(\theta_i - \theta_{i-1}\right)^2}$$

$$\hat{a} = \frac{k}{m} \qquad \frac{F}{m\cancel{x}} = \frac{kx}{m\cancel{x}}$$

$$\frac{F}{mx} = \frac{ma}{mx} = \frac{a}{x} = \frac{k}{m} = \frac{\ddot{x}}{x}$$

$\frac{k}{m}$ = "NORMALIZED ACCELERATION"

$$r_n^i = \frac{\left[\hat{a}R - g\sin\left(\theta_i\right)\right]\Delta t^2 + 2r_n^{i-1} - r_n^{i-2}}{1 + \hat{a}\Delta t^2 + \left(\theta_i - \theta_{i-1}\right)^2}$$

$$\sum_n r_n^2\ddot{\theta} + 2\dot{r}_n\ddot{r}_n\dot{\theta} = -\sum_n g r_n \cos\left(\theta + 2\pi\frac{n}{N}\right)$$

$$\ddot{\theta}\overbrace{\sum_n \dot{r}_n^{2}}^{b} + \dot{\theta}\overbrace{\sum_n 2 r_n \ddot{r}_n}^{c} = -\overbrace{\sum_n g r_n \cos\left(\theta + 2\pi \frac{n}{N}\right)}^{d}$$

USE θ_{i-1}

$$\ddot{\theta} \approx \frac{\theta_i - 2\theta_{i-1} + \theta_{i-2}}{\Delta t^2} \qquad \dot{\theta} = \frac{\theta_i - \theta_{i-1}}{\Delta t}$$

$$\ddot{\theta} b + \dot{\theta} c + d = 0$$

$$\frac{\theta_i}{\Delta t^{\cancel{2}}} b + \frac{-2\theta_{i-1} + \theta_{i-2}}{\Delta t^{\cancel{2}}} b + \frac{\theta_i}{\cancel{\Delta t}} c - \frac{\theta_{i-1}}{\cancel{\Delta t}} c + d\,\Delta t = 0$$

$$\frac{\theta_i}{\cancel{\Delta t}}\left(\frac{b}{\Delta t} + c\right) + \frac{\theta_{i-2} - 2\theta_{i-1}}{\Delta t} b - \theta_{i-1} c + d = 0$$

$$\theta_i = \frac{\theta_{i-1} c + \frac{2\theta_{i-1} - \theta_{i-2}}{\Delta t} - d}{\frac{b}{\Delta t} + c}$$

WHERE

$$c = \sum_n 2 r_n \left(\frac{r_n - 2r_{n-1} + r_{n-2}}{\Delta t^2}\right)$$

$$b = \sum_n \left(\frac{r_n - r_{n-1}}{\Delta t}\right)^2$$

$$d = \sum_n g r_n \cos\left(\theta_{i-1} + 2\pi \frac{n}{N}\right)$$

$$\mathcal{L} = \sum_{n=0}^{N-1} \frac{1}{2}m(\dot{r}_n^2 + r_n^2\dot{\theta}_n^2) - \frac{1}{2}k(R-r_n)^2 - mgr_n\sin(\theta + \frac{n}{N}2\pi)$$

E.L. FOR θ

$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right] = \frac{d}{dt}\sum_{n=0}^{N-1} mr_n^2\dot{\theta} = \frac{\partial\mathcal{L}}{\partial\theta}$$

$$= \sum_{n=0}^{N-1} -mgr_n\cos(\theta + \frac{n}{N}2\pi)$$

D_θ

$$\boxed{\sum_{n=0}^{N-1} m(2r_n\dot{r}_n\dot{\theta} + r_n^2\ddot{\theta}) = -\sum_{n=0}^{N-1} mgr_n\cos(\theta + \frac{n}{N}2\pi)}$$

EL FOR r_n:

$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{r}_n}\right] = m\ddot{r}_n = \frac{\partial\mathcal{L}}{\partial r_n}$$

D_{r_n}

$$= \boxed{mr_n\dot{\theta} + k(R-r_n) - mg\sin(\theta + \frac{n}{N}2\pi) = m\ddot{r}_n}$$

→ WHICH OF THESE DO I USE TO FIND WHICH VARIABLES?

→ FOR D_θ, USE FORWARD DIFFERENCE METHOD IN θ - THIS WILL KEEP θ_{i+1}' OUT OF COSINE

→ FOR D_{r_n}, USE BACKWARDS DIFFERENCE IN BOTH r AND θ

SOLVE D_{r_n} FOR $r_n^i(r_n^{i-1}, r_n^{i-1}, \theta_i, \theta_{i-1})$

SOLVE D_θ FOR $\theta_{i+1}(r_n$

D_{r_n}:

$$\cancel{m}r_n\frac{\theta_n - \theta_{n-1}}{\Delta t} + \frac{k}{m}(R-r_n) - \cancel{m}g\sin(\theta_n + 2\pi\frac{n}{N}) = \cancel{m}\frac{r_{n+1} - 2r_n + r_{n-1}}{\Delta t^2}$$

$$\hat{a}R\Delta t^2 + [(\theta_n - \theta_{n-1}) - \hat{a}\Delta t]\Delta t\, r_n - g\sin(\theta + 2\pi\frac{n}{N})\Delta t^2$$

$$= r_{n+1} - 2r_n + r_{n-1}$$

$$r_{n+1} = \left(\hat{a}R\cancel{\Delta t}\left([\theta_n - \theta_{n-1}] - \hat{a}\Delta t\right)r_n\right)\Delta t - g\sin(\ldots)\Delta t^2 + 2r_n - r_{n-1}$$

$$= \left(\hat{a}[R - r_n]\Delta t^2 + [\theta_n - \theta_{n-1}]\Delta t\right.$$

$$[\theta_n - \theta_{n-1}]r_n\Delta t + \omega^2\Delta t^2[R - r_n] - g\sin(\ldots)\Delta t^2 + 2r_n - r_{n-1} = r_{n+1}$$

$$r_{n+1} = 2r_n - r_{n-1} + (\theta_i - \theta_{i-1})r_n\Delta t + \left(\omega^2[R - r_n] - g\sin\theta_n^i\right)\Delta t^2$$

$$\theta_n^i = \theta_i + 2\pi\frac{n}{N} \quad ; \quad \omega = \sqrt{\frac{K}{M}}$$

Dθ

$$\sum_{n=0}^{N-1} \cancel{m}\left(2r_n\dot{r}_n\dot{\theta} + r_n^2\ddot{\theta}\right) = -\sum_{n=0}^{N-1} \cancel{m}gr_n\cos\left(\theta + \frac{n}{N}2\pi\right)$$

$$\underbrace{\left(\sum 2r_n\dot{r}_n\right)}_{\alpha}\dot{\theta} + \underbrace{\left(\sum r_n^2\right)}_{\beta}\ddot{\theta} = -g\sum r_n\cos(\theta_n)$$

$$\alpha\frac{\theta_i - \theta_{i-1}}{\Delta t} + \beta\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{\Delta t^2} = -g\sum r_n\cos(\theta_n^i)$$

$$\Delta t\,\alpha(\theta_i - \theta_{i-1}) + \beta(\theta_{i+1} - 2\theta_i + \theta_{i-1}) = -g\Delta t^2\sum r_n\cos(\theta_n^i)$$

$$\Delta t\,\alpha(\theta_i - \theta_{i-1}) + g\Delta t^2\underbrace{\sum r_n\cos(\theta_n^i)}_{} = -\beta(\theta_{i+1} - 2\theta_i + \theta_{i-1})$$

$$\Delta t\,\alpha\theta_i - \Delta t\,\alpha\theta_{i-1} + g\Delta t^2\gamma - 2\beta\theta_i + \beta\theta_{i-1} = -\beta\theta_{i+1}$$

$$\beta\theta_{i+1} = (2\beta - \Delta t\,\alpha)\theta_i + (\Delta t\,\alpha - \beta)\theta_{i-1} - g\gamma\Delta t^2$$

$$\theta_{i+1} = \frac{(2\beta - \Delta t\,\alpha)\theta_i + (\Delta t\,\alpha - \beta)\theta_{i-1} - g\gamma\Delta t^2}{\beta}$$

WHERE

$$\Delta t\,\alpha = \sum_{n=0}^{N-1} 2r_n^i(r_n^i - r_n^{i-1}) \qquad \beta = \sum_{n=0}^{N-1} r_n^{i\,2} \qquad \gamma = \sum_{n=0}^{N-1} r_n^i\cos\left(\theta_i + 2\pi\frac{n}{N}\right)$$

Drn:

$$\cancel{m} r_n \left(\frac{\theta_n - \theta_{n-1}}{\Delta t}\right)^2 + \frac{k}{m}(R - r_n) - \cancel{m} g \sin\left(\theta_n + 2\pi \frac{n}{N}\right) = \cancel{m}\,\frac{r_{n+1} - 2r_n + r_{n-1}}{\Delta t^2}$$

$$\hat{a} R \Delta t^2 + [(\theta_n - \theta_{n-1})^2 - \hat{a}\Delta t]\Delta t\, r_n - g \sin\left(\theta + 2\pi\frac{n}{N}\right)\Delta t^2$$

$$= r_{n+1} - 2r_n + r_{n-1}$$

$$r_{n+1} = \left(\hat{a} R \cancel{\Delta t}\left([\theta_n - \theta_{n-1}] - \hat{a}\Delta t\right) r_n\right)\Delta t - g\sin(\dots)\Delta t^2 + 2r_n - r_{n-1}$$

$$= \left(\hat{a}[R - r_n]\Delta t^2 + [\theta_n - \theta_{n-1}]\Delta t\right.$$

$$[\theta_n - \theta_{n-1}]^2 r_n \cancel{\Delta t} + \omega^2 \Delta t^2 [R - r_n] - g\sin(\dots)\Delta t^2 + 2r_n - r_{n-1} = r_{n+1}$$

$$r_{n+1} = 2r_n - r_{n-1} + (\theta_i - \theta_{i-1})^2 r_n \cancel{\Delta t} + \left(\omega^2[R - r_n] - g\sin\theta_n^i\right)\Delta t^2$$

$$\theta_n^i = \theta_i + 2\pi\frac{n}{N} \; ; \quad \omega = \sqrt{\frac{k}{m}}$$