# Superlinear Angular Momentum in Reaction Wheels

Seth Hinz

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## 1 Introduction

In several robotics applications, reaction wheels are used to control the rotation of a body. However, they do have some drawbacks - size, mass, and controllability are all important factors to consider. For my term project, I will be analyzing a reaction wheel with variable radius, which may offer mitigations to these shortcomings.

## 2 Background

Before we introduce a reaction wheel with variable radius, it is useful to examine the simpler fixed-radius case. To provide the largest moment of inertia, we will model the reaction wheel as a thin ring of mass $m$ at a radius $r$ from the axis of rotation (which, importantly, is the symmetry axis of the ring).

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From this model, we can calculate the moment of inertia via integrating over the mass:

$$I = \oint_{\text{ring}} \mathrm{d}m \tag{1}$$
$$= \int_0^{2\pi} (\frac{m}{2\pi r})(r^2) r \mathrm{d}\theta \tag{2}$$
$$= \frac{mr^2}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \tag{3}$$
$$= mr^2 \tag{4}$$

Using this result, we can now find the angular momentum of the system.

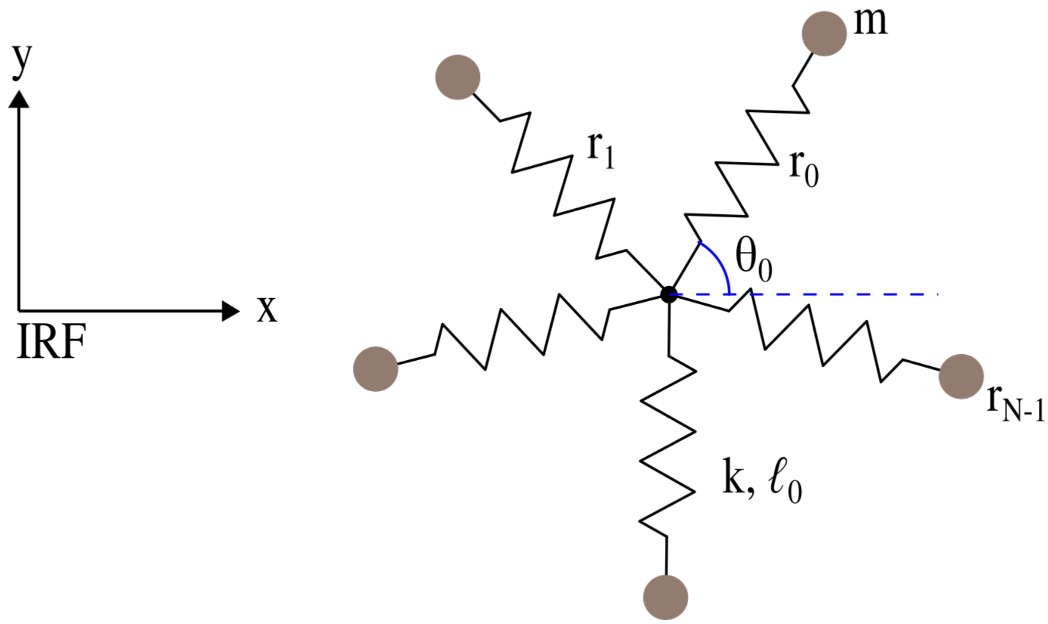
$$L = I\omega \tag{5}$$
$$= \omega mr^2 \tag{6}$$

Earlier, I stated that the controllability of a reaction wheel needs to be considered during the design process. Equation (6) shows why this is true - the angular momentum added to a body by a reaction wheel is only linear in the angular velocity. So, if a large amount of angular acceleration is needed quickly, a motor driving a reaction wheel must drastically alter its speed.

# 3 Variable Radius

The motivation for using a wheel with changing radius can now be illustrated. If the radial term in equation (6) was replaced by an increasing function in $\omega$ (namely, $L(\omega) = m\omega \cdot r(\omega)^2$), then we could push $L(\omega)$ far above linearity. Any motor speed decrease that maintains angular momentum will come at the cost of a larger torque, of course, but this may be a desirable tradeoff.

This insight led me to investigate a reaction wheel that uses discrete, spring-affixed masses about a central hub. In this model, each spring is flexible in only the radial direction, which reduces the number of degrees of freedom in this problem. Each appendage has mass $m$, and there are $N$ of them in total. The rotation of the system is measured via the angle of $r_0$, which is referred to as $\theta_0$.




One important thing to note is that this model cannot enforce all radii to be the same. $r_0$, the radius of the first arm, might be different than $r_1$ due to gravitational effects. Keeping this in mind, we will first model the most general version of this system that we can, and then simplify as needed.

We begin by finding the coordinates and velocities of the involved objects:

$$\theta_n = \theta_0 + 2\pi \frac{n}{N} \tag{7}$$

$$\dot{\theta_n} = \dot{\theta_0} \equiv \omega \tag{8}$$

$$x_i = r_n \cos(\theta_n) \tag{9}$$

$$y_i = r_n \sin(\theta_n) \tag{10}$$

$$\dot{x_n} = \dot{r_n} \cos(\theta_n) - r_n \sin(\theta_n)\dot{\theta_0} \tag{11}$$

$$\dot{y_n} = \dot{r_n} \sin(\theta_n) + r_n \cos(\theta_n)\dot{\theta_0} \tag{12}$$

Now, we can model the kinetic energy of the system.

$$T = \sum_{n=0}^{N-1} \frac{1}{2} m(\dot{x_n}^2 + \dot{y_n}^2) \tag{13}$$

$$= \frac{m}{2} \sum_{n=0}^{N-1} (\dot{r_n} \cos(\theta_n) - r_n \sin(\theta_n)\dot{\theta}_0)^2 + (\dot{r_n} \sin(\theta_n) + r_n \cos(\theta_n)\dot{\theta}_0)^2 \tag{14}$$

$$= \frac{m}{2} \sum_{n=0}^{N-1} \left( \dot{r_n}^2 + r_n^2 \dot{\theta_0}^2 \right) \tag{15}$$

Next, we find the potential energy:

$$U = \sum_{n=0}^{N-1} \left( \frac{1}{2} k(\ell_0 - r_n)^2 + mgr_n \sin(\theta_n) \right) \tag{16}$$

Finally, we can combine these to yield the lagrangian for this system.

$$\mathscr{L} = T - U \tag{17}$$

$$= \sum_{n=0}^{N-1} \frac{m}{2} \left( \dot{r_n}^2 + r_n^2 \dot{\theta_0}^2 \right) - \frac{1}{2} k(\ell_0 - r_n)^2 - mgr_n \sin(\theta_n) \tag{18}$$

Now, we can find the Euler-Lagrange equations. This system has $s = N + 1$ degrees of freedom. $N$ of these are given by the parameters $r_0, r_1, \ldots, r_{N-1}$, and the final parameter is $\theta_0$.

Although we do technically have $N + 1$ Euler-Lagrange equations, $N$ of them will be indistinguishable due to symmetry. We'll begin with the equation for the angle:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\partial \mathscr{L}}{\partial \dot{\theta}_0} \right) = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{n=0}^{N-1} m\dot{\theta}_0 r_n^2 \tag{19}$$

$$= m \sum_{n=0}^{N-1} \ddot{\theta}_0 r_n^2 + 2r_n \dot{\theta}_0 \dot{r_n} \tag{20}$$

$$= \frac{\partial \mathscr{L}}{\partial \theta_0} = -mg \sum_{n=0}^{N-1} r_n \cos(\theta_n)\dot{\theta}_0 \tag{21}$$

$$\tag{22}$$

Dividing out the common $m$ term, we see that the equation simplifies to the following.

$$\sum_{n=0}^{N-1} \ddot{\theta}_0 r_n^2 + 2 r_n \dot{\theta}_0 \dot{r_n} = -g \sum_{n=0}^{N-1} r_n \cos(\theta_n) \dot{\theta}_0 \tag{23}$$

We'll now compute the Euler-Lagrange equations for the remaining $N$ parameters - however, as mentioned, the symmetry of the problem means this is really just a single computation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{r_n}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{r_n}\right) = m\ddot{r_n} \tag{24}$$

$$= \frac{\partial \mathscr{L}}{\partial r_n} = \left(m\dot{\theta_0}^2 r_n + k(\ell_0 - r_n) - mg\sin(\theta_n)\right) \tag{25}$$

Motivated by the math of a simple spring oscillation, we'll introduce the fundamental frequency, $\omega_0 = \sqrt{\frac{k}{m}}$, into this calculation by dividing mass through.

$$\ddot{r_n} = \dot{\theta_0}^2 r_n + \frac{k}{m}(\ell_0 - r_n) - g\sin(\theta_n) \tag{26}$$

$$= \dot{\theta_0}^2 r_n + \omega_0^2(\ell_0 - r_n) - g\sin(\theta_n) \tag{27}$$

To summarize, our Euler-Lagrange equations are shown below.

$$\ddot{r_n} = \dot{\theta_0}^2 r_n + \omega_0^2(\ell_0 - r_n) - g\sin(\theta_n), \ \forall n \in [0, N-1] \tag{28}$$

$$\sum_{n=0}^{N-1} \ddot{\theta}_0 r_n^2 + 2 r_n \dot{\theta}_0 \dot{r_n} = -g \sum_{n=0}^{N-1} r_n \cos(\theta_n) \dot{\theta}_0 \tag{29}$$

# 4 Analysis

The equations that we just derived are a little too rich to solve by hand (for a third year, at least), but we can further simplify them to find some important truths.

For our analysis, we'll restrict N to being 1, and then use a small oscillation approximation to compute the system's properties. This may seem overly restrictive, but will provide some insight into how the changing spring angle affects its extension.

We begin by formulating our reduced lagrangian:

$$\mathscr{L}_r = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{2}k(\ell_0 - r)^2 - mgr\sin\theta \tag{30}$$

To use a small oscillations approximation, we'll first taylor expand around the global minimum of $U$, which we'll find now:

$$\frac{\partial U_r}{\partial \theta} = -mgr\cos(\theta) \equiv 0 \tag{31}$$

$$r \neq 0 \implies \cos(\theta) = 0 \tag{32}$$

$$\theta = \pm\frac{\pi}{2} \tag{33}$$

Obviously, the stable equilibrium will occur when the mass is at its lowest point (otherwise, the situation is analogous to an inverted pendulum). So, $\theta_0 = -\frac{\pi}{2}$ (note: $\theta_0$ is used to refer to the equilibrium angle in this section, but it is used in other sections to refer to the angle between the +x axis and the first arm.)

$$\frac{\partial U_r}{\partial r} = k(\ell_0 - r) - mg\sin(\theta) \equiv 0 \tag{34}$$

$$\implies r = \ell_0 - \frac{mg}{k}\sin(\theta) \tag{35}$$

$$\theta = -\frac{\pi}{2} \implies r = \ell_0 + \frac{mg}{k} \tag{36}$$

Thus, our stable equilibrium point is $(r_0, \theta_0) = (\ell_0 + \frac{mg}{k}, -\frac{pi}{2})$.

Now, we can take our taylor expansion of the lagrangian. This calculation was tedious and not very enlightening, so was not typeset - the rough work for this project is available in its github repository (github.com/shinzlet/reaction-wheel).

$$\mathscr{L}_r = \frac{m}{2}\left(\dot{r}^2 + r_0^2\dot{\theta}^2\right) - \frac{k}{2}(r - r_0)^2 - \frac{mgr_0}{2}(\theta + \frac{\pi}{2})^2 \tag{37}$$

Now, we can compute the simplified Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}_r}{\partial \dot{r}}\right) = m\ddot{r} \tag{38}$$

$$= \frac{\partial \mathscr{L}_r}{\partial r} = -k(r - r_0) \tag{39}$$

$$\implies \ddot{r} = -\omega_0^2(r - r_0) \tag{40}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}_r}{\partial \dot{\theta}}\right) = mr_0^2\ddot{\theta} \tag{41}$$

$$= \frac{\partial \mathscr{L}_r}{\partial \theta} = -mgr_0(\theta + \frac{\pi}{2}) \tag{42}$$

$$\implies r_0\ddot{\theta} = -g(\theta + \frac{\pi}{2}) \tag{43}$$

These expressions must be simplified via a change of variables:

$$\theta + \frac{\pi}{2} \to \phi \tag{44}$$

$$r - r_0 \to x \tag{45}$$

$$\implies \ddot{x} = -\omega_0^2 x \tag{46}$$

$$\implies \ddot{\phi} = -\frac{g}{r_0}\phi \tag{47}$$

These equations are very interesting, because they look identical to the motion of a spring at rest, and the motion of a planar pendulum, respectively. And, in fact, a superposition of those two systems would be a very good description of the $N = 1$ system.

To solve this, we'll use the standard perscription for small oscillations. Note that the constants are in different units - length ($A$) and angle ($B$).

$$x(t) \equiv A\cos(\omega t + \gamma) \tag{48}$$

$$\phi(t) \equiv B\cos(\Omega t + \delta) \tag{49}$$

Note that there are different frequencies for these - this is because our taylor expansion was, although standard, too simple to capture the coupling in this system. As a result, there might not be a common frequency.

$$
\begin{aligned}
\ddot{x} &= -\omega^2 A \cos(\omega t + \gamma) && (50)\\
&= -\omega_0^2 x && (51)\\
&= -\omega_0^2 A \cos(\omega t + \gamma) && (52)\\
\implies \omega^2 &= \omega_0^2 \text{ where } \omega_0 = \sqrt{\frac{k}{m}} && (53)\\
\implies \omega &= \pm\sqrt{\frac{k}{m}} && (54)
\end{aligned}
$$

$$
\begin{aligned}
\ddot{\phi} &= -\Omega^2 B \cos(\Omega t + \delta) && (55)\\
&= -\frac{g}{r_0} B \cos(\Omega t + \delta) && (56)\\
\implies \Omega^2 &= \frac{g}{r_0} && (57)\\
\implies \Omega &= \pm\sqrt{\frac{g}{\ell_0 + \frac{mg}{k}}} && (58)
\end{aligned}
$$

Finally, we can reverse the substitutions:

$$
\begin{aligned}
r(t) &= \ell_0 + \frac{mg}{k} + A\cos(\pm\sqrt{\frac{k}{m}}t + \gamma) && (59)\\
\theta(t) &= B\cos(\pm\sqrt{\frac{g}{\ell_0 + \frac{mg}{k}}}t + \delta) - \frac{pi}{2} && (60)
\end{aligned}
$$

# 5 Numerical Approximation

In the previous sections, we examined this system from an analytical perspective. However, finding exact solutions of the motion of this system is likely not possible (as is true of many systems). To better understand the behavior of a variable-radius reaction wheel, we'll turn to numerical simulations.

First, we must convert our euler-lagrange equations into a finite difference approximation. The angular equation used a forwards difference approximation of $\dot{theta}$.

$$\theta_{i+1} = \frac{(2\beta - \Delta t\alpha)\theta_i + (\Delta\alpha - \beta)\theta_{i-1} - g\gamma\Delta t^2}{\beta} \tag{61}$$

$$\text{where } \Delta t\alpha = \sum_{n=0}^{N-1} 2r_n^i(r_n^i - r_n^{i-1});\ \beta = \sum_{n=0}^{N-1} (r_n^i)^2 \tag{62}$$

$$\gamma = \sum_{n=0}^{N-1} r_n^i \cos(\theta_i + 2\pi\frac{n}{N}) \tag{63}$$

The r equations were transformed using the central difference approximation of $\ddot{r_i}$, and the backward difference approximation of $\dot{\theta}_0$. These choices seem arbitrary, but allow the $r_n$ term to be more easily solved for.

$$r_{n+1} = 2r_n - r_{n-1} + (\theta_i - \theta_{i-1})^2 r_n + \left(\omega^2(\ell_0 - R_n) - g\sin(\theta_n^i)\right)\Delta t^2 \tag{64}$$

$$\text{where } \theta_n^i = \theta_i + 2\pi\frac{n}{N} \tag{65}$$

$$\omega = \sqrt{\frac{k}{m}} \tag{66}$$

These equations are obviously unpleasant, but many of the sums can be computed in the same for loop when implemented in code.