Superlinear Angular Momentum in Reaction Wheels

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1 Introduction

In several robotics applications, reaction wheels are used to control the rotation of a body. However, they do have some drawbacks - size, mass, and controllability are all important factors to consider. For my term project, I will be analyzing a reaction wheel with variable radius, which may offer mitigations to these shortcomings.

2 Background

Before we introduce a reaction wheel with variable radius, it is useful to examine the simpler fixed-radius case. To provide the largest moment of inertia, we will model the reaction wheel as a thin ring of mass m at a radius r from the axis of rotation (which, importantly, is the symmetry axis of the ring).

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From this model, we can calculate the moment of inertia via integrating over the mass:

$$
\begin{align}
I &= \oint_{\text{ring}} \mathrm{d}m \tag{1}\\
&= \int_0^{2\pi} (\frac{m}{2\pi r})(r^2) r \mathrm{d}\theta \tag{2}\\
&= \frac{mr^2}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \tag{3}\\
&= mr^2 \tag{4}
\end{align}
$$

Using this result, we can now find the angular momentum of the system.

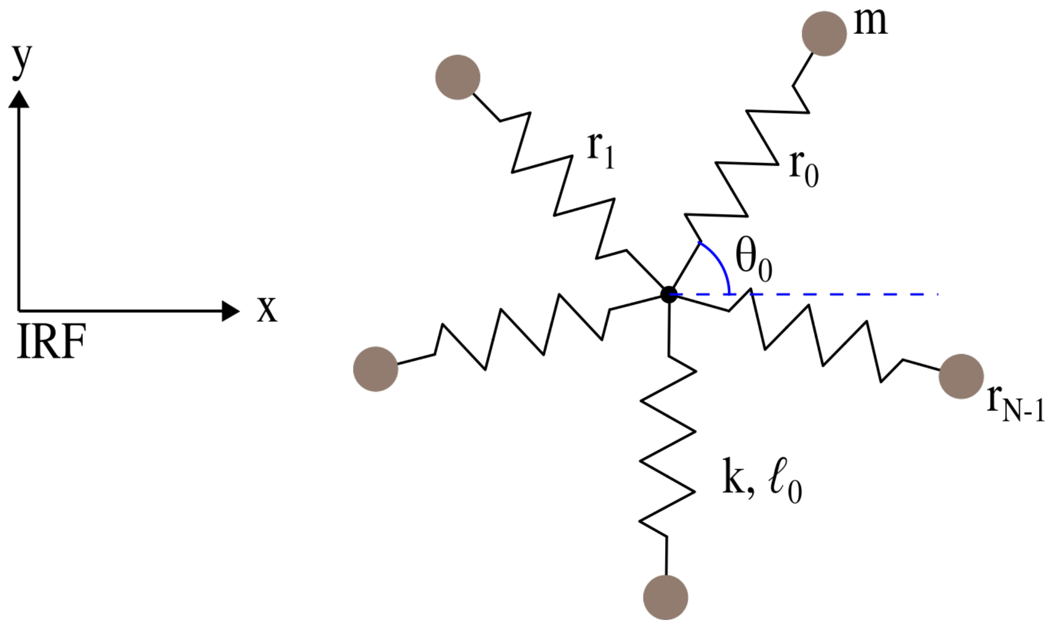
$$
\begin{align}
L &= I\omega \tag{5}\\
&= \omega m r^2 \tag{6}
\end{align}
$$

Earlier, I stated that the controllability of a reaction wheel needs to be considered during the design process. Equation (6) shows why this is true - the angular momentum added to a body by a reaction wheel is only linear in the angular velocity. So, if a large amount of angular acceleration is needed quickly, a motor driving a reaction wheel must drastically alter its speed.

3 Variable Radius

The motivation for using a wheel with changing radius can now be illustrated. If the radial term in equation (6) was replaced by an increasing function in ω (namely, $L(\omega) = m\omega \cdot r(\omega)^2$), then we could push $L(\omega)$ far above linearity. Any motor speed decrease that maintains angular momentum will come at the cost of a larger torque, of course, but this may be a desirable tradeoff.

This insight led me to investigate a reaction wheel that uses discrete, spring-affixed masses about a central hub. In this model, each spring is flexible in only the radial direction, which reduces the number of degrees of freedom in this problem. Each appendage has mass m, and there are N of them in total. The rotation of the system is measured via the angle of r_0, which is referred to as θ_0.



One important thing to note is that this model cannot enforce all radii to be the same. r_0, the radius of the first arm, might be different than r_1 due to gravitational effects. Keeping this in mind, we will first model the most general version of this system that we can, and then simplify as needed.

We begin by finding the coordinates and velocities of the involved objects:

$$\theta_n = \theta_0 + 2\pi\frac{n}{N} \tag{7}$$

$$\dot{\theta_n} = \dot{\theta_0} \equiv \omega \tag{8}$$

$$x_i = r_n\cos(\theta_n) \tag{9}$$

$$y_i = r_n\sin(\theta_n) \tag{10}$$

$$\dot{x_n} = \dot{r_n}\cos(\theta_n) - r_n\sin(\theta_n)\dot{\theta_0} \tag{11}$$

$$\dot{y_n} = \dot{r_n}\sin(\theta_n) + r_n\cos(\theta_n)\dot{\theta_0} \tag{12}$$

Now, we can model the kinetic energy of the system.

$$
\begin{aligned}
T &= \sum_{n=0}^{N-1} \frac{1}{2} m(\dot{x_n}^2 + \dot{y_n}^2) && (13) \\
&= \frac{m}{2} \sum_{n=0}^{N-1} (\dot{r_n} \cos(\theta_n) - r_n \sin(\theta_n)\dot{\theta_0})^2 + (\dot{r_n} \sin(\theta_n) + r_n \cos(\theta_n)\dot{\theta_0})^2 && (14) \\
&= \frac{m}{2} \sum_{n=0}^{N-1} \left(\dot{r_n}^2 + r_n^2 \dot{\theta_0}^2\right) && (15)
\end{aligned}
$$

Next, we find the potential energy:

$$
U = \sum_{n=0}^{N-1} \left(\frac{1}{2} k(\ell_0 - r_n)^2 + mgr_n \sin(\theta_n) \right) \tag{16}
$$

Finally, we can combine these to yield the lagrangian for this system.

$$
\begin{aligned}
\mathscr{L} &= T - U && (17) \\
&= \sum_{n=0}^{N-1} \frac{m}{2} \left(\dot{r_n}^2 + r_n^2 \dot{\theta_0}^2\right) - \frac{1}{2} k(\ell_0 - r_n)^2 - mgr_n \sin(\theta_n) && (18)
\end{aligned}
$$

Now, we can find the Euler-Lagrange equations. This system has $s = N + 1$ degrees of freedom. N of these are given by the parameters $r_0, r_1, \ldots, r_{N-1}$, and the final parameter is θ_0.

Although we do technically have $N + 1$ Euler-Lagrange equations, N of them will be indistinguishable due to symmetry. We'll begin with the equation for the angle:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathscr{L}}{\partial \dot{\theta_0}} \right) &= \frac{\mathrm{d}}{\mathrm{d}t} \sum_{n=0}^{N-1} m \dot{\theta_0} r_n^2 && (19) \\
&= m \sum_{n=0}^{N-1} \ddot{\theta_0} r_n^2 + 2 r_n \dot{\theta_0} \dot{r_n} && (20) \\
&= \frac{\partial \mathscr{L}}{\partial \theta_0} = -mg \sum_{n=0}^{N-1} r_n \cos(\theta_n) \dot{\theta_0} && (21) \\
& && (22)
\end{aligned}
$$

Dividing out the common m term, we see that the equation simplifies to the following.

$$\sum_{n=0}^{N-1} \ddot{\theta}_0 r_n^2 + 2r_n\dot{\theta}_0\dot{r_n} = -g\sum_{n=0}^{N-1} r_n \cos(\theta_n)\dot{\theta}_0 \tag{23}$$

We'll now compute the Euler-Lagrange equations for the remaining N parameters - however, as mentioned, the symmetry of the problem means this is really just a single computation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{r_n}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{r_n}\right) = m\ddot{r_n} \tag{24}$$

$$= \frac{\partial \mathscr{L}}{\partial r_n} = \left(m\dot{\theta_0}^2 r_n + k(\ell_0 - r_n) - mg\sin(\theta_n)\right) \tag{25}$$

Motivated by the math of a simple spring oscillation, we'll introduce the fundamental frequency, $\omega_0 = \sqrt{\frac{k}{m}}$, into this calculation by dividing mass through.

$$\ddot{r_n} = \dot{\theta_0}^2 r_n + \frac{k}{m}(\ell_0 - r_n) - g\sin(\theta_n) \tag{26}$$

$$= \dot{\theta_0}^2 r_n + \omega_0^2(\ell_0 - r_n) - g\sin(\theta_n) \tag{27}$$

To summarize, our Euler-Lagrange equations are shown below.

$$\ddot{r_n} = \dot{\theta_0}^2 r_n + \omega_0^2(\ell_0 - r_n) - g\sin(\theta_n),\ \forall n \in [0, N-1] \tag{28}$$

$$\sum_{n=0}^{N-1} \ddot{\theta}_0 r_n^2 + 2r_n\dot{\theta}_0\dot{r_n} = -g\sum_{n=0}^{N-1} r_n \cos(\theta_n)\dot{\theta}_0 \tag{29}$$

4 Analysis

The equations that we just derived are a little too rich to solve by hand (for a third year, at least), but we can further simplify them to find some important truths.

For our analysis, we'll set g to zero, which represents the behaviour of the system when it is coplanar with the ground. Furthermore, we'll assume that the springs are initially in equilibrium. This might seem like it is only an initial condition, but it is actually much more powerful. By symmetry, there is no difference between r_0 and r_1, or any other combination. So, if the initial conditions are the same, and gravity is zero, then $r_i(t) = r_j(t)\ \forall\ i, j, t$.

Let's write these changes into effect:

$$\ddot{r} = \dot{\theta_0}^2 r + \omega_0^2(\ell_0 - r) \tag{30}$$

$$N\left(\ddot{\theta_0} r_n^2 + 2r_n\dot{\theta_0}\dot{r_n}\right) = -gr\dot{\theta_0}\sum_{n=0}^{N-1}\cos(\theta_n) \tag{31}$$

To simplify further, note the following trig identity:

$$\sum_{n=0}^{N-1}\cos(\theta_0 + 2\pi\frac{n}{N}) = \cos(\theta_0)\ \forall N \in \mathbb{Z} \tag{32}$$

Thus, our reduced equations are as follows:

$$\ddot{r} = \dot{\theta_0}^2 r + \omega_0^2(\ell_0 - r) \tag{33}$$

$$-gr\dot{\theta_0}\cos(\theta_n) = N\left(\ddot{\theta_0} r_n^2 + 2r_n\dot{\theta_0}\dot{r_n}\right) \tag{34}$$

These are far more managable, but still nonlinear. To continue, we'll have to taylor expand around $\theta_0 = 0, r = \ell_0$ in several places.

First, we'll do this to the angular equation.

$$-gr\dot{\theta_0}\cos(\theta_n) \approx -gr\dot{\theta_0} \tag{35}$$

$$N\left(\ddot{\theta_0} r_n^2 + 2r_n\dot{\theta_0}\dot{r_n}\right) \approx N() \tag{36}$$