Promotion pattern on Young Tableaux with nested tableaux

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Abstract
This journal propose a bijection between 4-row Young tableaux and some webs for sl4 whose boundary verteices are all source. It aims to contribute to the definition of irreducible webs for sl4 and its bijection to 4-row Young Tableaux.

1 INTRODUCTION

The bijection between standard Young tableaux of shape (n, n, n), or 3-row Young tableaux, and irreducible webs for sl3 has been defined by Khovanov–Kuperberg's recursive algorithm as well as Tymoczko's simple mapping. In this paper, following Tymoczko's mapping with an intermediate construction, m-diagram, we will define and proof a bijection between standard Young tableaux of shape (n, n, n, n) and some webs in sl4 whose boundary vertices are all sources.

Definition of webs Webs in sl4 have the following properties:

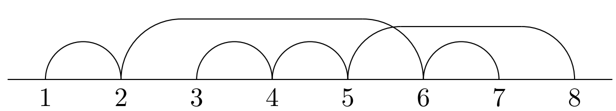
- they are planar directed graphs;
- each internal vertex is trivalent and has one un-directed/bi-directed edge and two directed edges either both directed in or both directed out;
- and each boundary vertex has degree one and is a source.

Bijection In this paper, we will show the mapping from standard Young tableaux of shape (n, n, n, n) to some webs in sl4 whose boundary vertices are all sources (Section?) and the mapping back respectively (Section ?). To give an intuition, here we provide a quick walk-through example and colloquial description of the bijective mapping.

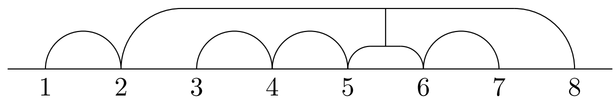
Example Consider a 4×2 standard Young tableau

1	3
2	4
5	6
7	8

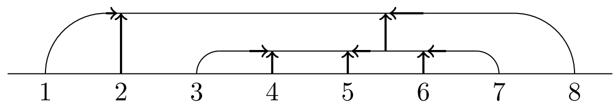
As in Tymoczko's mapping algorithm, we will use arc-diagram as an intermediate object. Arc-diagram is a diagram consist of a line with the numbers $1, 2, \ldots, N$ and collection of arcs drawn above it. To draw the arcs, read from the top to the bottom row, and then from left to right along each row, connecting the number i with an arc to the largest number on the row above i that is not yet connected to a number on i's row. The arc-diagram of the 4×2 Young tableau is as below.



Next, we transform the arc-diagram to a web. There are two steps: fixing quadrivalent/crossing vertices and fixing boundary vertices. First, we transform each quadrivalent vertex into two neighboring trivalent vertices. See below.

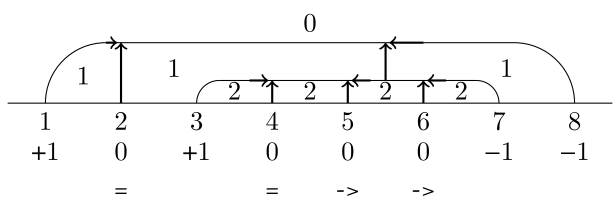


Second, for each boundary vertex that looks locally like a "V", we replace this neighborhood with a "Y". Finally, we add directions, where each boundary vertex is a source, to get the corresponding web of the Young tableau.



In the following sections(?), we will formally argue that such procedure is well-defined.

As the other direction of our bijection, the mapping from webs to standard Young tableau uses the notion of *path depth*, which is the minimum number of edges crossed by paths from a given face to the unbounded face of the web. We observe the change of path depth from the left to the right of each boundary vertex. And for boundary vertex with a path depth change of 0, we mark the directness of the first edge that we meet when we center at the boundary vertex's internal neighbor and going counter-clockwise from the edge in between the two.



In Section ?, we will show that:

$$\text{Boundary vertices corresponding to path depth (and edge direction)} \begin{cases} +1 \\ 0\text{ , undirected} \\ 0\text{ , directed} \\ -1 \end{cases} \text{should be put on the}$$

$$\begin{cases} \text{first} \\ \text{second} \\ \text{third} \\ \text{fourth} \end{cases} \text{row of the Young tableau.}$$

Equivalence class for sl4 With this bijection, we also observe the presumed correspondence of tableau promotions and web rotations that has been proved for 3-row case. In Section?, we proposed the assumption of an equivalence class for webs in sl4 that generates from such correspondence.

Future work There are still many waiting to be explored after this paper. First, this bijection is defined for all $4 \times n$ Young tableau and **some** webs for sl4; and second, we are **not yet sure** whether the **correspondence** between tableau promotions and web rotations really exists for 4-row case. Thus, for future research that aims to ultimately define a bijection between the tableaux and reduced webs for sl4, we ask two questions:

- What is the definition of of a reduced web for sl4? Could the webs that this mapping constructs from tableaux shed some lights on its definition?
- Does the correspondence exist? If so, is our proposed equivalence class for sl4 valid?

2 FROM YOUNG TABLEAUX TO WEBS

2.1 Young tableaux to arc-diagrams

The transformation from Young tableaux to our intermediate object, arc-diagrams, which is similar to the m-diagrams proposed in Tymoczko's paper, is defined as:

- Draw a line with integers $1, ..., N$, where N is the largest integer in the Young tableau;
- From the top to the bottom row, and from left to right:
  - For an integer i in the cell, find the largest unmatched integer j on the row above i such that $j < i$,
  - Draw an arc connecting i and j on the line.

Examples

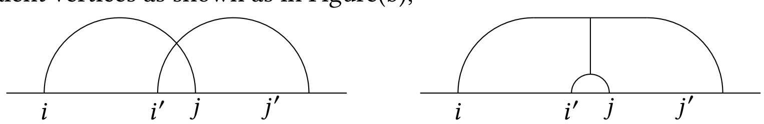
2.2 Arc-diagrams to webs

Next, we transform the arc-diagrams to webs. Again, recall the properties of webs for $sl4$:

- they are planar directed graphs;
- each internal vertex is trivalent and has one un-directed/bi-directed edge and two directed edges either both directed in or both directed out;
- and each boundary vertex has degree one and is a source.

There are three problems in arc-diagrams that we face when transforming them into webs: crossing/4-valent interior vertices, boundary vertices, and directions. And we propose the following procedures:

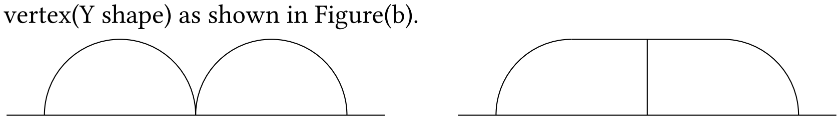
- *Transform crossings/4-valent interior vertices*: replace each of crossing vertex as shown in Figure(a) by two neighboring 3-valent vertices as shown as in Figure(b);



(a) (b)

Figure 2

- *Transform boundary vertices*: replace each boundary vertex with two arcs(V shape) as shown in Figure(a) by a 3-valent vertex(Y shape) as shown in Figure(b).



(a) (b)

Figure 3

Examples

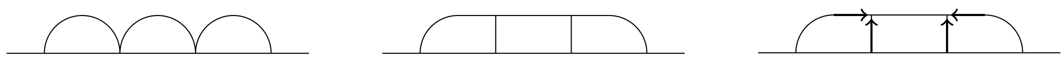
Webs obtained are valid In the following, we argue that such construction will always generate a valid web for $sl4$.

LEMMA 2.1. *Arc-diagrams after transformation of crossings and boundary vertices are planer graphs.*

Proof. Arc-diagrams are planer graphs with crossings regarded as vertices. Transformations do not add intersecting edges. □

Lemma 2.2. *Consider an arc-diagram with boundary vertices transformed. There exists one and only one way to set edges directions on the arc-diagram ignoring crossings that satisfies the webs' properties that interior vertices are all trivalent with one un-directed/bi-directed edge and two directed edges either both directed in or both directed out and that boundary vertices have one and only one edge directed out.*

Proof. Consider sets of arcs each pair of which shares a same boundary vertex. Call such sets arc sets. In standard Young tableaux of shape (n, n, n, n), there are n arc sets and each set contains 3 arcs and connects 4 numbers. In the lemma, since we ignore crossings which only exists between different arc sets, we analyze each arc set in the arc-diagrams independently.

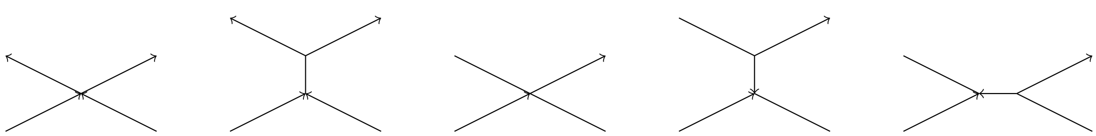


(a) (b) (c)

Figure(a) shows an arc set. Figure(b) is the arc set after transformation of boundary vertices. There is one and only one way to specify directions on the arcs such that boundary vertices have one and only one edge directed out and interior vertices are all trivalent with one un-directed/bi-directed edge and two directed edges either both directed in or both directed out, as shown in Figure(c). □

Lemma 2.3. *The transformation of a crossing/quadrivalent interior vertex v in an arc-diagram into a pair of trivalent vertices keeps the corresponding web unchanged outside of a small neighborhood of v.*

Proof. Consider arc-diagrams with boundary vertices transformed. There are two types of neighborhood around the crossings, one where two directed edges intersect at the crossing(Figure(a)) and another where one directed and one un-directed/bi-directed edge intersect at the crossing(Figure(c)). For the first type, the crossing vertex will be transformed as two neighboring 3-valent vertices connected by an un-directed/bi-directed edge, as shown in Figure(b). For the second type, the crossing vertex will be transformed as two neighboring 3-valent vertices connected by a directed edge. There are two possibilities depending on the relative position of the edges' endpoints, as shown in Figure(d),(e). We can see that all three transformations maintain the directions of the existing edges. Thus, the transformation keeps the corresponding web unchanged outside of a small neighborhood of a crossing vertex.



(a) (b) (c) (d) (e)

□

Therefore, the webs that we obtain from arc-diagrams are always valid.

3 FROM CONSTRUCTED WEBS TO YOUNG TABLEAUX

Path Depth To map from webs that we obtain from the previous section to Young tableaux, we use a property called path depth. As in Tymoczko's paper, it is defined as the following:

Definition 3.1. Path depth of an area A in a web is the minimum number of edges on the path from unbounded area to A. Denote as $d(A)$. For a number i on the line of a web that is in A, we can also denote its path depth as $d(i) = d(A)$.

Following Tymoczko's paper, we consider the difference of path depths of the left and right side of boundary vertices. More formally, we give the definition of path depth difference.

Definition 3.2. Path depth difference of a boundary vertex i of a web is the difference in path depths of areas on its left and its right. Or equivalently, let ϵ be a small number, the path depth difference of i is $d(i-\epsilon)-d(i+\epsilon)$.

Tymoczko has proved that path depth difference of boundary vertices can only take values from -1 , 0, and +1.
Examples
Incident edge directedness Another property that we use is the incident edge directedness of boundary vertices.

Definition 3.3. For a boundary vertex i of a web, consider its incident edge $e=(i,j)$ where j is the endpoint vertex of e other than the vertex on i. Incident edge directedness of i is the direction of the first edge encountered when moving counter-clockwise from the incident edge e of i at the endpoint j. Denote it as $D(i)$. In a web for $sl4$, $D(i)$ for any i can either be directed or un-directed/bi-directed.

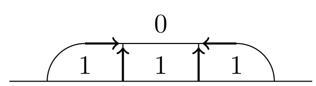
Examples
From webs to Young tableaux Now, we will use path depths and incident edge directedness to construct the mapping from Young tableaux to their corresponding webs.

Theorem 3.4. *Given a web W created with the procedure defined in the previous section,*

$$\textit{put boundary vertex } i \textit{ of } W \textit{ to row} \begin{cases}1\\4\end{cases} \textit{of the Young tableau if its path depth difference is} \begin{cases}+1\\-1\end{cases},$$

$$\textit{and to row} \begin{cases}2\\3\end{cases} \textit{if its path depth difference is 0 and its incident edge is} \begin{cases}\textit{un-directed/bi-directed}\\\textit{directed}\end{cases}$$

Proof. The first half of the theorem can be proved similarly as in Tymoczko's paper. Consider a single arc set as below.



We can see that

□