PROMOTION PATTERN ON YOUNG TABLEAUX WITH NESTED TABLEAUX

MEREDITH PAN

1 INTRODUCTION

The bijection of webs and Young Tableaux has caught interests by people in combinatorics as well as in representation theory. Khovanov–Kuperberg presented a recursive algorithm that provides a bijection between standard Young tableaux of the shape $3 \times n$, or 3-row Young tableaux, and irreducible webs for sl3. In correspondence, Tymoczko found a more explicit mapping with an intermediate construction, m-diagram, and showed that the correspondence of the Jeu-de-taquin promotion(for simplicity, we will use "promotion" and "Jeu-de-taquin promotion" interchangeably) on the Young Tableaux and the rotation of *webs* aligns with the mapping.

Now, we want to extend the result to the bijection between 4-row standard Young tableaux and irreducible webs for sl4. One requirement of our bijection is the correspondence of promotions and rotations. In this journal records a first attempt to approach such correspondence on specific cases.

2 PROBLEM FORMALIZATION AND STATEMENT

In this journal, we present a pattern of promotion performed on special-structured standard Young Tableaux.

Jeu-de-taquin promotion Recall that Jeu-de-taquin slide on Young tableaux is where an empty box percolates to the boundary of a tableau. In a single step on the configuration

$$\begin{array}{|c|c} \hline a & \multicolumn{1}{c}{} \\ \hline & \multicolumn{1}{|c|}{b} \\ \hline \end{array}$$

,the number a slides down if $a < b$ and the number b slides left if $b < a$. (Numbers outside of the tableau are considered to be ∞.) Jeu-de-taquin promotion, then, is the operation on standard tableaux obtained by

- erasing 1,
- performing jeu-de-taquin slides until a new Young tableaux is obtained,
- reduce all values by 1,
- and then adding n to the newly-empty spot.

Problem formalization In this journal, we focus specifically on standard Young Tableaux of shape $m \times k$ that have nested Young Tableaux of shape $m \times j$ on their left, and the nested Young Tableaux are filled with consecutive integers $1, \ldots, mj$, i.e., the mj smallest positive integers. We will show that the cells with consecutive integers and the ones with new numbers evolved in the bottom-right always form a rectangle of the shape $m \times k$ when combined together.
Example Consider a 3×4 Young tableaux

$$\begin{array}{|c|c|c|c|}\hline \cellcolor{yellow}1 & \cellcolor{yellow}3 & 7 & 9 \\ \hline \cellcolor{yellow}2 & \cellcolor{yellow}4 & 8 & 11 \\ \hline \cellcolor{yellow}5 & \cellcolor{yellow}6 & 10 & 12 \\ \hline \end{array}$$

that contains nested 3×2 Young tableaux on its left(in yellow). Now we perform a step-by-step Jeu-de-taquin promotion on it while tracking the movement of the cells in the nest Young tableaux.

$$\begin{array}{|c|c|c|c|}\hline \cellcolor{yellow}1 & \cellcolor{yellow}3 & 7 & 9 \\ \hline \cellcolor{yellow}2 & \cellcolor{yellow}4 & 8 & 11 \\ \hline \cellcolor{yellow}5 & \cellcolor{yellow}6 & 10 & 12 \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline & \cellcolor{yellow}3 & 7 & 9 \\ \hline \cellcolor{yellow}2 & \cellcolor{yellow}4 & 8 & 11 \\ \hline \cellcolor{yellow}5 & \cellcolor{yellow}6 & 10 & 12 \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline \cellcolor{yellow}2 & \cellcolor{yellow}3 & 7 & 9 \\ \hline & \cellcolor{yellow}4 & 8 & 11 \\ \hline \cellcolor{yellow}5 & \cellcolor{yellow}6 & 10 & 12 \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline \cellcolor{yellow}2 & \cellcolor{yellow}3 & 7 & 9 \\ \hline \cellcolor{yellow}4 & & 8 & 11 \\ \hline \cellcolor{yellow}5 & \cellcolor{yellow}6 & 10 & 12 \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline \cellcolor{yellow}2 & \cellcolor{yellow}3 & 7 & 9 \\ \hline \cellcolor{yellow}4 & \cellcolor{yellow}6 & 8 & 11 \\ \hline \cellcolor{yellow}5 & & 10 & 12 \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline \cellcolor{yellow}2 & \cellcolor{yellow}3 & 7 & 9 \\ \hline \cellcolor{yellow}4 & \cellcolor{yellow}6 & 8 & 11 \\ \hline \cellcolor{yellow}5 & 10 & & 12 \\ \hline \end{array}$$

$$\Rightarrow \begin{array}{|c|c|c|c|}\hline \cellcolor{yellow}2 & \cellcolor{yellow}3 & 7 & 9 \\ \hline \cellcolor{yellow}4 & \cellcolor{yellow}6 & 8 & 11 \\ \hline \cellcolor{yellow}5 & 10 & 12 & \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline \cellcolor{yellow}1 & \cellcolor{yellow}2 & 6 & 8 \\ \hline \cellcolor{yellow}3 & \cellcolor{yellow}5 & 7 & 10 \\ \hline \cellcolor{yellow}4 & 9 & 11 & \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline \cellcolor{yellow}1 & \cellcolor{yellow}2 & 6 & 8 \\ \hline \cellcolor{yellow}3 & \cellcolor{yellow}5 & 7 & 10 \\ \hline \cellcolor{yellow}4 & 9 & 11 & \cellcolor{green}12 \\ \hline \end{array}$$

Notice that the yellow cells from the nested Young tableau and the green cell of newly added 12 together form a 3×2 rectangle:

$$\begin{array}{|c|c|}\hline \cellcolor{yellow}1 & \cellcolor{yellow}2 \\ \hline \cellcolor{yellow}3 & \cellcolor{yellow}5 \\ \hline \cellcolor{yellow}4 & \cellcolor{green}12 \\ \hline \end{array}$$

Continue the promotions, we notice the same pattern in the resulted Young tableaux:

$$\begin{array}{|c|c|c|c|}\hline \cellcolor{yellow}1 & \cellcolor{yellow}4 & 5 & 7 \\ \hline \cellcolor{yellow}2 & 6 & 9 & \cellcolor{green}11 \\ \hline \cellcolor{yellow}3 & 8 & 10 & \cellcolor{green}12 \\ \hline \end{array}\text{, where } \begin{array}{|c|c|}\hline \cellcolor{yellow}1 & \cellcolor{yellow}4 \\ \hline \cellcolor{yellow}2 & \cellcolor{green}11 \\ \hline \cellcolor{yellow}3 & \cellcolor{green}12 \\ \hline \end{array};$$

$$\begin{array}{|c|c|c|c|}\hline \cellcolor{yellow}1 & \cellcolor{yellow}3 & 4 & 6 \\ \hline \cellcolor{yellow}2 & 5 & 8 & \cellcolor{green}10 \\ \hline 7 & 9 & \cellcolor{green}11 & \cellcolor{green}12 \\ \hline \end{array}\text{, where } \begin{array}{|c|c|}\hline \cellcolor{yellow}1 & \cellcolor{yellow}3 \\ \hline \cellcolor{yellow}2 & \cellcolor{green}10 \\ \hline \cellcolor{green}11 & \cellcolor{green}12 \\ \hline \end{array}.$$

2.1 Proof

An alternative definition of Jeu-de-taquin promotion Jeu-de-taquin promotion can also obtained by an alternative procedure:

- start from the cell at upper-left corner; compare the values below and to the right of the current cell, move along the cells with smaller values until reaching the cell at bottom-right corner; call the path with visited cells p.
- move each value of visited cell on p to its previously visited cell,
- reduce all values by 1,
- and finally add n to the newly-empty spot.

An illustration of step-by-step promotion on the previous would be (where red color marks path p):

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 7 & 9 \\ \hline 2 & 4 & 8 & 11 \\ \hline 5 & 6 & 10 & 12 \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline \cellcolor{red}1 & 3 & 7 & 9 \\ \hline \cellcolor{red}2 & \cellcolor{red}4 & 8 & 11 \\ \hline 5 & \cellcolor{red}6 & \cellcolor{red}10 & \cellcolor{red}12 \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline \cellcolor{red}2 & 3 & 7 & 9 \\ \hline \cellcolor{red}4 & \cellcolor{red}6 & 8 & 11 \\ \hline 5 & \cellcolor{red}10 & \cellcolor{red}12 & \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline \cellcolor{red}2 & 3 & 7 & 9 \\ \hline \cellcolor{red}4 & \cellcolor{red}6 & 8 & 11 \\ \hline 5 & \cellcolor{red}10 & \cellcolor{red}12 & \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 8 \\ \hline 3 & 5 & 7 & 10 \\ \hline 4 & 9 & 11 & \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 8 \\ \hline 3 & 5 & 7 & 10 \\ \hline 4 & 9 & 11 & 12 \\ \hline \end{array}$$

LEMMA 2.1. *Consider the positions of nested-Young tableau cells. In a promotion on any Young tableau Y promoted from the original one, cell position that changes from having a nested-Young tableau cell to not must be one that has non-nested-Young tableau cells both on its right and bottom in Y. For example in*

1	3	7	9
2	4	8	11
5	6	10	12

, the cell position that can change from having a nested-Young tableau cell to not is $(3, 2)$ on row 3 column 2.

PROOF. We give a proof by contradiction. Consider a cell position a that has nested-Young tableau cells on its right or bottom in Y:

a	b
c	

, where at least one of b, c is a nested-Young tableau cell. Without loss of generality, suppose b is a nested-Young tableau cell. It is obvious that nested-Young tableau cells always contain the smallest integers in Y, so $b < c$. Then, the position a will remain as a nested-Young tableau cell after promotion as b slides to it. Therefore, the position must have all non-nested-Young tableau cells on its right or bottom for it to change from having a nested-Young tableau cell to not. □

LEMMA 2.2. *In a promotion on any Young tableau Y promoted from the original one, one and only one cell position changes from having a nested-Young tableau cell to not, and one and only one cell position changes from not having newly added number to having.*

PROOF. We first show that first part of the lemma. Since path p goes from the top-left corner, which is always a nested-Young tableau cell, to the bottom-right corner, which is always not a nested-Young tableau cell assuming that the nested Young tableau is smaller than the whole, there is at least one edge e in path p that goes from a nested-Young tableau cell i to a not-nested-Young tableau cell j. On this edge, we will slide cell j to position of i and make it a not-nested-Young tableau cell. Thus, there is at least one cell position changes from having a nested-Young tableau cell to not. We also argue that there is only one since path p only has one such edge e. Numbers on p is strictly increasing as right and bottom of a number in standard Young tableau is bigger than itself. So the first time that p reaches a number that is larger than all nested-Young tableau numbers, the number will only get larger later which cannot be nested-Young tableau numbers.

Similarly, we can show that one and only one cell position changes from not having newly added number to having since p contains one and only one edge e that goes from a nested-Young tableau cell to a not-nested-Young tableau cell. □

LEMMA 2.3. *In a promotion, no cell position changes from not having a nested-Young tableau cell to having, and one and only one cell position changes from not having newly added number to having.*

THEOREM 2.4. *In a promotion on any Young tableau Y promoted from the original one, the cell position that changes from not having newly added number to having is always on the same row of the cell position that changes from having a nested-Young tableau cell to not.*

For example, notice that in the following Young tableaux, path p moves along a single row once reaching the last nested-Young tableau cell, which is also the one that will change to non-nested Young tableau cell after the promotion.

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 7 & 9 \\ \hline 2 & 4 & 8 & 11 \\ \hline 5 & 6 & 10 & 12 \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 8 \\ \hline 3 & 5 & 7 & 10 \\ \hline 4 & 9 & 11 & 12 \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline 1 & 4 & 5 & 7 \\ \hline 2 & 6 & 9 & 11 \\ \hline 3 & 8 & 10 & 12 \\ \hline \end{array}$$

Proof. We will present a proof by induction. We claim that in a promotion, only the row of the cell position that changes from having a nested-Young tableau cell to not slides to the left and all other cells remain in place.

As the base case, consider a $m \times k$ Young tableau with a nested $m \times j$ Young tableau.

$$m\left\{\underbrace{\begin{array}{|c|c|c|c|c|}\multicolumn{3}{c}{\overbrace{\hphantom{\cdots\ \cdots\ \cdots}}^{j}} & \multicolumn{2}{c}{} \\ \hline 1 & \cdots & \cdots & \cdots & \cdots \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \cdots & \cdots & mj & \cdots & n \\ \hline \end{array}}_{k}\right.$$

Argue that the promotion will only slide the row of the cell position that changes from having a nested-Young tableau cell to not. By Lemma 2.1, the only possible cell position to change is the one in row m column j. At (m, j), we determine the next step of p by comparing the number on its right and bottom. Since numbers outside of teh tableau are considered to be ∞, p steps to the right on cell $(m, j + 1)$. With the same logic, p goes along the bottom row and reach the bottom-right corner (m, k). So the statement is true in this case.

Then, for inductive steps, suppose that in every one of the first i promotions, only the row of the cell position that changes from having a nested-Young tableau cell to not slides. Consider the cell position t that changes from having a nested-Young tableau cell to not.

$$\begin{array}{|c|c|c|c|c|}\hline \cdots & t & a_1 & a_2 & \cdots \\ \hline \cdots & b_1 & b_2 & \cdots & \multicolumn{1}{c}{} \\ \cline{1-4} \end{array}$$

□

Since in every previous promotion, only one row slides to the left as a whole,