Expectation and Moments

Statistical Methods in Political Research I

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Expectation

- Readings: **BH**, Ch. 4 and p. 200; **DS**, Ch. 4; **W**, Ch. 3; **CB**, 2.2
- Summary of r.v. X:
  - What is the gain you expect from a lottery?
  - What is the number of Dems you expect in a sample?
  - What is the lifetime income you expect from an academic job?
- Expectation or expected value of X: Weighted average of X where the weights are the probability measure of events $X = x$
- For a discrete r.v. X,
$$\mathbb{E}[X] = \sum_x x f_X(x)$$
- For a continuous r.v. X,
$$\mathbb{E}[X] = \int_x x f_X(x) dx$$
- $X \sim \text{Bern}(p) \Rightarrow \mathbb{E}[X] = p$
- $X \sim \text{Unif}(0, 1) \Rightarrow \mathbb{E}[X] = 1/2$

Existence of Expectation

- Expectation does not always exist
- Existence of expectation: $\mathbb{E}[X]$ exists if and only if $\mathbb{E}[X_-] < \infty$ or $\mathbb{E}[X_+] < \infty$, where $X_- \equiv -\min\{X, 0\}$ and $X_+ \equiv \max\{X, 0\}$.
  1. $\mathbb{E}[X_-] < \infty$ and $\mathbb{E}[X_+] < \infty$: $-\infty < \mathbb{E}[X] < \infty$
  2. $\mathbb{E}[X_-] < \infty$ and $\mathbb{E}[X_-] = \infty$: $\mathbb{E}[X] = -\infty$
  3. $\mathbb{E}[X_+] = \infty$ and $\mathbb{E}[X_-] < \infty$: $\mathbb{E}[X] = \infty$
  4. $\mathbb{E}[X_+] = \infty$ and $\mathbb{E}[X_-] = \infty$: $\mathbb{E}[X]$ does not exist
- Expectation can be infinity, but its sign should be well defined

- X follows the standard Cauchy distribution:

$$f_X(x) = \frac{1}{\pi(1 + x^2)} \text{ for } -\infty < x < \infty$$

  - Valid p.d.f: $\int_{-\infty}^{\infty} f(x)dx = [\tan^{-1}(x)/\pi]_{-\infty}^{\infty} = \{\pi/2 - (-\pi/2)\}/\pi = 1$
  - $\int_0^{\infty} xf(x)dx = [\log(1 + x^2)/2]_0^{\infty} = \infty$
  - Similarly, $\int_{-\infty}^{0} -xf(x)dx = [\log(1 + x^2)/2]_{-\infty}^{0} = \infty$
  - Expectation does not exist for the Cauchy distribution

Indicator and Linearity

- Expectation of Indicator: For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let A be an event and define r.v. $1_A \equiv 1\{\omega \in A\}$. Then, $\mathbb{E}[1_A] = \mathbb{P}(A)$
- Corollary: Let $C \subset \mathbb{R}$. For r.v. X, define $1_C(X) \equiv 1\{X \in C\}$. Then, $\mathbb{E}[1_C(X)] = \mathbb{P}(X \in C)$
- Dice roll: X is the number on the face
  - Let $C \equiv \{2, 3, 4, 5\}$
  - $\mathbb{P}(X \in C) = 2/3$
  - $\mathbb{E}[1_C(X)] = 1 \times (4 \times 1/6) + 0 \times (2 \times 1/6) = 2/3$

- Linearity: Let X_1, X_2 be r.v.s. Then,

$$\mathbb{E}[aX_1 + bX_2 + c] = a\mathbb{E}[X_1] + b\mathbb{E}[X_2] + c$$

- Binomial expectation:
  - By definition of expactation,

$$\mathbb{E}[X] = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{n-x} = np$$

  - Binom(n, p) is the distribution of $\sum_{i=1}^{n} X_i$ where $X_i \overset{\text{i.i.d.}}{\sim}$ Bern(p)

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = np$$

Random Vectors

- Expectation of random vector: For a random vector $\mathbf{X}$, its expectation is defined as
$$\mathbb{E}[\mathbf{X}] \equiv (\mathbb{E}[X_1], \dots, \mathbb{E}[X_n])$$
where the expectation of X_i is over its marginal distribution

- Multinomial distribution $\mathbf{X} \sim \text{Multi}(n, \mathbf{p})$:
  1. Marginal distribution of X_1 is Binomial:
$$\begin{aligned}\mathbb{P}(X_1 = x_1) &= \sum_{x_2 \dots x_K} \frac{n!}{x_1! \dots x_K!} p_1^{x_1} \dots p_K^{x_K} \\ &= \frac{n!}{x_1!(n-x_1)!} p_1^{x_1} \sum_{x_2 \dots x_K} \frac{(n-x_1)!}{x_2! \dots x_K!} p_2^{x_2} \dots p_K^{x_K} \\ &= \frac{n!}{x_1!(n-x_1)!} p_1^{x_1} (1-p_1)^{n-x_1}\end{aligned}$$
  2. $\mathbb{E}[\mathbf{X}] = (np_1, \dots, np_K)$

Functions and Product

- Expectation of functions of r.v.: Let X be a r.v. and $g : \mathbb{R} \to \mathbb{R}$. Then,
$$\mathbb{E}[g(X)] = \begin{cases} \sum_x g(x) f_X(x) & (X \text{ discrete}) \\ \int_x g(x) f_X(x) dx & (X \text{ continuous}) \end{cases}$$
Proof. Directly follows from the fact that for any $C \subset \mathbb{R}$, $\mathbb{P}(g(X) \in C) = \mathbb{P}(X \in \{x \in \mathbb{R} \mid g(x) \in C\})$
- X follows a Geometric distribution, $X \sim \text{Geom}(p)$:
$$f_X(x) = (1-p)^{x-1} p, \text{ for } x = 1, 2, \ldots$$
  1. St. Petersburg paradox: $g(x) \equiv 2^x \Rightarrow \mathbb{E}[g(X)] = \infty$ if $p = 1/2$
  2. $\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$ in general: $\mathbb{E}[X] = 2$
- Lemma: Let X be a discrete r.v. whose support is the non-negative integers. Then, $\mathbb{E}[X] = \sum_{x=1}^{\infty} \mathbb{P}(X \geq x)$

- Product of independent r.v.s: Let $X_i, i = 1, \ldots, n$ are independent. Then, $\mathbb{E}[\prod_{i=1}^n X_n] = \prod_{i=1}^n \mathbb{E}[X_i]$
- $X_i \overset{\text{i.i.d.}}{\sim} \text{Bern}(p) \Rightarrow \mathbb{P}(X_1 = 1, \ldots, X_n = 1) = p^n$

Inequalities of Expectation

- If $X_1 \leq X_2$ with probability 1, i.e., $X_1(\omega) \leq X_2(\omega)$ for all $\omega \in \Omega$, then $\mathbb{E}[X_1] \leq \mathbb{E}[X_2]$
- If $a \leq X \leq b$ with probability 1, i.e., $a \leq X(\omega) \leq b$ for all $\omega \in \Omega$, then $a \leq \mathbb{E}[X] \leq b$

- Jensen's inequality: Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a concave (convex) function. Then, for a random vector X, $\mathbb{E}[g(\mathrm{X})] \leq (\geq) g(\mathbb{E}[\mathrm{X}])$
- Concave function: A function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is concave if and only if for every $a \in (0,1)$,
$$g\big(a\mathrm{x} + (1-a)\mathrm{y}\big) \geq a g(\mathrm{x}) + (1-a) g(\mathrm{y})$$
for any $x, y \in \mathbb{R}^n$
- Logarithm is common in statistics: $\mathbb{E}[\log(X)] < \log\big(\mathbb{E}[X]\big)$

Moments and Variance

- Moments of an r.v.: For an r.v. X and a positive integer k, $\mathbb{E}[X^k]$ is called the kth *moment* of X
- Existence of moments: If $\mathbb{E}[X^k]$ exists, $\mathbb{E}[X^l]$ exists for any $l < k$
- Central moments: $\mathbb{E}[(X - \mathbb{E}[X])^k]$ is called the kth *central moment* or the kth *moment of X about the mean*
- If the kth moment exists, the lth central moment exists for $l \leq k$

- Variance: The second centeral moment of X is called the *variance* of X, denoted by $\mathbb{V}(X) \equiv \mathbb{E}[(X - \mathbb{E}[X])^2]$
- Variance and moments: $\mathbb{V}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
- $\mathbb{V}(X) \geq 0$, with equality if and only if $\mathbb{P}(X = c) = 1$ for some c
- $Y = aX + b \Rightarrow \mathbb{V}(Y) = a^2\mathbb{V}(X)$
- Variance of Bern(p): $\mathbb{E}[X^2] - (\mathbb{E}[X])^2 = p - p^2 = p(1 - p)$

Covariance

- Covariance: For r.v.s X_1 and X_2, the *covariance* of X_1 and X_2, denoted by $\text{Cov}(X_1, X_2)$, is defined as:
$$\text{Cov}(X_1, X_2) \equiv \mathbb{E}[(X_1 - \mathbb{E}[X_1])(X_2 - \mathbb{E}[X_2])]$$
- Analogously to the variance, $\text{Cov}(X_1, X_2) = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2]$
- Correlation: The *correlation* of X_1 and X_2, denoted by $\rho(X_1, X_2)$, is defined as:
$$\rho(X_1, X_2) \equiv \frac{\text{Cov}(X_1, X_2)}{\sqrt{\mathbb{V}(X_1)\mathbb{V}(X_2)}}$$
- $\rho(X_1, X_2) = \text{Cov}\left((X_1 - \mathbb{E}[X_1])/\sqrt{\mathbb{V}(X_1)}, (X_2 - \mathbb{E}[X_2])/\sqrt{\mathbb{V}(X_2)}\right)$
- Covariance depends on the scale of r.v.s, but $|\rho(X_1, X_2)| \leq 1$

- X_1 and X_2 are *uncorrelated* if and only if $\text{Cov}(X_1, X_2) = 0$
- X_1 and X_2 are independent $\Rightarrow$ X_1 and X_2 are uncorrelated
- The converse does not necessarily hold:
  - $U \sim \text{Unif}(0, 1)$, $X_1 = \cos 2\pi U$ and $X_2 = \sin 2\pi U$
  - Clearly, X_1 and X_2 are not independent, but $\text{Cov}(X_1, X_2) = 0$
- Covariance and correlation indicate *linear* relationship b/w r.v.s

Variance-Covariance Matrix

- Trivially, $\mathbb{V}(X) = \mathbb{E}[(X - \mathbb{E}[X])(X - \mathbb{E}[X])] = \text{Cov}(X, X)$
- Variance-covariance Matrix: For r.v.s $X_1, \ldots, X_n$, we define the *(variance-)covariance matrix*, denoted by $\mathbb{V}(\mathrm{X})$ or Σ_{X}, as

$$\Sigma_{\mathrm{X}} \equiv \begin{pmatrix} \mathbb{V}(X_1) & \text{Cov}(X_1, X_2) & \cdots & \text{Cov}(X_1, X_n) \\ \text{Cov}(X_2, X_1) & \mathbb{V}(X_2) & \cdots & \text{Cov}(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(X_n, X_1) & \text{Cov}(X_n, X_2) & \cdots & \mathbb{V}(X_n) \end{pmatrix}$$

- In vector notation, $\Sigma_{\mathrm{X}} = \mathbb{E}[(\mathrm{X} - \mathbb{E}[\mathrm{X}])(\mathrm{X} - \mathbb{E}[\mathrm{X}])^\top]$
- Σ_{X} is positive semi-definite
- Σ_{X} is positive definite unless some r.v.s are constant $\Rightarrow$ invertible

- If $X_1, \ldots, X_n$ are i.i.d., Σ_{X} is diagonal
- $\mathbb{V}\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i^2 \mathbb{V}(X_i) + 2\sum_{i<j} a_i a_j \text{Cov}(X_i, X_j)$
- X_is are uncorrelated, $\mathbb{V}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathbb{V}(X_i)$
- If $X_1, \ldots, X_n$ are i.i.d., $\mathbb{V}\left(\sum_{i=1}^n X_i\right) = \text{tr}(\Sigma_{\mathrm{X}})$

Conditional Expectation

- Summarizing prediction of Y using X:
  - Given $X = x$, Y is predicted by its conditional *distribution*
  - Conditional *expectation* is used to summarize the prediction
- Conditional expectation: The *conditional expectation* of Y given X, denoted by $\mathbb{E}[Y \mid X]$, is the expectation of the conditional distribution of Y given X
- Conditional moments are defined with expectation replaced by conditional expectation, e.g. $\mathbb{V}(Y \mid X) \equiv \mathbb{E}[(Y - \mathbb{E}[Y \mid X])^2 \mid X]$
- *Conditional expectation is an r.v.*:
  - $\mathbb{E}[Y \mid X]$ is a function of X
  - $\mathbb{E}[Y \mid X]$ has a distribution defined by F_X
- *Conditional expectation is expectation*:
  - For any fixed x_1, $\mathbb{E}[Y \mid X = x_1]$ is expectation
  - All the properties of expectation hold for $\mathbb{E}[Y \mid X = x_1]$
- Uniform-Binomial example:

$$\mathbb{E}[X_1 \mid X_2] = \int_0^1 x_1 \frac{x_1^{X_2}(1 - x_1)^{n - X_2}}{B(X_2 + 1, n - X_2 + 1)} dx_1 = \frac{X_2 + 1}{n + 2}$$

Properties of Conditional Expectation

- Law of iterated expectations: Let X and Y be r.v.s. Then,
$$\mathbb{E}\left[\mathbb{E}[g(X,Y) \mid X]\right] = \mathbb{E}[g(X,Y)]$$
for any function g.
- Law of total variance: Let X and Y be r.v.s. Then,
$$\mathbb{E}\left[\mathbb{V}(Y \mid X)\right] + \mathbb{V}\left(\mathbb{E}[Y \mid X]\right) = \mathbb{V}(Y)$$
- Uniform-Binomial example:
$$\mathbb{E}[X_2] = \mathbb{E}[nX_1] = \frac{n}{2}$$
$$\mathbb{V}[X_2] = \mathbb{E}[nX_1(1 - X_1)] + \mathbb{V}(nX_1) = \frac{n}{6} + \frac{n^2}{12}$$
- Minimization of expected squared distance (regression): Conditional expectation is the “best” predictor in the sense that
$$\operatorname*{argmin}_c \mathbb{E}\left[(Y-c)^2 \mid X\right] = \mathbb{E}[Y \mid X]$$

Proof.
$$\mathbb{E}\left[(Y-c)^2 \mid X\right] = \mathbb{E}\left[(Y - \mathbb{E}[Y \mid X])^2 \mid X\right] + \underbrace{(\mathbb{E}[Y \mid X] - c)^2}_{=0 \text{ iff } c = \mathbb{E}[Y|X]}$$

Standard Gaussian Distribution

- X follows the standard multivariate Gaussian distribution:
  - Joint p.d.f.: For a vector of real numbers $\mathrm{x} \equiv x_1 \dots, x_K$,

$$f_{\mathrm{X}}(\mathrm{x}) = \frac{1}{(2\pi)^{K/2}} e^{-\frac{1}{2}\mathrm{x}^\top \mathrm{x}} = \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_k^2}{2}}$$

  - Denoted by: $\mathrm{X} \sim \mathcal{N}(0, \mathbf{I}_K)$
  - $X_1, \dots, X_K$ are independent
  - $K = 1$: The standard Gaussian (Normal) distribution $\mathcal{N}(0, 1)$

- Box-Muller Transfromation: Let U_1 and U_2 be independent uniform r.v.s. Define

$$X_1 = \sqrt{-2\log U_1} \cos(2\pi U_2),$$
$$X_2 = \sqrt{-2\log U_1} \sin(2\pi U_2).$$

  Then,

$$\mathrm{X} \sim \mathcal{N}(0, \mathbf{I}_2)$$

- Random number generator for the Gaussian distributions

Change of Variables

- **BH**, 8.1; **DS**, p. 172–3, 182–6
- Change of variables: Let X be a continuous random vector of length K and $\mathrm{Y} \equiv g(\mathrm{X})$ where $g : \mathbb{R}^K \to \mathbb{R}^K$ is one-to-one and differentiable. Then, the p.d.f. of Y is

$$f_{\mathrm{Y}}(\mathrm{y}) = f_{\mathrm{X}}\left(g^{-1}(\mathrm{y})\right) |\det\left(\mathbf{J}(\mathrm{y})\right)|$$

where $g^{-1} : \mathbb{R}^K \to \mathbb{R}^K$ is the inverse function of g.

$\mathbf{J}(\cdot)$ is the Jacobian (matrix) of g^{-1} defined as:

$$\mathbf{J}(\mathrm{y}) = \begin{pmatrix} \frac{\partial g_1^{-1}}{\partial y_1}(\mathrm{y}) & \cdots & \frac{\partial g_1^{-1}}{\partial y_K}(\mathrm{y}) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_K^{-1}}{\partial y_1}(\mathrm{y}) & \cdots & \frac{\partial g_K^{-1}}{\partial y_K}(\mathrm{y}) \end{pmatrix}$$

where $g_i^{-1}(\mathrm{y})$ is the ith element of $g^{-1}(\mathrm{y})$.

Univariate Change of Variables

- Univariate change of variables: Let X be a continuous r.v. and $Y \equiv g(X)$ where $g : \mathbb{R} \to \mathbb{R}$ is one-to-one and differentiable. Then, the p.d.f. of Y is

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{dg^{-1}}{dy}(y)\right|$$

- **Proof.**
  1. If $g(x)$ is one-to-one and differentiable, it is either strictly increasing or decreasing.
  2. First, we assume that it is strictly increasing. Then, the c.d.f. of Y is

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y) = \mathbb{P}\left(X \leq g^{-1}(y)\right) = F_X\left(g^{-1}(y)\right)$$

  3. So the p.d.f. of Y is

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X\left(g^{-1}(y)\right) \\ &= f_X\left(g^{-1}(y)\right)\frac{dg^{-1}}{dy}(y) \quad (\because \text{chain rule}) \end{aligned}$$

- **Proof, cont.**
  4. Because g is strictly increasing, we have $\frac{dg^{-1}}{dy}(y) > 0$ so that $\frac{dg^{-1}}{dy}(y) = \left|\frac{dg^{-1}}{dy}(y)\right|$.
  5. Second, we consider the case in which g is strictly decreasing. Then, the c.d.f. of Y is

$$\begin{aligned} F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y) &= \mathbb{P}\left(X \geq g^{-1}(y)\right) \\ &= 1 - \mathbb{P}\left(X \leq g^{-1}(y)\right) = 1 - F_X\left(g^{-1}(y)\right) \end{aligned}$$

     Note that the inequality is flipped because g is strictly decreasing.
  6. So the p.d.f. of Y is

$$\begin{aligned} f_Y(y) = \frac{d}{dy}F_Y(y) &= \frac{d}{dy}\left(1 - F_X\left(g^{-1}(y)\right)\right) \\ &= -f_X\left(g^{-1}(y)\right)\frac{dg^{-1}}{dy}(y) = f_X\left(g^{-1}(y)\right)\left(-\frac{dg^{-1}}{dy}(y)\right) \end{aligned}$$

  7. Because g is strictly decreasing, we have $\frac{dg^{-1}}{dy}(y) < 0$ so that $-\frac{dg^{-1}}{dy}(y) = \left|\frac{dg^{-1}}{dy}(y)\right|$.

Box-Muller Transformation

- Inverse of the Box-Muller transformation:

$$\begin{aligned} X_1 &= \sqrt{-2\log U_1}\cos(2\pi U_2) \\ X_2 &= \sqrt{-2\log U_1}\sin(2\pi U_2) \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} U_1 &= e^{-(X_1^2+X_2^2)/2} \\ U_2 &= \frac{1}{2\pi}\arctan\left(\frac{X_2}{X_1}\right) \end{aligned}$$

- The determinant of the Jacobian is:

$$\begin{aligned} \det\left(\mathbf{J}(\mathrm{X})\right) &= \det\begin{pmatrix} \frac{\partial U_1}{\partial X_1} & \frac{\partial U_1}{\partial X_2} \\ \frac{\partial U_2}{\partial X_1} & \frac{\partial U_2}{\partial X_2} \end{pmatrix} \\ &= \det\begin{pmatrix} -X_1 e^{-(X_1^2+X_2^2)/2} & -X_2 e^{-(X_1^2+X_2^2)/2} \\ \frac{-X_2/X_1^2}{2\pi\left(1+(X_2/X_1)^2\right)} & \frac{1/X_1}{2\pi\left(1+(X_2/X_1)^2\right)} \end{pmatrix} \\ &= \frac{-1-X_2^2/X_1^2}{2\pi\left(1+(X_2/X_1)^2\right)} e^{-(X_1^2+X_2^2)/2} \\ &= -\frac{1}{\sqrt{2\pi}} e^{-X_1^2/2} \times \frac{1}{\sqrt{2\pi}} e^{-X_2^2/2} \end{aligned}$$

Linear Transformation of Gaussian

- *Linear transformation*: Let X be a K-dimensional random vector. A linear transformation of X is
$$\mathsf{Y} = \mathsf{a} + \mathbf{A}\mathsf{X}$$
where $\mathbf{A}$ is a matrix with K columns
- Multivariate Gaussian: Let X follow the standard multivariate Gaussian distribution. Then, for a full rank $K \times K$ matrix $\mathbf{A}$ and an K dimensional vector μ, $\mathsf{Y} = \mu + \mathbf{A}\mathsf{X}$ has a p.d.f.:
$$f_{\mathsf{Y}}(\mathsf{y}) = \frac{1}{(2\pi)^{K/2} \det(\Sigma)^{1/2}} e^{-\frac{1}{2}(\mathsf{y}-\mu)^\top \Sigma^{-1}(\mathsf{y}-\mu)}$$
where $\Sigma = \mathbf{A}\mathbf{A}^\top$
Proof.
$$\mathsf{X} = \mathbf{A}^{-1}(\mathsf{Y} - \mu)$$
$$J = \mathbf{A}^{-1}$$
- $\mathbb{E}[\mathsf{Y}] = \mu$ and $\mathbb{V}(\mathsf{Y}) = \Sigma$
- Uncorrelated $\Leftrightarrow$ (pairwise) independent

Conditional and Marginal of Gaussian

- Let $(X_1, X_2)^\top \sim \mathcal{N}(\mu, \Sigma)$. The joint p.d.f. is:

$$f_{(X_1,X_2)}(x_1,x_2) = \frac{e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right\}}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

$$= \underbrace{\frac{e^{-\frac{1}{2(1-\rho^2)\sigma_2^2}\left(x_2-\mu_2-\rho\sigma_2\frac{x_1-\mu_1}{\sigma_1}\right)^2}}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}}}_{f_{X_2|X_1}(x_2|x_1)} \underbrace{\frac{e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}}{\sqrt{2\pi}\sigma_1}}_{f_{X_1}(x_1)}$$

- $X_1 \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$, $X_2 \mid X_1 \sim \mathcal{N}\left(\mu_2 + \frac{\rho\sigma_2}{\sigma_1}(X_1-\mu_1), (1-\rho^2)\sigma_2^2\right)$
- Both marginal and conditional distributions are Gaussian

- Regression $\mathbb{E}[X_2 \mid X_1] = \mu_2 + \frac{\rho\sigma_2}{\sigma_1}(X_1-\mu_1)$ is linear in X_1
  $\rightsquigarrow$ linear regression

Moment Generating Function

- Moment generating function: Let X be an r.v. The *moment generating function (m.g.f.)* of X, denoted by $M_X(t)$, is defined as
$$M_X(t) = \mathbb{E}[e^{tX}]$$
if $\mathbb{E}[e^{tX}]$ exists for all $t \in (-s, s)$ for some $s > 0$.

- If m.g.f. is given, higher order moments can be easily computed:
$$\frac{d^k}{dt^k} M_X(t)\bigg|_{t=0} = \mathbb{E}[X^k e^{0X}] = \mathbb{E}[X^k], \text{ for } k = 1, 2, \ldots$$

- If $X_1, \ldots, X_n$ are independent, the m.g.f. of $Y \equiv \sum_{i=1}^n X_i$ is
$$M_Y(t) = \mathbb{E}[e^{t\sum_{i=1}^n X_i}] = \mathbb{E}[\prod_{i=1}^n e^{tX_i}] = \prod_{i=1}^n \mathbb{E}[e^{tX_i}] = \prod_{i=1}^n M_{X_i}(t)$$

- M.g.f. uniquely determines the distribution: If r.v.s X_1 and X_2 have m.g.f.s and $M_{X_1}(t) = M_{X_2}(t)$ for all $t \in (-a, a)$ for some $a > 0$, then c.d.f.s $F_{X_1}(x) = F_{X_2}(x)$ for all x

M.g.f. of Gamma Distributions

- Square of a Gaussian r.v. follows a *Gamma distribution* (PS9)
- X follows a Gamma distribution:
  - P.d.f.:
$$f_X(x) = \frac{\beta^a}{\Gamma(a)} x^{a-1} e^{-\beta x} \quad \text{for} \quad x > 0$$
  - Parameters: Shape $a > 0$ and rate $\beta > 0$
  - Alternative parameterization: Shape $a > 0$ and scale $\theta = 1/\beta$
  - Gamma function: $\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$
  - Denoted by: $X \sim \text{Ga}(a, \beta)$
- M.g.f. of the Gamma distribution:
$$\mathbb{E}[e^{tX}] = \frac{\beta^a}{\Gamma(a)} \int_0^\infty x^{a-1} e^{-(\beta-t)x} dx = \frac{\beta^a}{\Gamma(a)} \frac{\Gamma(a)}{(\beta-t)^a} = \left(\frac{\beta}{\beta-t}\right)^a$$
- Expectation and variance: $\mathbb{E}[X] = a/\beta, \mathbb{V}[X] = a/\beta^2$
- Sum of independent Gamma r.v.s $X_i \sim \text{Ga}(a_i, \beta), i = 1, \ldots, n$:
$$M_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^n M_{X_i}(t) = \prod_{i=1}^n \left(\frac{\beta}{\beta-t}\right)^{a_i} = \left(\frac{\beta}{\beta-t}\right)^{\sum_{i=1}^n a_i}$$
$\Rightarrow \sum_{i=1}^n X_i \sim \text{Ga}(\sum_{i=1}^n a_i, \beta)$

Sample Moments

- Let $X_1, \ldots, X_n$ is a random sample from a distribution F_X
- In other words, $X_1, \ldots, X_n$ are *data*
- Sample moments: Let $X_1, \ldots, X_n$ be i.i.d. r.v.s. The kth *sample moment*, denoted by M_k, is defined as

$$M_k \equiv \frac{1}{n} \sum_{i=1}^{n} X_i^k$$

- Why this is important: We never observe F_X, hence neither $\mathbb{E}[X^k]$
- We use M_k as an *estimator*–a function of r.v.s, therefore r.v.
- Mean and variance of M_k for any F_X:

$$\mathbb{E}[M_k] = \mathbb{E}[X^k], \quad \mathbb{V}(M_k) = \frac{\mathbb{V}(X^k)}{n}$$

- In particular, for sample mean $\bar{X} \equiv \sum_{i=1}^{n} X_i/n$,

$$\mathbb{E}[\bar{X}] = \mathbb{E}[X], \quad \mathbb{V}(\bar{X}) = \frac{\mathbb{V}(X)}{n}$$

- If we specify F_X, we can derive the distribution of M_k

Sample Mean of Gaussian R.v.

- We want to find the distribution of the sum of independent r.v.s $\rightsquigarrow$ use m.g.f.!
- M.g.f. of the Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$:

$$\begin{aligned}
\mathbb{E}[e^{tX}] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2 + tx} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\left\{x-(\mu+\sigma^2 t)\right\}^2 + \mu t + \frac{1}{2}\sigma^2 t^2} dx \\
&= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\left\{x-(\mu+\sigma^2 t)\right\}^2} dx = e^{\mu t + \frac{1}{2}\sigma^2 t^2}
\end{aligned}$$

- M.g.f. of the sample mean:

$$\mathbb{E}[e^{t\bar{X}}] = \prod_{i=1}^{n} \mathbb{E}[e^{\frac{t}{n}X}] = e^{n\left\{\mu \frac{t}{n} + \frac{1}{2}\sigma^2 \left(\frac{t}{n}\right)^2\right\}} = e^{\mu t + \frac{1}{2}\left(\frac{\sigma^2}{n}\right) t^2}$$

$$\Rightarrow \bar{X} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$