Performance-Guaranteed Solutions for Multi-Agent Optimal Coverage Problems using Submodularity, Curvature, and Greedy Algorithms

Shirantha Welikala and Christos G. Cassandras

***Abstract*— We consider a class of multi-agent optimal coverage problems where the goal is to determine the optimal placement for a group of agents in a given mission space such that they maximize a joint "coverage" objective. This class of problems is extremely challenging due to the non-convex nature of the mission space and of the coverage objective. Motivated by this, we propose to use a *greedy algorithm* as a means of getting feasible coverage solutions efficiently. Even though such greedy solutions are suboptimal, the *submodularity* (diminishing returns) property of the coverage objective can be exploited to provide performance bound guarantees - not only for the greedy solutions but also for any subsequently improved solutions. Moreover, we show that improved performance bound guarantees (beyond the standard performance bound (1-1/e)) can be established using various *curvature measures* that further characterize the considered coverage problem. In particular, we provide a brief review of all existing popular curvature measures found in the submodular maximization literature, including a recent curvature measure that we proposed, and discuss in detail their applicability, practicality, and effectiveness in the context of optimal coverage problems. Moreover, we characterize the dependence of the effectiveness of different curvature measures (in providing improved performance bound guarantees) on the agent sensing capabilities. Finally, we provide several numerical results to support our findings and propose several potential future research directions.**

I. INTRODUCTION

Introduction paragraphs ...

Organization: The paper is organized as follows. We introduce the considered class of multi-agent optimal coverage problem in Section II. Some notations, preliminary concepts, and the proposed greedy solution are reported in Section III. Different curvature measures found in the literature, along with discussions on their applicability, practicality, and effectiveness in the context of optimal coverage problems, are provided in Section IV. Interesting observations, advantages, limitations, and potential future research directions are summarized in Section V. Several numerical results obtained from different multi-agent optimal coverage problems [10] are reported in Section VI before concluding the paper in Section VII.

Notation: The sets of real and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. $\mathbb{R}_{\geq 0}$ represents the set of non-negative real numbers, $\mathbb{R}^n$ denotes the set of n-dimensional real (column) vectors, $\mathbb{N}_n \triangleq \{1,2,\ldots,n\}$, $\mathbb{N}_n^0 \triangleq \mathbb{N}_n \cup \{0\}$, and $[a,b] \triangleq \{x : x \in \mathbb{R}, a \leq x \leq b\}$. $\|\cdot\|$ represents the Euclidean norm, $|\cdot|$ denotes the scalar absolute value or set cardinality (based on the type of the argument), $\lfloor \cdot \rfloor$ denotes the floor operator, and $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function. Given two sets A and B, the set subtraction operator is denoted as $A - B = A \backslash B = A \cap B^c$. 2^X denotes the power set of a set X and $\emptyset$ is the empty set.

II. MULTI-AGENT OPTIMAL COVERAGE PROBLEM

We begin by providing the details of the considered multi-agent optimal coverage problem. The goal of the considered coverage problem is to determine an optimal placement for a given team of agents (e.g., sensors, cameras, guards, etc.) in a given mission space that maximizes the probability of detecting events that occur randomly over the mission space.

We model the *mission space* Ω as a convex polytope in $\mathbb{R}^n$ that may also contain h polytopic (and possibly non-convex) *obstacles* $\{\Psi_i : \Psi_i \subset \Omega, i \in \mathbb{N}_h\}$. The characteristics of the obstacles are such that they: (1) limit the agent placement to the feasible space $\Phi \triangleq \Omega \backslash \cup_{i\in\mathbb{N}_h} \Psi_i$, (2) constrain the sensing capabilities of the agents via obstructing their line of sight, and (3) are in areas where no events of interest occur.

To model the likelihood of random *events* occurring over the mission space, an *event density function* $R : \Omega \to \mathbb{R}_{\geq 0}$ is used, where $R(x) = 0, \forall x \notin \Phi$ and $\int_\Omega R(x)dx < \infty$. Note that when no prior information is available on $R(x)$, one can use $R(x) = 1, \forall x \in \Phi$.

To detect these random events, N *agents* are to be placed inside the feasible space Φ. The placement of this team of N agents (i.e., the decision variable) is denoted in a compact form by $s = [s_1, s_2, \ldots, s_N] \in \mathbb{R}^{m\times N}$, where each $s_i, i \in \mathbb{N}_N$ represents an agent placement such that $s_i \in \Phi$.

The ability of an agent to *detect* events is limited by visibility obstruction from obstacles and the agent's sensing capabilities. For an agent placed at $s_i \in \Phi$, its *visibility region* is defined as

$$V(s_i) \triangleq \{x : (qx + (1-q)s_i) \in \Phi, \forall q \in [0,1]\}.$$

In terms of agent *sensing capabilities*, agents are assumed to be homogeneous, where each agent has a finite *sensing radius* $\delta \in \mathbb{R}_{\geq 0}$ and *sensing decay rate* $\lambda \in \mathbb{R}_{\geq 0}$. In particular, the probability of an agent placed at $s_i \in \Phi$ detecting an event occurring at $x \in \Phi$ is described by a *sensing function* defined as

$$p(x, s_i) \triangleq e^{-\lambda \|x - s_i\|} \cdot \mathbf{1}_{\{x \in V(s_i)\}}. \tag{1}$$

Figure **??** illustrates an example visibility region and the corresponding sensing function in a 2-D mission space.

Given the placement s of the team of agents, their combined ability to detect an event occurring at $x \in \Phi$ is characterized by a *detection function* $P(x, s)$. One popular detection function is the *joint detection function* given by

$$P_J(x, s) \triangleq 1 - \prod_{i\in\mathbb{N}_N} (1 - p(x, s_i)), \tag{2}$$

which represents the probability of detection by at least one agent (assuming agents detect events independently from each other). Another widely used detection function is the *max detection function* given by

$$P_M(x,s) \triangleq \max_{i \in \mathbb{N}_N} p(x,s_i), \tag{3}$$

which represents the maximum probability of detection by any agent. The following remark summarizes the pros and cons of using (2) or (3) as the detection function $P(x,s)$.

Remark 1: The joint detection function (2) aggregates the contributions of all agents and, thus, offers a comprehensive view of coverage. This method is suitable for applications that benefit from the collective capabilities of the agent team. However, it can be computationally demanding to compute. On the other hand, the max detection function (3) prioritizes the most effective agent at each point and, thus, offers a conservative estimate of coverage. This method is simpler and may be preferred in critical applications where ensuring the highest detection probability at every point is paramount. However, it can lead to under-utilization of the agent team as it does not fully account for the combined coverage provided by multiple sensors. Ultimately, the choice between these detection functions should consider the coverage application's comprehensiveness and reliability requirements, the nature of the agents, and the available computational resources.

Motivated by the contrasting nature of the joint and max detection functions (2)-(3), in this paper, we consider the detection function

$$P(x,s) \triangleq \theta P_J(x,s) + (1-\theta) P_M(x,s), \tag{4}$$

where $\theta \in [0,1]$ is a predefined weight.

Using the notions of event density function $R(x)$ and detection function $P(x,s)$ introduced above, the considered *coverage function* is defined as

$$H(s) \triangleq \int_{\Omega} R(x)P(x,s)dx. \tag{5}$$

Consequently, the considered multi-agent optimal coverage problem can be stated as

$$s^* = \underset{s:s_i \in \Phi, i \in \mathbb{N}_N}{\arg\max} \ H(s). \tag{6}$$

Continuous Optimization Approach: The optimal coverage problem (6) involves a non-convex feasible space and a non-convex, non-linear, and non-smooth objective function. While the prior is due to the presence of obstacles in the mission space, the latter is due to the nature of the event density, sensing, joint detection, and coverage function forms. Consequently, it is extremely difficult to solve this problem without using: (1) standard global optimization solvers that are computationally expensive, (2) systematic gradient-based solvers that require extensive domain knowledge, or (3) Voronoi partition techniques that require significant limiting assumptions (e.g., convexity [12] and connectivity [4]).

Combinatorial Optimization Approach: Motivated by the aforementioned challenges associated with different continuous optimization approaches, in this paper, we take a combinatorial optimization approach to the formulated the multi-agent optimal coverage problem (6). This requires reformulating (6) as a set function maximization problem.

First, we discretize the feasible space Φ formulating a *ground set* $X = \{x_l : x_l \in \Phi, l \in \mathbb{N}_M\}$. For this discretization, a grid can be placed inside the mission space, and then all the grid points except for those that fall inside obstacles can be considered as the ground set[1].

Second, upon formulating the ground set X, we define a *set variable* $S = \{s_i : i \in \mathbb{N}\}$ to represent the set of selected locations for agent placement. As we are interested in placing only N agents strictly in locations selected from the ground set X, we need to constrain this set variable S such that $S \in \mathscr{I}^N \triangleq \{Y : Y \subseteq X, |Y| \leq N\}$. It is worth noting that a set system of the form $(X, \mathscr{I}^N)$ is known as a *uniform matroid* of rank N, and a set constraint of the form $S \in \mathscr{I}^N$ is known as a uniform matroid constraint of rank N.

To represent the coverage function value of the agent placement defined by the set variable S, using the coverage function (5), we next define a *set coverage function* as:

$$H(S) \triangleq \int_{\Omega} R(x)P(x,S)dx, \tag{7}$$

where $P(x,S)$ represents a *set detection function*. Inspired by (2)-(4), this set detection function $P(x,S)$ is selected as:

$$P(x,S) \triangleq \theta P_J(x,S) + (1-\theta) P_M(x,S) \tag{8}$$

where

$$P_J(x,S) \triangleq 1 - \prod_{s_i \in S}(1 - p(x,s_i)), \tag{9}$$

$$P_M(x,S) \triangleq \max_{s_i \in S} p(x,s_i). \tag{10}$$

Finally, we restate the original multi-agent optimal coverage problem (6) as a set function maximization problem:

$$S^* = \underset{S \in \mathscr{I}^N}{\arg\max} \ H(S). \tag{11}$$

Since the size of the set variable search space of the formulated set function maximization problem (11) is combinatorial (in particular $|\mathscr{I}^N| = \binom{M}{N} = \frac{M!}{(M-N)!N!}$), obtaining an optimal solution (i.e., S^*) for it is impossible without significant simplifying assumptions. Therefore the overall goal is to obtain a candidate solution for (11) (say S^G) in an efficient manner with some guarantees on its coverage performance $H(S^G)$ with respect to optimal coverage performance $H(S^*)$.

One obvious approach to obtaining such a candidate solution efficiently is via a vanilla *greedy algorithm* as given

[1]If obstacle and/or grid resolution are large (compared to the mission space dimensions), the resulting grid-based ground set needs to be further optimized to ensure uniformity over feasible space. This can be achieved by assuming all points in the obtained ground set are occupied by a set of "virtual" agents and then executing a simplified multi-agent optimal coverage solver. The resulting virtual agent locations can then be treated as the ground set.

in Alg. 1. Note that it uses the *marginal coverage function* defined as

$$\Delta H(y|S^{i-1}) \triangleq H(S^{i-1}\cup\{y\}) - H(S^{i-1}) \tag{12}$$

to iteratively determine the optimal agent placements until N such agent placements have been chosen. Note that, in (12), we require $S^{i-1}\in\mathscr{I}^N$ and $S^{i-1}\cup\{y\}\in\mathscr{I}^N$, and clearly, $y\in X\backslash S^{i-1}$ (to ensure $\Delta H(y|S^{i-1})>0$). While the latter condition means that no two agents are allowed at the same place in the mission space, if necessary, by appropriately defining the ground set X, we can relax such placement constraints.

Let us define the notion of *marginal detection function* as

$$\Delta P(x,y|S^{i-1}) = P(x,S^{i-1}\cup\{y\}) - P(x,S^{i-1}). \tag{13}$$

This definition is motivated by the linear relationship (7) between the set coverage function $H(S)$ and the set detection function $P(x,S)$ with respect to the set argument S, because a similar linear relationship exists between the corresponding marginal functions $\Delta H(y|S)$ (12) and $\Delta P(x,y|S)$ (13). In the sequel, we exploit these linear relationships to conclude certain set function properties of $H(S)$ using those of $P(x,S)$ and $\Delta P(x,y|S)$.

Finally, we point out that, the notation $S^i \triangleq \{s^1,s^2,\ldots,s^i\}$ (with $S^0=\emptyset$ and $S^N=S^G$) used to represent the greedy solution obtained after i greedy iterations in Alg. 1 will be used more liberally for any $i\in\{0,1,2,\ldots,M\}$ in the sequel.

Algorithm 1 The greedy algorithm to solve (11)

$i=0$; $S^i=\emptyset$; ▷ Greedy iteration index and solution
for $i=1,2,3,\ldots,N$ **do**
  $s^i = \arg\max_{\{y:S^{i-1}\cup\{y\}\in\mathscr{I}^N\}} \Delta H(y|S^{i-1})$; ▷ New item
  $S^i = S^{i-1}\cup\{s^i\}$; ▷ Append the new item
end for
$S^G := S^N$; **Return** S^G;

III. The Greedy Solution with Performance Bound Guarantees

In this section, we will show that the greedy solution S^G obtained using Alg. 1 for the optimal coverage problem (11) is not only computationally efficient but also entertains performance guarantees with respect to the global optimal coverage performance $H(S^*)$. For this, we first need to introduce some standard set function properties.

For generality, let us consider an arbitrary set function $F:2^Y\to\mathbb{R}$ defined over a finite ground set Y. Similar to before, the corresponding marginal function is defined as

$$\Delta F(y|A) \triangleq F(A\cup\{y\}) - F(A) \tag{14}$$

to represent the gain of set function value due to the addition of an extra element $y\in Y\backslash A$ to the set A. Note that we can use this marginal function notation more liberally as $\Delta F(B|A)\triangleq f(A\cup B)-f(A)$ for any $A,B\subseteq Y$ (even allowing $B\not\subseteq Y\backslash A$).

Definition 1: [11] The set function $F:2^Y\to\mathbb{R}$ is:

1) *normalized* if $F(\emptyset)=0$;
2) *monotone* if $\Delta F(y|A)\geq 0$ for all y,A where $A\subset Y$ and $y\in Y\backslash A$, or equivalently, if $F(B)\leq F(A)$ for all B,A where $B\subseteq A\subseteq Y$;
3) *submodular* if $\Delta f(y|A)\leq \Delta f(y|B)$ for all y,A,B where $B\subseteq A\subset Y$ and $y\in Y\backslash A$, or equivalently, if $F(A\cup B)+F(A\cap B)\leq F(A)+F(B)$ for all $A,B\subseteq Y$,
4) a *polymatroid* set function if it is normalized, monotone and submodular [1], [3].

It is worth noting that the first condition outlined for the submodularity property in Def. 1-(3) is more commonly known as the *diminishing returns* property.

The following lemma and the theorem establish the polymatroid nature of the set coverage function $H(S)$ (7).

Lemma 1: With respect to a common ground set, any positive linear combination of arbitrary polymatroid set functions is also a polymatroid set function.

Proof: Consider n polymatroid set functions $F_1,F_2,\ldots,F_n$, where each is defined over a common ground set Y. Let $F\triangleq\sum_{i\in\mathbb{N}_n}\alpha_i F_i$, where $\alpha_i\geq 0,\forall i\in\mathbb{N}_n$. Clearly, $F(\emptyset)=0$ as $F_i(\emptyset)=0$ and $\alpha_i\geq 0,\forall i\in\mathbb{N}_n$. Therefore, F is normalized. With respect to any A,B such that $B\subseteq A\subseteq Y$, note that $F(B)\geq F(A)$ holds as $F_i(B)\leq F_i(A)$ and $\alpha_i\geq 0,\forall i\in\mathbb{N}_n$. Thus, F is monotone. Using the same arguments, the submodularity and, thus, the polymatroid nature of F can be established. ■

Theorem 1: The set coverage function $H(S)$ (7) is a polymatroid set function.

Proof: From (7) and (8), it is clear that the set coverage function $H(S)$ can be viewed as a linear combination of set detection function components $P_J(x,S)$ (9) and $P_M(x,S)$ (10) - with respect to the common ground set X. Therefore, in light of Lm. 1, to prove $H(S)$ a polymatroid set function, we only have to show $P_J(x,S)$ and $P_M(x,S)$ are polymatroid set functions with respect to any feasible space point $x\in\Phi$.

Let us first consider the set function $P_J(x,S)$ (9). By definition, $P_J(x,\emptyset)=0$, and thus, $P_J(x,S)$ is normalized. The corresponding marginal function (13) can be derived as:

$$\begin{aligned}\Delta P_J(x,y|A) &= P_J(x,A\cup\{y\})-P_J(x,A)\\ &= -\prod_{s_i\in A\cup\{y\}}(1-p(x,s_i))+\prod_{s_i\in A}(1-p(x,s_i))\\ &= p(x,y)\prod_{s_i\in A}(1-p(x,s_i)),\end{aligned} \tag{15}$$

for any $A\subset X$ and $y\in X\backslash A$. From (15), it is clear that $\Delta P_J(x,y|A)\geq 0$ for all $A\subset X$ and $y\in X\backslash A$. Therefore, $P_J(x,S)$ is monotone. Note also that the product term in (15) diminishes with the growth of the set A. This property can be used to conclude that $\Delta P_J(x,y|A)\leq\Delta P_J(x,y|B)$, for all y,A,B where $B\subseteq A\subset X$ and $y\in Y\backslash A$. Hence $P_J(x,S)$ is submodular. Therefore, $P_J(x,S)$ is a polymatroid set function.

Finally, let us consider the set function $P_M(x,S)$ (10). Again, by definition, $P_M(x,\emptyset)=0$ implying that $P_M(x,S)$ is normalized. The corresponding marginal function (13) can be derived as:

$$\begin{aligned}\Delta P_M(x,y|A) &= P_M(x,A\cup\{y\}) - P_M(x,A)\\ &= \max_{s_i\in A\cup\{y\}} p(x,s_i) - \max_{s_i\in A} p(x,s_i)\\ &= \max\{p(x,y) - \max_{s_i\in A} p(x,s_i), 0\},\end{aligned} \tag{16}$$

for any $A \subset X$ and $y \in X\backslash A$. From (16), it is clear that $\Delta P_M(x,y|A) \geq 0$ for all $A \subset X$ and $y \in X\backslash A$. Therefore, $P_M(x,S)$ is monotone. Note also that the first term inside the outer max operator in (16) diminishes with the growth of the set A. This property implies that $\Delta P_M(x,y|A) \leq \Delta P_M(x,y|B)$, for all y,A,B where $B \subseteq A \subset X$ and $y \in Y\backslash A$. Hence $P_M(x,S)$ is submodular. Consequently, $P_M(x,S)$ is a polymatroid set function. This completes the proof. ∎

As a direct result of this polymatroid nature of the set coverage function $H(S)$ (7), we can characterize the proximity of the performance of the greedy solution (i.e., $H(S^G)$) to that of the globally optimal solution (i.e., $H(S^*)$). For this characterization, we particularly use the notion of a *performance bound*, (denoted by β) defined as a theoretically established lower bound for the ratio $\frac{H(S^G)}{H(S^*)}$, i.e.,

$$\beta \leq \frac{H(S^G)}{H(S^*)}. \tag{17}$$

Having a performance bound β close to 1 implies that the performance of the greedy solution is close to that of the global optimal solution. Consequently, β can also serve as an indicator of the effectiveness of the greedy approach to solve the interested optimal coverage problem (11).

The seminal work [5] has established a performance bound (henceforth called the *fundamental performance bound*, and denoted by β_f) for polymatroid set function maximization problems, which, in light of Th. 1, is applicable to the optimal coverage problem (11) as:

$$\beta_f \triangleq 1 - \left(1 - \frac{1}{N}\right)^N \leq \frac{H(S^G)}{H(S^*)}. \tag{18}$$

Note that, while β_f decreases with the number of agents N, it is lower-bounded by $1-\frac{1}{e} \simeq 0.6321$, because $\lim_{N\to\infty}\beta_f = (1-\frac{1}{e})$. This implies that the coverage performance of the greedy solution will always be not worse than 63.21% of the maximum achievable coverage performance.

Moreover, as shown in [11], upon obtaining the greedy solution S^G from Alg. 1, any subsequently improved solution $\bar{S}^G \in \mathscr{I}^N$ (e.g., via a gradient process [6], [7] or an interchange scheme [9]), will have an improved performance bound $\bar{\beta}$ than the original performance bound β such that

$$\beta \leq \bar{\beta} \triangleq \beta * \frac{H(\bar{S}^G)}{H(S^G)} \leq \frac{H(\bar{S}^G)}{H(S^*)}.$$

Besides (and independently from) improving the original greedy solution, as we will see in the next section, an improved performance bound can also be achieved by exploiting certain characteristics (called *curvature measures*) of the interested set function maximization problem.

IV. Improved Performance Bound Guarantees using Curvature Measures

In this section, we will discuss several improved performance-bound guarantees that are applicable to the considered optimal coverage problem (11) and its greedy solution S^G given by Alg. 1. The goal is to obtain tighter performance bounds for S^G, i.e., closer to 1 compared to β_f in (18). This is important as such a performance bound will accurately characterize the proximity of S^G to S^*, and thus allow making informed decisions regarding spending extra resources (e.g., computational power, agents and sensing capabilities) to seek a further improved coverage solution beyond S^G.

As mentioned earlier, curvature measures are used to obtain such improved performance bounds. These curvature measures are dependent purely on the underlying objective function, the ground set, and the feasible space, which, in the considered optimal coverage problem, are $H(S)$, X, and $\mathscr{I}^N$, respectively. In this section, we will review five established curvature measures and their respective performance bounds, outlining their unique characteristics, strengths, and weaknesses in their application to the considered optimal coverage problem (11).

A. Total Curvature [2]

By definition, the *total curvature* of (11) is given by

$$\alpha_t \triangleq \max_{y\in X}\left[1 - \frac{\Delta H(y|X\backslash\{y\})}{\Delta H(y|\emptyset)}\right]. \tag{19}$$

The corresponding performance bound β_t is given by

$$\beta_t \triangleq \frac{1}{\alpha_t}\left[1 - \left(1 - \frac{\alpha_t}{N}\right)^N\right] \leq \frac{H(S^G)}{H(S^*)}. \tag{20}$$

From comparing (20) and (18), it is clear that when $\alpha_t \to 1$, the corresponding performance bound $\beta_t \to \beta_f$ (i.e., no improvement). However, on the other hand, when $\alpha_t \to 0$, the corresponding performance bound $\beta_t \to 1$ (i.e., a significant improvement). Moreover, it can be shown that β_t is monotonically decreasing in α_t. Using the above three facts and (19), it is easy to see that the improvement in the performance bound is proportional to the magnitude of:

$$\gamma_t \triangleq \min_{y\in X}\left[\frac{\Delta H(y|X\backslash\{y\})}{\Delta H(y|\emptyset)}\right]. \tag{21}$$

The diminishing returns (submodularity) property of H implies $\frac{\Delta H(y|X\backslash\{y\})}{\Delta H(y|\emptyset)} \leq 1, \forall y \in X$. Therefore, γ_t is large only when $\frac{\Delta H(y|X\backslash\{y\})}{\Delta H(y|\emptyset)} \simeq 1, \forall y \in X$. In other words, a significantly improved performance bound from the total curvature measure can only be obtained when H is just "weakly" submodular (i.e., when H is closer to being modular rather than submodular). This is also clear from simplifying the condition $\frac{\Delta H(y|X\backslash\{y\})}{\Delta H(y|\emptyset)} \simeq 1, \forall y \in X$ using (12), as it leads to

$$H(X) \simeq H(y) + H(X\backslash\{y\}), \quad \text{for all } y \in X \tag{22}$$

which holds whenever H is modular.

In particular, as H is the set coverage function (7), the above condition (22) holds (leading to improved performance bounds) when an agent deployed at any $y \in X$ and all other agents deployed at $X\backslash\{y\}$ contribute to the coverage objective independently in a modular fashion. This happens when the ground set X is very sparse and/or when the agents have significantly weak non-overlapping sensing capabilities (i.e., small range δ and high decay λ in (1)).

However, the condition (22) is easily violated (leading to poor performance bounds) if

$$H(X) \ll H(y) + H(X\backslash\{y\}), \quad \text{for some } y \in X.$$

To interpret this condition using (7), we need to consider the corresponding detection function (8) requirement:

$$P(x,X) \ll P(x,\{y\}) + P(x,X\backslash\{y\}), \quad \text{for some } y \in X$$

for a majority of $x \in \Phi$. Now, using (15) and (16), we get

$$\begin{aligned} 0 \ll\ & \theta(p(x,y)(1-\prod_{s_i\in X\backslash\{y\}}(1-p(x,s_i))) \\ & +(1-\theta)(p(x,y)-\max\{p(x,y)-\max_{s_i\in X\backslash\{y\}}p(x,s_i),0\}), \end{aligned}$$

where the second term can be further simplified to obtain:

$$\begin{aligned} 0 \ll\ & \theta(p(x,y)(1-\prod_{s_i\in X\backslash\{y\}}(1-p(x,s_i))) \\ & +(1-\theta)(\min\{\max_{s_i\in X\backslash\{y\}}p(x,s_i),p(x,y)\}). \end{aligned}$$

Since $\theta \in [0,1]$, we need to consider both terms above separately. However, both terms lead to the same condition (under which the above requirement holds):

$$0 \ll p(x,y) \text{ and } 0 \ll p(x,s_i), \text{ for some } s_i \in X\backslash\{y\}.$$

In all, the total curvature measure leads to poor performance bounds when there exists some $y \in X$ and $s_i \in X\backslash\{y\}$ so that

$$0 \ll p(x,y) \simeq p(x,s_i) \simeq 1,$$

for many feasible space locations $x \in \Phi$. Evidently, this requirement holds when the ground set X is dense and when the agents have significantly strong overlapping sensing capabilities (i.e., large range δ and small decay λ in (1)).

One final remark about the total curvature measure is that it requires an evaluation of $H(X)$ and $M(\triangleq |X|)$ evaluations of $H(X\backslash\{y\})$ terms (i.e., for all $y \in X$). In certain coverage applications, this might be ill-defined [6] and computationally expensive as often $H(S)$ is of the complexity $O(|S|)$.

B. Greedy Curvature [2]

The *greedy curvature* of (11) is given by

$$\alpha_g \triangleq \max_{0\leq i\leq N-1}\left[\max_{y\in X^i}\left(1-\frac{\Delta H(y|S^i)}{\Delta H(y|\emptyset)}\right)\right], \tag{23}$$

where $X^i \triangleq \{y : y \in X\backslash S^i, (S^i \cup \{y\}) \in \mathscr{I}^N\}$ (i.e., the set of feasible options at the $(i+1)^{\text{th}}$ greedy iteration). The corresponding performance bound β_g is given by

$$\beta_g \triangleq 1-\alpha_g\left(1-\frac{1}{N}\right) \leq \frac{H(S^G)}{H(S^*)}. \tag{24}$$

Note that β_g is a monotonically decreasing function in α_g, and due to the submodularity of H, $0 \leq \alpha_g \leq 1$. Consequently, as $\alpha_g \to 0$, $\beta_g \to 1$, and on the other hand, as $\alpha_g \to 1$, $\beta_g \to \frac{1}{N}$ (which may be worse than β_f, when $\frac{1}{N} < \beta_f$). Using these facts and (23), it is easy to see that the improvement in the performance bound is proportional to the magnitude of

$$\gamma_g \triangleq \min_{0\leq i\leq N-1}\left[\min_{y\in X^i}\left(\frac{\Delta H(y|S^i)}{\Delta H(y|\emptyset)}\right)\right]. \tag{25}$$

Similar to before, the diminishing returns property of H implies that γ_g is large only when $\frac{\Delta H(y|S^i)}{\Delta H(y|\emptyset)} \simeq 1, \forall y \in X^i, i \in \{0,1,2,...,N-1\}$. In other words, similar to the total curvature, the greedy curvature provides a significantly improved performance bound when H is weakly submodular.

In fact, as reported in [6], when H is significantly weakly submodular, it can provide better performance bounds even compared to those provided by the total curvature, i.e., $\beta_f \ll \beta_t \leq \beta_g \simeq 1$. This observation can be theoretically justified using (21) and (25) as follows. Due to submodularity, $\Delta H(y|X\backslash\{y\}) \leq \Delta H(y|S^i)$ for any y and S^i, and thus, $\gamma_t \leq \gamma_g$. This, with weak submodularity of H leads to $\alpha_t \geq \alpha_g \simeq 0$. Now, noticing that the growth of β_g is faster as $\alpha_g \to 0$ compared to that of β_t as $\alpha_t \to 0$, we get $\beta_f \ll \beta_t \leq \beta_g \simeq 1$.

We can follow the same steps and arguments as before to show that such improved performance bounds can only be achieved when the ground set is sparse and/or when the agents have weak sensing capabilities. On the other hand, when the ground set is dense and when the agents have strong sensing capabilities, greedy curvature provides poor performance bounds (often, it may even be worse than β_f).

However, compared to the total curvature, greedy curvature has some more redeeming qualities: it is always fully defined, and it can be computed efficiently using only the evaluations of H executed in the greedy algorithm.

C. Elemental Curvature [8]

The *elemental curvature* of (7) is given by

$$\alpha_e \triangleq \max_{\substack{(S,y_i,y_j):S\subset X,\\ y_i,y_j\in X\backslash S,\ y_i\neq y_j.}}\left[\frac{\Delta H(y_i|S\cup\{y_j\})}{\Delta H(y_i|S)}\right]. \tag{26}$$

The corresponding performance bound β_e is given by

$$\beta_e \triangleq 1-\left(\frac{\alpha_e+\alpha_e^2+\cdots+\alpha_e^{N-1}}{1+\alpha_e+\alpha_e^2+\cdots+\alpha_e^{N-1}}\right)^N \leq \frac{H(S^G)}{H(S^*)}. \tag{27}$$

It can be shown that β_e is a monotonically decreasing function in α_e, and due to the submodularity of H, $0 \leq \alpha_e \leq 1$. Consequently, when $\alpha_e \to 0$, $\beta_e \to 1$ and when $\alpha_e \to 1$, $\beta_e \to \beta_f$ (the latter is unlike β_g, but similar to β_t).

Since H is submodular, according to [5, Prop. 2.1], for all feasible (S,y_i,y_j) considered in (26), $\frac{\Delta H(y_i|S\cup\{y_j\})}{\Delta H(y_i|S)} \leq 1$. Therefore, based on (26), whenever there exists some feasible (S,y_i,y_j) such that $\frac{\Delta H(y_i|S\cup\{y_j\})}{\Delta H(y_i|S)} \simeq 1$, i.e. when H is weakly submodular (or, equivalently, closer to being modular) in that region, the elemental curvature measure will provide poor performance bounds (closer to β_f). This modularity argument is also evident from considering a simplified case of condition $\frac{\Delta H(y_i|S\cup\{y_j\})}{\Delta H(y_i|S)} \simeq 1$ assuming $S=\emptyset$, as it leads to

$$H(\{y_i,y_j\}) \simeq H(\{y_j\}) + H(\{y_i\}), \quad \text{for some } y_i, y_j \in X$$

which holds whenever H is modular.

As we observed before, the coverage function H shows such modular behaviors (leading to poor performance bounds $\beta_e \simeq \beta_f$) when the ground set X is very sparse and/or when agents have significantly weak non-overlapping sensing capabilities. It is worth highlighting that this particular behavior of elemental curvature contrasts from that of the previously discussed total curvature and greedy curvature - where weakly submodular scenarios (with agents having weak sensing capabilities) lead to significantly improved performance bounds $\beta_f \ll \beta_t \leq \beta_g \simeq 1$.

On the other hand, the elemental curvature provides an improved performance bound when $\frac{\Delta H(y_i|S\cup\{y_j\})}{\Delta H(y_i|S)} \ll 1$ over all feasible (S,y_i,y_j) considered in (26). To further interpret this condition, we need to consider the corresponding marginal detection function (13) requirement:

$$\Delta P(x,y_i|S\cup\{y_j\}) \ll \Delta P(x,y_i|S), \quad \forall (S,y_i,y_j) \tag{28}$$

for a majority of $x\in\Phi$. Since each $\Delta P = \theta\Delta P_J + (1-\theta)\Delta P_M$ where $\theta\in[0,1]$, let us first consider the requirement (28) with respect to the ΔP_J (i.e., when $\theta=1$ in (8)) using (15):

$$\begin{aligned}
&\Delta P_J(x,y_i|S\cup\{y_j\}) \ll \Delta P_J(x,y_i|S)\\
&\iff p(x,y_i)\prod_{s_i\in S\cup\{y_j\}}(1-p(x,s_i)) \ll p(x,y_i)\prod_{s_i\in S}(1-p(x,s_i))\\
&\iff 0 \ll p(x,y_i)p(x,y_j)\prod_{s_i\in S}(1-p(x,s_i)).
\end{aligned}$$

Clearly, this condition holds if for all feasible (S,y_i,y_j),

$$0 \ll p(x,y_i) \simeq p(x,y_j) \simeq 1 \text{ with } 0 \simeq p(x,s_i) \ll 1 \tag{29}$$

for some $s_i\in S$ over many feasible space locations $x\in\Phi$.

Now, let us consider the requirement (28) with respect to the ΔP_M (i.e., when $\theta=0$ in (8)) using (10):

$$\begin{aligned}
&\Delta P_M(x,y_i|S\cup\{y_j\}) \ll \Delta P_M(x,y_i|S)\\
&\iff \max_{s_i\in S\cup\{y_j,y_i\}} p(x,s_i) - \max_{s_i\in S\cup\{y_j\}} p(x,s_i)\\
&\qquad \ll \max_{s_i\in S\cup\{y_i\}} p(x,s_i) - \max_{s_i\in S} p(x,s_i)\\
&\iff \max_{s_i\in S\cup\{y_j,y_i\}} p(x,s_i) + \max_{s_i\in S} p(x,s_i)\\
&\qquad \ll \max_{s_i\in S\cup\{y_i\}} p(x,s_i) + \max_{s_i\in S\cup\{y_j\}} p(x,s_i).
\end{aligned} \tag{30}$$

For notational convenience, let $P_S \triangleq \max_{s_i\in S} p(x,s_i)$, $P_{y_i} \triangleq p(x,y_i)$ and $P_{y_j} \triangleq p(x,y_j)$. Then, (30) can be restated as

$$\begin{aligned}
&\max\{P_S,P_{y_i},P_{y_j}\} + P_S \ll \max\{P_S,P_{y_i}\} + \max\{P_S,P_{y_j}\}\\
&\iff \max\{2P_S, P_S+P_{y_i}, P_S+P_{y_j}\}\\
&\qquad \ll \max\{P_S+\max\{P_S,P_{y_j}\}, P_{y_i}+\max\{P_S,P_{y_j}\}\}\\
&\iff \max\{2P_S, P_S+P_{y_i}, P_S+P_{y_j}\}\\
&\qquad \ll \max\{\max\{2P_S,P_S+P_{y_j}\}, \max\{P_S+P_{y_i},P_{y_i}+P_{y_j}\}\}\\
&\iff \max\{2P_S, P_S+P_{y_i}, P_S+P_{y_j}\}\\
&\qquad \ll \max\{\max\{2P_S,P_S+P_{y_j},P_S+P_{y_i}\}, P_{y_i}+P_{y_j}\}\\
&\iff 0 \ll \max\{0, P_{y_i}+P_{y_j}-\max\{2P_S,P_S+P_{y_j},P_S+P_{y_i}\}\}\\
&\iff \max\{2P_S,P_S+P_{y_j},P_S+P_{y_i}\} \ll P_{y_i}+P_{y_j}\\
&\iff P_S+\max\{P_S,P_{y_j},P_{y_i}\} \ll P_{y_i}+P_{y_j}
\end{aligned}$$

Since P_{y_i} and P_{y_j} are interchangeable in the above expression, let us denote $P_y \triangleq P_{y_i} \simeq P_{y_j}$. This makes the above condition:

$$\begin{aligned}
&P_S+\max\{P_S,P_y\} \ll 2P_y\\
&\iff \max\{2P_S-2P_y, P_S-P_y\} \ll 0 \iff P_S \ll P_Y\\
&\iff \max_{s_i\in S} p(x,s_i) \ll p(x,y_i) \simeq p(x,y_j),
\end{aligned}$$

which leads to the same condition obtained in (29).

In all, the elemental curvature measure leads to significantly improved performance bounds when for all (S,y_i,y_j) such that $S\subset X, y_i,y_j\in X\backslash S$ and $y_i\neq y_j$,

$$0 \simeq p(x,s_i) \ll p(x,y_i) \simeq p(x,y_j) \simeq 1$$

for some $s_i\in S$ over many feasible space locations $x\in\Phi$. Clearly, this requirement holds when the ground set X is dense and when the agents have significantly strong overlapping sensing capabilities (i.e., large range δ and small decay λ in (1)).

Finally, note that the evaluation of the elemental curvature α_e (26) is computationally expensive (even compared to the total curvature) as it involves solving a set function maximization problem (notice the set variable S in (26)). However, as shown in [7], there may be special structural properties that can be exploited to obtain at least an upper bound on α_e, leading to a lower bound on β_e - which would still be a valid performance bound for the optimal coverage problem (11).

For example, it can be shown that, α_e (26) is bounded such that

$$\alpha_e \leq \max_{(S,y_i,y_j)} \int_\Omega \frac{\Delta P_J(x,y_i|S\cup\{y_j\})}{\Delta P_J(x,y_i|S)} + \frac{\Delta P_M(x,y_i|S\cup\{y_j\})}{\Delta P_M(x,y_i|S)} dx,$$

independently of the weight factor θ and the event density $R(x)$. Here, the first term inside the integral directly simplifies to $(1-p(x,y_j))$. However, the second term, even though it simplifies to (using the notations introduced earlier)

$$1 - \frac{P_{y_i}+P_{y_j}-P_S-\max\{P_S,P_{y_i},P_{y_j}\}}{\max\{0,P_{y_i}-P_S\}},$$

it cannot be further simplified or upper bounded - except for the obvious upper bound 1 due to the submodularity of

P_M. Therefore, a practical elemental curvature metric for the optimal coverage problem would be $\bar{\alpha}_e$ where:

$$\alpha_e \leq \bar{\alpha}_e \triangleq \max_{y_j \in X} \int_{\Omega} (2 - p(x,y_j))dx.$$

D. Partial Curvature [3]

The *partial curvature* of (11) is given by

$$\alpha_p = \max_{(S,y):y\in S\in \mathscr{I}^N} \left[1 - \frac{\Delta H(y|S\backslash\{y\})}{\Delta H(y|\emptyset)}\right]. \tag{31}$$

The corresponding performance bound β_p is given by

$$\beta_p \triangleq \frac{1}{\alpha_p}\left[1 - \left(1 - \frac{\alpha_p}{N}\right)^N\right] \leq \frac{H(S^G)}{H(S^*)}. \tag{32}$$

This partial curvature measure α_p (31) provides an alternative to the total curvature measure α_t (19). In particular, it addresses the potentially ill-defined nature of the $H(X)$ term involved in α_t (19). Consequently, α_p can be evaluated when the domain of H constrained, i.e., when $H : \mathscr{I} \to \mathbb{R}_{\geq 0}$ with some $\mathscr{I} \subset 2^X$.

The above β_p (32) is only valid under a few additional conditions on f, X and $\mathscr{I}^N$ (which are omitted here, but can be found in [3]). Note that we can directly compare α_t and α_p to conclude regarding the nature of the corresponding performance bounds β_t and β_p, as β_t (20) and β_p (32) has identical forms. The work in [3] has established that $\alpha_p \leq \alpha_t$, which implies that $\beta_p \geq \beta_t$, i.e., β_p is always tighter than β_t.

Note also that, similar to β_t, β_p will provide significantly improved performance bounds (i.e., $\beta_p \simeq 1$) when H is weakly submodular. As observed before, such a scenario occurs when the ground set is sparse and/or agent sensing capabilities are weak. On the other hand, again, similar to β_t, β_p will provide poor performance bounds (i.e., $\beta_p \simeq \beta_f$) when H is strongly submodular. This happens when the ground set is dense, and agent sensing capabilities are strong.

Unfortunately, similar to the elemental curvature α_e (26), evaluating the partial curvature α_p (31) involves solving a set function maximization problem (notice the set variable Y in (31)). Therefore, evaluating α_p is much more computationally expensive compared to evaluating α_t or α_g. However, like in the case of α_e, we can exploit some special structural properties of the optimal coverage problem to overcome this challenge. In particular, notice that

$$\begin{aligned}\alpha_p &= 1 - \min_{(S,y):y\in S\in \mathscr{I}^N} \frac{\Delta H(y|S\backslash\{y\})}{\Delta H(y|\emptyset)}\\ &= 1 + \max_{y\in X}\left(\frac{1}{H(\{y\})} \max_{B\in \mathscr{I}_{N-1}, y\notin B} [-\Delta H(y|B)]\right).\end{aligned}$$

Now, using the submodularity of $-\Delta H(y|B)$ (established in the following lemma), we can provide an upper bound for the inner submodular maximization problem [11, Lm. 1]. This can be used to compute an upper bound to α_p, which then will provide a lower bound to β_p.

Lemma 2: Given $y \in X$, the negated marginal coverage function -$\Delta H(y|B)$ is submodular with respect to $B \subset X\backslash\{y\}$.

Proof: To be included ∎

E. Extended Greedy Curvature [11]

The *extended greedy curvature*, as the name suggests, requires executing some extra greedy iterations in the greedy algorithm (i.e., Alg. 1). This is not an issue as Alg. 1 can be executed beyond N iterations until $M \triangleq |X|$ iterations - analogous to a scenario where more than N agents are to be deployed to the mission space in a greedy fashion.

To define the extended greedy curvature, we first need some additional notations. Recall that we used (S^i, s^i) to denote the greedy (set, element) observed at the i^{th} greedy iteration, where $i \in \mathbb{N}_M^0$. Let $m \triangleq \lfloor \frac{M}{N} \rfloor$, and for any $n \in \mathbb{N}_{m-1}^0$,

$$S_n^G \triangleq S^{(n+1)N}\backslash S^{nN} = \{s^{nN+1}, s^{nN+2}, \ldots, s^{nN+N}\}, \tag{33}$$

$$X_n \triangleq X\backslash S^{nN} \quad \text{and} \quad \mathscr{I}_n^N \triangleq \{S : S \subseteq X_n, |S| \leq N\}. \tag{34}$$

Simply, S_n^G is the $(n+1)^{\text{th}}$ block of size N greedy agent placements, and X_n is the the set of agent locations remaining after nN greedy iterations. Note that, $S_0^G = S^G, X_0 = X$ and $\mathscr{I}_0^N = \mathscr{I}^N$. Note also that, for any $n \in \mathbb{N}_{m-1}^0$, the set system $(X_n, \mathscr{I}_n^N)$ is a uniform matroid of rank N, and S_n^G is the greedy solution for $\arg\max_{S\in\mathscr{I}_n^N} H(S)$.

The extended greedy curvature of (11) is given by

$$\alpha_u \triangleq \min_{i\in Q} \alpha_u^i, \tag{35}$$

where $Q \subseteq \bar{Q} \triangleq \{i \in \mathbb{N}_M : i = nN+1, \ n \in \mathbb{N}_{m-1}^0 \text{ or } i = nN, \ n \in \mathbb{N}_m \text{ or } i = M\}$ and

$$\alpha_u^i \triangleq \begin{cases} H(S^{i-1}) + \max\limits_{S\in \mathscr{I}_{(i-1)/N}^N} \left[\sum_{y\in Y} \Delta H(y|S^{i-1})\right] \\ \qquad\qquad \text{if } i = nN+1, \ n \in \mathbb{N}_{m-1}^0, \\ H(S^{i-N}) + \frac{1}{\beta_f}\left[H(S^i) - H(S^{i-N})\right] \\ \qquad\qquad \text{if } i = nN, \ n \in \mathbb{N}_m, \\ H(S^i) \qquad\qquad \text{if } i = M. \end{cases} \tag{36}$$

Note that $\bar{Q}$ is a fixed set of greedy iteration indexes. For each $i \in \bar{Q}$, a corresponding α_u^i value can be computed using known byproducts generated during the execution of greedy iterations. Unlike $\bar{Q}$, Q is an arbitrary subset selected from $\bar{Q}$ based on the user preference. For example, one may choose $Q = \{1, N, N+1, 2N, 2N+1\}$ so that α_u value can be obtained upon executing only $N+1$ extra greedy iterations.

The performance bound β_u corresponding to the extended greedy curvature measure α_u is given by

$$\beta_u \triangleq \frac{H(S^G)}{\alpha_u} \leq \frac{H(S^G)}{f(Y^*)}. \tag{37}$$

****** Revised up to this point...**

From the review presented in the previous section, three main limitations of existing improved performance bounds (i.e., of β_t [2], β_g [2], β_e [8] and β_p [3]) can be identified:

1) **Computational complexity**: For example, β_e and β_p (i.e., the most recently proposed performance bounds) require solving hard combinatorial optimization problems.

2) **Inherent limitations**: For example, β_t, β_g and β_p inherently provide improved performance bounds only when the submodularity property (of f over X) is weak.
3) **Technical limitations**: For example, β_t and β_p have technical conditions that need to be satisfied (by f, X and $\mathscr{I}^N$ involved in (**??**)) to validate their usage.

To counter the limitations mentioned above, in this section, a new performance bound (denoted by β_u) is proposed for the greedy solution Y^G given by Alg. 1 for the class of problems in (**??**). Similar to the previously reviewed improved performance bounds, this new performance bound β_u is also defined through a corresponding (also new) curvature measure. In particular, we denote this new curvature measure as α_u and call it the *extended greedy curvature*.

As the name suggests, this new curvature measure α_u (and hence β_u) is derived exploiting the information computed when executing an extended number of greedy iterations (i.e., more than the usual N greedy iterations executed in Alg. 1). As we will see in the sequel, the exact number of extra greedy iterations required depends on the application and the user preference. Since running greedy iterations is computationally inexpensive, the complexity of computing β_u is much less than that of β_e or β_p and is in the same order of computing β_t or β_g. Moreover, as we will see in the sequel, unlike $\beta_t, \beta_g, \beta_e$ and β_p, β_u does not have any inherent or technical limitations.

A summary of our extended greedy curvature metric and the corresponding performance bound.

Lemma 3: Given the greedy solution performance $f(Y^G)$:

1) α is an upper bound for $f(Y^*)$ if and only if $\beta = \frac{f(Y^G)}{\alpha}$ is a valid performance bound for $f(Y^G)$.
2) β is a valid performance bound for $f(Y^G)$ if and only if $\alpha = \frac{1}{\beta} f(Y^G)$ is an upper bound for $f(Y^*)$.

Proof: Cases (a) and (b) can be proved using the relationships: $\alpha \geq f(Y^*) \iff \frac{f(Y^G)}{\alpha} \leq \frac{f(Y^G)}{f(Y^*)}$ and $\beta \leq \frac{f(Y^G)}{f(Y^*)} \iff f(Y^*) \leq \frac{1}{\beta} f(Y^G)$, respectively. ■

We now introduce some additional notations. Let $[0,k]$ be the set $\{0,1,2,\ldots,k\}$. Recall the (Z^i, z^i) notation introduced in Section **??** for $i \in [0,M]$. Using that, let us define

$$Y_n^G \triangleq Z^{(n+1)N} \backslash Z^{nN} = \{z^{nN+1}, z^{nN+2}, \ldots, z^{nN+N}\}, \tag{38}$$

for any $n \in [0, m-1]$ where $m \triangleq \lfloor \frac{M}{N} \rfloor$ ($\lfloor \cdot \rfloor$ denotes the floor operator). Simply, Y_n^G is the $(n+1)^{\text{th}}$ block of N greedily selected options. Hence, $|Y_n^G| = N$ and $Y_0^G = Y^G$. Along the same lines, let us also define

$$X_n \triangleq X \backslash Z^{nN} \quad \text{and} \quad \mathscr{I}_n^N \triangleq \{Y : Y \subseteq X_n, |Y| \leq N\}, \tag{39}$$

for any $n \in [0, m-1]$. Simply, X_n is the set of remaining available options after selecting n blocks of N greedy options (i.e., after nN greedy iterations). Hence, $X_0 = X$ and $\mathscr{I}_0^N = \mathscr{I}^N$. Similar to the set system $(X, \mathscr{I}^N)$ considered in (**??**), the set system $(X_n, \mathscr{I}_n^N)$ is a uniform matroid of rank N, for any $n \in [0, m-1]$.

Let us also consider a series of auxiliary set function maximization problems: $\{\mathbf{P}_n\}_{n \in [0,m-1]}$ where

$$\mathbf{P}_n : \qquad Y_n^* \triangleq \underset{Y \in \mathscr{I}_n^N}{\arg\max}\ \Delta f(Y | Z^{nN}). \tag{40}$$

According to Lemma **??**, the objective function of $\mathbf{P}_n$ (i.e., $\Delta f(Y|Z^{nN})$) is a polymatroid set function over X_n. This implies that $\mathbf{P}_n$ aims to find the optimal set Y_n^* that maximizes the polymatroid set function $\Delta f(Y|Z^{nN})$ over the uniform matroid $(X_n, \mathscr{I}_n^N)$. Hence, each $\mathbf{P}_n, n \in [0, m-1]$ falls into the same class of problems as in (**??**), and in fact, $\mathbf{P}_0$ is equivalent to (**??**) (i.e., $Y_0^* = Y^*$). Moreover, it is easy to see that Y_n^G introduced in (38) is the greedy solution to $\mathbf{P}_n$ in (40) for $n \in [0, m-1]$.

Next, we establish two lemmas that provide two different upper bounds for the global optimal performance of $\mathbf{P}_n$ in (40).

Lemma 4: For $n \in [0, m-1]$,

$$\Delta f(Y_n^* | Z^{nN}) \leq \max_{Y \in \mathscr{I}_n^N} \left[\sum_{y \in Y} \Delta f(y | Z^{nN}) \right]. \tag{41}$$

Proof: Due to the normalized and monotone nature of the set function f (see Def. 1(a)-(b)), we have $0 \leq f(A \cap B)$ for all $A, B \subseteq X$. Using this result in the second equivalent condition given for the submodularity property in Def. 1(c), we can write $f(A \cup B) \leq f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$, i.e., $f(A \cup B) \leq f(A) + f(B)$ for all $A, B \subseteq X$ (notice the resemblance with the triangle inequality). Based on this result, it is easy to see that any normalized monotone submodular set function f defined over a ground set X will follow the property:

$$f(A) \leq \sum_{a \in A} f(\{a\}), \quad \forall A \subseteq X. \tag{42}$$

As mentioned before, $\Delta f(Y|Z^{nN})$ is a normalized monotone submodular set function in Y over the ground set X_n (from Lemma **??**). Therefore, $\Delta f(Y|Z^{nN})$ should follow the property in (42), i.e.,

$$\Delta f(Y | Z^{nN}) \leq \sum_{y \in Y} \Delta f(y | Z^{nN}), \quad \forall Y \in \mathscr{I}_n^N, \tag{43}$$

(note that, according to (39), $Y \in \mathscr{I}_n^N \iff Y \subseteq X^n$). Now, taking the maximum of both sides of (43) over all possible $Y \in \mathscr{I}_n^N$, we get

$$\max_{Y \in \mathscr{I}_n^N} \Delta f(Y | Z^{nN}) \leq \max_{Y \in \mathscr{I}_n^N} \left[\sum_{y \in Y} \Delta f(y | Z^{nN}) \right]. \tag{44}$$

Finally, using (40), we can rewrite the left hand side (LHS) of the above expression as $\Delta f(Y_n^* | Z^{nN})$. This completes the proof. ■

Lemma 5: For $n \in [0, m-1]$,

$$\Delta f(Y_n^* | Z^{nN}) \leq \frac{1}{\beta_f} \left[f(Z^{(n+1)N}) - f(Z^{nN}) \right]. \tag{45}$$

Proof: Using (40), let us rewrite the LHS of (45) as

$$\Delta f(Y_n^* | Z^{nN}) = \max_{Y \in \mathscr{I}_n^N} \Delta f(Y | Z^{nN}). \tag{46}$$

Since $\Delta f(Y|Z^{nN})$ is a polymatroid set function in Y over the uniform matroid $(X_n, \mathscr{I}_n^N)$, the performance bound β_f given in (18) can be applied for a greedy solution of the above set function maximization problem (in right hand side of (46)). In fact, from the used notation, $Y = Y_n^G$ is the greedy solution that maximizes $\Delta f(Y|Z^{nN})$. Therefore, using the known performance bound β_f and the greedy solution performance $\Delta f(Y_n^G|Z^{nN})$ in Lemma 3(b), we obtain an upper bound to (46) as

$$\Delta f(Y_n^*|Z^{nN}) = \max_{Y\in\mathscr{I}_n^N} \Delta f(Y|Z^{nN}) \leq \frac{1}{\beta_f}\Delta f(Y_n^G|Z^{nN}). \quad (47)$$

Finally, we use (**??**) and (38) to simplify $\Delta f(Y_n^G|Z^{nN})$ as $\Delta f(Y_n^G|Z^{nN}) = f(Y_n^G \cup Z^{nN}) - f(Z^{nN}) = f(Z^{(n+1)N}) - f(Z^{nN})$. Substituting this result in (47) completes the proof. ■

The following lemma establishes an important equality condition that will be used later on.

Lemma 6: For $n \in [0, m-1]$,

$$\max_{Y\in\mathscr{I}_n^N} \Delta f(Y|Z^{(n+1)N}) = \max_{Y\in\mathscr{I}_{n+1}^N} \Delta f(Y|Z^{(n+1)N}). \quad (48)$$

Proof: Note that this result is non-trivial as the feasible spaces of the optimization problems on both sides of (48) are related such that $\mathscr{I}_n^N \supset \mathscr{I}_{n+1}^N$. Let us denote $Y = \{y_1, y_2, \ldots, y_N\} \in \mathscr{I}_n^N$ and rewrite $\Delta f(Y|Z^{(n+1)N})$ as a telescoping sum:

$$\begin{aligned}\Delta f(Y|Z^{(n+1)N}) =& f(\{y_1,\ldots,y_N\}\cup Z^{(n+1)N}) - f(Z^{(n+1)N})\\ =& f(\{y_1,\ldots,y_N\}\cup Z^{(n+1)N})\\ &- f(\{y_1,\ldots,y_{N-1}\}\cup Z^{(n+1)N}) + \cdots\\ &\cdots + f(\{y_1\}\cup Z^{(n+1)N}) - f(Z^{(n+1)N})\\ =& \sum_{i=1}^{N}\Delta f(y_i|\{y_1,\ldots,y_{i-1}\}\cup Z^{(n+1)N}). \quad (49)\end{aligned}$$

Note that $\Delta f(y_i|\{y_1,\ldots,y_{i-1}\}\cup Z^{(n+1)N}) = 0$ for any $y_i \in Z^{(n+1)N}$ and $\Delta f(y_i|\{y_1,\ldots,y_{i-1}\}\cup Z^{(n+1)N}) > 0$ for any $y_i \in X\backslash Z^{(n+1)N}$. Therefore, according (49), when maximizing the set function $\Delta f(Y|Z^{(n+1)N})$ with respect to Y, selecting $Y \subset X\backslash Z^{(n+1)N}$ (a.k.a. $Y \in \mathscr{I}_{n+1}^N$) is sufficient as opposed to selecting $Y \subseteq X\backslash Z^{nN}$ (a.k.a. $Y \in \mathscr{I}_n^N$). Hence (48) holds. ■

Now, we establish a lemma that provides an upper bound for the performance of the global optimal solution of (**??**).

Lemma 7: For $n \in [0, m-1]$,

$$f(Y^*) \leq f(Z^{nN}) + \Delta f(Y_n^*|Z^{nN}). \quad (50)$$

Proof: Since $\Delta f(Y|Z^{nN})$ is a monotone set function in Y over the ground set X_n (from Lemma **??**, for any $n \in [0, m-1]$),

$$\begin{aligned}\Delta f(Y_n^*|Z^{nN}) \leq& \Delta f(Y_n^*\cup Y_n^G|Z^{nN}),\\ =&\Delta f(Y_n^G|Z^{nN}) + \Delta f(Y_n^*\cup Y_n^G|Z^{nN}) - \Delta f(Y_n^G|Z^{nN})\\ =&\Delta f(Y_n^G|Z^{nN}) + f(Y_n^*\cup Y_n^G\cup Z^{nN}) - f(Z^{nN})\\ &- f(Y_n^G\cup Z^{nN}) + f(Z^{nN}) \quad \text{(using (??))}\\ =&\Delta f(Y_n^G|Z^{nN}) + f(Y_n^*\cup Y_n^G\cup Z^{nN}) - f(Y_n^G\cup Z^{nN})\\ =&\Delta f(Y_n^G|Z^{nN}) + \Delta f(Y_n^*|Y_n^G\cup Z^{nN}) \quad \text{(using (??))}\\ =&\Delta f(Y_n^G|Z^{nN}) + \Delta f(Y_n^*|Z^{(n+1)N}), \quad \text{(using (38))}\end{aligned}$$

i.e.,

$$\Delta f(Y_n^*|Z^{nN}) \leq \Delta f(Y_n^G|Z^{nN}) + \Delta f(Y_n^*|Z^{(n+1)N}). \quad (51)$$

Note that $\Delta f(Y_n^*|Z^{(n+1)N}) \leq \max_{Y\in\mathscr{I}_n^N} \Delta f(Y|Z^{(n+1)N})$ as $Y_n^* \in \mathscr{I}_n^N$. Using this result in (51), we can write

$$\begin{aligned}\Delta f(Y_n^*|Z^{nN}) \leq& \Delta f(Y_n^G|Z^{nN}) + \max_{Y\in\mathscr{I}_n^N}\Delta f(Y|Z^{(n+1)N})\\ =&\Delta f(Y_n^G|Z^{nN}) + \max_{Y\in\mathscr{I}_{n+1}^N}\Delta f(Y|Z^{(n+1)N}) \quad \text{(From Lm. 6)}\\ =&\Delta f(Y_n^G|Z^{nN}) + \Delta f(Y_{n+1}^*|Z^{(n+1)N}). \quad \text{(using (40))}\end{aligned}$$

Replacing n with k, the above result can be written as

$$\Delta f(Y_k^*|Z^{kN}) \leq \Delta f(Y_k^G|Z^{kN}) + \Delta f(Y_{k+1}^*|Z^{(k+1)N}), \quad (52)$$

for $k \in [0, m-1]$. Now summing (52) for $k \in [0, n-1]$ we get

$$\Delta f(Y_0^*|Z^0) \leq \sum_{k=0}^{n-1}\Delta f(Y_k^G|Z^{kN}) + \Delta f(Y_n^*|Z^{nN}). \quad (53)$$

Note that $Y_0^* = Y^*$ in (**??**) and $Z^0 = \emptyset$ by definition. Thus, $\Delta f(Y_0^*|Z^0) = f(Y^*)$. Further, using (**??**) and (38), we can show that

$$\begin{aligned}\sum_{k=0}^{n-1}\Delta f(Y_k^G|Z^{kN}) =& \sum_{k=0}^{n-1}\Delta f(Z^{(k+1)N}\backslash Z^{kN}|Z^{kN})\\ =& \sum_{k=0}^{n-1} f(Z^{(k+1)N}) - f(Z^{kN})\\ =& f(Z^{nN}).\end{aligned}$$

Therefore, using the above two results in (53), we now can obtain (50), which holds for any $n \in [0, m-1]$ (equality holds when $n = 0$). ■

F. Extended Greedy Curvature Based Performance Bound

We now define the proposed extended greedy curvature measure α_u as

$$\alpha_u \triangleq \min_{i\in Q} \alpha_u^i, \quad (54)$$

where $Q \subseteq \bar{Q} \triangleq \{1, N, N+1, 2N, 2N+1, \ldots, (m-1)N+1, mN, M\}$ and

$$\alpha_u^i \triangleq \begin{cases} f(Z^{i-1}) + \max\limits_{Y\in\mathscr{I}_{(i-1)/N}^N}\left[\sum_{y\in Y}\Delta f(y|Z^{i-1})\right] \\ \qquad\qquad \text{if } i = 1, N+1, 2N+1, \ldots, (m-1)N+1,\\ f(Z^{i-N}) + \frac{1}{\beta_f}\left[f(Z^i) - f(Z^{i-N})\right] \\ \qquad\qquad\qquad\qquad \text{if } i = N, 2N, \ldots, mN,\\ f(Z^i) \qquad\qquad\qquad\quad \text{if } i = M. \end{cases} \quad (55)$$

We point out that $\bar{Q}$ is a fixed set of greedy iteration indexes upon each of which a corresponding α_u^i value can be computed using already known information. This is because all the $f(\cdot)$ and $\Delta f(\cdot|\cdot)$ terms required to evaluate any α_u^i form in (55) are automatically computed during the execution of first i greedy iterations. Hence, α_u^i sequence over $i \in \bar{Q}$ can be thought of as a sequence of byproducts generated during (in parallel with) the execution of greedy iterations. In contrast to $\bar{Q}$, Q is an arbitrary subset selected from $\bar{Q}$ based on the user preference. For example, one can simply set $Q = \{1, N, N+1, 2N\}$ so that α_u value can be obtained upon executing only N extra greedy iterations (i.e., $2N$ greedy iterations in total).

The performance bound β_u corresponding to the extended greedy curvature measure α_u is given in the following theorem.

Theorem 2: For the submodular maximization problem in (**??**), the greedy solution Y^G given by Alg. 1 satisfies the performance bound β_u where

$$\beta_u \triangleq \frac{f(Y^G)}{\alpha_u} \leq \frac{f(Y^G)}{f(Y^*)}. \tag{56}$$

Proof: To prove this result, according to Lemma 3(a) and (54), we only need to show that $f(Y^*) \leq \alpha_u^i$ for all $i \in \bar{Q}$ (note also that $Q \subseteq \bar{Q}$). We do this in three steps.

First, by adding the main results established in Lemma 7 and Lemma 4 (i.e., (50) and (41), respectively) we get

$$f(Y^*) \leq f(Z^{nN}) + \max_{Y \in \mathscr{I}_n^N} \left[\sum_{y \in Y} \Delta f(y | Z^{nN}) \right], \tag{57}$$

for $n \in [0, m-1]$. Now, replacing the variable n with i using the substitution $i = nN + 1$ we obtain $f(Y^*) \leq \alpha_u^i$ for $i \in \{1, N+1, 2N+1, \ldots, (m-1)N+1\}$.

Second, using the main results of Lemmas 5 and 7, we get

$$f(Y^*) \leq f(Z^{nN}) + \frac{1}{\beta_f} \left[f(Z^{(n+1)N}) - f(Z^{nN}) \right], \tag{58}$$

for $n \in [0, m-1]$. Now, replacing the variable n with i using the substitution $i = nN + N$ we obtain $f(Y^*) \leq \alpha_u^i$ for $i \in \{N, 2N, 3N, \ldots, mN\}$.

Finally, we use the monotonicity property of f and the fact that $Y^* \subseteq Z^M = X$ to obtain $f(Y^*) \leq f(Z^i) = \alpha_u^i$ for $i = M$. ■

As mentioned earlier, the set Q used for computing α_u in (54) can be smaller compared to the set $\bar{Q}$ (specially if the computational cost associated with running extra greedy iterations is a concern). However, based on (54) and (56), it is easy to show that the performance bound β_u is a *monotone set function* in $Q \subseteq \bar{Q}$. This implies that a superset of Q will always provide a higher (or at least equal) β_u value compared to the β_u value corresponding to the set Q.

Let us now consider a case where the submodularity property of f is weak (i.e., f is close to being a modular set function, see also Rm. **??**). In a such setting, the α_u^1 value (from (55), with $i = 1$) can be simplified as

$$\begin{aligned} \alpha_u^1 &= f(Z^0) + \max_{Y \in \mathscr{I}_0^N} \left[\sum_{y \in Y} \Delta f(y | Z^0) \right] = \max_{Y \in \mathscr{I}^N} \left[\sum_{y \in Y} f(\{y\}) \right] \\ &= \max_{Y \in \mathscr{I}^N} f(Y) + \varepsilon = f(Y^*) + \varepsilon, \end{aligned} \tag{59}$$

where $\varepsilon \geq 0$ is a parameter that represents the strength of the submodularity property ($\varepsilon = 0$ if f is modular). The above expression implies that the evaluated α_u^1 value will be a tight upper bound for $f(Y^*)$ as f become more modular (i.e., as $\varepsilon \to 0$). Therefore, in a such setting, the corresponding performance bound β_u will also be tight (close to 1). In that sense, β_u behaves similar to the performance bounds $\beta_t, \beta_g, \beta_p$ discussed in the previous section.

Remark 2: The strength of the submodularity can formally be defined as an additive constant $\varepsilon_a \in \mathbb{R}_{\geq 0}$ or a multiplicative constant $\varepsilon_m \in \mathbb{R}_{\geq 0}$ applicable for the submodularity inequality provided in Def. 1(c). Based on this definition, it can be shown that the parameter ε we introduced in (59) (to represent the strength of the submodularity) satisfies $\varepsilon \geq \varepsilon_a$ and $\frac{\varepsilon}{f(Y^*)} \geq \varepsilon_m$.

On the other hand, let us now consider a case where the submodularity property of f is strong. In a such setting, according to the "diminishing returns" view of the submodularity property (see Def. 1(c)), $f(Z^i)$ should saturate quickly with respect to i. Keeping this in mind, let us consider the α_u^{2N} value (from (55), with $i = 2N$) that can be simplified as

$$\begin{aligned} \alpha_u^{2N} &= f(Z^N) + \frac{1}{\beta_f} \left[f(Z^{2N}) - f(Z^N) \right] \\ &= f(Y^G) + \frac{1}{\beta_f} \left[f(Z^{2N}) - f(Z^N) \right]. \end{aligned}$$

Note that when the submodularity property of f is strong, the above difference term $\left[f(Z^{2N}) - f(Z^N) \right]$ will become small. Therefore, $\alpha_u^{2N} \to f(Y^G)$ and the corresponding performance bound $\beta_u = \frac{f(Y^G)}{\alpha_u^{2N}} \to 1$ revealing a tight performance bound. In that sense, β_u behaves similar to the performance bound β_e discussed in the previous section.

Therefore, β_u is designed to have the best of both worlds while also being computationally inexpensive and having no additional technical limitations on its applicability. In the next section, we confirm these conclusions using numerical results generated from several different experiments.

V. Discussion

Additional remarks about the extended greedy curvature based performance bound:

Theoretical/practical implications, limitations and potential future research directions...

VI. Case Studies

Results from coverage control experiments, ...

Discuss potential applications from other domains like learning, data mining

VII. Conclusion

References

[1] E Boros, K Elbassioni, V Gurvich, and L Khachiyan. An Inequality for Polymatroid Functions and its Applications. *Discrete Applied Mathematics*, 131(2):255–281, 2003. doi:10.1016/S0166-218X(02)00455-9.

[2] Michele Conforti and Gérard Cornuéjols. Submodular Set Functions, Matroids and The Greedy Algorithm: Tight Worst-Case Bounds and Some Generalizations of The Rado-Edmonds Theorem. *Discrete Applied Mathematics*, 7(3):251–274, 1984.

[3] Yajing Liu, Edwin K P Chong, and Ali Pezeshki. Improved Bounds for The Greedy Strategy in Optimization Problems with Curvature. *Journal of Combinatorial Optimization*, 37(4):1126–1149, 2018.

[4] Yajing Liu, Edwin K P Chong, Ali Pezeshki, and Zhenliang Zhang. Submodular Optimization Problems and Greedy Strategies: A Survey. *arXiv e-prints*, page 1905.03308, 2019. URL: http://arxiv.org/abs/1905.03308, arXiv:1905.03308.

[5] G. L. Nemhauser, L A. Wolsey, and M.L. Fisher. An Analysis of Approximations for Maximizing Submodular Set Functions—I. *Mathematical Programming*, 14(1):265–294, 1978.

[6] Chuangchuang Sun, Shirantha Welikala, and Christos G. Cassandras. Optimal Composition of Heterogeneous Multi-Agent Teams for Coverage Problems with Performance Bound Guarantees. *Automatica*, 117:108961, 2020. doi:10.1016/j.automatica.2020.108961.

[7] Xinmiao Sun, Christos G Cassandras, and Xiangyu Meng. Exploiting Submodularity to Quantify Near-Optimality in Multi-Agent Coverage Problems. *Automatica*, 100:349–359, feb 2019.

[8] Zengfu Wang, Bill Moran, Xuezhi Wang, and Quan Pan. Approximation for Maximizing Monotone Non-Decreasing Set Functions with A Greedy Method. *Journal of Combinatorial Optimization*, 31(1):29–43, 2016.

[9] Shirantha Welikala and Christos G. Cassandras. Asymptotic Analysis for Greedy Initialization of Threshold-Based Distributed Optimization of Persistent Monitoring on Graphs. In *Proc. of 21st IFAC World Congress*, volume 53, pages 3433–3438, 2020. URL: https://www.sciencedirect.com/science/article/pii/S2405896320322734, doi:10.1016/j.ifacol.2020.12.1670.

[10] Shirantha Welikala and Christos G. Cassandras. Distributed Non-Convex Optimization of Multi-Agent Systems Using Boosting Functions to Escape Local Optima. *IEEE Trans. on Automatic Control*, 2020. doi:10.1109/TAC.2020.3034869.

[11] Shirantha Welikala, Christos G. Cassandras, Hai Lin, and Panos J. Antsaklis. A New Performance Bound for Submodular Maximization Problems and Its Application to Multi-Agent Optimal Coverage Problems. *Automatica*, 144:110493, 2022. doi:10.1016/j.automatica.2022.110493.

[12] Dengxiu Yu, Hao Xu, C L Philip Chen, Wenjie Bai, and Zhen Wang. Dynamic Coverage Control Based on K-Means. *IEEE Trans. on Industrial Electronics*, 69(5):5333–5341, 2022. doi:10.1109/TIE.2021.3080205.